# HSSP Class 1

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## 1 INTRODUCTION

The main purpose of the class is to do calculations in Calculus. Throughout the study of Calculus, mathematicians have encountered many seemingly "inexplicable" identities, the most famous of which is

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

In this class we will learn about a marvelous idea of Euler to give an explanation of this identity.

## 2 FACTORIZATION OF POLYNOMIALS

In high school we learned the basic rules of how to factor polynomials. For example, to factor the polynomial $x^2 - x - 2$, we follow the steps:

(1) Write down the **equation** $x^2 - x - 2 = 0$.

(2) **Solve** the equation. In this case, the solutions are $x = 2$ and $x = -1$.

(3) Now we can write down the **identity**

$$x^2 - x - 2 = (x-2)(x-(-1)) = (x-2)(x+1)$$

One of the most important properties of polynomials is that the last formula is an **identity**. In other words, if we plug any value of $x$ into the left side of the equation, and plug the same value of $x$ into the right side of the equation, then we will get the same output.

We rewrite the steps above in a general fashion. All polynomials can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

Where the $a_i$ are numbers called **Coefficients**, and $x$ is the **Variable**. To factor $f$, we follow these steps:

(1) Find the **zeros** of the polynomials $z_1, z_2, \cdots, z_m$; these are the solutions to the equation $f(x) = 0$.

(2) We now have the identity

$$f(x) = a_n(x - z_1)(x - z_2)\cdots(x - z_m)\ldots\ldots(1)$$

.

We can get lots of information about $z_i$ from this identity. If we compare the **degree** of the polynomials on both side(the highest power $x$ is raised to), we get $n = m$. In other words, **The number of zeros of any polynomial is equal to its degree**. For example, if we see say $x^3 - x - 1 = 0$, then we immediately know that it has three zeros. In fact, they are

$$x_1 = 1.32472, x_2 = -0.662359 + 0.56228i, x_3 = -0.662359 - 0.56228i$$

# 3 FACTORING $sin(x)$: A FIRST ATTEMPT

Legends say that Euler applied the same process to $f(x) = \sin(x)$. The roots of $\sin(x)$ are $n\pi$ for $n = \cdots, -2, -1, 0, 1, 2, \cdots$; plugging this into (1),

$$\sin(x) \stackrel{?}{=} ax(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)\cdots$$

However, Euler realized that something is wrong. First of all, in (1), $a = a_n$ is the coefficient of the highest degree term $x^n$ of $f(x)$. But $\sin(x)$ has infinitely many zeros, so $a$ is the coefficient of $x^\infty$. But what is exactly this coefficient?

A second, more serious problem occurs when Euler tried to do calculations to verify that the identity indeed holds. He plugged in the value $x = 1$; the left hand side is $sin(1)$. But the right hand side becomes

$$a(1-\pi)(1+\pi)(1-2\pi)(1+2\pi)\cdots$$

Now, $1 + n\pi$ grows at the same speed as $n$, so the product goes to infinity. Thus,

$$\sin(1) = a\infty$$

This is absurd.

Finally, Euler realized that the rule for factoring polynomials cannot be extended to all functions. For example,

$$f(x) = \frac{1}{x+1}$$

has no zero at all; thus, if $f$ can be factored, then it must be a constant, contradiction. This further challenges the correctness of Euler's factorization.

However, Euler did not give up after this attempt. Instead, he tried to do something else.