

Improved bounds for 1-independent percolation on \mathbb{Z}^n

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Abstract

A 1-independent bond percolation model on a graph G is a probability distribution on the spanning subgraphs of G in which, for all vertex-disjoint sets of edges S_1 and S_2 , the states of the edges in S_1 are independent of the states of the edges in S_2 . Such a model is said to percolate if the random subgraph has an infinite component with positive probability. In 2012 the first author and Bollobás defined $p_{\max}(G)$ to be the supremum of those p for which there exists a 1-independent bond percolation model on G in which each edge is present in the random subgraph with probability at least p but which does not percolate.

A fundamental and challenging problem in this area is to determine the value of $p_{\max}(G)$ when G is the lattice graph \mathbb{Z}^2 . Since $p_{\max}(\mathbb{Z}^n) \leq p_{\max}(\mathbb{Z}^{n-1})$, it is also of interest to establish the value of $\lim_{n \rightarrow \infty} p_{\max}(\mathbb{Z}^n)$. In this paper we significantly improve the best known upper bound on this limit and obtain better upper and lower bounds on $p_{\max}(\mathbb{Z}^2)$. In proving these results, we also give an upper bound on the critical probability for a 1-independent model on the hypercube graph to contain a giant component asymptotically almost surely.

1 Introduction

A *percolation model* on a (possibly infinite) graph G is a probability distribution on the subgraphs of G . We say that a vertex or edge of G is *open* if it is present in the random subgraph of G associated with the percolation model, and *closed* otherwise. A *bond percolation model* on G is a probability distribution on the spanning subgraphs of G , meaning all vertices are open and the random subgraph depends only on the edges. An *independent* bond percolation model on G is a bond percolation model in which each edge is open independently. The focus of this paper is on the weaker condition of *1-independence*: a bond percolation model on G is said to be *1-independent* if, for any

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two disjoint sets of edges S_1 and S_2 of G such that no edge in S_1 shares a vertex with an edge in S_2 , the states (i.e. open or closed) of the edges in S_1 are independent of the states of the edges in S_2 . If G is an infinite graph, then we say that a percolation model on G *percolates* if the associated random subgraph of G has an infinite component with positive probability. Note that by a slightly modified version of Kolmogorov's zero-one law, if a percolation model on a locally-finite infinite graph G percolates, then in fact the random subgraph has an infinite component with probability one.

The study of 1-independent bond percolation models is motivated by their use as a tool to obtain bounds on critical probabilities of independent models via renormalisation (see, for example, [3] and Sections 3.5 and 6.2 of [6]). They have also been combined with renormalisation techniques to analyse bond percolation models with dependencies over a greater range (see [3] again for an example).

For an infinite graph G , we wish to obtain results on whether or not a 1-independent model on G in which each edge is open with some probability $p \in (0, 1)$ percolates. For any given p there will be (uncountably) many different 1-independent models, but we might expect that if p is sufficiently large, then any 1-independent model on G will percolate. In this spirit, in [2] the first author and Bollobás defined $\mathcal{D}_{\geq p}(G)$ to be the class of 1-independent bond percolation models on G in which each edge is open with probability at least p . They then defined

$$p_{\max}(G) = \sup \{p : \text{some model in } \mathcal{D}_{\geq p}(G) \text{ does not percolate}\}.$$

Note that in this definition it is equivalent to ask for a model in which each edge is open with probability exactly p . Indeed, if $\mu \in \mathcal{D}_{\geq p}(G)$ does not percolate, then the model in which we sample from μ and then independently delete each edge e with probability $1 - p/\mathbb{P}_\mu(e \text{ open})$ is a 1-independent bond percolation model which does not percolate, and in which each edge is open with probability exactly p .

The value of $p_{\max}(G)$ for different infinite graphs G has been studied, along with various other properties of 1-independent bond percolation models, in [2, 3, 11, 13, 18]. In particular, in [2] the authors determine $p_{\max}(T)$ for all locally finite infinite trees T in terms of a parameter known as the *branching number* of T . They go on to establish that $p_{\max}(G) \geq \frac{1}{2}$ for all locally finite connected infinite graphs G , and they construct such a G which achieves this lower bound.

The case where G is the lattice graph \mathbb{Z}^2 has been of particular interest, with the first author and Bollobás asking for the value of $p_{\max}(\mathbb{Z}^2)$.

Question 1.1 ([2]). *What is the value of $p_{\max}(\mathbb{Z}^2)$?*

Considering \mathbb{Z}^n more generally, they noted that $p_{\max}(\mathbb{Z}^n)$ is decreasing in n (since if a model does not percolate on \mathbb{Z}^n then it does not percolate on any \mathbb{Z}^{n-1} subspace), and asked for the limit of this value as $n \rightarrow \infty$.

Question 1.2 ([2]). *What is the value of $\lim_{n \rightarrow \infty} p_{\max}(\mathbb{Z}^n)$?*

The best known lower bound on both $p_{\max}(\mathbb{Z}^2)$ and $\lim_{n \rightarrow \infty} p_{\max}(\mathbb{Z}^n)$ comes from a construction of Day, Falgas-Ravry, and Hancock in [11] which gives

$$p_{\max}(\mathbb{Z}^n) \geq 4 - 2\sqrt{3} = 0.535898 \dots$$

for all $n \geq 2$. It was conjectured by Falgas-Ravry and Pfenninger in [13] that there exists some $n \geq 3$ for which this is an equality. In [11] the authors also give a separate construction which yields

$$p_{\max}(\mathbb{Z}^2) \geq p_{\text{site}}^2 + \frac{1}{2}(1 - p_{\text{site}}),$$

where $p_{\text{site}} = p_{\text{site}}(\mathbb{Z}^2)$ denotes the critical probability for independent site percolation on \mathbb{Z}^2 , that is, the supremum of the p such that percolation does not occur in the percolation model on \mathbb{Z}^2 in which vertices are open independently with probability p and edges are open if both their endpoints are open. The best known rigorous lower bound on p_{site} is 0.556 due to van den Berg and Ermakov [24]. Substituting this into the above gives $p_{\max}(\mathbb{Z}^2) \geq 0.531136$ which is slightly worse than the bound given by the first construction. However, using the non-rigorous estimate $p_{\text{site}} \approx 0.592746$ (see, for example, [29]) gives an improved bound of $p_{\max}(\mathbb{Z}^2) \geq 0.554974$.

The previous best known upper bound on $p_{\max}(\mathbb{Z}^2)$ was 0.8639 due to the first author, Bollobás, and Walters in [3], and no better upper bounds on $\lim_{n \rightarrow \infty} p_{\max}(\mathbb{Z}^n)$ were known. In this paper, we improve both upper bounds as follows.

Theorem 1.3. $p_{\max}(\mathbb{Z}^2) \leq 0.8457$.

Theorem 1.4. $\lim_{n \rightarrow \infty} p_{\max}(\mathbb{Z}^n) \leq 0.5847$.

We also improve the lower bound for $p_{\max}(\mathbb{Z}^2)$.

Theorem 1.5. $p_{\max}(\mathbb{Z}^2) \geq \frac{1}{32}(35 - 3\sqrt{33}) = 0.555197 \dots$

Note that this new bound is an improvement even on the non-rigorous bound stated above. However, it is likely that the true value of $p_{\max}(\mathbb{Z}^2)$ is still some distance from this, as suggested by the following high confidence result which gives an even better bound (see Section 4).

