HSSP Class 5

1 A flaw in our argument

So Euler's formula is correct, and we got remarkable results from it! Everything seems to be good, right? But it turns out that there is one small problem that we overlooked.

Recall the first rule of factorization we stated:

Rule 1 *For a polynomial $f(x)$, (1) Find its* ***zeros****(also called* ***roots****); let's denote them by $z_1 \cdots z_n$.*

(2) Write down the identity

$$f(x) = a_n(x - z_1) \cdots (x - z_n)$$

We know the two sides are equal for all x.

Now let's use the rules to factor $f(x) = x^2 + x + 1$. In step (1) we should find its zeros; in other words, we solve

$$x^2 + x + 1 = 0$$

Plugging into the quadratic formula, we get

$$x_{1,2} = \frac{-1 \pm \sqrt{-3}}{2}$$

Note how there is a negative number under the square root.

In middle school, you might be taught that if the determinant is less than zero, then there is "no solution". This is correct if we restrict to real numbers. But as mathematics develop, mathematicians gradually realize that taking the square root of negative numbers is necessary in many situations. Eventually, a whole new theory is developed to describe such new numbers; they are called the **complex numbers**.

2 Arithmetic of complex number

As we all know, there is no real number a such that $a^2 = -1$. Thus, we need a "new number" to describe $\sqrt{-1}$. Mathematicians denote this number by the letter i. The number i adds and multiplies just like any real number, with the exception that $i^2 = -1$. For example,

$$(i + 1) + 2 = i + 3$$

$$(i + 1)(i + 2) = -1 + 3i + 2 = 3i + 1$$

With the new symbol i, we can now take the square root of any negative real number a. In fact,

$$\sqrt{-a} = \sqrt{a}i$$

Because

$$(\sqrt{a}i)^2 = ai^2 = -a$$

For example, the roots of $f(x) = x^2 + x + 1$ can be written as

$$x_{1,2} = \frac{-1 \pm \sqrt{3}i}{2}$$

3 Complete rule of factorization

As we see from the example of $f(x) = x^2 + x + 1$, the rule of factorization is not complete without complex numbers. And in fact, we can show that the rule is complete with complex numbers.

Rule 2 (The complete rule of factorization) *The following criterion factors all polynomials:*

(1) Find its ***complex zeros****; let's denote them by $z_1 \cdots z_n$.*

(2) Write down the identity

$$f(x) = a_n(x - z_1) \cdots (x - z_n)$$

We know the two sides are equal for all x.

For example,

$$x^2 + x + 1 = (x - \frac{1 + \sqrt{3}i}{2})(x - \frac{1 - \sqrt{3}i}{2})$$

4 Factorization of $sin(x)$, revisited

The rule we used to factor $sin(x)$ is flawed! Indeed, we were only considering the **real roots** of $sin(x)$. Now we need to find the complex ones as well. But there is a very serious problem: $sin(x)$ is undefined for complex x! For real numbers x, $sin(x)$ is defined as the ratio between two edges of a right triangle with one acute angle x. But there is simply no angle with degree i, or any number other than the reals.

Yet Euler made an important discovery that forever changed mathematics.

Rule 3 (Euler's formula) *The function $sin(x)$ can be defined for all complex numbers $x = a + bi$. In fact,*

$$\sin(a + bi) = \frac{e^{-b}(\cos(a) + \sin(a)i) - e^{b}(\cos(a) - \sin(a)i)}{2i}$$

which only involve exponential and trigonometry of real numbers.

This rule may look like insanity now; yet next class we'll see how this formula makes sense.