HSSP Class 3

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1 REVIEW

In the last two classes, we saw two attempts of Euler to generalize the rule for factoring polynomials. The first "formula" is

$$\sin x = cx(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)\cdots$$

While the second one is

$$\sin x = x\prod_{n=1}^{\infty}(1-\frac{x}{n\pi})(1+\frac{x}{n\pi})$$

As we explained last class, the top formula is wrong in the sense that the right hand side "goes to infinity". But what exactly do we mean by that? And how is the second formula different from the first? We'll look into this today.

2 DIVERGENCE AND CONVERGENCE

Below is a "proof" that $1 > 2$:

$$(1+2+3+4+\cdots) > (2+4+\cdots) = 2(1+2+\cdots)$$

Cancelling $1+2+\cdots$ from both side, we get $1 > 2$.

Of course, this is absurd. The reason is that the sum $1+2+\cdots$ **diverges**. That is to say, if we add the infinite summands one by one, we will not reach a definite value; in this case, our sum goes to infinity. To say that a sum diverges is to say that the sum does not make sense, and any calculations involving divergent sums is mathematically invalid.

On the other hand, sums like

$$\frac{1}{1*2}+\frac{1}{2*3}+\cdots$$

and

$$\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\cdots$$

both **converges**. That is to say, if we add the infinite summands one by one, in the end we will approach a definite value. In other words, an infinite sum $a_1+a_2+\cdots$ converges if and only if

$$\lim_{n\to\infty} a_1+a_2+\cdots+a_n$$

exists. In fact, the top sum is equal to 1 while the bottom sum is equal to $e = 2.718\cdots$.

Here are some rules for testing convergence and divergence:

Rule 1 (Converge to 0). *If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$($a_n$ will be very close to 0 when n is large).*

Rule 2 (Comparison). *If $\sum_{n=1}^{\infty} b_n$ converges, and $|a_n| \leq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ converges.*

Rule 3 (p-test). *For $p > 1$, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges; For $p \leq 1$, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.*

As an exercise, show that $\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$ converges, by applying the comparison rule together with the p-test for $p = 2$.

3 INFINITE PRODUCT

The **infinite product** of infinitely many numbers $z_1, z_2, \cdots$ is defined as

$$P = \prod_{n=1}^{\infty} z_n = \lim_{n\to\infty} z_1 z_2 \cdots z_n$$

The notion of convergence and divergence is almost the same as infinite sums, except for one complication: if none of a_i is zero, but the product goes to zero, then we consider the product divergent.

Example:

$$\prod_{n=1}^{\infty} 1 = 1$$

$$\prod_{n=1}^{\infty} (-1) \text{ diverges}$$

$$\prod_{n=1}^{\infty} \frac{n}{n+1} \text{ diverges}$$

3.1 Rules for testing convergence

The notion of convergence and divergence is similar to the same terminology in infinite sums. Only convergent series make sense in actual calculations, as most rules of arithmetic fail when there is "infinity" involved. But you might wonder why do we exclude the zero case in the infinite product. The intuition is that in arithmetic involving multiplication, 0 is kind of a exception. For example,

$$ab = ac \implies b = c$$

holds in all cases except when $a = 0$.

Another reason is that infinite series can be related to infinite sums through the **logarithm** operation. Recall that the function $\ln : \mathbb{R}^+ \to \mathbb{R}$ is defined as

$$\ln(x) = \int_1^x \frac{1}{t} dt$$

Or alternatively, the function that turns multiplication to addition

$$\ln(xy) = \ln(x) + \ln(y)$$

Rule 4. *$P = \prod_{n=1}^{\infty} a_n$ converges if and only if either one of the a_1 is zero, or $Q = \sum_{n=1}^{\infty} \log |a_n|$ converges and only finitely many a_i are negative.*

The $P = 0$ case is excluded because it would imply $Q = -\infty$.

From this rule, and the tests of convergence and divergence for infinite series, we can derive the following tests for the convergence and divergence of infinite series.

Rule 5 (Analog of Converge to 0 rule). *If* $P = \prod_{n=1}^{\infty} a_n$ *converges and none of* a_i *is zero, then* $\lim_{n\to\infty} a_n = 1$

A slightly more complicated argument shows that

Rule 6 (Analog of Comparison rule). *If* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\prod_{n=1}^{\infty}(1 - a_n)$ *also converges.*

4 SHOWING THAT EULER'S FORMULA MAKES SENSE

Now we apply the rules to the expansions

$$\sin x = cx(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)\cdots$$

$$\sin x = x\prod_{n=1}^{\infty}(1-\frac{x}{n\pi})(1+\frac{x}{n\pi})$$

and shows why Euler chose the second over the first.

Using Rule 5, as $|x - n\pi| \to \infty$, the first "formula" is diverges, thus cannot be used to get a meaningful result. On the other hand, using the identity

$$(1-\frac{x}{n\pi})(1+\frac{x}{n\pi}) = (1-\frac{x^2}{n^2\pi^2})$$

The second formula becomes

$$\sin x = x\prod_{n=1}^{\infty}(1-\frac{x^2}{n^2\pi^2})$$

Now if $x = n\pi$, then both sides are zero; otherwise, by p-series test,

$$\sum_{n=1}^{\infty}\frac{x^2}{n^2\pi^2}$$

converges. Therefore, by Rule 5, the infinite product in the formula converges, and we can actually do computation with this formula.