Result 1.6. *With high confidence (p -value $< 10^{-11}$) we have $p_{\max}(\mathbb{Z}^2) \geq 0.5921$.*

The proof of Theorem 1.3 uses a 2×2 renormalisation argument similar to that employed in [3]. We obtain a better bound than was derived in that paper by using a different condition for two renormalised sites to be joined by an edge. To calculate a lower bound on the probability that each edge is present, we solve two linear programming problems that are relaxations of the 1-independence constraints on a 4×2 grid.

The proof of Theorem 1.4 relies on the following result which translates a condition on 1-independent bond percolation models on the hypercube to one on the lattice. Here we denote by Q_n the n -dimensional hypercube graph which has vertex set $\{0, 1\}^n$ and in which vertices are joined by an edge if they differ in exactly one coordinate.

Theorem 1.7. *Let $p \in (0, 1]$ and $k \in \mathbb{N}$ be constants and let $P > (1 - p)^{2^{k-1}} \varphi$, where $\varphi = \frac{1}{2}(\sqrt{5} + 1)$ is the golden ratio. Suppose that for any model in $\mathcal{D}_{\geq p}(Q_k)$, the probability that the graph is connected is at least P . Then for large enough n , every model in $\mathcal{D}_{\geq p}(\mathbb{Z}^n)$ percolates. In particular,*

$$\lim_{n \rightarrow \infty} p_{\max}(\mathbb{Z}^n) \leq p.$$

In order to extract a concrete bound from this result we need the following lemma, which states that $p = 0.5847$ and $k = 6$ satisfy the conditions of the theorem.

Lemma 1.8. *Let φ be the golden ratio. Then there exists a constant $P > (1 - 0.5847)^{32}\varphi$ such that in any model in $\mathcal{D}_{\geq 0.5847}(Q_6)$, the probability that the graph is connected is at least P .*

Together, these results give Theorem 1.4. The approach we use to prove Theorem 1.7 also yields the following result about the critical probability for the asymptotically almost sure existence of a giant component in the hypercube graph Q_n under 1-independent bond percolation models, which may be of independent interest.

Theorem 1.9. *Let $p \in (0, 1]$ and $k \in \mathbb{N}$ be constants and let $P > (1 - p)^{2^{k-1}}\varphi$, where φ is the golden ratio. Suppose that for any model in $\mathcal{D}_{\geq p}(Q_k)$, the probability that the graph is connected is at least P . Then there exists $C > 0$ such that for $n \in \mathbb{N}$, and any model in $\mathcal{D}_{\geq p}(Q_n)$, there is a component containing at least $C \cdot 2^n$ vertices with probability at least $1 - e^{-\Omega(n^2)}$.*

Motivated by the statement of this theorem, let

$$p_{\text{giant}} = \inf \left\{ p : \exists C > 0 : \lim_{n \rightarrow \infty} \inf_{\mu \in \mathcal{D}_{\geq p}(Q_n)} \mathbb{P}_\mu(Q_n \text{ has a component of size } \geq C \cdot 2^n) = 1 \right\}$$

be the threshold p for the asymptotically almost sure existence of a giant component in Q_n under any model in $\mathcal{D}_{\geq p}(Q_n)$. Then Theorem 1.9 together with Lemma 1.8 imply that $p_{\text{giant}} \leq 0.5847$. The best known lower bound on p_{giant} comes from the simple construction used in the proof of Theorem 1.4 in [2], which shows that $p_{\text{giant}} \geq \frac{1}{2}$.

It is interesting to compare the known bounds on these 1-independent threshold probabilities with their analogues in the independent setting. Firstly, the celebrated Harris–Kesten theorem [15, 17] determines the threshold for percolation in the independent bond model on \mathbb{Z}^2 to be exactly $1/2$, whereas Result 1.6 suggests that $p_{\text{max}}(\mathbb{Z}^2) \geq 0.5921$. Turning to p_{giant} , Erdős and Spencer showed in [12] that for each fixed $\varepsilon > 0$, the random subgraph of Q_n in which each edge is open independently with probability $p = (1 - \varepsilon)/n$ has maximum component size $o(2^n)$ asymptotically almost surely, while Ajtai, Komlós, and Szemerédi showed in [1] that under the same model with $p = (1 + \varepsilon)/n$ where $\varepsilon > 0$ is fixed, there is a component of size $\Omega(2^n)$. The component structure has since been studied when $\varepsilon \rightarrow 0$ at different rates, for example in [5, 7, 8, 9, 26, 27] (see also [25] or Section 13.4 of [16] for a survey). The threshold probability for percolation of the independent model on \mathbb{Z}^n is $(1 + o(1))/2n$; see [16] (particularly Section 15.5) for the history of this fact and a survey of more precise results.

After outlining some notation and preliminary results, in Section 2 we prove Theorem 1.9 via a series of propositions and lemmas. That section contains many of the key results and ideas used in the proof of Theorem 1.7, which is dealt with in Section 3. The proof of Lemma 1.8 is handled in Section 3.1, and Section 4 contains the proof of Theorem 1.5 and the explanation of Result 1.6. We prove Theorem 1.3 in Section 5, and finally in Section 6 we discuss some possible avenues for future enquiry.

1.1 Notation and preliminaries

We use standard graph theoretic notation throughout. Although we often work on subgraphs of Q_n , the distance between two vertices will always be the Hamming distance, or equivalently, the distance between the vertices in Q_n . For a vertex v of a subgraph G of Q_n , define the second neighbourhood of v in G to be the set of vertices whose shortest path to v in G has length exactly 2, and write this as $N_G^2(v)$.

For two tuples $u \in \mathbb{Z}^i$ and $v \in \{0, 1\}^j$ where $i, j \in \mathbb{N}$, we will write $u \parallel v$ to mean the concatenation of u and v ; if $j = 1$, so that $v = (0)$ or $v = (1)$, then we will write $u \parallel 0$ or $u \parallel 1$ respectively. For $n \in \mathbb{N}$, we will write $[n]$ for the set $\{1, 2, \dots, n\}$.

We now state the Chernoff bound we will use throughout this paper (see [21] for a discussion of this result).

Lemma 1.10 (Chernoff bound). *Let $n \in \mathbb{N}$, let $p \in [0, 1]$, and let $X \sim \text{Bin}(n, p)$. Then for all $\varepsilon \in [0, 1]$,*

$$\mathbb{P}(X \leq (1 - \varepsilon)np) \leq e^{-\varepsilon^2 np/2}.$$

We also use the following simple corollary of Markov's inequality.

