HSSP Class 2

July 18, 2019

1 REVIEW

Recall the rules for factoring a polynomial $f(x)$:

Rule 1. *(1) Find its **zeros**(also called **roots**); let's denote them by $z_1 \cdots z_n$.*
(2) Write down the identity

$$f(x) = a_n(x - z_1) \cdots (x - z_n)$$

We know the two sides are equal for all x.

Euler tried to apply this to $sin(x)$, and wrote down the "identity"

$$\sin(x) = cx(x - \pi)(x + \pi)(x - 2\pi)(x + 2\pi) \cdots$$

However, he was confronted with the following problems:
(1) For $x = 1$, the right hand side is infinity; the same problem occurs for other values of x.
(2) The value of c should be the coefficient of the "highest degree term", but this doesn't make sense for $\sin(x)$. Therefore, Euler could not determine what c is.
(3) The same rule fails for other functions including $\frac{1}{x}$, which has no zero at all.

2 FACTORING $sin(x)$: A SECOND ATTEMPT

To address these problems, Euler made adjustions to the rules of factoring polynomial! More precisely, he rewrote the rules in an **equivalent form**. Let's use a quadratic polynomial $ax^2 + bx + c$ as an example. Using the original rule of factorization,

$$ax^2 + bx + c = a(x - z_1)(x - z_2)$$

Now, we divide a constant out of each term as follows:

$$x - z_1 = -z_1(1 - \frac{x}{z_1})$$

$$x - z_2 = -z_2(1 - \frac{x}{z_2})$$

Multiplying them, we get

$$f(x) = a(-z_1)(-z_2)(1 - \frac{x}{z_1})(1 - \frac{x}{z_2})$$

Now using a rule called **Vieta's formula**, or from the fact $z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$, we can actually know that

$$az_1z_2 = c$$

Thus we get a new identity, which will be our **new rule**

$$f(x) = c(1 - \frac{x}{z_1})(1 - \frac{x}{z_2})$$

But there is one complication: what if some of the z_i are 0? To tackle this, we separate the terms corresponding to these zero roots. Suppose $z_1 = 0$. Then, we must have

$$a * 0^2 + b * 0 + c = 0$$

Thus $c = 0$ and $z_2 = \frac{-b}{a}$. In this case

$$ax^2 + bx + c = ax^2 + bx = -bx(1 - \frac{x}{z_2})$$

If both roots are zero, then $f(x) = ax^2$.

For example, we have

$$x^2 - 100 = -100(1 - \frac{x}{10})(1 + \frac{x}{10})$$

$$4x^2 + 200x = 200x(1 + \frac{x}{50})$$

We can do the same thing for higher degree polynomials, and get the following more general rule:

Rule 2. *To factor a polynomial $f(x)$, suppose it has m roots equal to zero and $n - m$ non-zero roots $z_1 \cdots z_{n-m}$. Then*

$$f(x) = cx^m(1 - \frac{x}{z_1})(1 - \frac{x}{z_2}) \cdots (1 - \frac{x}{z_{n-m}})$$

where c is a real number.

For example, $x^3 - 9x$ can be written as

$$x^3 - 9x = -9x(1 - \frac{x}{3})(1 + \frac{x}{3})$$

2.1 Applying the new rule to $\sin x$

Euler tried to apply this new rule, and got:

$$\sin(x) = cx(1 - \frac{x}{\pi})(1 + \frac{x}{\pi})(1 - \frac{x}{2\pi})(1 + \frac{x}{2\pi}) \cdots$$

or more compactly,

$$\sin(x) = cx \prod_{n=1}^{\infty}(1 - \frac{x}{n\pi})(1 + \frac{x}{n\pi})$$

Now the question is: does the three problems that occur with the last formula still exist? i.e, we must show that

(1) c can be found.

(2) The "infinite product" on the right side can be actually evaluated to some finite value.

(3) The rule holds "in general".

We will answer (2) next class, and address (3) later. For now we answer (1). We have

$$\frac{\sin(x)}{x} = a_1(1 - \frac{x}{\pi})(1 + \frac{x}{\pi}) \cdots$$

Now we plug in $x = 0$. Ostensibly, the fraction on the right side is $\frac{0}{0}$, which is undefined. But if we actually draw a graph of $\frac{\sin(x)}{x}$, it will look like

