HSSP Class 4

1 REVIEW

In the last three classes, we saw how Euler wrote down his **product formula**

$$\sin(x) = x \prod_{n=1}^{\infty} (1 - \frac{x^2}{n^2 \pi^2})$$

We numerically verified that the right hand side looks like a good fit, then theoretically showed that we can actually do Calculation with this formula. In this class, we will see what the formula can tell us.

2 WALLIS' FORMULA

One of the most intuitive thing to do is to plug in $x = \frac{\pi}{2}$. Let's see what we can get

$$1 = \frac{\pi}{2} \prod_{n=1}^{\infty} \frac{4n^2 - 1}{4n^2}$$

This is called **Wallis' formula**. Rearranging this, we get

$$\frac{\pi}{2} = \frac{2 * 2}{1 * 3} * \frac{4 * 4}{3 * 5} * \cdots$$

Now, if we allow ourselves to be a bit informal, we can write the right hand side as

$$\frac{\pi}{2} = (\frac{2 * 4 * \cdots}{1 * 3 * 5 * \cdots})^2$$

In other word, *the ratio between the product of the even numbers and the odd numbers is* $\sqrt{\frac{\pi}{2}}$. Let's try another value, $x = \frac{\pi}{3}$. If we do a similar calculation, we get

$$\frac{2\pi}{3\sqrt{3}} = \frac{3 * 3}{2 * 4} * \frac{6 * 6}{5 * 7} * \cdots$$

This gives us relation between the product of numbers divisible by 3 and the product of numbers not divisible by 3.

3 EULER'S FORMULA

Euler took a different approach in analyzing his formula. He noticed that the right hand side is a polynomial. So why not write the left hand side as a polynomial as well. The question is, how can we write $\sin(x)$ as a polynomial?

The answer to this question was discovered a few years before Euler wrote down his product formula. It's called the **Taylor Expansion**:

Rule 1 (Taylor Expansion)**.** *For any function f, there is exactly one way f can be written as an infinite polynomial. Pick an $a \in \mathbb{R}$. Then,*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!} + \cdots$$

Where $f^{(n)}$ is the function obtained by differentiating f for n times.

In this class we will only need the special case $a = 0$, also called the **MacLaurin Expansion**:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots$$

Note that this is the standard form of a polynomial in x, only that it has infinitely many terms! We can manipulate it like normal polynomials. In this case, we have

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$$

Plugging into the product formula,

$$x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots = x\prod_{n=1}^{\infty}(1 - \frac{x^2}{n^2\pi^2})$$

Now we **compare Coefficients** of the x^3 term:

$$-\frac{1}{6} = -\frac{1}{1^2\pi^2} - \frac{1}{2^2\pi^2} - \cdots$$

Rearranging, we get

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

which is exactly what Euler wanted to show!