

# Counting graphic sequences via integrated random walks

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## Abstract

Given an integer  $n$ , let  $G(n)$  be the number of integer sequences  $n - 1 \geq d_1 \geq d_2 \geq \dots \geq d_n \geq 0$  that are the degree sequence of some graph. We show that  $G(n) = (c + o(1))4^n/n^{3/4}$  for some constant  $c > 0$ , improving both the previously best upper and lower bounds by a factor of  $n^{1/4}$  (up to polylog-factors).

Additionally, we answer a question of Royle, extend the values of  $n$  for which  $G(n)$  is known exactly from  $n \leq 290$  to  $n \leq 1651$  and determine the asymptotic probability that the integral of a (lazy) simple symmetric random walk bridge remains non-negative.

## 1 Introduction

Given a graph  $G$  and a vertex  $v \in V(G)$ , the *degree* of  $v$  is the number of edges incident to  $v$ , and the *degree sequence* of  $G$  is the non-increasing sequence of its vertex degrees. We consider the following very natural question: over all graphs on  $n$  vertices, how many different degree sequences are there?

Since the degree of a vertex is at most  $n - 1$  and at least 0, a simple upper bound follows by bounding the number of integer sequences  $n - 1 \geq d_1 \geq d_2 \geq \dots \geq d_n \geq 0$ . A ‘stars-and-bars’ argument shows that there are  $\binom{2n-1}{n-1} = \Theta(4^n/\sqrt{n})$  such sequences, but not all of them are degree sequences of graphs. Sequences which are the degree sequence of some graph are called *graphic sequences*. A famous result of Erdős and Gallai [9]

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provides necessary and sufficient conditions for a sequence to be graphical and various other characterisations are known [15, 17].

Let  $G(n)$  be the number of graphic sequences of length  $n$  (or equivalently the number of degree sequences across graphs on  $n$  vertices). The best known bounds on  $G(n)$  were given by Burns [6] who showed that

$$\frac{c_1 4^n}{n} \leq G(n) \leq \frac{4^n}{\sqrt{n} \log^{c_2} n}$$

for some constants  $c_1, c_2 > 0$  and for all  $n \in \mathbb{N}$ . To the best of our knowledge, these were the best known asymptotics before our work, and this has been explicitly mentioned as an open problem in several computational papers (e.g. [24, 39]).

Our main result pinpoints the asymptotics for  $G(n)$ .

**Theorem 1.1.** *The number graphic sequences of length  $n$  is  $G(n) = (c_{\text{deg}} + o(1))4^n/n^{3/4}$ , where  $c_{\text{deg}} > 0$  is a constant.*

This also answers in the affirmative a question of Royle [32] who asked<sup>1</sup> whether the ratio  $G(n)/G(n-1)$  tends to 4.

**The value of  $c_{\text{deg}}$  and a connection to random walks.** We can express the value of the constant  $c_{\text{deg}}$  in terms of the hitting probabilities of a particular random walk, which arises from our proof strategy. In particular, computational estimates give  $c_{\text{deg}} \approx 0.099094$  (see Section 5.3).

We will prove Theorem 1.1 by viewing a sequence  $n-1 \geq d_1 \geq \dots \geq d_n \geq 0$  as a path on a grid from  $(0, n)$  to  $(n-1, 0)$ , as depicted in Figure 1(a). We then count the number of graphic sequences of length  $n$  by sampling such a path uniformly at random, and computing the probability that the path satisfies the conditions given by Erdős and Gallai for a sequence to be graphical.

To be more precise, a sequence is graphical if and only if the sum of the degrees is even and it satisfies the dominating condition given in (2.1). Our aim is to show that the probability that a random sequence satisfies the dominating condition is asymptotically  $4\sqrt{\pi}c_{\text{deg}}n^{-1/4}$ . Then, we show that, asymptotically, half of the sequences that satisfy the dominating condition have even sum, and the result then follows from the fact that we considered  $\binom{2n-1}{n-1} \approx (2\sqrt{\pi})^{-1}4^n n^{-1/2}$  sequences in total.

Via a number of reformulations (see Section 2.1), the probability that a uniformly random sequence satisfies the dominating condition turns out to be the probability that a particular integrated random walk bridge stays non-negative. Let  $Y = (Y_k)_{k \geq 0}$  be a random walk that has increments that take the value 1 with probability  $\frac{1}{4}$ , the value  $-1$  with probability  $\frac{1}{4}$  and the value 0 otherwise, so that  $Y$  is a *lazy simple symmetric random walk*. Let  $A_k = \sum_{j=1}^k Y_j$  be its area process. Our probability of interest is *the probability that  $A_1, \dots, A_{n-1} \geq 0$ , conditional on the event that  $Y_{n-1} \in \{0, -1\}$* . To introduce its asymptotic value, we need some additional notation. Let  $\zeta_1 = \inf\{k \geq 1 : Y_k = 0, A_k \leq 0\}$  be the first visit of  $Y$  to 0 at which  $(A_k)_{k=1}^\infty$  hits  $(-\infty, 0]$  and let  $\rho = \mathbb{P}(A_{\zeta_1} = 0)$ . We

<sup>1</sup>Royle actually asked whether  $G'(n)/G'(n-1) \rightarrow 4$  where  $G'(n)$  is the number of degree sequences of graphs without isolated vertices. However  $G'(n) = G(n) - G(n-1)$ , so this question is equivalent to the question of whether  $G(n)/G(n-1)$  tends to 4.

prove the following result about lazy simple symmetric random walk bridges which may be of independent interest.

**Proposition 1.2.** *We have that*

$$n^{1/4}\mathbb{P}(A_1, \dots, A_n \geq 0 \mid Y_n = 0) \rightarrow \frac{\Gamma(3/4)}{\sqrt{2\pi(1-\rho)}}$$

as  $n \rightarrow \infty$ .

The probability that a random process does not go negative is also called the *persistence probability*. The persistence probability of integrated random processes was first studied by Sinai [35] in 1992, who showed that the persistence probability of an  $n$ -step simple symmetric random walk (SSRW) is  $\Theta(n^{-1/4})$ . The sharp asymptotics (including the constant) follow from a result by Vysotsky [38, Theorem 1]. His work on random walk bridges implies that the persistence probability of an  $n$ -step SSRW bridge is  $\Theta(n^{-1/4})$  [38, Proposition 1]. The sharp asymptotics for SSRW bridges are a natural next question, which we answer in Proposition 6.3 for the SSRW bridge and in Proposition 1.2 for the lazy variant.

We use Proposition 1.2 to show that  $G(n) = (c_{\text{deg}} + o(1))4^n/n^{3/4}$  for

$$c_{\text{deg}} = \frac{\Gamma(3/4)}{4\pi\sqrt{2(1-\rho)}}.$$

The probability generating function of the area of the first excursion of  $Y$  away from 0 satisfies a recursive equation which allows us to find that  $\rho$  is approximately 0.5158026, and plugging this into the equation gives  $c_{\text{deg}} \approx 0.099094$ .

**Related counting problems.** Much more is known about related counting problems, such as the number of graphs with a given degree sequence [4, 42] and a variant of our problem where the sequence does not need to be non-increasing (e.g. [36]).

The number  $T(n)$  of out-degree sequences for  $n$ -vertex tournaments, also called score sequences, has received particular interest, and the problem of determining  $T(n)$  can be traced back to MacMahon in 1920 [25]. Following work of Moser [28], Erdős and Moser (see [27]), and Kleitman [22], it was shown that  $T(n) = \Theta(4^n/n^{5/2})$  by Winston and Kleitman [41] (lower bound) and Kim and Pittel [21] (upper bound). Recently, Kolesnik [23] determined the exact asymptotics, showing that there is a constant  $c \approx 0.392$  such that  $T(n) = (c + o(1))4^n/n^{5/2}$ .

Another well-studied variant is the fraction  $p(N)$  of partitions of an integer  $N$  that are graphical. This corresponds to the variant of our problem where we fix the number of edges of the graph, rather than the number of vertices. In 1982, Wilf conjectured that  $p(N) \rightarrow 0$  as  $N \rightarrow \infty$ . Pittel [30] resolved this problem in the affirmative using a Kolmogorov zero-one law, and Erdős and Richmond [10] showed a lower bound of  $p(N) \geq \pi/\sqrt{6N}$  for sufficiently large even  $N$ . The best-known upper bound is  $p(N) = O(N^{-\alpha})$  for  $\alpha \approx 0.003$  from Melczer, Michelen and Mukherjee [26].

**Exact enumeration.** We also give an improved algorithm for the exact enumeration of graphic sequences and calculate the number of graphic sequences of length  $n$  for all  $n$  up to 1651.

There is previous work on enumerating graphic sequences [18, 19, 20, 33, 40], and the numbers were known up to  $n = 118$  [39] as OEIS sequence A004251, although the numbers  $G'(n)$  of zero-free graphic sequences were known up to  $n = 290$  [40], and these imply the values of  $G(n)$  up to the same number by the observation in footnote 1.

**Paper overview.** In Section 2, we provide the reformulation of our problem in terms of integrated random walks. In Section 3, we first show that  $G(n) = \Theta(4^n/n^{3/4})$ , and then prove Theorem 1.1 and Proposition 1.2, up to technical lemmas that we postpone to Section 4. In Section 5 we introduce an improved algorithm for computing  $G(n)$  exactly for small  $n$  and discuss computational results such as the approximation of  $\rho$ . We conclude with some open problems in Section 6. An overview of the notation used throughout the paper is given in Appendix A.

## 2 Reformulation and notation

In order to prove our results, we need a suitable criterion for when a sequence of non-negative integers is the degree sequence of a simple graph. For a given sequence of non-negative integers  $d_1 \geq d_2 \geq \dots \geq d_n \geq 0$ , let  $d'_i$  be the number of  $j$  with  $d_j \geq i$  and set

$$\begin{aligned} s_i &= (n-1) - d_i, \\ s'_i &= n - d'_i. \end{aligned}$$

Let  $\ell$  be the largest  $j$  such that  $d_j \geq j$ . Using these definitions we can now give the version of the Erdős–Gallai Theorem that we will use.

**Theorem 2.1** (Variant of Erdős–Gallai). *A sequence of integers  $d_1 \geq d_2 \geq \dots \geq d_n \geq 0$  is the degree sequence of a simple graph if and only if the sum  $\sum_{i=1}^n d_i$  is even and for all  $k \leq \ell$*

$$\sum_{i=1}^k s_i \geq \sum_{i=1}^k s'_i. \quad (2.1)$$

We remark that this is just one of many similar characterizations of graphic sequences and that this form follows from the classical statement of the Erdős–Gallai Theorem by rearranging terms and observing that only the first  $\ell$  conditions need be checked (see e.g. [16, 29, 31, 34]).

We will call (2.1) the *dominating condition*, so that a sequence of non-negative integers is a graphic sequence if it satisfies the dominating condition and its sum is even. As might be expected, it turns out that about half of the sequences that satisfy the dominating condition have an even sum although, surprisingly, the exact proportion seems to converge to  $1/2$  quite slowly as  $n \rightarrow \infty$  (see Section 5.2). We will first focus on counting the number of sequences that satisfy the dominating condition, and we will handle the parity condition in Section 4.3 (and briefly in the proof of Proposition 3.3).

## 2.1 It's a walk!

We now describe a way of associating a random walk to a uniformly random graphic sequence in such a way that we can easily check whether the sequence is graphic using only the walk.

**Lemma 2.2.** *Let  $(Y_i)_{i \geq 1}$  be a lazy simple symmetric random walk and let  $A_k = \sum_{i=1}^k Y_i$ . Then the probability that a uniformly random sequence  $n-1 \geq d_1 \geq \dots \geq d_n \geq 0$  satisfies the dominating condition (2.1) is equal to*

$$\mathbb{P}(A_1, \dots, A_{n-1} \geq 0 \mid Y_{n-1} \in \{0, -1\}). \quad (2.2)$$

The probability that it is graphic is

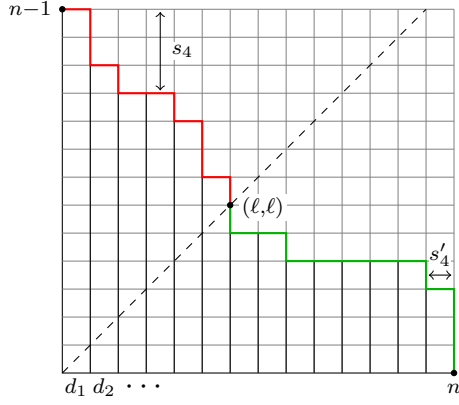
$$\mathbb{P}(A_1, \dots, A_{n-1} \geq 0, A_{n-1} \in 2\mathbb{N} \mid Y_{n-1} \in \{0, -1\}). \quad (2.3)$$

This lemma already gives some intuition for why  $G(n) = \Theta(4^n/n^{3/4})$ . Note that there are  $\binom{2n-1}{n} = \Theta(4^n/\sqrt{n})$  sequences  $n-1 \geq d_1 \geq \dots \geq d_n \geq 0$ , so we want (2.2) to be  $\Theta(n^{-1/4})$ . Firstly, note that we only need to check that  $A_k \geq 0$  for  $k$  such that  $Y_k = 0$  (and for  $k = n-1$ ), because  $A_k$  is monotone on excursions of  $Y$  away from 0. Up to time  $n$ , the walk  $Y$  visits zero  $\Theta(n^{1/2})$  times (in probability) so the area process restricted to times  $k$  when  $Y_k = 0$  is a random process with  $\Theta(n^{1/2})$  steps. If this process were a random walk, then a result from Feller's book [11, Theorem XII.7.1a] would tell us that the probability that it stays non-negative is  $\Theta((n^{1/2})^{-1/2}) = \Theta(n^{-1/4})$ . We also require that the total area  $A_{n-1}$  is even, but intuitively this should happen with probability roughly 1/2. Of course, the lengths (and hence the areas) of the excursions are not independent and we cannot apply Feller's result, but this heuristic does at least suggest the right order for (2.2).

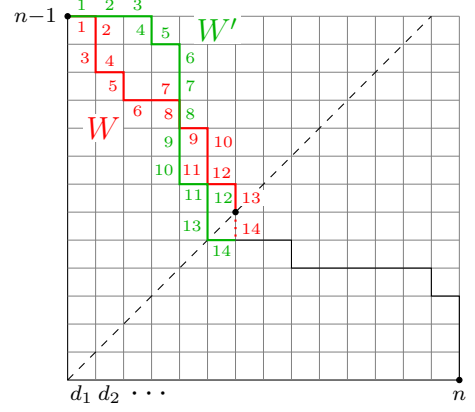
We will now prove Lemma 2.2. In order to get from a uniformly random non-increasing sequence to a lazy random walk conditioned to end in  $\{0, -1\}$ , we need to make a few reformulations.

**Step 1: View a sequence as a lattice path.** We start by viewing the non-increasing sequence as a lattice path  $P$  from  $(0, n-1)$  to  $(n, 0)$  which only takes steps to the right and downwards (see Figure 1a). Informally, we put  $n$  stacks of heights  $d_1, d_2, \dots, d_n$  respectively next to each other and let the  $P$  be the path from  $(0, n-1)$  to  $(n, 0)$  that traces the outline of the stacks. To be precise, the path begins with  $n-1-d_1$  steps downwards before taking a step right. For each  $2 \leq i \leq n$ , the walk takes  $d_{i-1}-d_i$  steps downwards on the line  $x = i-1$  before taking a step right, and it ends with  $d_n$  steps downwards to end at the point  $(n, 0)$ . There must be a unique  $\ell$  such that the walk goes through the point  $(\ell, \ell)$ , and it is not hard to see that  $\ell$  is the largest  $j$  such that  $d_j \geq j$ .

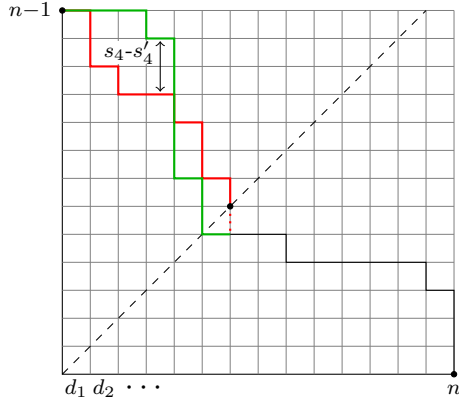
**Step 2: Cut the path in half and reflect the latter half.** Starting from  $(0, n-1)$ , the path takes  $n-1$  steps to reach the point  $(\ell, \ell)$  as each step decreases  $y-x$  by 1, and we define a path  $W$  which starts with these  $n-1$  steps before ending with one final step down. We now define another path  $W'$  starting at  $(0, n-1)$  using the other end of the path  $P$ . Starting at  $(n, 0)$ , walk backwards along the path  $P$  to  $(\ell, \ell)$  and, at each step, add the "reflected" step to  $W'$ : if the step on  $P$  is vertical, add a right step to  $W'$ ; otherwise add a down step to  $W'$ . This gives two paths  $W$  and  $W'$  which take  $n$  steps



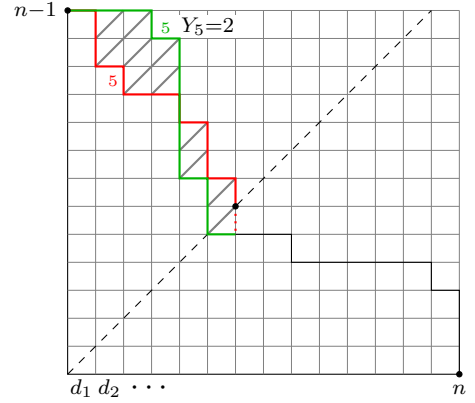
(a) Step 1: View a sequence as a lattice path.



(b) Step 2: Cut the path in half and reflect the latter half.



(c) Step 3: The dominating condition corresponds to the signed area between the pair of paths never going negative.



(d) Step 4: The cumulative distance process between the two paths gives the area.

Figure 1: By using a number of reformulations, we show that the probability that a uniformly random sequence  $n - 1 \geq d_1 \geq d_2 \geq \dots \geq d_n \geq 0$  satisfies the dominating condition is equal to the probability that the integral of a lazy simple symmetric random walk with  $n - 1$  steps, conditioned to end in 0 or  $-1$ , does not go negative.

and both end at  $(\ell, \ell - 1)$  (see Figure 1b, in which the steps on the two paths have been numbered).