Lemma 1.11. *Let X be a random variable taking values in the interval $[0, N]$ for some $N \in (0, \infty)$, and suppose that $\mathbb{E}(X) \geq cN$ for some $c \in (0, 1]$. Then for all $\varepsilon \in (0, 1)$,*

$$\mathbb{P}(X \geq (1 - \varepsilon)cN) \geq c\varepsilon.$$

Proof. Applying Markov's inequality we have

$$\begin{aligned} \mathbb{P}(X \geq (1 - \varepsilon)cN) &\geq 1 - \mathbb{P}(N - X \geq N - (1 - \varepsilon)cN) \\ &\geq 1 - \frac{N - \mathbb{E}(X)}{N - (1 - \varepsilon)cN} \\ &= \frac{\mathbb{E}(X)/N - (1 - \varepsilon)c}{1 - (1 - \varepsilon)c} \\ &\geq \frac{c\varepsilon}{1 - (1 - \varepsilon)c} \\ &\geq c\varepsilon. \end{aligned} \quad \square$$

2 Giant component in Q_n

We begin by studying the giant component in the hypercube Q_n , and prove Theorem 1.9. A key step in the proof is a renormalisation argument which reduces the problem to one concerning a more general class of percolation models where both the vertices and edges may be open or closed, but which still maintains that the states of subgraphs induced on disjoint subsets of vertices are independent. In return for allowing vertices to be closed, edges in the renormalised models have a higher probability of being open when both endpoints are open. Formally, we will consider percolation models from the following class.

Definition 2.1. For a (possibly infinite) graph G , and for $p_v \in (0, 1]$ and $p_e \in [0, 1]$, define $\mathcal{D}(G, p_v, p_e)$ to be the family of percolation models on G in which

- (i) vertices are open with probability at least p_v ;
- (ii) conditioned on their endpoints being open, edges are open with probability at least p_e ; and
- (iii) for all pairs of vertex-disjoint subgraphs G_1 and G_2 of G , the states of the vertices and edges in G_1 are independent of the states of the vertices and edges in G_2 .

The main result on such percolation models is the following proposition, proved in Section 2.2, from which Theorem 1.9 will follow almost immediately after a suitable renormalisation.

Proposition 2.2. *Let $p_v \in (0, 1]$ and $p_e \in (\frac{18}{19}, 1]$ be constants. Then there exists a positive constant C such that for all $n \in \mathbb{N}$, under every model in $\mathcal{D}(Q_n, p_v, p_e)$, there is a component containing at least $C \cdot 2^n$ vertices with probability at least $1 - e^{-\Omega(n^2)}$.*

In the next section we will prove Theorem 1.9 assuming Proposition 2.2, then in Section 2.2 we will return to prove the proposition via a series of lemmas. In order to apply Theorem 1.9 we will make use of Lemma 1.8, which shows that the conditions of Theorem 1.9 hold when $p = 0.5847$, and which is proved in Section 3.1.

2.1 Proof of Theorem 1.9

We now give the proof of Theorem 1.9 assuming Proposition 2.2 holds.

Proof of Theorem 1.9. Let $n \geq k$ be an integer. For a given 1-independent bond percolation model μ_n on Q_n in which each edge is open with probability p , we will inductively define coupled percolation models μ_{n-k}, \dots, μ_1 on Q_{n-k}, \dots, Q_1 respectively, starting with μ_{n-k} . We say a vertex $v \in Q_{n-k}$ is open under μ_{n-k} if the hypercube $H_v = \{v \parallel a : a \in \{0, 1\}^k\} \subseteq Q_n$ is connected under μ_n . An edge of Q_{n-k} between two open vertices u and v is chosen to be open under μ_{n-k} if at least one of the edges between the two hypercubes H_u and H_v is open under μ_n (so that the $(k+1)$ -dimensional cube $H_u \cup H_v$ is connected). For $i \in \{k, \dots, n-2\}$, define μ_{n-i-1} to be the model on Q_{n-i-1} in which a vertex v is open if the edge between the (open) vertices $v \parallel 0$ and $v \parallel 1$ (in Q_{n-i}) is open under μ_{n-i} , and an edge uv between two open vertices is open if the edge between $u \parallel 0$ and $v \parallel 0$ or the edge between $u \parallel 1$ and $v \parallel 1$ is open.

For each $i \geq k$, define $q_i = (1-p)^{2^i}$, so that $q_{i+1} = q_i^2$ for these i . Let $s_k = P$, and for $i \geq k$, let $s_{i+1} = s_i^2 - q_i$. Note that $s_i^2 > q_i \varphi^2 > 0$ for all $i \geq k$, where φ is the golden ratio. Indeed, $s_k^2 = P^2 > q_k \varphi^2$ by assumption, and if $s_{i-1}^2 > q_{i-1} \varphi^2$ for $i \geq k+1$, then $s_i^2 = (s_{i-1}^2 - q_{i-1})^2 > q_{i-1}^2 (\varphi^2 - 1)^2 = q_i \varphi^2$. Thus, for all $i \geq k$, we can define $r_i = q_i / s_i^2 \in [0, 1]$.

Claim 2.3. *For all $k \leq i \leq n-1$, we have $\mu_{n-i} \in \mathcal{D}(Q_{n-i}, s_i, 1 - r_i)$.*

Proof. We prove the claim by induction on i , considering first the case $i = k$. Since μ_n is a 1-independent bond percolation model, in μ_{n-k} vertices are open independently, and they are each open with probability at least $s_k = P$ by assumption. Let uv be an edge of Q_{n-k} and consider the hypercubes in Q_n corresponding to u and v . Write A for the event that all the edges in Q_n between these hypercubes are closed. Since all these edges are open independently, the probability of A occurring is q_k . Thus we have

$$\begin{aligned}\mathbb{P}(uv \text{ open} \mid u, v \text{ open}) &= \mathbb{P}(\{u, v \text{ open}\} \cap \{uv \text{ open}\}) / \mathbb{P}(u, v \text{ open}) \\ &\geq (\mathbb{P}(u, v \text{ open}) - \mathbb{P}(A)) / \mathbb{P}(u, v \text{ open}) \\ &\geq 1 - q_k / P^2 \\ &= 1 - r_k.\end{aligned}$$

Finally, if G_1 and G_2 are vertex-disjoint subgraphs of Q_{n-k} , then the states of the vertices and edges in each are determined by the states of the edges in two vertex-disjoint subgraphs of Q_n . Since μ_n is a 1-independent model, the states of the vertices and edges in G_1 and G_2 are independent of each other and hence $\mu_{n-k} \in \mathcal{D}(Q_{n-k}, s_k, 1 - r_k)$.

Now let $k + 1 \leq i \leq n - 1$ and assume that $\mu_{n-i+1} \in \mathcal{D}(Q_{n-i+1}, s_{i-1}, 1 - r_{i-1})$. This means each vertex in Q_{n-i+1} is open independently with probability at least s_{i-1} , and given that two endpoints of an edge of Q_{n-i+1} are open, the edge itself is open with probability at least $1 - r_{i-1}$. It follows that the probability that a vertex of Q_{n-i} is open is at least $(1 - r_{i-1})s_{i-1}^2 = s_i$. If G_1 and G_2 are vertex-disjoint subgraphs of Q_{n-i} , then the states of the vertices and edges in each are determined by the states of the vertices and edges in two vertex-disjoint subgraphs of Q_{n-i+1} . Since $\mu_{n-i+1} \in \mathcal{D}(Q_{n-i+1}, s_{i-1}, 1 - r_{i-1})$, we deduce that the states of the edges and vertices in G_1 and G_2 are independent of each other.