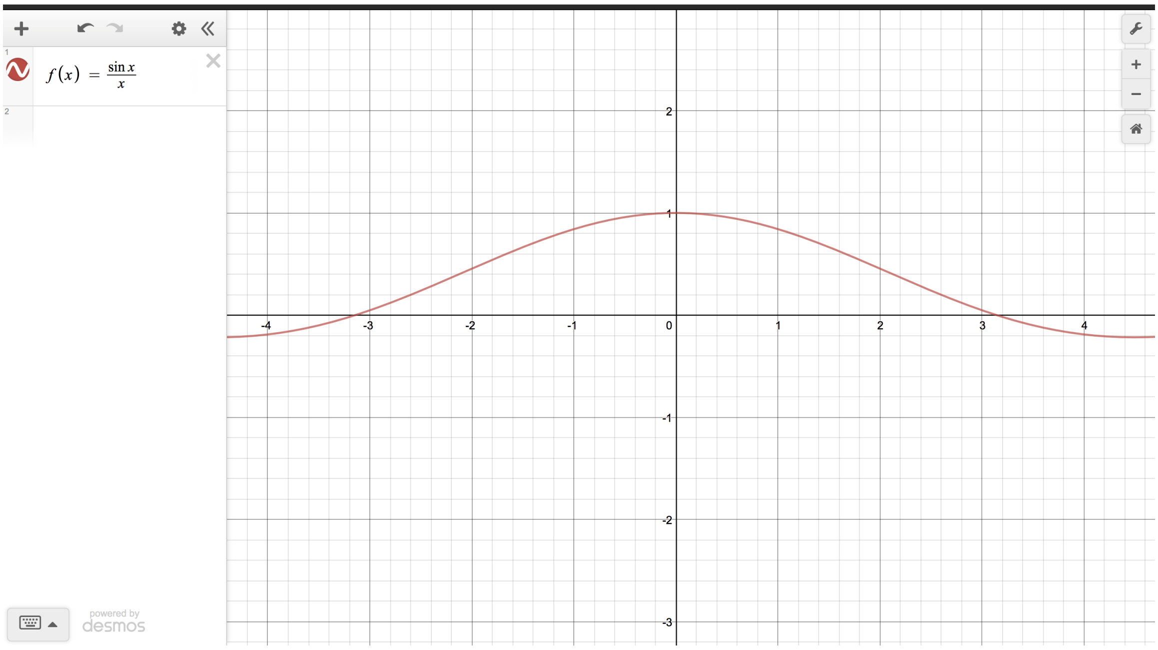

Figure 1: f(x) = $\sin(x)/x$

Observe how the graph behaves near 0; therefore, it makes sense to define $\frac{\sin(x)}{x} = 1$. An alternative way to see this is

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

On the other hand, the right side is just a_1, as all terms in the product becomes 1. Therefore

$$1 = a_1$$

Thus, Euler finally managed to write down his formula

$$\sin(x) = x \prod_{n=1}^{\infty} (1 - \frac{x}{n\pi})(1 + \frac{x}{n\pi})$$

Some experiment shows that this is a marvelous fit! For example, if we approximate the infinite product by taking 100 terms(like what we do with infinite sums), and graph the resulting function, we will get

Note how the two curves nearly overlap. Thus, there is a good chance that our formula is correct. But it takes more than approximation to convince mathematicians; for example, the product

$$\prod_{n=1}^{\infty} (1 + \frac{n}{100000000})$$

will "look like" 1 if we take the fist ten terms, but eventually goes to infinity. How do we show that our product is exactly $\sin(x)$ for all x? That's our focus for the rest of the class.

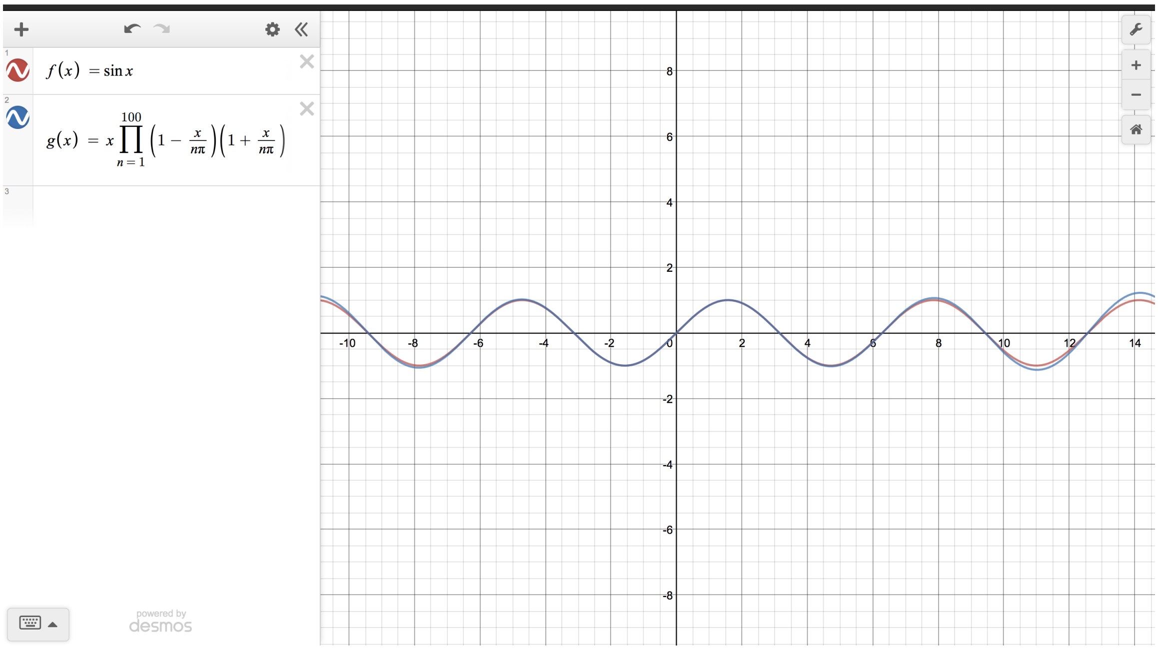

Figure 2: Comparison of $\sin(x)$(Red) and the product formula(Blue)