**Step 3: The dominating condition corresponds to the signed area between the pair of paths never going negative.** If the walk  $W$  (resp.  $W'$ ) has a horizontal line from  $(i - 1, a)$  to  $(i, a)$ , then  $s_i = n - 1 - a$  (resp.  $s'_i = n - 1 - a$ ). In particular, if  $W$  is at  $(i, a)$  after  $k$  steps, the sum  $\sum_{j=1}^i s_j$  is exactly the area enclosed by the walk  $W$ , the line  $y = n - 1$  and the line  $x = i$ . Similarly, if  $W'$  is at  $(i, a)$  after  $k$  steps, the sum  $\sum_{j=1}^i s'_j$  is exactly the area enclosed by the walk  $W'$ , the line  $y = n - 1$  and the line  $x = i$ . Therefore, if  $W$  and  $W'$  are both at  $(i, a)$  after  $k$  steps, then the signed area between  $W'$  and  $W$  is exactly the sum  $\sum_{j=1}^i (s_j - s'_j)$  (see Figure 1c). The dominating condition checks if this sum is non-negative for all  $i$ .

**Step 4: The cumulative distance process between the two paths gives the area.** Define  $Z_1, \dots, Z_n$  by setting  $Z_i$  equal to  $+1$  if the walk  $W$  goes down at the  $i$ th step, and  $-1$  if it goes right. Similarly, define  $Z'_1, \dots, Z'_n$  by setting  $Z'_i$  equal to  $-1$  if the walk  $W'$  goes down at the  $i$ th step, and  $+1$  otherwise. Then,  $Y_i = \frac{1}{2} \sum_{j=1}^i (Z_j + Z'_j)$  keeps track of the (signed) number of diagonal right/up steps from the walk  $W$  to the walk  $W'$ . When the two walks coincide at time  $k$ , there will be a diagonal line through every box between the two walks, and hence the number of diagonal lines is equal to the signed area between  $W'$  and  $W$  up to time  $k$  (see Figure 1d). We claim that we only need to check the dominating condition when the two walks coincide. Indeed, between times at which the walks coincide, the area process is monotone so if the area process first becomes negative at some time  $i$  where the walks do not coincide, the process will still be negative when the walks next coincide. However, the number of diagonal lines used up to time  $k$  is the integral of  $Y$  up to time  $k$ , and it suffices to check that condition  $\sum_{i=1}^k Y_i \geq 0$  at all times  $k$  where the walks coincide. Since the integral of  $Y$  is also monotone on excursions away from zero (i.e. between times when  $W$  and  $W'$  coincide), the condition (2.1) is equivalent to  $\sum_{i=1}^k Y_i \geq 0$  for all  $k \leq n$ . But  $Y_n = 0$ , so it is in fact sufficient that

$$\sum_{i=1}^k Y_i \geq 0 \quad \text{for all } k \leq n - 1. \quad (2.4)$$

**Step 5: The parity condition corresponds to the integral of  $Y$  being even** We now claim that

$$\sum_{i=1}^n d_i \equiv \sum_{i=1}^{n-1} Y_i \pmod{2}. \quad (2.5)$$

Split the sum  $\sum_{i=1}^n d_i$  into  $A_{\text{red}} = \sum_{i=1}^{\ell} d_i$  and  $A_{\text{green}} = \sum_{i=\ell+1}^n d_i$ . In Figure 1a,  $A_{\text{red}}$  is the area below the red curve (left of  $(\ell, \ell)$ ) and  $A_{\text{green}}$  is the area below the green curve (right of  $(\ell, \ell)$ ). Clearly,  $A_{\text{red}}$  is also the area under the walk  $W$ . The area under the walk  $W'$  is given by  $A_{\text{green}} + \ell(\ell - 1)$ . Hence, the signed area enclosed by  $W'$  and  $W$  is given by

$$(A_{\text{green}} + \ell(\ell - 1)) - A_{\text{red}} \equiv A_{\text{green}} + A_{\text{red}} \pmod{2}.$$

It was shown in the previous step that the signed area enclosed by  $W'$  and  $W$  is given by  $\sum_{i=1}^n Y_i = \sum_{i=1}^{n-1} Y_i$ , and the claim follows.



**Step 6: The distance process between a random pair of paths is a conditioned lazy random walk.** Finally, we need to understand the distribution of  $Y$  when we sample a sequence  $n - 1 \geq d_1 \geq \dots \geq d_n \geq 0$  uniformly at random. First, observe that Step 2 gives a bijection between such non-increasing sequences and pairs of lattice paths  $(W, W')$  of  $n$  steps that start at  $(0, n - 1)$  and end at the same point, and such that  $W$  ends by taking a downwards step. These are in turn in bijection with pairs of lattice paths  $(W, W')$  with  $n - 1$  steps that start at  $(0, n - 1)$  and for which the corresponding walk  $(Y_k)_{k=0}^{n-1}$  has  $Y_{n-1} \in \{0, -1\}$ . Therefore, sampling a uniformly random sequence corresponds to sampling a uniformly random pair of such paths. If we ignore the requirement that  $Y_{n-1} \in \{0, -1\}$ , then for any  $j$ ,  $Z_j$  and  $Z'_j$  have opposite signs with probability  $1/2$ , and both have value  $+1$  (or  $-1$ ) with probability  $1/4$ , making  $Y$  a lazy simple symmetric random walk. To recover the actual distribution of (the first  $n - 1$  steps of)  $Y$ , we need to restrict to the paths for which  $Y_{n-1} \in \{0, 1\}$ , and we simply condition on this being the case.

We remark that steps 1 through 5 give a deterministic mapping which maps a sequence  $n - 1 \geq d_1 \geq \dots \geq d_n \geq 0$  to a walk  $(Y_i)_{i=1}^n$  and we have shown that we can check if the sequence is graphic by checking certain properties of the walk. In Section 5, we use this reformulation (without introducing randomness) to enumerate the number of graphic sequences for small  $n$ .

### 3 Proof of main result

In this section, we prove Theorem 1.1, which is a direct consequence of the following two results.

**Proposition 3.1.** *A uniformly random sequence  $n - 1 \geq d_1 \geq \dots \geq d_n \geq 0$  has probability*

$$(1 + o(1)) \frac{\Gamma(3/4)}{\sqrt{2\pi(1 - \rho)}} n^{-1/4}$$

*of satisfying the dominating condition (2.1).*

**Lemma 3.2.** *A uniformly random sequence  $n - 1 \geq d_1 \geq \dots \geq d_n \geq 0$  which satisfies the dominating condition (2.1), has probability  $1/2 + o(1)$  of being a graphic sequence (equivalently, of having  $\sum_{i=1}^n d_i$  even).*

The remainder of this section is structured as follows. In Section 3.1, we show that  $G(n) = \Theta(4^n n^{-3/4})$ . Then, in Section 3.2 we introduce a lemma (due to Burns [6]) on exchangeable random sequences that turns out to be very useful in our proofs. Finally, in Section 3.3, we use this lemma to prove Proposition 1.2, which states that as  $n \rightarrow \infty$ ,

$$n^{1/4} \mathbb{P}(A_1, \dots, A_n \geq 0 \mid Y_n = 0) \rightarrow \frac{\Gamma(3/4)}{\sqrt{2\pi(1 - \rho)}}.$$

We then deduce Proposition 3.1 from this in Section 4.2. The proofs of Proposition 3.1, Lemma 3.2, and the lemmas used in the proof of Proposition 1.2 are postponed to Section 4. An overview of the notation is given in Appendix A.



### 3.1 First asymptotics

In this section, we determine the asymptotics of the number of sequences that satisfy the dominating condition, as well as  $G(n)$ , up to a constant factor.

**Proposition 3.3.** *The number of graphic sequences of length  $n$  is  $G(n) = \Theta(4^n/n^{3/4})$ .*

The following proposition is a key ingredient in the proof of the above proposition.

**Proposition 3.4.** *For  $(Y_i)_{i=1}^n$  the lazy SSRW and  $A_k = \sum_{i=1}^k Y_i$  its area process,*

$$\mathbb{P}(A_1, \dots, A_n \geq 0 \mid Y_n = 0) = \Theta(n^{-1/4}).$$

This proposition is a direct consequence of results from Vysotsky [38] on positivity of integrated (right-continuous) random walks. Let  $Y = (Y_k)_{k \geq 0}$  be a random walk and let  $\mathcal{D}(Y)$  be the times for which the walk has positive probability of being at 0, i.e.  $\mathcal{D}(Y) = \{n : \mathbb{P}(Y_n = 0) > 0\}$ . If the walk  $Y$  is integer valued, we will say it is *right-continuous* (or *upwards skip-free*) if  $Y_{i+1} \leq Y_i + 1$  almost surely.

**Theorem 3.5** (Proposition 1 of [38]). *Let  $Y = (Y_k)_{k \geq 0}$  be an integer valued random walk starting at 0 with  $\mathbb{E}[Y_1] = 0$  and  $\text{Var}(Y_1) < \infty$ . Write  $A_k = \sum_{i=1}^k Y_i$ . Then, for  $n \in \mathcal{D}(Y)$ , we have*

$$\mathbb{P}(A_1, \dots, A_n > 0 \mid Y_n = 0) = O(n^{-1/4}).$$

Furthermore, if  $Y$  is also right-continuous, then for  $n \in \mathcal{D}(Y)$ ,

$$\mathbb{P}(A_1, \dots, A_n > 0 \mid Y_n = 0) = \Theta(n^{-1/4}).$$

Proposition 3.4 follows by combining this theorem with some simple observations. Indeed, the lower bound follows since the lazy SSRW is clearly right-continuous and the event  $\{A_1, \dots, A_n > 0\}$  is a subset of the event  $\{A_1, \dots, A_n \geq 0\}$ . For the upper bound, consider only the walks beginning with an upwards step and then a downwards step. After these first two steps the walk behaves like a lazy SSRW of length  $n - 2$  conditioned to end at 0, but the partial sums are one higher. This means the event  $\{\sum_{i=1}^k Y_i > 0, k = 1, \dots, n\}$  is exactly the event  $\{\sum_{i=3}^k Y_i \geq 0, k = 3, \dots, n\}$ . A simple calculation shows that the probability the walk goes up and then down given that  $Y_n = 0$  is at least  $1/16$  for all  $n \geq 2$ . It follows that

$$\mathbb{P}(A_1, \dots, A_n > 0 \mid Y_n = 0) \geq \frac{1}{16} \mathbb{P}(A_1, \dots, A_{n-2} \geq 0 \mid Y_{n-2} = 0),$$

which shows the upper bound.

We cannot immediately apply the proposition to calculate the order of growth of the number of graphic sequences as we need to condition on the walk ending at either 0 or  $-1$ , not only 0. The following lemma shows that this change in conditioning only changes the probability by a constant factor.

**Lemma 3.6.** *For all  $n \geq 1$ ,*

$$\frac{1}{2} \leq \frac{\mathbb{P}(A_1, \dots, A_n \geq 0 \mid Y_n \in \{0, -1\})}{\mathbb{P}(A_1, \dots, A_n \geq 0 \mid Y_n = 0)} \leq 1.$$

The proof of Lemma 3.6 is elementary, but for the sake of brevity we postpone it to Section 4.

*Proof of Proposition 3.3.* By Lemma 2.2, the probability that a uniformly random sequence  $n-1 \geq d_1 \geq \dots \geq d_n \geq 0$  satisfies the dominating condition (2.1) is equal to  $\mathbb{P}(A_1, \dots, A_{n-1} \geq 0 \mid Y_{n-1} \in \{0, -1\})$ , which is  $\Theta(n^{-1/4})$  by Proposition 3.4 and Lemma 3.6. Hence, the number of sequences which satisfy the dominating condition is  $\Theta(4^n/n^{3/4})$ .

Clearly,  $G(n) = O(4^n/n^{3/4})$  is immediate, and we only need to show a corresponding lower bound. Let  $H(n)$  denote the number of sequences  $(d_i)_{i=1}^n$  that satisfy the dominating condition for which  $\sum_{i=1}^n d_i$  is odd. The number of sequences of length  $n$  that satisfy the dominating condition is therefore  $G(n) + H(n) = \Theta(4^n/n^{3/4})$ . Each sequence  $(d_i)_{i=1}^n$  in  $G(n)$  gives rise to a unique sequence in  $G(n+1)$  by appending  $d_{n+1} = 0$ . Separately, each sequence  $(d_i)_{i=1}^n$  in  $H(n)$  gives rise to a unique sequence in  $G(n+1)$  by adding an extra 1 to the graphic sequence (in the appropriate place). Hence,

$$G(n+1) \geq \max\{G(n), H(n)\} \geq \frac{1}{2}(G(n) + H(n)).$$

Hence,  $G(n) \geq \frac{1}{2}(G(n-1) + H(n-1)) = \Theta(4^n/n^{3/4})$  as required.  $\square$

### 3.2 A useful lemma

We make use of the following lemma that appears in Sections 2.3–2.4 of [6]. We include a proof in the appendix for completeness.

**Lemma 3.7.** *Let  $x = (x_1, \dots, x_n) \in \mathbb{R}_{\geq 0}^n$ , let  $\sigma$  be a uniformly random permutation of  $[n]$  and let  $s = (s_1, \dots, s_n)$  be an independent uniformly random element of  $\{-1, 1\}^n$ . Then*

$$\mathbb{P}\left(\sum_{i=1}^k s_i x_{\sigma(i)} \geq 0 \text{ for all } k \in [n]\right) \geq \frac{(2n-1)!!}{2^n n!}.$$

*Moreover, equality holds if for all distinct  $A, A' \subseteq [n]$ , the corresponding sums are also distinct:  $\sum_{i \in A} x_i \neq \sum_{i \in A'} x_i$ .*

We will apply this lemma to sequences of exchangeable random variables of which the law is invariant under sign changes of the elements, so we use the following equivalent form of the lemma.

**Lemma 3.8.** *Let  $(X_1, \dots, X_n)$  be a random variable in  $\mathbb{R}^n$  such that for any  $\sigma \in S_n$  and any  $s \in \{-1, 1\}^n$ ,*

$$(X_1, \dots, X_n) \stackrel{d}{=} (s_1 X_{\sigma(1)}, \dots, s_n X_{\sigma(n)}).$$

*Then,*

$$\mathbb{P}\left(\sum_{i=1}^k X_i \geq 0 \text{ for all } k \in [n]\right) \geq \frac{(2n-1)!!}{2^n n!},$$

*and equality holds if, almost surely, for all distinct  $A, A' \subseteq [n]$ , the corresponding sums are also distinct:  $\sum_{i \in A} X_i \neq \sum_{i \in A'} X_i$ .*

To see that Lemma 3.8 indeed implies Lemma 3.7, observe that for  $x$ ,  $s$  and  $\sigma$  as in the statement of Lemma 3.7, the random variable  $(X_1, \dots, X_n) := (s_1 x_{\sigma(1)}, \dots, s_n x_{\sigma(n)})$  satisfies the conditions of Lemma 3.8.

*Proof of Lemma 3.8.* We observe that, for  $\sigma$  a uniformly random permutation of  $[n]$  and  $s = (s_1, \dots, s_n)$  an independent uniformly random element of  $\{-1, 1\}^n$ ,

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^k X_i \geq 0 \text{ for all } k \in [n]\right) &= \mathbb{P}\left(\sum_{i=1}^k s_i X_{\sigma(i)} \geq 0 \text{ for all } k \in [n]\right) \\ &= \mathbb{E}\left[\mathbb{P}\left(\sum_{i=1}^k s_i X_{\sigma(i)} \geq 0 \text{ for all } k \in [n] \mid X_1, \dots, X_n\right)\right], \end{aligned}$$

so that the result follows by applying Lemma 3.7 to

$$\mathbb{P}\left(\sum_{i=1}^k s_i X_{\sigma(i)} \geq 0 \text{ for all } k \in [n] \mid X_1, \dots, X_n\right). \quad \square$$

We make the following observation about the value of the lower bound in Lemmas 3.7 and 3.8. Note that by taking either an even or odd number of terms in Wallis's product formula

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdots$$

one obtains alternately lower and upper bounds for  $\frac{\pi}{2}$ . Rearranging these inequalities gives that for all  $n \geq 1$

$$\frac{1}{\sqrt{\pi(n+1/2)}} \leq \frac{(2n-1)!!}{2^n n!} = \binom{2n}{n} \frac{1}{4^n} \leq \frac{1}{\sqrt{\pi n}}. \quad (3.1)$$

### 3.3 Proof of Proposition 1.2

Let  $Y^{br} = (Y_k^{br})_{k=1}^n$  be the first  $n$  steps of the lazy SSRW  $Y = (Y_k)_{k \geq 1}$  conditioned on the event  $Y_n = 0$ , and let  $N_n$  be the number of times that the walk  $Y^{br}$  hits 0. Condition on the event that  $N_n = N$ . Then the walk  $Y^{br}$  has  $N$  excursions with areas  $X_1^{br}, X_2^{br}, \dots, X_N^{br}$  say. Since the walk  $A_k^{br} = \sum_{i=1}^k Y_i^{br}$  is monotone during the individual excursions, the walk  $S^{br} = (S_k^{br})_{k=1}^N$ , defined by  $S_k^{br} = \sum_{i=1}^k X_i^{br}$ , is never negative if and only if the walk  $A^{br}$  is never negative.

Ideally, we would like to use Lemma 3.8 to estimate the probability that  $S^{br}$  never goes negative. Indeed, the summands of  $S^{br}$  are exchangeable and their law is invariant under sign changes. However, the probability given by Lemma 3.8 is only exact when distinct sums of elements in  $(X_i^{br} : 1 \leq i \leq N_n)$  are distinct almost surely, and this is in general not the case. We therefore perturb each  $X_i^{br}$  by a small random amount, so that the sums are distinct almost surely and the probability from Lemma 3.8 is exact. To be precise, let  $\varepsilon_i \sim \text{Uniform}[-\frac{1}{2n}, \frac{1}{2n}]$  and define  $\tilde{X}_i^{br} = X_i^{br} + \varepsilon_i$  and  $\tilde{S}_k^{br} = \sum_{i=1}^k \tilde{X}_i^{br}$ . Now,  $\tilde{S}^{br}$  satisfies the conditions of the equality in Lemma 3.8 so the probability that it never goes negative (conditional on the event  $N_n = N$ ) is exactly

$$\frac{(2N-1)!!}{2^N N!}.$$

The random variable  $n^{-1/2}N_n$  converges in distribution to a Rayleigh( $\sqrt{2}$ ) distribution, which we write as  $2\sqrt{E}$  where  $E \sim \text{Exp}(1)$  has a standard exponential distribution, so one hopes that we can replace  $N$  with  $2\sqrt{nE}$  in the probability above. It requires some extra work to ensure that  $n^{-1/2}N_n$  does not put too much mass near 0, but Lemma 4.3 says that we do indeed get what one would expect. That is,

$$\mathbb{P}(\tilde{S}_1^{br}, \dots, \tilde{S}_{N_n}^{br} \geq 0) = \mathbb{E} \left[ \frac{(2N_n - 1)!!}{2^{N_n} N_n!} \right] \sim n^{-1/4} \frac{\Gamma(3/4)}{\sqrt{2\pi}}.$$

Now, note that our perturbations are so small that  $\tilde{S}_i^{br} < 0$  whenever  $S_i^{br} < 0$ , so to compute the probability that  $S^{br}$  never goes negative we can use the following equality

$$\mathbb{P}(\tilde{S}_1^{br}, \dots, \tilde{S}_{N_n}^{br} \geq 0) = \mathbb{P}(S_1^{br}, \dots, S_{N_n}^{br} \geq 0) \mathbb{P}(\tilde{S}_1^{br}, \dots, \tilde{S}_{N_n}^{br} \geq 0 \mid S_1^{br}, \dots, S_{N_n}^{br} \geq 0).$$

We already know  $\mathbb{P}(\tilde{S}_1^{br}, \dots, \tilde{S}_{N_n}^{br} \geq 0)$ , so our result will follow if we can get a good grasp on the term  $\mathbb{P}(\tilde{S}_1^{br}, \dots, \tilde{S}_{N_n}^{br} \geq 0 \mid S_1^{br}, \dots, S_{N_n}^{br} \geq 0)$ . The key to this is the observation that, provided  $S^{br}$  does not go negative, the only times where  $\tilde{S}^{br}$  can go negative are at the points that  $S^{br}$  is equal to zero.