It just remains to check that an edge uv of Q_{n-i} is present with probability at least $1 - r_i$ given that u and v are open. Let $u_b = u \parallel b$ and $v_b = v \parallel b$ for each $b \in \{0, 1\}$. The edge uv is open in Q_{n-i} exactly when the edges u_0u_1 , v_0v_1 , and at least one of u_0v_0 and u_1v_1 are open in Q_{n-i+1} . Let B be the event that u_0 , u_1 , v_0 and v_1 are all open in Q_{n-i+1} . Then

$$\begin{aligned}\mathbb{P}(uv \text{ open} \mid u, v \text{ open}) &= \mathbb{P}(\{uv \text{ open}\} \cap \{u, v \text{ open}\}) / \mathbb{P}(u, v \text{ open}) \\ &= \mathbb{P}(\{u_0u_1, v_0v_1 \text{ open}\} \cap \{u_0v_0 \text{ or } u_1v_1 \text{ open}\}) / \mathbb{P}(u_0u_1, v_0v_1 \text{ open}) \\ &= \mathbb{P}(\{u_0u_1, v_0v_1 \text{ open}\} \cap \{u_0v_0 \text{ or } u_1v_1 \text{ open}\} \mid B) / \mathbb{P}(u_0u_1, v_0v_1 \text{ open} \mid B) \\ &\geq \frac{\mathbb{P}(u_0u_1, v_0v_1 \text{ open} \mid B) - \mathbb{P}(u_0v_0, u_1v_1 \text{ closed} \mid B)}{\mathbb{P}(u_0u_1, v_0v_1 \text{ open} \mid B)} \\ &\geq 1 - \frac{r_{i-1}^2}{(1 - r_{i-1})^2} \\ &= 1 - \frac{q_{i-1}^2 / s_{i-1}^4}{(s_{i-1}^2 - q_{i-1})^2 / s_{i-1}^4} \\ &= 1 - r_i.\end{aligned}$$

Hence, $\mu_{n-i} \in \mathcal{D}(Q_{n-i}, s_i, 1 - r_i)$ as required. \square

Claim 2.4. *We have $r_i \rightarrow 0$ as $i \rightarrow \infty$.*

Proof. For $i \geq k$, we have

$$\frac{1}{r_{i+1}} = \frac{s_{i+1}^2}{q_{i+1}} = \left(\frac{s_i^2 - q_i}{q_i} \right)^2 = \left(\frac{1}{r_i} - 1 \right)^2.$$

If $1/r_i = \varphi^2 + \varepsilon$ for some $\varepsilon > 0$, then

$$\frac{1}{r_{i+1}} = (\varphi^2 - 1 + \varepsilon)^2 = (\varphi + \varepsilon)^2 \geq \varphi^2 + 2\varphi\varepsilon.$$

In particular, this can be iterated to obtain

$$\frac{1}{r_{i+j}} \geq \varphi^2 + (2\varphi)^j \varepsilon.$$

By assumption $1/r_k = P^2/(1-p)^{2^k} = \varphi^2 + \varepsilon$ for some $\varepsilon > 0$ and, as $2\varphi > 1$, we have $r_i \rightarrow 0$ as claimed. \square

Let I be a constant which is large enough so that $r_I < \frac{1}{19}$. Then for $n > I$, we have $\mu_{n-I} \in \mathcal{D}(Q_{n-I}, s_I, 1-r_I)$ where $s_I > 0$ and $1-r_I > \frac{18}{19}$, so by Proposition 2.2 there exists a constant C (independent of n and μ_n) such that, under μ_{n-I} , there is a component in Q_{n-I} containing at least $C \cdot 2^{n-I}$ vertices with probability at least $1 - e^{-\Omega(n^2)}$. By the construction of $\mu_{n-k}, \dots, \mu_{n-I}$, the existence of a component of size s in Q_{n-I} under μ_{n-I} implies the existence of a component of size at least $s \cdot 2^I$ in Q_n under μ_n . It follows that with probability at least $1 - e^{-\Omega(n^2)}$, the random subgraph of Q_n under μ_n has a component containing at least $C \cdot 2^n$ vertices, which completes the proof of the theorem. \square

2.2 Proof of Proposition 2.2

In this section we will prove Proposition 2.2 via a series of lemmas. Throughout this section, when considering a model in $\mathcal{D}(Q_n, p_v, p_e)$ for some $n \in \mathbb{N}$ and $p_v, p_e \in [0, 1]$ we will write G for the associated random subgraph of Q_n , and H for the random subgraph of Q_n induced by the open vertices (thus G is the subgraph of H obtained by deleting closed edges). For any vertex v of Q_n , we will denote by H_v the subgraph of H induced by vertices with the same first coordinate as v . For each $n \in \mathbb{N}$, order the vertices of Q_n deterministically, for example using the lexicographic ordering. We will assume that subsets of $V(Q_n)$ inherit this ordering. In particular, for $v \in V(Q_n)$ we will often work with the first $\lceil p_v^2 n^2 / 4 \rceil$ vertices in $N_{H_v}^2(v)$ when using the fixed global ordering, and we will denote this set by $I(v)$ provided it exists. We make the following definitions.

Definition 2.5. Let $n \in \mathbb{N}$, $p_v \in (0, 1]$, $p_e \in (0, 1]$, and $\varepsilon \in (0, 1)$. Assume we are given a percolation model in $\mathcal{D}(Q_n, p_v, p_e)$.

- (i) A vertex v of Q_n is *vertex-good* if it is open and $|N_{H_v}^2(v)| \geq p_v^2 n^2 / 4$. In this case we denote by $I(v)$ the first $\lceil p_v^2 n^2 / 4 \rceil$ vertices of $N_{H_v}^2(v)$.

- (ii) A vertex v of Q_n is ε -edge-good if it is vertex-good and at least $(1-\varepsilon)(2p_e-1)p_v^2n^2/4$ vertices of $I(v)$ are joined to v by a path of length 2 in G .
- (iii) An unordered pair $\{u, v\}$ of distinct vertices of Q_n is ε -vertex-good if u and v are vertex-good and there are at least $(1-\varepsilon)p_v^2n^2/4$ vertex-disjoint paths in H , each of length at most 15, between $I(u)$ and $I(v)$.
- (iv) An unordered pair $\{u, v\}$ of distinct vertices of Q_n is ε -edge-good if u and v are vertex-good and there are at least $(1-\varepsilon)(15p_e-14)p_v^2n^2/4$ vertex-disjoint paths in G between $I(u)$ and $I(v)$.

Observe that parts (ii) and (iv) of this definition are only substantive when $p_e > 1/2$ and $p_e > 14/15$ respectively, and note that whether or not a vertex v is vertex-good or a pair of vertices $\{u, v\}$ is ε -vertex-good is a property of the sites only. Also, whether or not v is vertex-good or ε -edge-good depends only on the sites and bonds in the copy of Q_{n-1} inside Q_n defined by vertices with the same first coordinate as v .

We will prove Proposition 2.2 by first showing that for all $\varepsilon \in (0, 1)$, with probability at least $1 - e^{-\Omega(n^2)}$, every pair of distinct vertex-good vertices within distance 9 of one another form an ε -edge-good pair. We then show that there exists ε such that if this event occurs, then any two ε -edge-good vertices at distance at most 9 from one another are in the same component in G . Next, by showing that for all $\varepsilon \in (0, 1)$, with probability at least $1 - e^{-\Omega(n^2)}$, every vertex in Q_n has an ε -edge-good vertex within Q_n -distance 4, we show that there exists ε such that with probability at least $1 - e^{-\Omega(n^2)}$ all ε -edge-good vertices are in the same component. To complete the proof of the proposition we show that for all $\varepsilon \in (0, 1)$, with probability at least $1 - e^{-\Omega(n^2)}$ a constant fraction of the vertices in Q_n are ε -edge-good. Parts of the arguments that follow, particularly in the proofs of Lemma 2.8 and Lemma 2.12, are based on arguments of McDiarmid, the fourth author, and Withers in [20].

Lemma 2.6. *For all $p_v \in (0, 1]$, $p_e \in (0, 1]$, and $n \geq 12$, and all models in the class $\mathcal{D}(Q_n, p_v, p_e)$, the probability that a given vertex is vertex-good is at least $p_v^3/3$.*

Proof. Let $n \geq 12$ and fix a model in $\mathcal{D}(Q_n, p_v, p_e)$. Fix a vertex $v \in V(Q_n)$, condition on it being open, and denote the size of its second neighbourhood in H_v by $X_v = |N_{H_v}^2(v)|$. Let $u \in N_{Q_n}^2(v)$ have the same first coordinate as v , and let w be a common neighbour of u and v in Q_n . Since vertices of Q_n are open independently, the probability that u and w are open given that v is open is at least p_v^2 . Thus, the probability that u is in $N_{H_v}^2(v)$ is at least p_v^2 and $\mathbb{E}[X_v \mid v \text{ open}] \geq p_v^2 \binom{n-1}{2}$. Hence, applying Lemma 1.11 and noting that $p_v^2 n^2/4 \leq 2p_v^2 \binom{n-1}{2}/3$ for $n \geq 12$, we have

$$\begin{aligned} \mathbb{P}(v \text{ vertex-good} \mid v \text{ open}) &\geq \mathbb{P}(X_v \geq 2p_v^2 \binom{n-1}{2}/3 \mid v \text{ open}) \\ &\geq p_v^2/3. \end{aligned}$$

It follows that the probability that v is vertex-good is at least $p_v^3/3$ as required. \square

Lemma 2.7. *Let $p_v \in (0, 1]$, $p_e \in (\frac{1}{2}, 1]$, and $\varepsilon \in (0, 1)$ be constants. Then there exists a constant $c = c(p_v, p_e, \varepsilon) > 0$ such that for all $n \geq 12$ and all models in $\mathcal{D}(Q_n, p_v, p_e)$, the probability that a given vertex is ε -edge-good is at least c .*

Proof. Let $n \geq 12$, fix a model in $\mathcal{D}(Q_n, p_v, p_e)$ and $v \in V(Q_n)$, and condition on the event that v is vertex-good. Let X_v be the number of elements of $I(v)$ which are joined to v by a path of length 2 in G . Given a vertex $u \in I(v)$, let w be an open neighbour of both u and v . By a union bound, the probability that at least one of the edges uw and wv is closed is at most $2(1 - p_e)$, so the probability that both of them are open is at least $2p_e - 1$. Hence, $\mathbb{E}[X_v \mid v \text{ is vertex-good}] \geq (2p_e - 1) \lceil p_v^2 n^2 / 4 \rceil$. Applying Lemmas 1.11 and 2.6 we have

$$\begin{aligned} \mathbb{P}(v \text{ } \varepsilon\text{-edge-good}) &= \mathbb{P}(X_v \geq (1 - \varepsilon)(2p_e - 1)p_v^2 n^2 / 4 \mid v \text{ vertex-good}) \\ &\quad \cdot \mathbb{P}(v \text{ vertex-good}) \\ &\geq \varepsilon(2p_e - 1) \cdot p_v^3 / 3, \end{aligned}$$

which completes the proof of the lemma. \square

Lemma 2.8. *Let $p_v \in (0, 1]$, $p_e \in (0, 1]$ and $\varepsilon \in (0, 1)$ be constants. Then, in any model in $\mathcal{D}(Q_n, p_v, p_e)$, all pairs $\{u, v\}$ of distinct, vertex-good vertices of Q_n at distance at most 9 from one another are ε -vertex-good with probability at least $1 - e^{-\Omega(n^2)}$.*

Proof. Note that we may assume n is large. Fix a model in $\mathcal{D}(Q_n, p_v, p_e)$, and let u and v be distinct vertices of Q_n at distance at most 9 from one another. We will show that the probability that u and v are both vertex-good, but the pair $\{u, v\}$ is not ε -vertex-good is at most $e^{-\Omega(n^2)}$. Since there are at most 4^n pairs of vertices in Q_n , a union bound will complete the proof.

Roughly, we will choose large sets $A \subseteq I(u)$ and $B \subseteq I(v)$, pair up the vertices in A and B , and then to each pair (a, b) associate many paths from a to b in Q_n . The paths for a given pair will be vertex-disjoint except at their endpoints so the internal vertices will all be open independently. This means it is highly likely that for at least one of these paths all the vertices will be open. The paths for different pairs will be vertex-disjoint and so there will be a vertex-open path for each pair independently. Since A and B are large, a Chernoff bound will complete the proof.

We identify Q_n with $\mathcal{P}(n)$ in the natural way and assume without loss of generality that $u = \emptyset$ and $v = [d]$, where $d \leq 9$ is the distance between u and v . Start by revealing the states of all vertices at distance (in Q_n) at most 2 from u or v . If u or v is not vertex-good, then there is nothing to prove and we are done. Otherwise we have large sets $I(u)$ and $I(v)$ of vertices that are at distance 2 in H from u and v respectively. Let $N = N(p_v, \varepsilon)$ be a large constant, and construct sets $A \subseteq I(u)$ and $B \subseteq I(v)$ as follows.

1. Remove from $I(u)$ or $I(v)$ any sets which differ from u or v in any of the elements $1, \dots, 13N + 9$. Let the new sets be A_0 and B_0 respectively.
2. Arbitrarily delete elements from the larger of A_0 and B_0 until they are of equal size to obtain A and B .

We first show that A and B are large. As elements of $I(u)$ or $I(v)$ differ from u and v in exactly two elements, the first step above removes at most $(13N + 9)n$ elements from each

of $I(u)$ and $I(v)$. As $|I(u)| = |I(v)| = \lceil p_v^2 n^2 / 4 \rceil$, we have $|A| = |B| = \min\{|A_0|, |B_0|\} \geq (1 - \varepsilon/2)p_v^2 n^2 / 4$ for large enough n .

Arbitrarily pair up elements of A and B to obtain $(a_1, v \cup b_1), \dots, (a_\ell, v \cup b_\ell)$ where $\ell = |A| = |B|$ and $a_i \in A$ and $v \cup b_i \in B$ for $i = 1, \dots, \ell$. Here each a_i and b_i is a pair of elements in $\{13N + 10, \dots, n\}$. To each pair $(a, b) = (a_i, b_i)$ we associate an integer in $[13]$ which we will use to construct the paths associated to the pair. Given a pair (a, b) , label the elements of a as α_1 and α_2 , and the elements of b as β_1 and β_2 , where we assume that $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$. If $\alpha_1, \alpha_2, \beta_1$, and β_2 are all distinct, then they have six possible orderings when they are sorted into ascending order, and we associate each order with a distinct number in $[6]$ arbitrarily. If $\alpha_1 = \beta_1$ but $\alpha_2 \neq \beta_2$, then there are two possible orderings and we associate them to 7 and 8. Continuing in this manner, we can associate a unique integer in $[13]$ to each of the 13 possible orderings.