Suppose that  $S^{br}$  is equal to zero  $M_n$  times, namely at  $\xi_1, \dots, \xi_{M_n}$ , and let us also set  $\xi_0 = 0$ . By definition,  $\tilde{S}_{\xi_k}^{br} = \sum_{i=1}^{\xi_k} \varepsilon_i$ , and the increment between times  $\xi_{k-1}$  and  $\xi_k$  is exactly  $\eta_k = \sum_{i=\xi_{k-1}+1}^{\xi_k} \varepsilon_i$ . The sequence  $(\eta_k)_{1 \leq k \leq M_n}$  is an exchangeable sequence and its law is invariant under sign changes of the elements. Distinct sums of elements in the set are distinct almost surely and we can therefore use Lemma 3.8 to deduce that

$$\mathbb{P}(\tilde{S}_1^{br}, \dots, \tilde{S}_{N_n}^{br} \geq 0 \mid S_1^{br}, \dots, S_{N_n}^{br} \geq 0, M_n = M) = \frac{(2M - 1)!!}{2^M M!}.$$

This reduces our task to understanding the distribution of  $M_n$  conditional on the event that  $\{S_1^{br}, \dots, S_{N_n}^{br} \geq 0\}$ , and we will show that it converges in distribution to  $G \sim \text{Geom}(\rho)$ . For this it is enough to show that for  $\ell \geq 1$ ,

$$\mathbb{P}(M_n \geq \ell + 1 \mid S_1^{br}, \dots, S_{N_n}^{br} \geq 0, M_n \geq \ell) \rightarrow \rho. \quad (3.2)$$

Firstly, note that

$$\rho_n := \mathbb{P}(M_n \geq 1 \mid S_1^{br}, \dots, S_{N_n}^{br} \geq 0)$$

converges to  $\rho$  as  $n \rightarrow \infty$  by Lemma 4.8. If  $M_n \geq \ell$ , then let  $\zeta_\ell$  be the time at which  $Y^{br}$  and  $A^{br}$  hit 0 simultaneously for the  $\ell$ th time. Then,  $Y^{br}$  and  $A^{br}$  restricted to  $[\zeta_\ell, n]$  are distributed as a lazy simple symmetric random walk bridge with  $n - \zeta_\ell$  steps and its area process respectively, so

$$\mathbb{P}(M_n \geq \ell + 1 \mid S_1^{br}, \dots, S_{N_n}^{br} \geq 0, M_n \geq \ell, \zeta_\ell) = \rho_{n-\zeta_\ell}.$$

Lemma 4.6 implies that (conditional on the event  $\{S_1^{br}, \dots, S_{N_n}^{br} \geq 0\}$ )  $\zeta_{M_n} = O(n^{-1/2})$  with high probability, so  $n - \zeta_\ell \sim n$  and (3.2) follows from  $\rho_n \rightarrow \rho$ . This implies that

$$\mathbb{P}(\tilde{S}_1^{br}, \dots, \tilde{S}_{N_n}^{br} \geq 0 \mid S_1^{br}, \dots, S_{N_n}^{br} \geq 0) \rightarrow \mathbb{E} \left[ \frac{(2G - 1)!!}{2^G G!} \right],$$

and a simple calculation shows that this equals  $\sqrt{1 - \rho}$ . Putting this all together gives

$$\mathbb{P}(S_1^{br}, \dots, S_{N_n}^{br} \geq 0) = \frac{\mathbb{P}(\tilde{S}_1^{br}, \dots, \tilde{S}_{N_n}^{br} \geq 0)}{\mathbb{P}(\tilde{S}_1^{br}, \dots, \tilde{S}_{N_n}^{br} \geq 0 \mid S_1^{br}, \dots, S_{N_n}^{br} \geq 0)} \sim \frac{\Gamma(3/4)}{\sqrt{2\pi(1 - \rho)}} n^{-1/4}.$$

## 4 The postponed proofs

We will first introduce a coupling between simple symmetric random walks (resp. bridges) and lazy simple symmetric random walks (resp. bridges) that turns out to be useful in our proofs. Let  $(\hat{Y}_i)_{i \geq 0}$  be a simple symmetric random walk. Then, if we set  $Y_i = \frac{1}{2}\hat{Y}_{2i}$  for each  $i$  we see that  $(Y_i)_{i \geq 0}$  has the law of a lazy simple symmetric random walk. Moreover, since  $\hat{Y}$  is zero only at even times, the zeroes of  $\hat{Y}$  are in one-to-one correspondence with the zeroes of  $Y$ , and in particular, if  $(\hat{Y}_i^{br})_{0 \leq i \leq 2n}$  is a simple symmetric random walk bridge and  $Y_i^{br} = \frac{1}{2}\hat{Y}_{2i}^{br}$ , we have that  $(Y_i^{br})_{0 \leq i \leq n}$  is a lazy simple symmetric random walk bridge.

To prove Lemma 3.6 we first show that the probability of event  $\{A_1, \dots, A_n \geq 0\}$  is monotone with respect to the value of  $Y_n$ .

**Lemma 4.1.** *For all  $-n \leq k \leq k' \leq n$ ,*

$$\mathbb{P}(A_1, \dots, A_n \geq 0 \mid Y_n = k) \leq \mathbb{P}(A_1, \dots, A_n \geq 0 \mid Y_n = k').$$

*Proof.* It suffices to prove the claim when  $k' = k + 1$ . We will use the coupling between the SSRW and the lazy SSRW introduced at the beginning of this section. Consider the SSRW  $\hat{Y} = (\hat{Y}_k)_{k=1}^{2n}$  conditioned on the event  $\hat{Y}_{2n} = 2k$ , and define  $Y$  by  $Y_i := \frac{1}{2}\hat{Y}_{2i}$ . Then  $Y$  is distributed as a lazy SSRW with  $n$  steps conditioned on the event  $Y_n = k$ .

Now, observe that the increments of  $\hat{Y}$  conditioned on  $\hat{Y}_{2n} = 2k$  are distributed as a uniform ordering of  $n + k$  instances of 1 and  $n - k$  instances of  $-1$ . We consider a modified sequence obtained from the original sequence by picking a uniformly random  $-1$  and changing it to a  $+1$ , so that the modified sequence has the law of the increments of a SSRW of  $2n$  steps conditioned to end in  $2k + 2$ , which, again via the coupling, corresponds to a lazy SSRW of  $n$  steps conditioned to end in  $k + 1$ . We see that our modification increased one increment of  $Y$  by 1, and since the event  $\{A_1, \dots, A_n \geq 0\}$  is increasing in the increments of  $Y$ , the result follows.  $\square$

We are now ready to prove Lemma 3.6.

*Proof of Lemma 3.6.* We abbreviate  $E = \{A_1, \dots, A_n \geq 0\}$ , and start with the lower bound.

$$\begin{aligned} \mathbb{P}(E \mid Y_n \in \{0, -1\}) &= \frac{\mathbb{P}(E \mid Y_n = 0)\mathbb{P}(Y_n = 0) + \mathbb{P}(E \mid Y_n = -1)\mathbb{P}(Y_n = -1)}{\mathbb{P}(Y_n \in \{0, -1\})} \\ &\geq \frac{\mathbb{P}(E \mid Y_n = 0)\binom{2n}{n}}{\binom{2n+1}{n}} \\ &= \frac{n+1}{2n+1}\mathbb{P}(E \mid Y_n = 0) \\ &\geq \frac{1}{2}\mathbb{P}(E \mid Y_n = 0). \end{aligned}$$

We now turn to the upper bound, for which we use Lemma 4.1.

$$\begin{aligned} \mathbb{P}(E \mid Y_n \in \{0, -1\}) &= \frac{\mathbb{P}(E \mid Y_n = 0)\mathbb{P}(Y_n = 0) + \mathbb{P}(E \mid Y_n = -1)\mathbb{P}(Y_n = -1)}{\mathbb{P}(Y_n \in \{0, -1\})} \\ &\leq \frac{\mathbb{P}(E \mid Y_n = 0)(\mathbb{P}(Y_n = 0) + \mathbb{P}(Y_n = -1))}{\mathbb{P}(Y_n \in \{0, -1\})} \\ &= \mathbb{P}(E \mid Y_n = 0). \end{aligned} \quad \square$$

## 4.1 Lemmas for Proposition 1.2

As before, let  $N_n = |\{1 \leq i \leq n : Y_i^{br} = 0\}|$  be the number of returns to 0 of  $Y^{br}$  up to time  $n$ . We have the following lemma.

**Lemma 4.2.** *As  $n \rightarrow \infty$ ,  $n^{-1/2}N_n \xrightarrow{d} 2\sqrt{E}$  where  $E \sim \text{Exp}(1)$  has a standard exponential distribution. Moreover, for any  $\gamma \geq 0$ ,*

$$\mathbb{P}(N_n < \gamma n^{1/2}) \leq \frac{\gamma^2}{2}.$$

*Proof.* We fix an  $n$  and a  $k \leq n$ . We will count the number of Bernoulli bridges with  $2n$  steps and at least  $k$  returns to 0 (i.e. at least  $k$  excursions away from 0). For a Bernoulli bridge with  $2n$  steps and at least  $k$  excursions, flip the last  $k$  excursions so that they are positive. Then remove the last step of each of the last  $k$  excursions. Each of the removed steps was downward, so we now obtain a path of length  $2n - k$  that ends at level  $k$ . We can recover the original position of the removed steps: the  $i$ th removed step should be included after the last time the path is at level  $i$ . Therefore, each path of length  $2n - k$  that ends at level  $k$  corresponds to  $2^k$  bridges with more than  $k$  zeroes, so the number of bridges with  $2n$  steps and more than  $k$  zeroes equals  $2^k \binom{2n-k}{n}$ . Thus the probability that a simple symmetric random walk bridge with  $2n$  steps returns to 0 at least  $k$  times equals

$$\frac{2^k \binom{2n-k}{n}}{\binom{2n}{n}} = \frac{2^k n(n-1) \cdots (n-k+1)}{(2n)(2n-1) \cdots (2n-k+1)} = \prod_{i=1}^{k-1} \left(1 - \frac{i}{2n-i}\right).$$

By the coupling between a lazy random walk bridge on  $[n]$  and a simple symmetric random walk bridge on  $[2n]$  that preserves the number of zeroes, we then see that for  $k = O(\sqrt{n})$ ,

$$\log \mathbb{P}(N_n \geq k) = \log \prod_{i=1}^{k-1} \left(1 - \frac{i}{2n-i}\right) = - \sum_{i=1}^{k-1} \left(\frac{i}{2n} + O(i^2/n^2)\right) = -\frac{k^2}{4n} + O(n^{-1/2}),$$

so  $\mathbb{P}(N_n \geq tn^{1/2}) \rightarrow e^{-t^2/4} = \mathbb{P}(E > t^2/4) = \mathbb{P}(2\sqrt{E} > t)$  where  $E \sim \text{Exp}(1)$  has an exponential distribution. We also see that, for  $0 \leq k \leq n$ ,

$$\mathbb{P}(N_n \geq k) = \prod_{i=1}^{k-1} \left(1 - \frac{i}{2n-i}\right) \geq 1 - \sum_{i=1}^{k-1} \frac{i}{2n-i} \geq 1 - \frac{k(k-1)/2}{2n-k} \geq 1 - \frac{k^2}{2n}.$$

Hence, as the above inequality is trivially true for  $k > n$ ,

$$\mathbb{P}(N_n < \gamma n^{1/2}) = 1 - \mathbb{P}(N_n \geq \lfloor \gamma n^{1/2} \rfloor) \leq \frac{\lfloor \gamma n^{1/2} \rfloor^2}{2n} \leq \frac{\gamma^2}{2}. \quad \square$$

We use Lemma 4.2 in the following lemma which considers the asymptotic value of  $\mathbb{P}(\tilde{S}_1^{br}, \dots, \tilde{S}_{N_n}^{br} \geq 0) = \mathbb{E}\left[\frac{(2N_n-1)!!}{2^{N_n}N_n!}\right]$ .

**Lemma 4.3.** *As  $n \rightarrow \infty$ ,*

$$\mathbb{E}\left[\frac{(2N_n-1)!!}{2^{N_n}N_n!}\right] \sim n^{-1/4} \frac{\Gamma(3/4)}{\sqrt{2\pi}}.$$

*Proof.* Fix some  $\delta \in (0, 1)$ . We split the expectation according to the contribution with  $N_n < \delta n^{1/2}$  and  $N_n \geq \delta n^{1/2}$ . By Lemma 4.2,  $\mathbb{P}(N_n < \gamma n^{1/2}) \leq \gamma^2/2$  for all  $n$  and we also recall that

$$\frac{1}{\sqrt{\pi(m+1/2)}} \leq \frac{(2m-1)!!}{2^m m!} \leq \frac{1}{\sqrt{\pi m}}.$$

Hence,

$$\begin{aligned} n^{1/4} \mathbb{E} \left[ \frac{(2N_n - 1)!!}{2^{N_n} N_n!} \mathbb{1}_{\{N_n < \delta n^{1/2}\}} \right] &\leq n^{1/4} \sum_{i=0}^{\infty} \mathbb{P}(2^{-i-1} \delta n^{1/2} \leq N_n < 2^{-i} \delta n^{1/2}) \frac{1}{\sqrt{\pi 2^{-i-1} \delta n^{1/2}}} \\ &\leq \sum_{i=0}^{\infty} \frac{1}{2} (2^{-i} \delta)^2 \cdot (\pi 2^{-i-1} \delta)^{-1/2} = O(\delta^{3/2}). \end{aligned}$$

Since  $n^{1/4}/\sqrt{\pi N_n}$  is bounded for  $N_n \geq \delta n^{1/2}$  and  $n^{-1/2} N_n$  converges in distribution to  $2\sqrt{E}$  where  $E \sim \text{Exp}(1)$ ,

$$n^{1/4} \mathbb{E} \left[ \frac{(2N_n - 1)!!}{2^{N_n} N_n!} \mathbb{1}_{\{N_n \geq \delta n^{1/2}\}} \right] = \mathbb{E} \left[ \frac{n^{1/4}}{\sqrt{\pi N_n + O(1)}} \mathbb{1}_{\{n^{-1/2} N_n \geq \delta\}} \right] \rightarrow \mathbb{E} \left[ \frac{\mathbb{1}_{\{2\sqrt{E} \geq \delta\}}}{\sqrt{2\pi\sqrt{E}}} \right]$$

as  $n \rightarrow \infty$ . Now

$$\mathbb{E} \left[ \frac{\mathbb{1}_{\{2\sqrt{E} \geq \delta\}}}{\sqrt{2\pi\sqrt{E}}} \right] = \mathbb{E} \left[ \frac{\mathbb{1}_{\{E \geq \delta^2/4\}}}{\sqrt{2\pi} E^{1/4}} \right] = \frac{1}{\sqrt{2\pi}} \int_{\delta^2/4}^{\infty} x^{-1/4} e^{-x} dx = \frac{\Gamma(3/4)}{\sqrt{2\pi}} + O(\delta^{3/2}).$$

Hence,

$$n^{1/4} \mathbb{E} \left[ \frac{(2N_n - 1)!!}{2^{N_n} N_n!} \right] \rightarrow \frac{\Gamma(3/4)}{\sqrt{2\pi}} + O(\delta^{3/2})$$

as  $n \rightarrow \infty$ . As  $\delta$  was arbitrary, the result now follows.  $\square$

We now want to show that

$$\mathbb{P}(\exists k \in (n^{1/2}, n] : A_k^{br} = Y_k^{br} = 0 \mid A_1^{br}, \dots, A_n^{br} \geq 0) = o(1),$$

which is the content of Lemma 4.6. For the proof of this, we will need Lemma 4.4, which is a local limit theorem for the position and area of a lazy SSRW, and Lemma 4.5, which we will use to control the probability that the integral and position of an unconditioned lazy SSRW hit 0 simultaneously at a late time.

**Lemma 4.4.** *We have*

$$\lim_{n \rightarrow \infty} \sup_{a, b} |n^2 \mathbb{P}(Y_n = a, A_n = b) - \phi(n^{-1/2} a, n^{-3/2} b)| = 0$$

where the supremum runs over all  $(a, b) \in \mathbb{Z}^2$  and

$$\phi(x, y) = \frac{2\sqrt{3}}{\pi} \exp(-4x^2 + 12xy - 12y^2).$$

The proof of this lemma is postponed to Appendix C.



**Lemma 4.5.** *There exists a constant  $C$  such that for all  $n \geq 1$ ,*

$$\mathbb{P}(A_n = Y_n = 0, A_1, \dots, A_n \geq 0) \leq Cn^{-5/2}.$$

With Lemma 4.4 in hand, the proof of Lemma 4.5 is a direct adaptation of the proof of the upper bound of Theorem 1 of [3] on simple symmetric random walks. For the sake of completeness, we have included a proof in Appendix D.