Given a pair (a, b) and $j \in [N]$, define a path $P_{a,j}$ from a to $v \cup b$ as follows. Let $k \in [13]$ be the integer associated to (a, b) as above, and let $x = (k - 1)N + j + 9$. Begin with the vertex a followed by $a \cup \{x\}$, $a \cup \{\beta_1, x\}$, and $a \cup b \cup \{x\}$, ignoring duplicate vertices. From here we add the elements of $v = [d]$ one by one in increasing order until we reach $a \cup b \cup v \cup \{x\}$. Finally, we add the vertices $b \cup v \cup \{\alpha_2, x\}$, $b \cup v \cup \{x\}$, and $b \cup v$. Note that the path contains at most 16 vertices, with the exact number depending on $|a \cap b|$ and d .

Claim 2.9. *Let $a, a' \in A$ and $j, j' \in [N]$ be such that $a \neq a'$. Then the paths $P_{a,j}$ and $P_{a',j'}$ are vertex-disjoint, and if $j \neq j'$, then the paths $P_{a,j}$ and $P_{a,j'}$ are vertex-disjoint except at their endpoints.*

Proof. Let $v \cup b \in B$ be the vertex paired with a , and let $k \in [13]$ be the integer associated to the pair (a, b) . We start with the second part of the claim. Every vertex in the path $P_{a,j}$ except the two end vertices contains a unique element $x = (k - 1)N + j + 9$ in $[10, 13N + 9]$, and so $P_{a,j}$ and $P_{a,j'}$ can only share an internal vertex if $(k - 1)N + j + 9 = (k - 1)N + j' + 9$, which implies $j = j'$. Moreover, it cannot be the case that an internal vertex of $P_{a,j}$ is an endpoint of $P_{a,j'}$ as the endpoints do not contain any element of $[10, 13N + 9]$.

Turning to the first part of the claim, let z be a vertex on the path $P_{a,j}$. We will show that the set a is uniquely determined. Suppose first that z contains an element $x \in [10, 13N + 9]$. Since $x > 9$, we have $x \notin v$ and since $x \leq 13N + 9$, we have $x \notin a \cup b$. Hence, $x = (k - 1)N + j + 9$ and we can read off the value of k . If $z \cap v = \emptyset$, then z is one of $a \cup \{x\}$, $a \cup \{\beta_1, x\}$, or $a \cup b \cup \{x\}$. Using the value of k and the size of z we can deduce which case we are in, and further, which of the elements form the pair a . Similarly, if $v \subseteq z$, then z is one of $a \cup b \cup v \cup \{x\}$, $b \cup v \cup \{\alpha_2, x\}$, or $b \cup v \cup \{x\}$, and using the value of k and the size of z we can deduce which case we are in, and further, which of the elements form the pair b (which then determines a). Finally, if $\emptyset \neq z \cap v \neq v$ then $z \setminus v = a \cup b \cup \{x\}$ and again we can determine where we are in the path (from $|z \cap v|$) and the value of a from k . Finally, we consider the case where z does not contain an element $x \in [10, 13N + 9]$. This means $z = a$ or $z = b \cup v$, and a is clearly uniquely determined in both cases. \square

For each $a \in A$, let C_a be the event that all the vertices are open in at least one

of the paths $P_{a,j}$, $j \in [N]$. By construction, every vertex on the paths except for the endpoints are at distance at least 3 from u and v , and they are still open independently with probability at least p_v . The paths contain at most 14 internal vertices and hence each path is open with probability at least p_v^{14} . As the paths are disjoint except at their endpoints, the vertices in each path are open independently and, by choosing N large enough relative to ε and p_v , the event C_a has probability at least $1 - \varepsilon/2$ for all a .

By Claim 2.9, for distinct $a, a' \in A$ the events C_a and $C_{a'}$ depend on disjoint sets of vertices and are independent. By a Chernoff bound, C_a holds for at least $(1 - \varepsilon)p_v^2 n^2/4$ values of a with probability at least $1 - e^{-\Omega(n^2)}$. There are at most 4^n pairs of vertices in Q_n , so by a union bound, every pair $\{u, v\}$ of distinct, vertex-good vertices in Q_n at distance at most 9 from one another is ε -vertex-good with probability at least $1 - e^{-\Omega(n^2)}$, as required. \square

Lemma 2.10. *Let $p_v \in (0, 1]$, $p_e \in (\frac{14}{15}, 1]$, and $\varepsilon \in (0, 1)$ be constants. Then, under every model in $\mathcal{D}(Q_n, p_v, p_e)$, every pair of distinct, vertex-good vertices of Q_n at distance at most 9 from one another form an ε -edge-good pair with probability at least $1 - e^{-\Omega(n^2)}$.*

Proof. Let $n \in \mathbb{N}$ and fix a model in $\mathcal{D}(Q_n, p_v, p_e)$. Reveal the graph H (i.e., the states of the vertices, but not the states of the edges) and suppose that every pair of distinct vertex-good vertices at distance at most 9 from one another form an $(\varepsilon/2)$ -vertex-good pair. By Lemma 2.8, the probability that this does not hold is at most $e^{-\Omega(n^2)}$. Let u and v be distinct vertex-good vertices in H at distance at most 9 from one another in Q_n . We will show that the probability that the pair $\{u, v\}$ is not ε -edge-good (given the graph H) is at most $e^{-\Omega(n^2)}$, then since there are at most 4^n choices for u and v , the probability that there are two distinct vertex-good vertices which do not form an ε -edge-good pair is also at most $e^{-\Omega(n^2)}$.

By our assumption on H , the pair $\{u, v\}$ is an $(\varepsilon/2)$ -vertex-good pair so there is a set of $\lceil (1 - \varepsilon/2)p_v^2 n^2/4 \rceil$ vertex-disjoint paths in H , each of length at most 15, between $I(u)$ and $I(v)$. Since the paths are vertex-disjoint and we have only conditioned on the realisation of H , it follows from the fact that the model is in $\mathcal{D}(Q_n, p_v, p_e)$ that each path is open independently. Further, the probability that a given path is open is at least $15p_e - 14$, and applying a Chernoff bound shows that $\{u, v\}$ is an ε -edge-good pair with probability at least $1 - e^{-\Omega(n^2)}$, as required. \square

In the proof of the next lemma we will make use of the following theorem of Wilson which gives the maximum number of edge-disjoint copies of K_4 that can be packed into K_n (see also [10]).

Theorem 2.11 ([28]). *If n is sufficiently large, then the maximum cardinality of a set of 4-subsets of $[n]$ which pairwise intersect in at most one element is $\lfloor \frac{n}{4} \lfloor \frac{n-1}{3} \rfloor \rfloor - 1$ if $n \equiv 7, 10 \pmod{12}$ and $\lfloor \frac{n}{4} \lfloor \frac{n-1}{3} \rfloor \rfloor$ otherwise.*

In fact, it will be sufficient for our purposes that this maximum cardinality is $\Omega(n^2)$, and it is not difficult to construct suitable sets of this size.