**Lemma 4.6.** *We have that*

$$\mathbb{P}(\exists k \in (n^{1/2}, n]: A_k^{br} = Y_k^{br} = 0 \mid A_1^{br}, \dots, A_n^{br} \geq 0) = O(n^{-3/4}).$$

*Proof.* Define  $p_n = \mathbb{P}(A_1, \dots, A_n \geq 0, Y_n = 0)$ . Note that for any  $k$ ,

$$\begin{aligned} \mathbb{P}(A_k^{br} = Y_k^{br} = 0 \mid A_1^{br}, \dots, A_n^{br} \geq 0) &= \frac{1}{p_n} \mathbb{P}(A_k = Y_k = 0, A_1, \dots, A_n \geq 0, Y_n = 0) \\ &= \frac{1}{p_n} \mathbb{P}(A_k = Y_k = 0, A_1, \dots, A_k \geq 0) \\ &\quad \times \mathbb{P}(A_{k+1}, \dots, A_n \geq 0, Y_n = 0 \mid A_k = Y_k = 0) \\ &= \frac{p_{n-k}}{p_n} \mathbb{P}(A_k = Y_k = 0, A_1, \dots, A_k \geq 0). \end{aligned}$$

By our earlier results (Proposition 3.4),  $p_n = \Theta(n^{-3/4})$  and by Lemma 4.5, the final factor is  $O(k^{-5/2})$ . Therefore, there exists a  $C$  such that for each  $k \leq n$ ,

$$\mathbb{P}(A_k^{br} = Y_k^{br} = 0 \mid A_1^{br}, \dots, A_n^{br} \geq 0) \leq Ck^{-5/2}(n - k + 1)^{-3/4}n^{3/4}.$$

Then, the result follows from the union bound by observing that

$$\sum_{n^{1/2} < k \leq n/2} Ck^{-5/2}(n - k + 1)^{-3/4}n^{3/4} \leq 2^{3/4}C \sum_{k > n^{1/2}} k^{-5/2} = O((n^{1/2})^{-3/2}) = O(n^{-3/4})$$

and

$$\sum_{k=[n/2]}^n Ck^{-5/2}(n - k + 1)^{-3/4}n^{3/4} = O(n^{-7/4} \sum_{j=1}^{[n/2]+1} j^{-3/4}) = O(n^{-7/4}n^{1/4}) = O(n^{-3/2}). \quad \square$$

We now turn to Lemma 4.8 which shows that  $\rho_n \rightarrow \rho$ . For the proof of this we need the following result.

**Lemma 4.7.** *Let  $q_n = \mathbb{P}(A_1, \dots, A_n \geq 0 \mid Y_n = 0)$ . Then, uniformly over all  $m \leq n^{1/2}$ , we have  $q_{n-m}/q_n \rightarrow 1$  as  $n \rightarrow \infty$ .*

*Proof.* We define  $\varphi_\ell(k) = \mathbb{P}(Y_\ell = -k)$ , which is also the probability that a lazy SSRW starting at  $k$  is at 0 at time  $\ell$ . Let  $0 < m < b < n$  and let  $Z$  be a random variable that depends only on  $\mathcal{F}_{n-b} = \sigma(Y_1, \dots, Y_{n-b})$ . We first show that

$$\mathbb{E}[Z \mid Y_n = 0] = \mathbb{E}\left[Z \cdot \frac{\varphi_b(Y_{n-b})}{\varphi_{b-m}(Y_{n-b})} \frac{\varphi_{n-m}(0)}{\varphi_n(0)} \mid Y_{n-m} = 0\right]. \quad (4.1)$$

For  $b' < n'$  and any  $Z'$  that depends only on  $\mathcal{F}_{b'}$ ,

$$\begin{aligned}\mathbb{E}[Z' \mid Y_{n'} = 0] &= \frac{\mathbb{E}[Z' \mathbb{1}\{Y_{n'} = 0\}]}{\mathbb{P}(Y_{n'} = 0)} \\ &= \frac{\mathbb{E}[\mathbb{E}[Z' \mathbb{1}\{Y_{n'} = 0\} \mid \mathcal{F}_{b'}]]}{\mathbb{P}(Y_{n'} = 0)} \\ &= \frac{\mathbb{E}[Z' \mathbb{P}(Y_{n'} = 0 \mid \mathcal{F}_{b'})]}{\mathbb{P}(Y_{n'} = 0)} = \mathbb{E}\left[Z' \cdot \frac{\varphi_{n'-b'}(Y_{b'})}{\varphi_{n'}(0)}\right].\end{aligned}$$

Applying this to  $Z' = Z \frac{\varphi_b(Y_{n-b})}{\varphi_{b-m}(Y_{n-b})} \frac{\varphi_{n-m}(0)}{\varphi_n(0)}$ ,  $n' = n - m$  and  $b' = n - b$ , we find that

$$\begin{aligned}\mathbb{E}[Z' \mid Y_{n-m} = 0] &= \mathbb{E}\left[Z' \cdot \frac{\varphi_{b-m}(Y_{n-b})}{\varphi_{n-m}(0)}\right] \\ &= \mathbb{E}\left[Z \cdot \frac{\varphi_b(Y_{n-b})}{\varphi_{b-m}(Y_{n-b})} \frac{\varphi_{n-m}(0)}{\varphi_n(0)} \cdot \frac{\varphi_{b-m}(Y_{n-b})}{\varphi_{n-m}(0)}\right] \\ &= \mathbb{E}\left[Z \cdot \frac{\varphi_b(Y_{n-b})}{\varphi_n(0)}\right] \\ &= \mathbb{E}[Z \mid Y_n = 0]\end{aligned}$$

as desired.

We assume  $m \leq n^{1/2}$  and let  $b = b(n) \in \mathbb{N}$  be such that  $b(n) \sim n^{7/9}$ . Next, we provide bounds on  $\frac{\varphi_b(Y_{n-b})}{\varphi_{b-m}(Y_{n-b})} \frac{\varphi_{n-m}(0)}{\varphi_n(0)}$ . We first note that for  $|k| \leq \ell$  we have

$$\begin{aligned}\varphi_\ell(k) &= \binom{2\ell}{k+\ell} 4^{-\ell} = \binom{2\ell}{\ell} 4^{-\ell} \cdot \prod_{j=1}^k \left(1 - \frac{2j-1}{\ell+j}\right) \\ &= \frac{1}{\sqrt{\pi\ell} + O(1)} \cdot \exp\left(\sum_{j=1}^k \log\left(1 - \frac{2j-1}{\ell+j}\right)\right) \\ &= \frac{1}{\sqrt{\pi\ell}} e^{O(1/\ell)} \cdot \exp\left(-\sum_{j=1}^k \left(\frac{2j-1}{\ell} + O(j^2/\ell^2)\right)\right) \\ &= \frac{1}{\sqrt{\pi\ell}} e^{-k^2/\ell + O(k^3/\ell^2) + O(1/\ell)}.\end{aligned}$$

Hence,  $\varphi_\ell(k)/\varphi_{\ell'}(k) \rightarrow 1$  provided  $\ell \rightarrow \infty$ ,  $\ell/\ell' \rightarrow 1$ ,  $k^2(\ell - \ell')/\ell\ell' \rightarrow 0$  and  $k^3/\ell^2 \rightarrow 0$ . Now define

$$\begin{aligned}\underline{\delta}_n &= \min_{|k| \leq n^{1/2}} \frac{\varphi_b(k)}{\varphi_{b-m}(k)} \frac{\varphi_{n-m}(0)}{\varphi_n(0)} \\ \bar{\delta}_n &= \max_{|k| \leq n^{1/2}} \frac{\varphi_b(k)}{\varphi_{b-m}(k)} \frac{\varphi_{n-m}(0)}{\varphi_n(0)}.\end{aligned}$$

and note that  $\underline{\delta}_n \rightarrow 1$  and  $\bar{\delta}_n \rightarrow 1$  as  $n \rightarrow \infty$  for our choice of  $m$  and  $b$ . Indeed,  $\log \underline{\delta}_n, \log \bar{\delta}_n = O(m/b + mn/b^2 + n^{3/2}/b^2) \rightarrow 0$ .

Applying (4.1) with  $Z = \mathbb{1}_{\{A_1, \dots, A_{n-b} \geq 0, |Y_{n-b}| \leq n^{1/2}\}}$  gives

$$\begin{aligned}
\mathbb{P}(A_1, \dots, A_{n-b} \geq 0 \mid Y_n = 0) &\leq \mathbb{P}(A_1, \dots, A_{n-b} \geq 0, |Y_{n-b}| \leq n^{1/2} \mid Y_n = 0) \\
&\quad + \mathbb{P}(|Y_{n-b}| > n^{1/2} \mid Y_n = 0) \\
&\leq \bar{\delta}_n \mathbb{P}(A_1, \dots, A_{n-b} \geq 0, |Y_{n-b}| \leq n^{1/2} \mid Y_{n-m} = 0) \\
&\quad + \mathbb{P}(|Y_{n-b}| > n^{1/2} \mid Y_n = 0) \\
&\leq \bar{\delta}_n \mathbb{P}(A_1, \dots, A_{n-b} \geq 0 \mid Y_{n-m} = 0) \\
&\quad + \mathbb{P}(|Y_{n-b}| > n^{1/2} \mid Y_n = 0).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{P}(A_1, \dots, A_{n-b} \geq 0 \mid Y_n = 0) &\geq \mathbb{P}(A_1, \dots, A_{n-b} \geq 0, |Y_{n-b}| \leq n^{1/2} \mid Y_n = 0) \\
&\geq \underline{\delta}_n \mathbb{P}(A_1, \dots, A_{n-b} \geq 0, |Y_{n-b}| \leq n^{1/2} \mid Y_{n-m} = 0) \\
&\geq \underline{\delta}_n \mathbb{P}(A_1, \dots, A_{n-b} \geq 0 \mid Y_{n-m} = 0) \\
&\quad - \underline{\delta}_n \mathbb{P}(|Y_{n-b}| > n^{1/2} \mid Y_{n-m} = 0).
\end{aligned}$$

We note that

$$\begin{aligned}
\mathbb{P}(|Y_{n-b}| > n^{1/2} \mid Y_n = 0) &= \mathbb{P}(|Y_b| > n^{1/2} \mid Y_n = 0) = O(n^{-\omega(1)}), \\
\mathbb{P}(|Y_{n-b}| > n^{1/2} \mid Y_{n-m} = 0) &= \mathbb{P}(|Y_{b-m}| > n^{1/2} \mid Y_{n-m} = 0) = O(n^{-\omega(1)})
\end{aligned}$$

uniformly in all  $m \leq n^{1/2}$ , where we use that on the event  $\{Y_n = 0\}$ , it holds that  $(Y_i)_{i=1}^n$  and  $(Y_{n-i})_{i=1}^n$  have the same law and the fact that  $n^{1/2} \sim n^\varepsilon b^{1/2}$  for some  $\varepsilon > 0$  (and large enough  $n$ ). Therefore, using that  $q_n = \Theta(n^{-1/4})$  by Proposition 3.4,

$$\mathbb{P}(A_1, \dots, A_{n-b} \geq 0 \mid Y_n = 0) = (1 + o(1)) \mathbb{P}(A_1, \dots, A_{n-b} \geq 0 \mid Y_{n-m} = 0). \quad (4.2)$$

To finish the proof, it suffices to show that

$$\frac{\mathbb{P}(A_1, \dots, A_{n-b} \geq 0 \mid Y_n = 0)}{\mathbb{P}(A_1, \dots, A_n \geq 0 \mid Y_n = 0)} \rightarrow 1 \quad (4.3)$$

as  $n \rightarrow \infty$ . Indeed, if we set  $b = \lfloor n^{7/9} \rfloor$ , we see that

$$\begin{aligned}
\frac{q_{n-m}}{q_n} &= \frac{\mathbb{P}(A_1, \dots, A_{n-m} \geq 0 \mid Y_{n-m} = 0)}{\mathbb{P}(A_1, \dots, A_n \geq 0 \mid Y_n = 0)} \\
&= \frac{\mathbb{P}(A_1, \dots, A_{n-m} \geq 0 \mid Y_{n-m} = 0)}{\mathbb{P}(A_1, \dots, A_{n-b} \geq 0 \mid Y_{n-m} = 0)} \frac{\mathbb{P}(A_1, \dots, A_{n-b} \geq 0 \mid Y_{n-m} = 0)}{\mathbb{P}(A_1, \dots, A_{n-b} \geq 0 \mid Y_n = 0)} \\
&\quad \times \frac{\mathbb{P}(A_1, \dots, A_{n-b} \geq 0 \mid Y_n = 0)}{\mathbb{P}(A_1, \dots, A_n \geq 0 \mid Y_n = 0)},
\end{aligned}$$

so (4.3) would imply that the first and the third factor in the product tend to 1. The second factor tends to 1 by (4.2).

To obtain (4.3) we need to show that

$$\mathbb{P}(\exists k \in (n-b, n]: A_k^{br} < 0 \mid A_1^{br}, \dots, A_{n-b}^{br} \geq 0) \rightarrow 0$$

as  $n \rightarrow \infty$ . We know by our earlier bounds (Proposition 3.4) that there is a  $c > 0$  such that for all  $n$  large enough

$$\mathbb{P}(A_1^{br}, \dots, A_n^{br} \geq 0) \geq cn^{-1/4},$$

so it is sufficient to show that

$$\mathbb{P}(\exists k \in (n-b, n]: \text{sgn}(A_{k-1}^{br}) \neq \text{sgn}(A_k^{br})) = o(n^{-1/4}).$$

Note that

$$\begin{aligned} \mathbb{P}(\exists k \in (n-b, n]: \text{sgn}(A_{k-1}^{br}) \neq \text{sgn}(A_k^{br})) \\ \leq \mathbb{P}(|A_n^{br}| \leq n^{6/5}) + \mathbb{P}\left(\max_{n-b+1 \leq i \leq n} |Y_i^{br}| > n^{6/5}/b\right) \\ \leq \mathbb{P}(|A_n^{br}| \leq n^{6/5}) + \mathbb{P}\left(\max_{1 \leq i \leq b} |Y_i^{br}| > n^{6/5}/b\right) \end{aligned}$$

by the union bound and the fact that  $(Y_i^{br})_{1 \leq i \leq n}$  and  $(Y_{n+1-i}^{br})_{1 \leq i \leq n}$  have the same law. By Lemma 4.4,

$$\mathbb{P}(|A_n^{br}| \leq n^{6/5}) = \frac{\mathbb{P}(|A_n| \leq n^{6/5}, Y_n = 0)}{\mathbb{P}(Y_n = 0)} = \frac{O(n^{6/5}/n^2)}{\Theta(n^{-1/2})} = o(n^{-1/4}).$$

Furthermore, by the reflection principle for the simple symmetric random walk  $\hat{Y}$ , we have that, for any  $k$

$$\mathbb{P}\left(\max_{1 \leq i \leq 2b} \hat{Y}_i \geq k\right) \leq 2\mathbb{P}(\hat{Y}_{2b} \geq k)$$

and under the usual coupling between  $(Y_k)_{1 \leq k \leq b}$  and  $(\hat{Y}_k)_{1 \leq k \leq 2b}$  we have that

$$\mathbb{P}\left(\max_{1 \leq i \leq b} |Y_i| > k\right) = 2\mathbb{P}\left(\max_{1 \leq i \leq b} Y_i > k\right) \leq 2\mathbb{P}\left(\max_{1 \leq i \leq 2b} \hat{Y}_i \geq 2k\right) \leq 4\mathbb{P}(\hat{Y}_{2b} \geq 2k),$$

so observing that  $n^{6/5}/b = n^\varepsilon b^{1/2}$  for some  $\varepsilon > 0$  implies that

$$\mathbb{P}\left(\max_{1 \leq i \leq b} |Y_i| > n^{6/5}/b\right) = n^{-\omega(1)}.$$

Then,  $\mathbb{P}(Y_n = 0) = \Theta(n^{-1/2})$  implies that also

$$\mathbb{P}\left(\max_{1 \leq i \leq b} |Y_i^{br}| > n^{6/5}/b\right) = n^{-\omega(1)}.$$

So it follows that

$$\mathbb{P}(\exists k \in (n-b, n]: \text{sgn}(A_{k-1}^{br}) \neq \text{sgn}(A_k^{br})) = o(n^{-1/4})$$

as claimed.  $\square$

Recall that  $\rho_n = \mathbb{P}(M_n \geq 1 \mid S_1^{br}, \dots, S_{N_n}^{br} \geq 0)$  where  $M_n = \#\{i > 0 : S_i^{br} = 0\}$ . We will use the preceding lemma to prove Lemma 4.8.

**Lemma 4.8.** *We have  $\rho_n \rightarrow \rho$  as  $n \rightarrow \infty$ .*

*Proof.* By definition,

$$\rho_n = \mathbb{P}(\exists k \in [n]: A_k^{br} = Y_k^{br} = 0 \mid A_1^{br}, \dots, A_n^{br} \geq 0).$$

Let  $\zeta_1 = \zeta_1(Y^{br}) = \min\{k > 0 : Y_k^{br} = 0, A_k^{br} \leq 0\}$ , with the convention that  $\min \emptyset = \infty$ . On  $\{A_1^{br}, \dots, A_n^{br} \geq 0\}$  it holds that  $\{\exists k \in [n]: A_k^{br} = Y_k^{br} = 0\} = \{\zeta_1 \leq n\}$  and we first show that, under this conditioning, with high probability  $\zeta_1$  is either at most  $n^{1/2}$  or larger than  $n$ . We will then show that we may stop conditioning on  $\{A_1^{br}, \dots, A_n^{br} \geq 0\}$  if we instead consider the event  $\{\zeta_1 \leq n^{1/2}, A_{\zeta_1}^{br} = 0\}$  (and accept a  $o(1)$  term). We then further show that the fact that we are considering  $Y^{br}$  instead of  $Y$  makes a negligible difference. After making these changes, the random variable  $\zeta_1$  does not depend on  $n$  and the result follows.