Lemma 2.12. *Let $p_v \in (0, 1]$ and $p_e \in (\frac{18}{19}, 1]$ be constants. Then there exists a constant $\varepsilon \in (0, 1)$ such that, under every model in $\mathcal{D}(Q_n, p_e, p_v)$, the probability that all ε -edge-good vertices in Q_n are in the same component in G is at least $1 - e^{-\Omega(n^2)}$.*

Proof. Let $\varepsilon \in (0, 1)$ be a constant small enough that $(1 - \varepsilon)(19p_e - 16) > 2$. Let $n \in \mathbb{N}$ and fix a model in $\mathcal{D}(Q_n, p_v, p_e)$. Note that we may assume n is large in terms of p_v , p_e , and ε . We will start by showing that if u and v are distinct vertices of Q_n at distance at most 9 from one another such that u and v are ε -edge-good vertices and $\{u, v\}$ is an ε -edge-good pair, then u and v are in the same component in G .

Indeed, let u and v be such vertices, and note that they are also both vertex-good. As $\{u, v\}$ is ε -edge-good, there is a set S of at least $(1 - \varepsilon)(15p_e - 14)p_v^2 n^2 / 4$ vertex-disjoint paths from $I(u)$ to $I(v)$ in G . Let T_1 be the set of pairs (a, b) such that a and b are the endpoints of a path in S , where $a \in I(u)$ and $b \in I(v)$. Arbitrarily pair up the remaining elements of $I(u)$ and $I(v)$ and let the set of all these pairs be T , so that $|T| = \lceil p_v^2 n^2 / 4 \rceil$.

Let T_2 be the set of pairs in T whose first entry has a path to u in G of length 2, and let T_3 be the set of pairs whose second entry has a path to v in G of length 2. Then, since u and v are ε -edge-good vertices, we have $|T_2|, |T_3| \geq (1 - \varepsilon)(2p_e - 1)p_v^2 n^2 / 4$. Thus,

$$\begin{aligned} |T_1| + |T_2| + |T_3| &\geq (1 - \varepsilon)[2(2p_e - 1) + (15p_e - 14)]p_v^2 n^2 / 4 \\ &= (1 - \varepsilon)(19p_e - 16)p_v^2 n^2 / 4. \end{aligned}$$

By our choice of ε , we have $(1 - \varepsilon)(19p_e - 16) > 2$, so if n is large enough in terms of p_v , p_e , and ε , then there exists a pair $(a, b) \in T_1 \cap T_2 \cap T_3$. It follows that u and v are in the same component in G since there is a walk from one to the other via this a and b .

We now prove the following claim.

Claim 2.13. *With probability at least $1 - e^{-\Omega(n^2)}$, every vertex in Q_n has an ε -edge-good vertex within distance 4.*

Proof. First note that any set of vertices in Q_n which are pairwise at distance at least 5 from one another are ε -edge-good independently. Fix a vertex v in Q_n and assume without loss of generality that $v = \mathbf{0}$. By Theorem 2.11 there exists a set S of vertices at distance 4 from v in Q_n which are pairwise at distance at least 6 in Q_n with $|S| = \Omega(n^2)$. The vertices in S are ε -edge-good independently, and by Lemma 2.7 (noting that we may assume that n is large), there exists a constant $c > 0$ such that each of them is ε -edge-good with probability at least c . Hence, the probability that at least one of the vertices in S is ε -edge-good is at least $1 - e^{-\Omega(n^2)}$, from which the claim follows by a union bound. \square

Now suppose that every pair of distinct, vertex-good vertices of Q_n at distance at most 9 from one another form an ε -edge-good pair and every vertex in Q_n has an ε -edge-good vertex within distance 4. We will show that under these assumptions, if u and v are distinct ε -edge-good vertices of Q_n , then they are in the same component of G . Since our assumptions hold with probability $1 - e^{-\Omega(n^2)}$, this will complete the proof.

Fix a path between u and v in Q_n , say $x_0 x_1 \dots x_k$ where $x_0 = u$ and $x_k = v$. If $k \leq 9$, then u and v are two ε -edge-good, and therefore vertex-good, vertices within distance 9.

Hence, they form an ε -edge-good pair and we are done by the first part of the proof. If instead $k > 9$, then each of x_5, \dots, x_{k-5} are within distance 4 of ε -edge-good vertices, say y_5, \dots, y_{k-5} respectively. Then x_0 and y_5 are both vertex-good vertices, and since they are within distance 9 of each other, they form an ε -edge-good pair. By the first part of the proof, they are in the same component in G . By similar logic, so are x_k and y_{k-5} , and y_i and y_{i+1} for each $i \in \{5, \dots, k-6\}$. Hence, u and v are in the same component and we are done. \square

We are now ready to prove Proposition 2.2.

Proof of Proposition 2.2. By Lemma 2.12 it is sufficient to show that for all constants $\varepsilon \in (0, 1)$ there exists a positive constant C such that for all $n \in \mathbb{N}$ and models in $\mathcal{D}(Q_n, p_e, p_v)$, there are at least $C \cdot 2^n$ ε -edge-good vertices in Q_n with probability at least $1 - e^{-\Omega(n^2)}$. Fix an $\varepsilon \in (0, 1)$, let $n \geq 6$, and consider a model in $\mathcal{D}(Q_n, p_e, p_v)$.

We observed in the proof of Claim 2.13 that any set of vertices which are pairwise at distance at least 5 in Q_n are ε -edge-good independently of one another, and by Lemma 2.7 (noting that we may assume that n is large) there exists a constant $c > 0$ such that the probability that a given vertex of Q_n is ε -edge-good is at least c . Consider the fifth power $Q_n^{(5)}$ of Q_n , that is the graph with vertex set $V(Q_n)$ and edges between vertices which are at distance at most 5 from one another in Q_n . This graph is Δ -regular for some $\Delta = O(n^5)$, so by Brooks' theorem we can properly vertex-colour $Q_n^{(5)}$ with Δ colours.

Each colour class consists of vertices which are pairwise at distance at least 5 from one another in Q_n , so they are ε -edge-good independently of one another. Hence, by a Chernoff bound, there exists a constant $C' = C'(p_v, p_e, \varepsilon) > 0$ such that if D is a colour class with $|D| \geq n^2$, then at least $C'|D|$ of the vertices in D are ε -edge-good with probability at least $1 - e^{-\Omega(n^2)}$. Since $\Delta = O(n^5)$ it follows by a union bound that, with probability at least $1 - e^{-\Omega(n^2)}$, in every colour class of size at least n^2 at least a C' proportion of the vertices are ε -edge-good. The total number of vertices in colour classes of size less than n^2 is at most $\Delta \cdot n^2 = O(n^7)$, so there exists a constant $C > 0$ such that with probability at least $1 - e^{-\Omega(n^2)}$, at least a C proportion of the vertices of Q_n are ε -edge-good, as required. \square

3 Percolation in \mathbb{Z}^n

In this section we build on the methods and results of Section 2 to prove Theorem 1.7. We also prove Lemma 1.8, which we need in order to apply the theorem, in Section 3.1. As was the case with Theorem 1.9, our proof of Theorem 1.7 rests on a renormalisation argument which reduces the problem to one concerning percolation models of the form given in Definition 2.1. We now state the analogue of Proposition 2.2 for this setting, before proceeding to prove Theorem 1.7 from the proposition by mimicking the proof of Theorem 1.9 from Proposition 2.2. Following this we will prove the proposition.