Observe that

$$\begin{aligned} \rho_n &= \mathbb{P}(\zeta_1 \leq n \mid A_1^{br}, \dots, A_n^{br} \geq 0) \\ &= \mathbb{P}(\zeta_1 \leq n^{1/2} \mid A_1^{br}, \dots, A_n^{br} \geq 0) + \mathbb{P}(\zeta_1 \in (n^{1/2}, n] \mid A_1^{br}, \dots, A_n^{br} \geq 0) \end{aligned}$$

and by Lemma 4.6,

$$\begin{aligned} \mathbb{P}(\zeta_1 \in (n^{1/2}, n] \mid A_1^{br}, \dots, A_n^{br} \geq 0) \\ \leq \mathbb{P}(\exists k \in (n^{1/2}, n]: A_k^{br} = Y_k^{br} = 0 \mid A_1^{br}, \dots, A_n^{br} \geq 0) = O(n^{-3/4}). \end{aligned}$$

It remains to show that  $\mathbb{P}(\zeta_1 \leq n^{1/2} \mid A_1^{br}, \dots, A_n^{br} \geq 0) \rightarrow \rho$ . Recall that  $q_n = \mathbb{P}(A_1^{br}, \dots, A_n^{br} \geq 0)$ . Then, for any  $k \in [n]$ ,

$$\begin{aligned} \mathbb{P}(A_1^{br}, \dots, A_n^{br} \geq 0 \mid \zeta_1 = k) &= \mathbb{P}(A_1^{br}, \dots, A_k^{br} \geq 0 \mid \zeta_1 = k) \\ &\quad \times \mathbb{P}(A_{k+1}^{br}, \dots, A_n^{br} \geq 0 \mid \zeta_1 = k, A_k^{br} = 0) \\ &= q_{n-k} \mathbb{P}(A_1^{br}, \dots, A_k^{br} \geq 0 \mid \zeta_1 = k) \\ &= q_{n-k} \mathbb{P}(A_{\zeta_1}^{br} = 0 \mid \zeta_1 = k). \end{aligned}$$

where we used that, on the event  $\{\zeta_1 = k, A_k^{br} = 0\}$ , the restrictions of  $Y^{br}$  and  $A^{br}$  to  $[k, n]$  have the joint law of a lazy simple symmetric random walk bridge on  $[n - k]$  and its area process respectively. Therefore,

$$\mathbb{P}(\zeta_1 \leq n^{1/2} \mid A_1^{br}, \dots, A_n^{br} \geq 0) = \sum_{k=1}^{\lfloor n^{1/2} \rfloor} \mathbb{P}(\zeta_1 = k) \frac{q_{n-k}}{q_n} \mathbb{P}(A_{\zeta_1}^{br} = 0 \mid \zeta_1 = k).$$

Lemma 4.7 shows that  $\frac{q_{n-k}}{q_n} \rightarrow 1$  as  $n \rightarrow \infty$  uniformly over all  $k \leq n^{1/2}$ , so

$$\begin{aligned} \rho_n &= \sum_{k=1}^{\lfloor n^{1/2} \rfloor} \mathbb{P}(\zeta_1 = k) \mathbb{P}(A_{\zeta_1}^{br} = 0 \mid \zeta_1 = k) + o(1) \\ &= \mathbb{P}(\zeta_1 \leq n^{1/2}, A_{\zeta_1}^{br} = 0) + o(1). \end{aligned}$$

We now show that removing the condition that  $\{Y_n = 0\}$  has only a negligible effect on the probability. Fix  $0 < \varepsilon < 1/4$  and note that

$$\begin{aligned} \mathbb{P}(\zeta_1 \leq n^{1/2}, A_{\zeta_1}^{br} = 0) &= \mathbb{P}\left(\zeta_1 \leq n^{1/2}, A_{\zeta_1}^{br} = 0, |Y_{\lfloor n^{1/2} \rfloor}^{br}| < n^{1/4+\varepsilon}\right) + o(1) \\ &= \mathbb{E}\left[\mathbb{1}_{\{\zeta_1 \leq n^{1/2}, A_{\zeta_1} = 0, |Y_{\lfloor n^{1/2} \rfloor}| < n^{1/4+\varepsilon}\}} \frac{\varphi_{n-\lfloor n^{1/2} \rfloor}(Y_{\lfloor n^{1/2} \rfloor})}{\varphi_n(0)}\right] + o(1), \end{aligned}$$

where we have used the function  $\varphi$  from the proof of Lemma 4.7 and that the event

$$\left\{\zeta_1 \leq n^{1/2}, A_{\zeta_1}^{br} = 0, |Y_{\lfloor n^{1/2} \rfloor}^{br}| < n^{1/4+\varepsilon}\right\}$$

is measurable with respect to  $\sigma(Y_1^{br}, \dots, Y_{\lfloor n^{1/2} \rfloor}^{br})$ . From the proof of Lemma 4.7 we have that

$$\frac{\varphi_{n-\lfloor n^{1/2} \rfloor}(a)}{\varphi_n(0)} \rightarrow 1$$

uniformly over all  $|a| < n^{1/4+\varepsilon}$ , so

$$\begin{aligned} \mathbb{P}(\zeta_1 \leq n^{1/2}, A_{\zeta_1}^{br} = 0) &= \mathbb{P}(\zeta_1 \leq n^{1/2}, A_{\zeta_1} = 0, |Y_{\lfloor n^{1/2} \rfloor}| < n^{1/4+\varepsilon}) + o(1) \\ &= \mathbb{P}(\zeta_1 \leq n^{1/2}, A_{\zeta_1} = 0) + o(1). \end{aligned}$$

Now observe that, under the law of  $Y$ , we have that  $\zeta_1$  is a random variable on  $\mathbb{N}$  that does not depend on  $n$ , so  $\mathbb{P}(\zeta_1 \leq n^{1/2}, A_{\zeta_1} = 0) \rightarrow \rho$  as  $n \rightarrow \infty$  and the statement follows.  $\square$

## 4.2 Conditioning on ending in 0 or $-1$

In this section, we show how Proposition 3.1 follows from Proposition 1.2. By Lemma 2.2, we need to show that for  $(Y_k)_{k \geq 1}$  a lazy SSRW, and  $A_k = \sum_{i=1}^k Y_i$ , we have that

$$\mathbb{P}(A_1, \dots, A_{n-1} \geq 0 \mid Y_{n-1} \in \{0, -1\}) \sim n^{-1/4} \frac{\Gamma(3/4)}{\sqrt{2\pi(1-\rho)}}.$$

*Proof of Proposition 3.1.* First, observe that by Proposition 1.2 and the fact that  $\mathbb{P}(Y_n = 0) \sim \frac{1}{\sqrt{\pi n}}$ ,

$$\mathbb{P}(A_1, \dots, A_{n-1} \geq 0, Y_{n-1} = 0) \sim n^{-3/4} \frac{\Gamma(3/4)}{\pi \sqrt{2(1-\rho)}}.$$

We also need to calculate

$$\mathbb{P}(A_1, \dots, A_{n-1} \geq 0, Y_{n-1} = -1).$$

Note that

$$\begin{aligned} \mathbb{P}(A_1, \dots, A_n \geq 0, Y_n = 0) &= \frac{1}{4} \mathbb{P}(A_1, \dots, A_{n-1} \geq 0, Y_{n-1} = -1) \\ &\quad + \frac{1}{4} \mathbb{P}(A_1, \dots, A_{n-1} \geq 0, Y_{n-1} = 1) \\ &\quad + \frac{1}{2} \mathbb{P}(A_1, \dots, A_{n-1} \geq 0, Y_{n-1} = 0). \end{aligned} \tag{4.4}$$

Both  $\mathbb{P}(A_1, \dots, A_n \geq 0, Y_n = 0)$  and  $\mathbb{P}(A_1, \dots, A_{n-1} \geq 0, Y_{n-1} = 0)$  are asymptotically equal to  $n^{-3/4} \frac{\Gamma(3/4)}{\pi\sqrt{2(1-\rho)}}$ , so if we show that

$$\mathbb{P}(A_1, \dots, A_{n-1} \geq 0, Y_{n-1} = -1) \sim \mathbb{P}(A_1, \dots, A_{n-1} \geq 0, Y_{n-1} = 1), \quad (4.5)$$

we can deduce that

$$\mathbb{P}(A_1, \dots, A_{n-1} \geq 0, Y_{n-1} = -1) \sim n^{-3/4} \frac{\Gamma(3/4)}{\pi\sqrt{2(1-\rho)}}.$$

Then, the fact that  $\mathbb{P}(Y_{n-1} \in \{0, -1\}) \sim \frac{2}{\sqrt{\pi n}}$  implies that

$$\mathbb{P}(A_1, \dots, A_{n-1} \geq 0 \mid Y_{n-1} \in \{0, -1\}) \sim n^{-1/4} \frac{\Gamma(3/4)}{\sqrt{2\pi(1-\rho)}},$$

as required.

We now prove (4.5). Let  $\mathcal{Y}_-$  be the set of paths  $(y_0, y_1, \dots, y_{n-1})$  starting at 0 with steps in  $\{-1, 0, 1\}$  which satisfy

$$\left\{ \sum_{i=1}^k y_i \geq 0 \text{ for all } k \in [n-1], y_{n-1} = -1 \right\},$$

and let  $\mathcal{Y}_+$  be the set of paths  $(y_0, y_1, \dots, y_{n-1})$  starting at 0 with steps in  $\{-1, 0, 1\}$  which satisfy

$$\left\{ \sum_{i=1}^k y_i \geq 0 \text{ for all } k \in [n-1], y_{n-1} = 1 \right\}.$$

We have

$$\mathbb{P}(A_1, \dots, A_{n-1} \geq 0, Y_{n-1} = -1) = \mathbb{P}(Y \in \mathcal{Y}_-)$$

and

$$\mathbb{P}(A_1, \dots, A_{n-1} \geq 0, Y_{n-1} = 1) = \mathbb{P}(Y \in \mathcal{Y}_+).$$

We will define an injective map  $f$  from  $\mathcal{Y}_-$  to  $\mathcal{Y}_+$  such that  $\mathbb{P}(Y = y) = \mathbb{P}(Y = f(y))$  for all  $y \in \mathcal{Y}_-$ . Then,

$$\begin{aligned} \mathbb{P}(Y \in \mathcal{Y}_-) &= \sum_{y \in \mathcal{Y}_-} \mathbb{P}(Y = y) \\ &= \sum_{y \in \mathcal{Y}_-} \mathbb{P}(Y = f(y)) \\ &= \sum_{y \in f(\mathcal{Y}_-)} \mathbb{P}(Y = y) \\ &= \mathbb{P}(Y \in \mathcal{Y}_+) - \mathbb{P}(Y \in \mathcal{Y}_+ \setminus f(\mathcal{Y}_-)). \end{aligned}$$

so we will be done if we can show that  $\mathbb{P}(Y \in \mathcal{Y}_+ \setminus f(\mathcal{Y}_-)) = o(n^{-3/4})$ . We let  $f$  be the map that changes the sign of the last excursion away from 0. To be precise, for  $y \in \mathcal{Y}_-$ , let  $\tau_{max} = \tau_{max}(y) = \max\{k \leq n-1 : y_k = 0\}$  and let

$$f(y)_i = \begin{cases} y_i, & \text{if } i \leq \tau_{max}; \\ -y_i, & \text{if } i > \tau_{max}. \end{cases}$$



It is immediate that  $f$  has the claimed properties. In particular,  $f(\mathcal{Y}_-) \subseteq \mathcal{Y}_+$  because for  $y \in \mathcal{Y}_-$ , it holds that  $f(y)_k \geq y_k$  for all  $k$ . Moreover,  $\mathcal{Y}_+ \setminus f(\mathcal{Y}_-)$  consists of the paths  $y$  that end at 1 for which the area at time  $n - 1$  is smaller than twice the area of the last excursion away from 0, that is,

$$\mathbb{P}(Y \in \mathcal{Y}_+, Y \notin f(\mathcal{Y}_-)) = \mathbb{P}\left(\sum_{i=1}^k Y_i \geq 0 \text{ for } k \in [n-1], Y_{n-1} = 1, \sum_{i=1}^{n-1} Y_i < 2 \sum_{i=\tau_{max}}^{n-1} Y_i\right).$$

We see that

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^k Y_i \geq 0 \text{ for } k \in [n-1], Y_{n-1} = 1, \sum_{i=1}^{n-1} Y_i < 2 \sum_{i=\tau_{max}}^{n-1} Y_i\right) \\ \leq 4\mathbb{P}\left(\sum_{i=1}^k Y_i \geq 0 \text{ for } k \in [n], Y_n = 0, \sum_{i=1}^n Y_i < 2 \sum_{i=\tau_{max}}^n Y_i\right), \end{aligned}$$

by conditioning on the value of the  $n$ th increment (similar to the calculation (4.4)) and

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^k Y_i \geq 0 \text{ for } k \in [n], Y_n = 0, \sum_{i=1}^n Y_i < 2 \sum_{i=\tau_{max}}^n Y_i\right) \\ \leq \mathbb{P}\left(0 \leq \sum_{i=1}^n Y_i < 2 \sum_{i=\tau_{max}}^n Y_i \mid Y_n = 0\right) \mathbb{P}(Y_n = 0). \end{aligned}$$

Since  $\mathbb{P}(Y_n = 0) = \Theta(n^{-1/2})$ , we are done if we can show that

$$\mathbb{P}\left(0 \leq \sum_{i=1}^n Y_i^{br} < 2 \sum_{i=\tau_{max}}^n Y_i^{br}\right) = o(n^{-1/4}).$$

Pick any  $0 < \varepsilon < 1/4$ . We compute

$$\begin{aligned} \mathbb{P}\left(0 \leq \sum_{i=1}^n Y_i^{br} < 2 \sum_{i=\tau_{max}}^n Y_i^{br}\right) &\leq \mathbb{P}\left(\sum_{i=\tau_{max}}^n Y_i^{br} \geq n^{1+\varepsilon}\right) + \mathbb{P}\left(0 \leq \sum_{i=1}^n Y_i^{br} < 2n^{1+\varepsilon}\right) \\ &= \mathbb{P}\left(\sum_{i=1}^{\tau_1} Y_i^{br} \geq n^{1+\varepsilon}\right) + \mathbb{P}\left(0 \leq \sum_{i=1}^n Y_i^{br} < 2n^{1+\varepsilon}\right), \end{aligned}$$

where  $\tau_1 = \tau_1(Y^{br})$  is the first return time of  $Y^{br}$  to 0. We trivially have that  $\sum_{i=1}^{\tau_1} Y_i^{br} \leq (\tau_1)^2$ , and it is easy to see that

$$\mathbb{P}(\tau_1 = k) = \frac{\frac{1}{2k-1} \binom{2k-1}{k} \binom{2n-2k}{n-k}}{\binom{2n}{n}} \sim \frac{1}{4\sqrt{\pi}} k^{-3/2} n^{1/2} (n-k)^{-1/2},$$

as  $k, n \rightarrow \infty$ . Hence,

$$\mathbb{P}(\tau_1 \geq n^{1/2+\varepsilon/2}) = O(n^{-1/4-\varepsilon/4}) = o(n^{-1/4}).$$

Moreover, by Lemma 4.4, we see that

$$\mathbb{P}\left(0 \leq \sum_{i=1}^n Y_i^{br} < 2n^{1+\varepsilon}\right) = O(n^{1+\varepsilon}n^{-3/2}) = o(n^{-1/4}),$$

so we conclude that

$$\mathbb{P}(Y \in \mathcal{Y}_+, Y \notin f(\mathcal{Y}_-)) = o(n^{-3/4}).$$

This proves (4.4) and

$$\mathbb{P}\left(\sum_{i=1}^k Y_i \geq 0 \text{ for all } k \in [n], Y_n = 0\right) \sim n^{-3/4} \frac{\Gamma(3/4)}{\pi \sqrt{2(1-\rho)}}. \quad \square$$

### 4.3 About half of the sums are even

So far we have looked at the probability that a sequence  $n-1 \geq d_1 \geq \dots \geq d_n \geq 0$  satisfies the dominating condition (2.1). In this section, we consider the parity condition that  $d_1 + \dots + d_n$  is even and prove Lemma 3.2.

Among the sequences  $n-1 \geq d_1 \geq \dots \geq d_n \geq 0$  that satisfy the dominating condition (2.1), let  $\mathcal{E}$  denote the set of such sequences for which the sum  $d_1 + \dots + d_n$  is even, and  $\mathcal{O}$  the sequences for which the sum is odd. We will define a partial matching between  $\mathcal{E}$  and  $\mathcal{O}$  such that the number of unmatched elements of  $\mathcal{E} \cup \mathcal{O}$  is  $o(|\mathcal{E}| + |\mathcal{O}|)$ . This will immediately imply Lemma 3.2.

Each sequence  $n-1 \geq d_1 \geq \dots \geq d_n \geq 0$  corresponds to a right/down path from  $(0, n-1)$  to  $(n, 0)$  taking  $2n-1$  steps. This corresponds to a sequence  $(B_1, \dots, B_{2n-1})$  where  $B_i \in \{\rightarrow, \downarrow\}$  takes the value  $\rightarrow$  if the path goes right (which it does  $n$  times) and  $\downarrow$  if the path goes down (which it does  $n-1$  times). We say  $j \in [2n-2]$  with  $j \equiv 0 \pmod{2}$  is a *switch position* for the sequence  $(B_1, \dots, B_{2n-1})$  if  $(B_j, B_{j+1}) \in \{(\rightarrow, \downarrow), (\downarrow, \rightarrow)\}$ . Switching the sequence  $(B_1, \dots, B_{2n-1})$  at position  $j$  refers to replacing  $(B_j, B_{j+1})$  with the unique element in  $\{(\rightarrow, \downarrow), (\downarrow, \rightarrow)\} \setminus \{(B_j, B_{j+1})\}$ , resulting in some new sequence  $(B'_1, \dots, B'_{2n-1})$ . We make two observations:

- First, for any  $j \in [2n-2]$  with  $j \equiv 0 \pmod{2}$ , the position  $j$  is a switch position for the sequence  $B$  if and only if  $j$  is a switch position for the sequence  $B'$ . Moreover, switching at position  $j$  is self-inverse: switching  $B'$  at position  $j$  gives  $B$ .
- If  $B$  corresponds to a sequence  $n-1 \geq d_1 \geq \dots \geq d_n \geq 0$ , then the sequence  $B'$  also corresponds to a sequence  $n-1 \geq d'_1 \geq \dots \geq d'_n \geq 0$ , where for some  $k \in [n]$

$$d'_i \in \begin{cases} \{d_i\}, & \text{if } i \neq k; \\ \{d_i - 1, d_i + 1\}, & \text{if } i = k. \end{cases}$$

In particular, the parities of  $\sum_i d_i$  and  $\sum_i d'_i$  are different.

It is not necessarily the case that performing a switch on a sequence in  $\mathcal{E}$  results in a sequence in  $\mathcal{O}$  (or vice versa) as the switched sequence may violate the dominating condition (2.1). Therefore, we will only define the matching between subsets  $\mathcal{E}' \subseteq \mathcal{E}$  and  $\mathcal{O}' \subseteq \mathcal{O}$ , where we choose  $\mathcal{E}'$  and  $\mathcal{O}'$  so that for all sequences in  $\mathcal{E}' \cup \mathcal{O}'$ , have some slack in the dominating condition (2.1). We will show that  $|\mathcal{E}' \cup \mathcal{O}'| = (1 + o(1))|\mathcal{E} \cup \mathcal{O}|$ , for which we need the following two lemmas.

**Lemma 4.9.** *As  $n \rightarrow \infty$ ,*

$$\mathbb{P}(A_{\lfloor n/2 \rfloor}, \dots, A_{n-1} \geq 1 \mid A_1, \dots, A_{n-1} \geq 0, Y_{n-1} \in \{-1, 0\}) = 1 - o(1).$$

*Proof.* By Proposition 3.1,  $\mathbb{P}(A_1, \dots, A_{n-1} \geq 0, Y_{n-1} \in \{-1, 0\}) = \Theta(n^{-3/4})$ , so we are done if we show that

$$\mathbb{P}(A_i = 0 \text{ for some } i \geq \lfloor n/2 \rfloor, A_1, \dots, A_{n-1} \geq 0, Y_{n-1} \in \{-1, 0\}) = o(n^{-3/4}).$$

But by conditioning on the value of the  $n$ th increment, we see that

$$\begin{aligned} & \mathbb{P}(A_i = 0 \text{ for some } i \geq \lfloor n/2 \rfloor, A_1, \dots, A_{n-1} \geq 0, Y_{n-1} \in \{-1, 0\}) \\ & \leq 4\mathbb{P}(A_i = 0 \text{ for some } i \geq \lfloor n/2 \rfloor, A_1, \dots, A_{n-1} \geq 0, Y_n = 0), \end{aligned}$$

and Lemma 4.6 and Proposition 1.2 imply that the right-hand side is  $o(n^{-3/4})$  as claimed.  $\square$

Informally, this lemma states that switching the sequence ‘near the last checks’ is unlikely to affect whether the dominating condition (2.1) holds. We next show that all but a negligible fraction of the sequences have switch positions ‘near the last checks’ (which is the near the middle of  $B$ ).

We say a sequence  $(B_1, \dots, B_{2n-1})$  is *k-switchable* if it has a switch position at some even  $i \in [n - 2k, n - 2]$ .

**Lemma 4.10.** *For any  $k \in [n-1]$ , the number of sequences  $(B_1, \dots, B_{2n-1}) \in \{\rightarrow, \downarrow\}^{2n-1}$  that are not  $k$ -switchable is at most  $2^{-k}4^n$ .*

*Proof.* We choose a sequence  $(B_1, \dots, B_{2n-1}) \in \{\rightarrow, \downarrow\}^{2n-1}$  by independently choosing each  $B_i$  uniformly at random. The probability that  $B$  has a switch at an even position  $j \in [2n - 1]$  is  $\frac{1}{2}$ , and the events are independent for different even values of  $j$ . The probability that there are  $k$  independent failures is hence  $2^{-k}$ .  $\square$

We now have all the necessary set-up to conclude the proof.

*Proof of Lemma 3.2.* Set  $k = \lfloor n/4 \rfloor$ . Let  $\mathcal{F}$  denote the set of sequences  $d \in \mathcal{E} \cup \mathcal{O}$  for which the right/down-sequence  $(B_1, \dots, B_{2n-1})$  that corresponds it (as defined earlier this section) is not  $k$ -switchable, or for which the area process takes the value 0 at some point in the last  $2k$  steps. Then, applying Lemma 4.9 and Lemma 4.10, we find that  $|\mathcal{F}| = o(|\mathcal{E} \cup \mathcal{O}|)$ .

We set  $\mathcal{O}' = \mathcal{O} \setminus \mathcal{F}$  and  $\mathcal{E}' = \mathcal{E} \setminus \mathcal{F}$  and define a matching between  $\mathcal{O}'$  and  $\mathcal{E}'$  as follows. Let  $d \in \mathcal{E}' \cup \mathcal{O}'$  and let  $B \in \{\rightarrow, \downarrow\}^{2n-1}$  be the corresponding right/down-path. With  $(\ell, \ell)$  as the unique grid point at which the path crosses the diagonal, as defined in Section 2, we note that  $(B_1, \dots, B_{n-1})$  is a path from  $(0, n-1)$  to  $(\ell, \ell)$ . We will attempt to change the parity by making a switch as close to the end  $(\ell, \ell)$  of this path as possible. By choice of  $\mathcal{F}$ , there is a switch position at some even  $j \in [n - 2k, n - 2]$ . We perform the switch at the last such position. This only affects the last at most  $2k$  positions of the area process, which can only be increased by 1, reduced by 1 or stay the same. Therefore, the dominating condition (2.1) is not affected (by our choice of  $\mathcal{F}$ ). This defines a matching with the desired properties.  $\square$

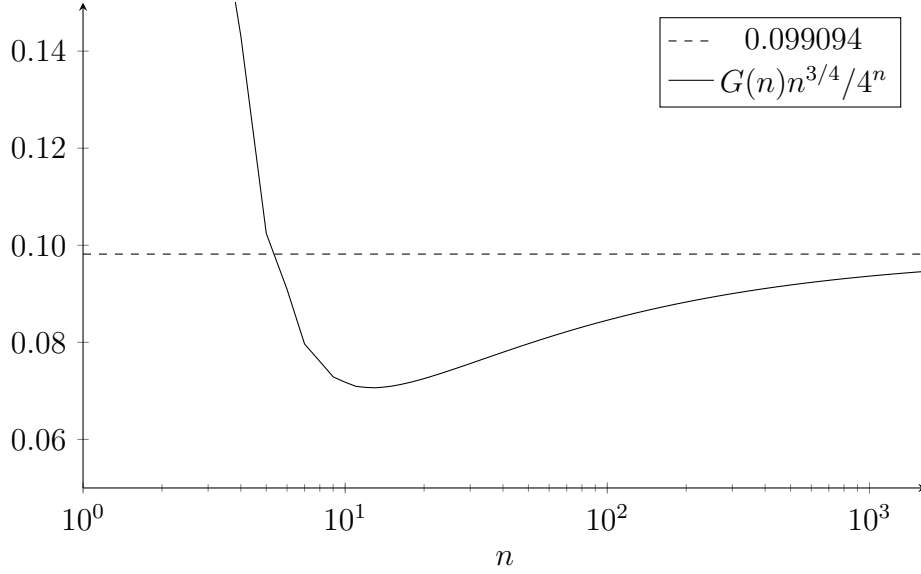


Figure 2: The graph depicts our numerical estimation of  $c_{\text{deg}}$  (dotted line) via the approximation of  $\rho$ , and a numerical estimation based on the exact values of  $G(n)$  for small  $n$ .

## 5 Computational results

In this section we give a new recursion which we have used to calculate  $G(n)$  for many new values, and a description of how we numerically estimated the value of  $\rho$  (and  $c_{\text{deg}}$ ). We also make the surprising observation that, while roughly half the sequences which satisfy the dominating condition have even sum, the convergence to a half is rather slow.

### 5.1 Determining the exact values of $G(n)$ for small $n$

We give a simple recursion to count the number of graphic sequences. In Section 2.1, steps 1 through 5 define a deterministic mapping from a sequence  $n-1 \geq d_1 \geq \dots \geq d_n \geq 0$  to a (non-random) walk  $Y$  (with steps in  $\{-1, 0, 1\}$  and ending in 0 or  $-1$ ), such that the sequence  $d$  satisfies the dominating condition (2.1) if and only if  $\sum_{j=1}^k Y_j \geq 0$  for all  $1 \leq k \leq n-1$ . Moreover, as explained in Step 5, we have  $\sum_{i=1}^n d_i \equiv \sum_{i=1}^{n-1} Y_i \pmod{2}$ . We will use this to count the number of graphic sequences.

Note that a single walk  $Y$  corresponds to  $2^z$  sequences, where  $z$  is the number of zero steps that the sample takes (see Figure 1). We will weight each walk  $Y$  accordingly when counting.

Let  $F(N, y, a)$  be the weighted number of walks  $(Y_i)_{i=1}^N$  which start at  $y$ , take  $N$  steps, end in  $\{-1, 0\}$  and satisfy  $a + \sum_{i=1}^k Y_i \geq 0$  for all  $0 \leq k \leq N$  and  $a + \sum_{i=1}^N Y_i \equiv 0 \pmod{2}$ . We think of  $a$  as the starting value of the area process, or the area from any earlier steps of the walk. We see that by taking  $N = n-1$ ,  $y = 0$  and  $a = 0$ , we count exactly the walks which correspond to sequences which are graphic. Hence, the number of graphic sequences of length  $n$  is  $F(n-1, 0, 0)$ , and we can calculate this using a recursion for  $F$  as follows.

First, we have a boundary condition that  $F(N, y, a) = 0$  whenever  $a < 0$ . If this is not the case, we consider the three options for the first step in the walk and how to complete

the walk after this step. A completion would need to take  $N - 1$  steps in every case, but the starting position and the area will change. The walk could step up, after which a completion starts at  $y + 1$  with area  $a + y + 1$ ; the walk could step down, in which case a completion starts at  $y - 1$  with area  $a + y - 1$ ; or the walk could take a step of size 0, in which case a completion starts at  $y$  with area  $a + y$ . Since in the last case the walk takes a step of size 0 and we are counting the weighted number of completions, this last case must be counted twice. This leads to the following recursion:

$$F(N, y, a) = F(N - 1, y + 1, a + y + 1) + F(N - 1, y - 1, a + y - 1) + 2F(N - 1, y, a + y),$$

with boundary condition  $F(N, y, a) = 0$  whenever  $a < 0$ , and initial condition  $F(0, y, a) = 1$  if  $y \in \{0, -1\}$ ,  $a \geq 0$  and  $a$  even, and  $F(0, y, a) = 0$  otherwise.

To calculate the number of graphic sequences, we calculated every value of  $F$  for a given value of  $N$  using the pre-computed values for  $N - 1$ , then calculated the values for  $N + 1$  using the values for  $N$  and continued this until the calculation ran out of memory. At each step, we get one more value of  $G(n)$  by reading off the value of  $F(n - 1, 0, 0)$ . For this approach to work, we need  $F$  to take only finitely many values for each value of  $N$ , and we only need to calculate  $F(N, y, a)$  for small (depending on  $N$ ) values of  $|y|$  and  $a$ . First, observe that  $F(N, y, a) = 0$  whenever  $y > N$  or  $y < -N - 1$  as the walk needs to end in  $\{0, -1\}$ . We now consider for which values of  $a$  we need to calculate  $F(N, y, a)$ . Clearly, the boundary condition means we do not need to consider  $a < 0$ . Moreover, if  $y < 0$ , we can ignore all  $a < y(y + 1)/2$  as the area process will go negative at some point. This just leaves large  $a$ .

We claim that, given that the walk starts at  $y$ , in  $N$  steps the area can decrease by at most

$$a' = \frac{N^2 - 2Ny + 2N - y^2 + \mathbb{1}_{\text{odd}}(N - y)}{4}.$$

In particular, this means that if  $a \geq a'$ , the area will always stay non-negative (provided it is already) and  $F$  only depends on the parity of  $a$ . Hence, if  $c = \max\{0, a'\}$  and  $a \geq c$ , the value  $F(N, y, a)$  is either  $F(N, y, c)$  or  $F(N, y, c + 1)$ .

Let us briefly justify the claim. First, consider the amount the area process can decrease when the walk starts at 0 and returns to 0. The best thing for the walk to do is for it to take  $\lfloor N/2 \rfloor$  steps downwards, then  $\lfloor N/2 \rfloor$  steps upwards. If  $N$  is odd, one extra step of size 0 should be inserted between the downwards and upwards steps. This reduces the area process by

$$\sum_{i=1}^{\lfloor N/2 \rfloor} i + \mathbb{1}_{\text{odd}}(N) \lfloor N/2 \rfloor + \sum_{i=1}^{\lfloor N/2 \rfloor - 1} i = \frac{N^2 - \mathbb{1}_{\text{odd}}(N)}{4}.$$

If  $0 \leq y \leq N - 1$ , the biggest reduction comes from taking  $y + 1$  steps downwards to  $-1$ , then doing the above steps with the remaining steps (but starting and ending at  $-1$ ). This reduces the area by

$$-\sum_{i=-1}^{y-1} i + \frac{(N - y - 1)^2 - \mathbb{1}_{\text{odd}}(N - y - 1)}{4} + (N - y - 1) = a'.$$

When  $y = N$ , the same equation holds. When  $y < 0$ , then the best thing to do is to reserve  $-y-1$  steps which will be used as upwards steps at the end. For the first  $N+y+1$  steps, one should aim to minimise the area while returning to  $y$ . This gives a total of

$$\frac{(N+y+1)^2 - \mathbb{1}_{\text{odd}}(N+y+1)}{4} - y(N+y+1) + - \sum_{i=y+1}^{-1} i = a'.$$

The final optimisation we implement is to consider the function  $f$  defined by

$$f(N, y, a) = \begin{cases} F(N, y, a) - F(N, y, a-1), & a \neq 0; \\ F(N, y, 0), & a = 0. \end{cases}$$

This satisfies a similar recursion to the function  $F$ , but the numbers are generally smaller and this helps with the memory usage.

We wrote a program in Go which uses this recursion to calculate the number of graphic sequences and ran this on a node with 512 GiB of RAM until the program ran out of memory. This produced the first 1392 numbers (starting from  $n = 0$ ). Of course, we are only interested in calculating  $F(N, 0, 0)$  and we do not need to calculate all values of  $F(N, \cdot, \cdot)$ . Even calculating  $F(N+1, 0, 0)$  only needs three values of  $F(N, \cdot, \cdot)$  (and one is trivially 0). We therefore ran the program a second time now we had established at what point the program ran out of memory and stopped the recursion just before this point. We then attempted to calculate  $F(N+1, 0, 0), F(N+2, 0, 0), \dots$  by only calculating the values  $F$  on demand (and keeping the calculated values in a map). We note that if  $F(N, \cdot, \cdot)$  is known entirely, calculating  $F(M, 0, 0)$  this way is clearly more efficient when  $M$  is not much bigger than  $N$ , but it is less efficient when  $M$  is much larger than  $N$ , and in particular, doing this from the start is less efficient than calculating the full layers. This second step added another 260 values, so that we know now  $G(n)$  for all  $n \leq 1651$ . The code (along with the computed values) is attached to the arXiv submission, and the new values have been added to the OEIS entry [A004251](#). The sequence [A095268](#) counts the number  $G'(n)$  of zero-free graphic sequences, which is given by  $G(n) - G(n-1)$ , and we have also updated this sequence.

## 5.2 A surprising observation on the parity condition

By changing the initial condition, the above recursion can be changed to count  $H(n)$ , the sequences which satisfy the dominating condition and have odd parity. Lemma [3.2](#) shows that  $H(n) \sim G(n)$ , and one might naturally assume that  $|G(n) - H(n)|$  is exponentially smaller than  $G(n)$  (and  $H(n)$ ), but this does not appear to be the case. Surprisingly, numerical estimates suggest that  $G(n) - H(n) = \Theta(4^n/n^{5/2})$  which is only a factor of  $\Theta(n^{7/4})$  smaller than  $G(n)$ .

We remark that if we ignore the dominating condition and just count the number of sequences  $n-1 \geq d_1 \geq \dots \geq d_n \geq 0$  for which  $\sum d_i$  is even and for which  $\sum d_i$  is odd we get

$$\frac{1}{2} \left( \binom{2n-1}{n-1} + \binom{n-1}{\lfloor n/2 \rfloor} \right) \quad \text{and} \quad \frac{1}{2} \left( \binom{2n-1}{n-1} - \binom{n-1}{\lfloor n/2 \rfloor} \right)$$

respectively. The difference between these two quantities is only  $O(2^n)$ , which is exponentially smaller than either.

The partial matching we used to prove Lemma 3.2 suggests a possible explanation for this. In the proof, we switched at the last switchable position before  $n - 1$ . Adapting the proof of the main result in [3] to *lazy* SSRW bridges yields that  $\mathbb{P}(A_{n-1} = 0 \mid Y_{n-1} = 0, A_1, \dots, A_{n-1} \geq 0) = \Theta(n^{-7/4})$ , and a sequence with  $A_{n-1} = 0$  cannot be ‘switched down’. Assuming that roughly half of them would be switched down (with the other half switched up), this yields a proportion  $\Omega(n^{-7/4})$  of the even sequences which do not have an odd counterpart. Of course, there are also odd sequences without an even counterpart (e.g. those with  $A_{n-1} = 1$  and  $A_{n-2} = 0$  cannot be switched down), but this observation makes it at least reasonable that the order of the difference is  $\Theta(4^n/n^{5/2})$ .

### 5.3 Estimating the constant $\rho$

We now give a method for estimating  $\rho$ . Recall that  $(Y_k)$  is the lazy simple symmetric walk and  $(S_k)$  is the random walk formed by summing the (signed) excursion areas  $X_k$  of  $(Y_k)$ . We first need to find the distribution of the excursion areas  $X_k$ . We define the following (partial) generating function

$$g(x, y) = \sum_{i, j > 0} \mathbb{P}(\text{excursion is of area } i \text{ and length } j) x^i y^j.$$

Note that this is really only part of the full generating function, which is  $\frac{y}{2} + g(x, y) + g(x^{-1}, y)$ , to take into account excursions below the axis, and also the probability  $\frac{1}{2}$  event that the lazy random walk does not move on the first step (and hence the excursion area is zero).

**Lemma 5.1.** *The generating function  $g(x, y)$  satisfies the recursive equation*

$$g(x, y) = \frac{xy^2}{16(1 - \frac{xy}{2} - g(x, xy))}.$$

We note that as all terms of  $g(x, y)$  have a positive power of  $y$ , terms in  $g(x, xy)$  have higher  $x$ -degree than their counterparts in  $g(x, y)$ , and hence this equation allows one to recursively evaluate  $g(x, y)$  to any order. For example,

$$g(x, y) = \frac{xy^2}{16} + \frac{x^2y^3}{32} + \frac{x^3y^4}{64} + \frac{x^4(y^4+2y^5)}{256} + \frac{x^5(y^5+y^6)}{256} + \frac{x^6(2y^5+3y^6+2y^7)}{1024} + O(x^7).$$

*Proof.* To have a positive excursion, the first step of the lazy random walk must be up (probability  $\frac{1}{4}$ ). Then it either goes down (probability  $\frac{1}{4}$ ) giving a term  $\frac{1}{16}xy^2$  in  $g(x, y)$ , or it follows some non-negative excursion before returning to height 1. Let the excursion have area  $X_1$  and length  $T_1$  (so that the walk is now at 1 with total area  $X_1 + T_1 + 1$  and length  $T_1 + 1$ ). The rest of the excursion is then equivalent to one starting at 0 at time  $T_1$  with an initial step up, and say this excursion has area  $X_2$  and length  $T_2$ . The sum  $\sum \mathbb{P}(X_1, T_1) x^{X_1} z^{T_1}$  is just  $\frac{z}{2} + g(x, z)$  (as we allow the trivial area zero excursion here), and the sum  $\sum \mathbb{P}(X_2, T_2) x^{X_2} y^{T_2}$  is just  $4g(x, y)$  (as we automatically can assume the first step is up). Overall we get an excursion of length  $T_1 + T_2$  and area  $(X_1 + T_1) + X_2$ . Setting



$z = xy$  and multiplying these (together with a factor of  $\frac{1}{4}$  for the initial step up) gives the remaining terms of  $g(x, y)$ . Hence,

$$g(x, y) = \frac{xy^2}{16} + \frac{1}{4}\left(\frac{xy}{2} + g(x, xy)\right) \cdot 4g(x, y),$$

and the lemma follows from rearranging this equation.  $\square$

Now by setting  $y = 1$  we get a generating function for the (positive) excursion areas, namely

$$g(x, 1) = \frac{x}{16} + \frac{x^2}{32} + \frac{x^3}{64} + \frac{3x^4}{256} + \frac{x^5}{128} + \frac{7x^6}{1024} + \frac{21x^7}{4096} + \frac{37x^8}{8192} + \frac{31x^9}{8192} + O(x^{10}).$$

Note that  $g(1, 1) = \frac{1}{4}$  is the probability of a positive excursion.

The next task is to estimate or bound  $\rho$ , which is the probability that the random walk  $S_k$  with step sizes  $X_k$  hits 0 before becoming negative. To do this we construct a finite state Markov chain with states  $\{-, 0, 1, 2, \dots, n-1, \star\}$  where the states  $-$  and  $\star$  represent the walk going negative and reaching a state at least  $n$  respectively. The states  $-$  and  $\star$  are absorbing, and otherwise we add a random variable distributed like a signed excursion area. If adding this, takes the walk negative or at least  $n$ , the walk moves to  $-$  or  $\star$  as necessary. We start the Markov chain at 0 and run until we either hit 0 again, or one of the states  $-$  or  $\star$ . By a simple coupling argument it is clear that

$$\mathbb{P}(\text{hit } 0) \leq \rho \leq \mathbb{P}(\text{hit } 0) + \mathbb{P}(\text{hit } \star).$$

Taking  $n$  sufficiently large gives us reasonable bounds on  $\rho$ .

As an example, taking  $n = 2$  we have states  $\{-, 0, 1, \star\}$  and transition matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{16} & \frac{3}{16} \\ \frac{3}{16} & \frac{1}{16} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Writing  $h_{ij}$  for the probability of hitting  $j$  starting at  $i$ , we have

$$h_{10} = \frac{1}{16} + \frac{1}{2}h_{10}, \quad h_{1\star} = \frac{1}{4} + \frac{1}{2}h_{1\star},$$

giving  $h_{10} = \frac{1}{8}$ ,  $h_{1\star} = \frac{1}{2}$ . Then

$$h_{00} = \frac{1}{2} + \frac{1}{16}h_{10} = \frac{65}{128}, \quad h_{0\star} = \frac{3}{16} + \frac{1}{16}h_{1\star} = \frac{7}{32}.$$

Hence,

$$\frac{65}{128} \leq \rho \leq \frac{65}{128} + \frac{7}{32} = \frac{93}{128}.$$

Using this method with  $n = 2^{18}$  gives a lower bound of 0.51580258, which appears to be very close to the true value of  $\rho$ . However the upper bound of 0.54543568 obtained by this method seems to still be very far from the truth.

If we make the assumption that hitting 0 before going negative is a decreasing function of the starting point, we can amalgamate the states  $n-1$  and  $\star$  in the above model, giving transitions out of that state as if they were from  $n-1$ . Unfortunately we do not have a proof that  $h_{i0}$  is decreasing in  $i$ , so this does not give a rigorous bound on  $\rho$ . Nevertheless,

for  $n = 2^{18}$  we obtain a (non-rigorous) upper bound of 0.51580289 by this method, which does seem much closer to the true value. An even less rigorous estimate can be obtained by applying Richardson extrapolation to  $h_{00}$  in terms of  $1/n$ , which gives the estimate

$$\rho \approx 0.515802638089141858504490255841,$$

and corresponds to a value of

$$c_{\text{deg}} \approx 0.099094083237488745361449340935.$$

These approximations do not appear to correspond to any number with a simple closed form expression.

## 6 Concluding remarks

We have given a precise asymptotic for the number of graphical sequences. Similar asymptotics are known for tournaments, but not for other natural classes such as uniform hypergraphs and digraphs. Another interesting direction with open problems is the asymptotics of the number of graphical partitions of an integer  $N$ .

We outline some more directions for future research below.

**An upper bound in Lemmas 3.7 and 3.8** For  $x \in \mathbb{R}_{>0}^n$ , let

$$A(x) = \left\{ (\sigma, s) : \sigma \in S_n, s \in \{-1, 1\}^n, \sum_{i=1}^k s_i x_{\sigma(i)} \geq 0 \text{ for all } k \in [n] \right\}.$$

Lemma 3.7 states that  $|A(x)| \geq (2n-1)!!$ , with equality for all  $x$  for which all sums  $\sum_{i \in S} x_i$  are distinct (for distinct  $S \subseteq [n]$ ).

**Conjecture 6.1.** *For  $x \in \mathbb{R}_{>0}^n$ ,  $|A(x)|$  is maximized when  $x_1 = \dots = x_n$ .*

Rephrased in probabilistic terms, we conjecture that the simple symmetric random walk has the highest probability of staying non-negative, amongst all random processes with exchangeable increments  $(X_1, \dots, X_n) \in (\mathbb{R} \setminus \{0\})^n$  of which the law is invariant under sign changes of the elements.

This conjecture would imply (see (3.1)) that for all exchangeable  $(X_1, \dots, X_n) \in (\mathbb{R} \setminus \{0\})^n$  of which the law is invariant under sign changes of the elements,

$$\frac{1}{\sqrt{\pi(n+1/2)}} \leq \mathbb{P} \left( \sum_{i=1}^k X_i \geq 0 \text{ for all } k \in [n] \right) \leq \frac{\sqrt{2}}{\sqrt{\pi n}}.$$

Using Lemma 2.2 and Lemma 4.2, this would immediately imply Proposition 3.4, which yields a direct proof for  $G(n) = \Theta(n^{-3/4} 4^n)$  that does not rely on the results in [38]. We also think the conjecture is of independent interest, in particular because it would imply that for this very general class of random processes, the order of the probability of staying positive does not depend on the law, or even the tail behaviour of the increments.

**Uniformly random graphic sequences** A natural next question, that we intend to answer in future work, is the (asymptotic) law of a uniformly random graphic sequence of length  $n$ . We now make some observations using our reformulation and discuss a potential strategy to answer this question. Firstly, note that a uniform lattice path has the law of a simple symmetric random walk bridge with  $2n$  steps that straddles the line  $y = n - x$ , and in particular, for large  $n$ , its fluctuations away from the line  $y = n - x$  are of order  $n^{1/2}$ . In fact, for any  $\delta > 0$ , it is exponentially unlikely that at some point the lattice path is at distance more than  $\delta n$  from the line  $y = n - x$ . This means that even after conditioning on the lattice path encoding a graphic sequence (i.e. conditioning on an event with probability  $\Theta(n^{-1/4})$ ), it is exponentially unlikely that at some point the lattice path is at distance more than  $\delta n$  from the line  $y = n - x$ , and in general, the large deviations of a uniformly random graphic sequence are completely described by the large deviations of a uniformly random lattice path. (Observe that this implies that a uniform graphic sequence is very different from the degree sequence of a uniform graph, of which the corresponding lattice path stays close to the horizontal line  $y = n/2$ .)

Therefore, to observe the difference between a uniform graphic sequence and a uniform lattice path, we will need to consider their more fine-grained behaviour, for example by studying the scaling limit of their fluctuations around the line  $y = n - x$ . We conjecture the following.

**Conjecture 6.2.** *Let  $D_1 \geq \dots \geq D_n$  be a uniformly random graphic sequence of length  $n$ . Then, there exists a random continuous function  $D$  from  $[0, 1]$  to  $\mathbb{R}$  such that*

$$(n^{-1/2}(D_{\lfloor tn \rfloor} - (1 - t)n), 0 \leq t \leq 1) \xrightarrow{d} (D_t, 0 \leq t \leq 1)$$

*in the uniform topology.*

We expect  $D$  to have two characterizations. Firstly, it can be defined as a Brownian bridge conditioned to satisfy a continuous version of the dominating condition (2.1). Secondly, via a continuous version of the reformulation described in Section 2.1,  $D$  can be constructed via two paths, for which the distance between them is distributed as a conditioned Brownian bridge and their midpoint is determined by a Brownian motion (the Brownian motion plays the role of the ‘lazy steps’ that can go either right or down). This result would add graphic sequences to the long and varied list of uniformly random combinatorial structures with a ‘Brownian’ scaling limit; examples are numerous models of trees of which the scaling limit can be described by a Brownian excursion (see the survey paper [12]), mappings with a limit described by the Brownian bridge [1], various classes of maps with limits encoded by the Brownian snake (see the survey paper [14]) and pattern-avoiding permutations with limits described by a Brownian excursion (see [5] and references therein).

Such scaling limits are interesting in their own right, but can also be exploited to answer questions about the corresponding combinatorial class, such as ‘what proportion of Cayley trees of size  $n$  have height exceeding  $tn^{1/2}$ ?’ or ‘in what proportion of maps from  $[n]$  to  $[n]$  is the average distance to a cycle larger than  $tn^{1/2}$ ?’.

**Persistence probabilities of integrated random processes** The study of the probability that integrated random walks and random walk bridges stay non-negative started

with the work of Sinai on the SSRW [35] and has attracted a lot of attention in the past decade ([3, 7, 8, 13, 37, 38]; see also the survey paper [2]). However, all work on integrated random walk bridges only finds the right order of the persistence probability [3, 38] and for random walks the sharp asymptotics (including the value of the constant) are only known under a  $(2 + \delta)$ -moment condition on the step distribution. Our methods completely carry over to the setting of SSRW bridges, yielding the following result of independent interest. Let  $\hat{Y}$  be a SSRW and let  $\hat{A}$  be its area process.

**Proposition 6.3.** *We have that*

$$n^{1/4} \mathbb{P}(\hat{A}_1, \dots, \hat{A}_{2n} \geq 0 \mid \hat{Y}_{2n} = 0) \rightarrow \frac{\Gamma(3/4)}{\sqrt{2\pi(1 - \hat{\rho})}}$$

as  $n \rightarrow \infty$ , for  $\hat{\rho}$  the probability that the random walk with steps distributed as the signed area of the excursions of a SSRW hits 0 before going negative.

Adapting the method to numerically estimate the value of  $\rho$  described in Section 5.3 shows that  $\hat{\rho} \approx 0.0773408571485249705089600725$ .

It is possible that our techniques can be generalized to other models of random bridges. However, there are some difficulties to overcome, both conceptual and technical. Our proof relies heavily on Lemma 3.8, which requires the areas of different excursions of the bridge to be exchangeable and for their law to be invariant under sign changes. When only considering random walk bridges, these conditions combined restrict the method to (scalings of) symmetric processes with steps in  $\{-1, 0, 1\}$ , so new ideas are needed to adapt the method to other random walk bridges.

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## A Overview of notation

The following notation is used in the paper.

- $G(n)$  = the number of graphic sequences of length  $n$ .
- $Y = (Y_k)_{k \geq 0}$  = lazy random walk, that is, a random process with independent and identically distributed increments that take the value 0 with probability  $1/2$ ,  $-1$  with probability  $1/4$  and  $+1$  with probability  $1/4$ .
- $A_k = \sum_{i=0}^k Y_i$ , so  $(A_k)_{k \geq 0}$  is an integrated random walk.
- $X_i$  = area of the  $i$ th excursion of  $Y$ .
- $S_k = \sum_{i=1}^k X_i$ . We can see  $S$  as a ‘subsequence’ of  $A$ , where  $A \geq 0$  if and only if  $S \geq 0$ .
- $\zeta_1 = \inf\{k \geq 1 : Y_k = 0, A_k \leq 0\}$ .
- $\rho = \mathbb{P}(A_{\zeta_1} = 0)$  = probability that  $(A_k)_{k \geq 0}$  hits zero before the first time it goes negative.
- $Y^{br} = (Y_k^{br})_{k=0}^n$  = lazy random walk bridge, that is,  $(Y_k)_{k=0}^n$  conditioned on  $Y_n^{br} = 0$ .
- $A^{br}, X^{br}, S^{br}$  the counterparts of  $A, X, S$  respectively, with  $Y$  replaced by  $Y^{br}$  in the definition.
- $N_n$  = number of times  $Y^{br}$  hits 0 after time 0 (so  $N_n \in \{1, \dots, n\}$ ).
- $M_n$  = the number of times  $S^{br}$  hits zero.
- $\tilde{S}^{br}, \tilde{X}^{br}$  = perturbed variants of  $S^{br}$  and  $X^{br}$  respectively.
- $\xi_i$  = the  $i$ th time  $S^{br}$  hits zero.
- $\rho_n = \mathbb{P}(M_n \geq 1 \mid S_1^{br}, \dots, S_{N_n}^{br} \geq 0)$ .

## B Proof of Lemma 3.7

**Lemma 3.7.** *Let  $x = (x_1, \dots, x_n) \in \mathbb{R}_{>0}^n$ , let  $\sigma$  be a uniformly random permutation of  $[n]$  and let  $s = (s_1, \dots, s_n)$  be an independent uniformly random element of  $\{-1, 1\}^n$ . Then*

$$\mathbb{P}\left(\sum_{i=1}^k s_i x_{\sigma(i)} \geq 0 \text{ for all } k \in [n]\right) \geq \frac{(2n-1)!!}{2^n n!}.$$

*Moreover, equality holds if for all distinct  $A, A' \subseteq [n]$ , the corresponding sums are also distinct:  $\sum_{i \in A} x_i \neq \sum_{i \in A'} x_i$ .*

We include the proof of Lemma 3.7 from Burns [6] below for the convenience of the reader.

For  $x \in \mathbb{R}_{>0}^n$ , let

$$A(x) = \{(\sigma, s) : \sigma \in S_n, s \in \{-1, 1\}^n, \sum_{i=1}^k s_i x_{\sigma(i)} \geq 0 \text{ for all } k \in [n]\}.$$

We will show that

$$|A(x)| \geq (2n - 1)!!$$

and that equality holds if, for all distinct  $S, S' \subseteq [n]$ , the corresponding sums are also distinct.

We call a vector  $y \in \mathbb{R}_{>0}^n$  *rapidly decreasing* if

$$y_i > y_{i+1} + \cdots + y_n$$

for all  $i \in [n]$ . We will first use induction on  $n$  to show that  $|A(y)| = (2n - 1)!!$  for all rapidly decreasing  $y \in \mathbb{R}_{>0}^n$ . It is clear that  $A(y) = \{(\text{Id}, 1)\}$  when  $n = 1$ , and the claim holds.

Suppose that we have shown the claim for some  $n \geq 1$ , and let  $y \in \mathbb{R}_{>0}^{n+1}$ . Since the sequence is rapidly decreasing, the pair  $(\sigma, s)$  is in  $A(y)$  if and only if  $s(i) = 1$  for all  $i \in [n+1]$  such that  $\sigma(i)$  is the lowest number seen so far, i.e.  $\sigma(i) < \sigma(j)$  for all  $j < i$ . Let  $\alpha = \sigma^{-1}(n+1)$ . If  $\alpha \neq 1$ , then the term  $s_\alpha y_{n+1}$  has no impact on whether the sequence is valid, while if  $\alpha = 1$ , we require  $s_1 = 1$  and then we need the remainder of the sequence to satisfy the condition. More formally, let  $y'$ ,  $\sigma'$  and  $s'$  be what is left after removing  $y_{n+1}$ ,  $n+1$  and  $s_\alpha$  respectively. That is

$$y' = (y_1, \dots, y_n), \quad \sigma'(i) = \sigma(i + \mathbb{1}_{\geq \alpha}(i)), \quad s' = (s_1, \dots, s_{\alpha-1}, s_{\alpha+1}, \dots, s_{n+1}).$$

If  $\alpha = 1$ , then  $(\sigma, s) \in A(y)$  if and only if  $s_1 = 1$  and  $(\sigma', s') \in A(y')$ . If  $\alpha \neq 1$ , then  $(\sigma, s) \in A(y)$  if and only if  $(\sigma', s') \in A(y')$ . Hence, there are  $2n+1$  pairs  $(\sigma, s)$  in  $A(y)$  for each pair  $(\sigma', s')$  in  $A(y')$ . Clearly,  $y'$  is also rapidly decreasing and the result follows by induction.

For  $x \in \mathbb{R}^n$  and  $S \subseteq [n]$ , we write  $x_S = \sum_{i \in S} x_i$ . We call  $x \in \mathbb{R}_{>0}^n$  *sum-distinct* if  $x_S \neq x_{S'}$  for all distinct subsets  $S, S' \subseteq [n]$ . Let  $x \in \mathbb{R}_{>0}^n$  be sum-distinct. We will construct  $y \in \mathbb{R}_{>0}^n$  that is rapidly decreasing and for which  $|A(x)| = |A(y)|$ .

Since  $x$  is sum-distinct, we can define a total order  $<_x$  on the power set  $\mathcal{P}([n])$  by  $S <_x S'$  if and only if  $x_S < x_{S'}$ . We note that for a given  $\sigma$  and  $s$ , the condition  $\sum_{i=1}^k s_i x_{\sigma(i)} \geq 0$  is equivalent to the condition  $x_{S_-} <_x x_{S_+}$  where  $S_\pm = \{\sigma(i) : i \leq k, s_i = \pm 1\}$ , and hence  $A(x)$  only depends on  $x$  through  $<_x$ .

To get to the vector  $y$ , we will increase  $x_1$  until it is larger than  $x_2 + \cdots + x_n$ , and then increase  $x_1$  and  $x_2$  until we also have  $x_2 > x_3 + \cdots + x_n$  etc. We will do this in a series of steps so that each increase only changes the total ordering by a “small” amount, and the following claim shows this preserves  $|A(x)|$ . We defer the proof of this claim to the end of this section and first finish the current proof.

**Claim B.1.** *Let  $x$  and  $x'$  be sum-distinct and assume there is a unique pair  $\{L, R\}$  of disjoint subsets for which  $L <_x R$  yet  $R <_{x'} L$ . Then  $|A(x)| = |A(x')|$ .*

Suppose that we already have  $x_j > x_{j+1} + \dots + x_n$  for all  $j < i$ , and we wish to extend this to include  $j = i$  as well. We will slowly increase  $x_1, x_2, \dots, x_i$  so that we only change one disjoint inequality on the power set at a time and we can use the claim above to show that  $|A(x)|$  does not change. We first ensure that no signed sums are the same by slightly perturbing  $x$  by a small amount, which we choose to be small enough to not change the order  $<_x$ . Indeed, if

$$\varepsilon = \min_{S \neq S'} |x_S - x_{S'}| = \min_{S \neq S'} |x_{S \setminus S'} - x_{S' \setminus S}|,$$

then perturbing each entry of  $x$  by less than  $\varepsilon/n$  cannot possibly change the order  $<_x$ . Let  $z$  be a random vector formed by adding a small independent  $\text{Uniform}[0, \varepsilon/n]$  random variable to each entry of  $x$ , and note that  $<_x = <_z$ , so  $|A(x)| = |A(z)|$ . Almost surely there are no two pairs of disjoint sets  $(A, B)$  and  $(C, D)$  such that  $z_A - z_B = z_C - z_D$ , and we can order the pairs  $(A_t, B_t)$  of disjoint subsets of  $[n] \setminus [i]$  such that  $z_{A_t} - z_{B_t}$  is increasing in  $t$ . Suppose  $(A_\tau, B_\tau)$  is the first pair for which  $z_{A_\tau} - z_{B_\tau} > z_i$ . If there is no such pair, then we already have  $z_i > z_{i+1} + \dots + z_n$  (by choosing  $A = [n] \setminus [i]$  and  $B = \emptyset$ ). Let  $\delta$  be any value in the interval  $(z_{A_\tau} - z_{B_\tau} - z_i, z_{A_{\tau+1}} - z_{B_{\tau+1}} - z_i)$  (or in  $(z_{A_\tau} - z_{B_\tau} - z_i, \infty)$  if there is no pair  $(A_{\tau+1}, B_{\tau+1})$ ), and consider the vector

$$z^{(2)} = (z_1 + 2^{i-1}\delta, z_2 + 2^{i-2}\delta, \dots, z_i + \delta, z_{i+1}, z_{i+2}, \dots, z_n).$$

This is again sum-distinct and there is a unique pair  $(L, R) = (B_\tau \cup \{i\}, A_\tau)$  such that  $L <_z R$  but  $R <_{z^{(2)}} L$ , so  $|A(z^{(2)})| = |A(z)| = |A(x)|$ . It also still has the property that  $z_j^{(2)} > z_{j+1}^{(2)} + \dots + z_n^{(2)}$  for  $j < i$ . It may not yet have the property that  $z_i^{(2)} > z_{i+1}^{(2)} + \dots + z_n^{(2)}$ , but we do have  $z_i^{(2)} > z_{A_\tau} - z_{B_\tau}$ , and we can choose  $\delta^{(2)}$  in the interval  $(z_{A_{\tau+1}} - z_{B_{\tau+1}} - z_i^{(2)}, z_{A_{\tau+2}} - z_{B_{\tau+2}} - z_i^{(2)})$  and repeat to get  $z^{(3)}, z^{(4)}, \dots$ . The process terminates at some  $k$  when  $z_i^{(k)} > z_{i+1}^{(k)} + \dots + z_n^{(k)}$ , and we take this to be our new  $x$ . Repeating this process for  $i = 1, 2, \dots, n-1$  in turn, gives a rapidly decreasing vector  $y$  with  $|A(y)| = |A(x)|$ , as required.

We still need to prove the claim for  $x$  which are not sum-distinct. As before let

$$\varepsilon = \min_{x_S \neq x_{S'}} |x_S - x_{S'}| = \min_{x_S \neq x_{S'}} |x_{S \setminus S'} - x_{S' \setminus S}|.$$

If each entry is perturbed by less than  $\varepsilon/n$  to get a vector  $z$ , then

$$\sum_{i=1}^k s_i x_{\sigma(i)} \geq 0 \iff \sum_{i=1}^k s_i z_{\sigma(i)} > -\varepsilon \iff \sum_{i=1}^k s_i z_{\sigma(i)} \geq 0.$$

Therefore,  $A(x) \supseteq A(z)$ . If we get  $z$  by adding a small  $\text{Uniform}[0, \varepsilon/n]$  random variable to each entry of  $x$ , then  $z$  is almost surely sum-distinct and the result follows since  $|A(x)| \geq |A(z)| = (2n-1)!!$  almost surely.

We now return to the proof of Claim B.1.

*Proof of Claim B.1.* Let  $k = |L \cup R|$ , and consider the function  $f: S_n \times \{-1, 1\}^n \rightarrow S_n \times \{-1, 1\}^n$  which acts as follows. For any  $(\sigma, s) \in S_n \times \{-1, 1\}^n$ , the function  $f$  maps  $(\sigma, s)$  to  $(\sigma', s')$  where  $\sigma'$  is the permutation which is reversed on the first  $k$  inputs, and

$s'$  is formed by reversing the order of the first  $k$  entries in  $s$  and negating them. That is,  $\sigma'(i) = \sigma(k+1-i)$  for  $i \in [k]$  and  $\sigma'(i) = \sigma(i)$  for  $i \in [k+1, n]$ , and  $s'_i = -s_{k+1-i}$  for  $i \in [k]$  and  $s'_i = s_i$  for  $i \in [k+1, n]$ .

The function  $f$  is clearly self-inverse and hence bijective, and we will show that  $f(A(x) \setminus A(x')) \subseteq A(x') \setminus A(x)$ . By switching the roles of  $x$  and  $x'$  and of  $L$  and  $R$ , it follows that  $f(A(x') \setminus A(x)) \subseteq A(x) \setminus A(x')$ , and so  $|A(x)| = |A(x')|$ .

Suppose that  $(\sigma, s) \in A(x) \setminus A(x')$  and let  $f(\sigma, s) = (\sigma', s')$ . It is obvious that  $(\sigma', s') \notin A(x)$  as

$$\sum_{i=1}^k s'_i x_{\sigma'(i)} = - \sum_{i=1}^k s_i x_{\sigma(i)} < 0.$$

Since  $(\sigma, s) \notin A(x')$ , there must be at least one  $\ell$  for which  $\sum_{i=1}^{\ell} s_i x'_{\sigma(i)} < 0$ . We first show that there is exactly one choice for  $\ell$ , and that it is  $k$ . Let  $S_{\pm} = \{\sigma(i) : i \in [\ell], s_i = \pm 1\}$ . Then

$$\begin{aligned} \sum_{i=1}^{\ell} s_i x_{\sigma(i)} &= x_{S_+} - x_{S_-} > 0, \\ \sum_{i=1}^{\ell} s_i x'_{\sigma(i)} &= x'_{S_+} - x'_{S_-} < 0. \end{aligned}$$

In other words,  $S_- <_x S_+$  and  $S_+ <_{x'} S_-$ . Since  $S_+ \cap S_- = \emptyset$ , we find that  $(S_-, S_+) = (L, R)$  and the only option for  $\ell$  is  $k$ .

Using this we can check that  $(\sigma', s') \in A(x')$ . For  $j \in [k]$ , we have

$$\begin{aligned} \sum_{i=1}^j s'_i x'_{\sigma'(i)} &= - \sum_{i=k-j+1}^k s_i x'_{\sigma(i)} \\ &= - \sum_{i=1}^k s_i x'_{\sigma(i)} + \sum_{i=1}^{k-j} s_i x'_{\sigma(i)}. \end{aligned}$$

Both of these terms are non-negative since  $\sum_{i=1}^{\ell} s_i x'_{\sigma(i)} < 0$  if and only if  $\ell = k$ . Similarly, for  $j > k$ , we have

$$\begin{aligned} \sum_{i=1}^j s'_i x'_{\sigma'(i)} &= \sum_{i=1}^k s'_i x'_{\sigma'(i)} - \sum_{i=1}^k s_i x'_{\sigma(i)} + \sum_{i=1}^j s_i x'_{\sigma(i)} \\ &= -2 \sum_{i=1}^k s_i x'_{\sigma(i)} + \sum_{i=1}^j s_i x'_{\sigma(i)} \geq 0. \end{aligned}$$

Hence,  $(\sigma', s') \in A(x') \setminus A(x)$ . □

## C Proof of Lemma 4.4

**Lemma 4.4.** *We have*

$$\lim_{n \rightarrow \infty} \sup_{a, b} |n^2 \mathbb{P}(Y_n = a, A_n = b) - \phi(n^{-1/2}a, n^{-3/2}b)| = 0$$

where the supremum runs over all  $(a, b) \in \mathbb{Z}^2$  and

$$\phi(x, y) = \frac{2\sqrt{3}}{\pi} \exp(-4x^2 + 12xy - 12y^2).$$

The proof is an adaptation of the proof of Proposition 1 in [3].

Let  $a, b \in \mathbb{Z}$ . We will use Fourier inversion to estimate  $\mathbb{P}(Y_n = a, A_n = b)$ . We let  $f_n: \mathbb{R}^2 \rightarrow \mathbb{C}$  given by

$$(t_1, t_2) \mapsto \mathbb{E}[e^{i(t_1 Y_n + t_2 A_n)}]$$

be the characteristic function of  $(Y_n, A_n)$ , so that by 2-dimensional Fourier inversion, we have

$$\mathbb{P}(Y_n = a, A_n = b) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f_n(t_1, t_2) e^{-i(t_1 a + t_2 b)} dt_1 dt_2. \quad (\text{C.1})$$

We observe that, for  $L_1, L_2, \dots$  i.i.d. random variables distributed as the steps of  $Y$  (i.e.  $\mathbb{P}(L_i = 0) = \frac{1}{2}$ ,  $\mathbb{P}(L_i = -1) = \mathbb{P}(L_i = 1) = \frac{1}{4}$ ) we have

$$L_1 \stackrel{d}{=} \frac{1}{2}(B_1 + B_2)$$

for  $B_1$  and  $B_2$  two i.i.d. Bernoulli random variables with  $\mathbb{P}(B_1 = 1) = \mathbb{P}(B_2 = -1) = \frac{1}{2}$ . Therefore, the characteristic function of  $L_1$  satisfies

$$\mathbb{E}[e^{itL_1}] = \mathbb{E}[e^{itB_1/2}]^2 = \cos^2\left(\frac{t}{2}\right).$$

Moreover, note that

$$(Y_n, A_n) \stackrel{d}{=} (\sum_{k=1}^n L_k, \sum_{k=1}^n (n-k+1)L_k) \stackrel{d}{=} (\sum_{k=1}^n L_k, \sum_{k=1}^n kL_k),$$

so

$$f_n(t_1, t_2) = \prod_{k=1}^n \mathbb{E}[e^{i(t_1 + kt_2)L_k}] = \prod_{k=1}^n \cos^2\left(\frac{t_1 + kt_2}{2}\right).$$

In particular, we observe that for  $n \geq 2$ , the absolute value of the integrand in (C.1) is equal to 1 if and only if  $t = (t_1, t_2) = (0, 0)$  and is strictly smaller otherwise. We will examine the contribution to the integral of  $t$  in a small region around  $(0, 0)$ , and we will show that the contribution is negligible outside of that region.

Define  $T_1 = \{(t_1, t_2) \in [-\pi, \pi]^2 : |t_1| + n|t_2| \geq \pi\}$ . Then, for  $n \geq 2$ , it is not too hard to see that there exists a  $c < 1$  such that  $|\cos(\frac{t_1 + kt_2}{2})| \leq c$  for at least half of the values of  $k = 1, \dots, n$ . Hence, on  $T_1$ ,

$$|f_n(t_1, t_2)| \leq c^n = \exp(-\Omega(n)).$$

Now define  $T_2 = \{(t_1, t_2) : n^{-1/3} \leq |t_1| + n|t_2| < \pi\}$ . We observe that we have the bound  $\cos^2 x \leq e^{-x^2}$  for  $|x| \leq \frac{\pi}{2}$  and

$$\sum_{k=1}^n (t_1 + kt_2)^2 = nt_1^2 + n(n+1)t_1 t_2 + \frac{1}{6}n(n+1)(2n+1)t_2^2. \quad (\text{C.2})$$

Hence, on  $T_2$ , we have  $\sum_{k=1}^n (t_1 + kt_2)^2 = n(t_1 + (n+1)t_2/2)^2 + n(n^2 - 1)t_2^2/12 = \Omega(n^{1/3})$  as  $\max\{|nt_2|, |t_1 + (n+1)t_2/2|\} = \Omega(n^{-1/3})$ . Thus

$$|f_n(t_1, t_2)| \leq \exp\left(-\frac{1}{4} \sum_{k=1}^n (t_1 + kt_2)^2\right) = \exp(-\Omega(n^{1/3})).$$

We deduce that

$$\frac{n^2}{(2\pi)^2} \iint_{T_1 \cup T_2} f(t_1, t_2) e^{-i(t_1 a + t_2 b)} dt_1 dt_2 = o(1)$$

as  $n \rightarrow \infty$ , uniformly in  $a$  and  $b$ .

Finally consider  $T_3 = \{t : |t_1| + n|t_2| < n^{-1/3}\}$ . Now for small  $x$ ,  $\cos^2 x = \exp(-x^2 + O(x^4))$ , and so (C.2) implies that on  $T_3$  we have

$$f_n(t_1, t_2) = \exp\left(-\frac{1}{4}\left(nt_1^2 + n^2 t_1 t_2 + \frac{n^3}{3} t_2^2 + O(n^{-1/3})\right)\right).$$

Writing  $t_1 = n^{-1/2} s_1$ ,  $t_2 = n^{-3/2} s_2$ ,  $v_1 = n^{-1/2} a$  and  $v_2 = n^{-3/2} b$ , we have

$$\begin{aligned} & \frac{n^2}{(2\pi)^2} \iint_{T_3} f(t_1, t_2) e^{-i(t_1 a + t_2 b)} dt_1 dt_2 \\ &= \frac{n^2}{(2\pi)^2} \iint_{T_3} \exp\left(-\frac{1}{4}\left(nt_1^2 + n^2 t_1 t_2 + \frac{n^3}{3} t_2^2 + O(n^{-1/3})\right)\right) e^{-i(t_1 a + t_2 b)} dt_1 dt_2 \\ &= \frac{1}{(2\pi)^2} \iint_{|s_1| + |s_2| \leq n^{1/6}} \exp\left(-\frac{1}{4}\left(s_1^2 + s_1 s_2 + \frac{1}{3} s_2^2 + O(n^{-1/3})\right)\right) e^{-i(s_1 v_1 + s_2 v_2)} ds_1 ds_2 \\ &\rightarrow \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} \exp\left(-\frac{1}{4}\left(s_1^2 + s_1 s_2 + \frac{1}{3} s_2^2\right)\right) e^{-i(s_1 v_1 + s_2 v_2)} ds_1 ds_2 \end{aligned}$$

as  $n \rightarrow \infty$  uniformly over all  $v_1$  and  $v_2$ . Now

$$\begin{aligned} & \frac{1}{(2\pi)^2} \iint \exp\left(-\frac{1}{4}\left(s_1^2 + s_1 s_2 + \frac{1}{3} s_2^2\right)\right) e^{-i(s_1 v_1 + s_2 v_2)} ds_1 ds_2 \\ &= \frac{1}{(2\pi)^2} \iint \exp\left(-\frac{1}{2} s^T R^{-1} s\right) e^{-i s^T v} ds_1 ds_2 = \frac{\sqrt{\det R}}{2\pi} \exp\left(-\frac{1}{2} v^T R v\right), \end{aligned}$$

where

$$s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad R^{-1} = \frac{1}{2} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{2}{3} \end{pmatrix}, \quad R = \begin{pmatrix} 8 & -12 \\ -12 & 24 \end{pmatrix}, \quad \det R = 48.$$

Therefore, we deduce that

$$n^2 \mathbb{P}(Y_n = a, A_n = b) \rightarrow \frac{\sqrt{\det R}}{2\pi} \exp\left(-\frac{1}{2} v^T R v\right) = \frac{2\sqrt{3}}{\pi} \exp(-4v_1^2 + 12v_1 v_2 - 12v_2^2)$$

as  $n \rightarrow \infty$ , uniformly over all  $(v_1, v_2)$ , as required.

## D Proof of Lemma 4.5

**Lemma 4.5.** *There exists a constant  $C$  such that for all  $n \geq 1$ ,*

$$\mathbb{P}(A_n = Y_n = 0, A_1, \dots, A_n \geq 0) \leq Cn^{-5/2}.$$

With Lemma 4.4 in hand, the proof of Lemma 4.5 is a direct adaptation of the proof of the upper bound of Theorem 1 of [3] for simple symmetric random walks. We include the proof for our case for completeness. Let  $\bar{Y} = \bar{Y}(Y, n)$  be the adjoint process of  $Y$  on  $[n]$ , i.e. for  $i \in [n]$ , we set  $\bar{Y}_i := Y_n - Y_{n-i}$  and let  $\bar{A}$  be the area process of  $\bar{Y}$ .

Denote  $Y_k = \sum_{i=1}^k L_i$  for all  $k$ , set  $b = \lfloor n/4 \rfloor$ , and define the events

$$\begin{aligned}\Omega_n^+ &= \{A_1, \dots, A_b \geq 0\} \in \sigma(L_1, \dots, L_b), \\ \bar{\Omega}_n^+ &= \{\bar{A}_1, \dots, \bar{A}_b \geq 0\} \in \sigma(L_{n-b+1}, \dots, L_n).\end{aligned}$$

Observe that  $\bar{Y}$  has the same law as  $Y$ , so  $\Omega_n^+$  and  $\bar{\Omega}_n^+$  are independent and have equal probability. Moreover, by Theorem 1 and 2 in [37],  $\mathbb{P}(\Omega_n^+) = \Theta(n^{-1/4})$ .

Furthermore, on the event  $\{Y_n = A_n = 0\}$ , we see that  $\bar{A}_k = A_{n-k}$  for any  $k \in [n]$ , so

$$\begin{aligned}\mathbb{P}(Y_n = A_n = 0, A_1, \dots, A_n \geq 0) &\leq \mathbb{P}(\Omega_n^+ \cap \bar{\Omega}_n^+ \cap \{Y_n = A_n = 0\}) \\ &= \mathbb{P}(\Omega_n^+)^2 \mathbb{P}(Y_n = A_n = 0 \mid \Omega_n^+ \cap \bar{\Omega}_n^+) \\ &= \Theta(n^{-1/2}) \mathbb{P}(Y_n = A_n = 0 \mid \Omega_n^+ \cap \bar{\Omega}_n^+).\end{aligned}$$

Now, we observe that  $\Omega_n^+ \cap \bar{\Omega}_n^+$  only depends on  $L_1, \dots, L_b$  and  $L_{n-b+1}, \dots, L_n$ , so

$$\begin{aligned}\mathbb{P}(Y_n = A_n = 0 \mid \Omega_n^+ \cap \bar{\Omega}_n^+) &\leq \sup_{(\ell_i)} \mathbb{P}(Y_n = A_n = 0 \mid L_i = \ell_i \text{ for } i \in \{1, \dots, b\} \cup \{n-b+1, \dots, n\}),\end{aligned}$$

where the supremum is over all choices of  $\ell_1, \dots, \ell_b, \ell_{n-b+1}, \dots, \ell_n \in \{-1, 0, 1\}$ . We see that given the values of  $L_1, \dots, L_b$  and  $L_{n-b+1}, \dots, L_n$ , the processes  $(Y, A)$  restricted to  $b+1, \dots, n-b$  have the same distribution as a lazy random walk and its integrated counterpart with both processes started at some different point, which implies that

$$\mathbb{P}(Y_n = A_n = 0 \mid \Omega_n^+ \cap \bar{\Omega}_n^+) \leq \sup_{y, a} \mathbb{P}(Y_{n-2b} = y, A_{n-2b} = a),$$

which is  $\Theta(n^{-2})$  by Lemma 4.4. The result now follows.