Proposition 3.1. *Let $p_v \in (0, 1]$ and $p_e \in (\frac{18}{19}, 1]$ be constants. Then, for large enough n , all models in $\mathcal{D}(\mathbb{Z}^2 \times \{0, 1\}^{n-2}, p_v, p_e)$ percolate.*

Proof of Theorem 1.7. To prove the theorem it is sufficient to show that if n is large, then every 1-independent bond percolation model on $\mathbb{Z}^2 \times \{0, 1\}^{n-2}$ in which each edge is open with probability p percolates. Let n be large and let μ_n be such a model on $\mathbb{Z}^2 \times \{0, 1\}^{n-2}$. Define a percolation model μ_{n-k} on $\mathbb{Z}^2 \times \{0, 1\}^{n-k-2}$ by defining a vertex v to be open if the hypercube $H_v = \{v \parallel a : a \in \{0, 1\}^k\}$ is connected under μ_n , and defining an edge of $\mathbb{Z}^2 \times \{0, 1\}^{n-k-2}$ between two open vertices u and v to be open if at least one of the 2^k edges in $\mathbb{Z}^2 \times \{0, 1\}^{n-2}$ between H_u and H_v is open under μ_n . For $i \in \{k, \dots, n-3\}$, recursively define a model μ_{n-i-1} on $\mathbb{Z}^2 \times \{0, 1\}^{n-i-3}$ by defining a vertex v to be open if the edge between $v \parallel 0$ and $v \parallel 1$ is open in $\mathbb{Z}^2 \times \{0, 1\}^{n-i-2}$ under μ_{n-i} , and setting an edge uv to be open if the edge between $u \parallel 0$ and $v \parallel 0$ or the edge between $u \parallel 1$ and $v \parallel 1$ is open in $\mathbb{Z}^2 \times \{0, 1\}^{n-i-2}$ under μ_{n-i} .

Let $q = 1 - p$, and define q_i , s_i , and r_i for $i \geq k$ as in the proof of Theorem 1.9. Then we again have that $r_i \rightarrow 0$ as $i \rightarrow \infty$ and, for all $k \leq i \leq n-2$, we have $s_i > 0$ and $\mu_{n-i} \in \mathcal{D}(\mathbb{Z}^2 \times \{0, 1\}^{n-i-2}, s_i, 1 - r_i)$. Thus, we can take I to be a constant large enough that $r_I < \frac{1}{19}$, so that for $n \geq I + 2$ we have $\mu_{n-I} \in \mathcal{D}(\mathbb{Z}^2 \times \{0, 1\}^{n-I-2}, s_I, 1 - r_I)$ where $s_I > 0$ and $1 - r_I > \frac{18}{19}$, and Proposition 3.1 applies. Hence, μ_{n-I} percolates. By the construction of μ_{n-I} from μ_n , the existence of an infinite component in $\mathbb{Z}^2 \times \{0, 1\}^{n-I-2}$ under μ_{n-I} , implies the existence of an infinite component in $\mathbb{Z}^2 \times \{0, 1\}^{n-2}$ under μ_n , and hence μ_n percolates, which completes the proof of the theorem. \square

We now turn our attention to Proposition 3.1.

Proof of Proposition 3.1. To simplify the notation, we will prove the equivalent statement that for large enough n , all models in $\mathcal{D}(\mathbb{Z}^2 \times \{0, 1\}^{n-1}, p_v, p_e)$ percolate. To begin, apply Lemma 2.12 to find $\varepsilon \in (0, 1)$ such that for each $n \in \mathbb{N}$, under every model in $\mathcal{D}(Q_n, p_e, p_v)$, with probability at least $1 - e^{-\Omega(n^2)}$ all ε -edge-good vertices in Q_n are in the same component. Now let n be large and consider $\mu \in \mathcal{D}(\mathbb{Z}^2 \times \{0, 1\}^{n-1}, p_v, p_e)$. Let G be the random subgraph of $\mathbb{Z}^2 \times \{0, 1\}^{n-1}$ associated with μ . To each edge $e \in E(\mathbb{Z}^2)$, we associate the natural copy of Q_n in $\mathbb{Z}^2 \times \{0, 1\}^{n-1}$, which we denote by C_e . The subgraph of G induced on $V(C_e)$ follows a model in $\mathcal{D}(Q_n, p_v, p_e)$, so we can define a bond percolation model $\nu = \nu(\mu)$ on \mathbb{Z}^2 coupled to μ by declaring $e \in E(\mathbb{Z}^2)$ to be open if the following two conditions hold:

- (a) in the subgraph of G induced on $V(C_e)$, all ε -edge-good vertices of C_e are in the same component; and
- (b) if we partition $V(C_e)$ into two equal parts according to the first two coordinates of the vertices, then there are ε -edge-good vertices of C_e in both halves.

Since $\mu \in \mathcal{D}(\mathbb{Z}^2 \times \{0, 1\}^{n-1}, p_v, p_e)$, it is clear that ν is a 1-independent bond percolation model on \mathbb{Z}^2 . By our choice of ε , for each edge e condition (a) holds with probability at least $1 - e^{-\Omega(n^2)}$. For condition (b), recall that sets of vertices of Q_n which are pairwise at distance at least 5 are ε -edge-good independently. It is straightforward to see that there exists such a subset of $V(C_e)$ in which $\Omega(n^2)$ vertices are taken from each half of the cube, so applying Lemma 2.7 and using a Chernoff bound on each half we find that for each edge e condition (b) holds with probability at least $1 - e^{-\Omega(n^2)}$. Thus ν is a

1-independent bond percolation model on \mathbb{Z}^2 in which each edge is open with probability at least $1 - e^{-\Omega(n^2)}$. Since $p_{\max}(\mathbb{Z}^2) < 1$, for example by Theorem 1.3 or as shown in [3] and in [18], we see that ν percolates if n is large enough in terms of p_v and p_e .

It remains to show that percolation of ν implies percolation of μ . We claim that for any collection of connected, open edges under ν , the ε -edge-good vertices in the associated hypercubes are in the same component under μ . An infinite component under ν contains infinitely many disjoint edges and the hypercubes of each of these must contain two ε -edge-good vertices, so this immediately implies that μ percolates when ν percolates. To prove the claim, suppose that $v \in V(\mathbb{Z}^2 \times \{0, 1\}^{n-1})$ is ε -edge-good when considered as a vertex in C_e for some $e \in E(\mathbb{Z}^2)$. Let f be another edge of \mathbb{Z}^2 such that $v \in V(C_f)$ (so that e and f have a common vertex in \mathbb{Z}^2). Since the definition of a vertex in a copy of Q_n being ε -edge-good depends only on the Q_{n-1} subgraph with the same first coordinate as that vertex, it follows that v is also ε -edge-good when considered as a vertex of C_f . Now if e and f are distinct open edges of \mathbb{Z}^2 with a common vertex, then by condition (b) applied to e or f there exists an ε -edge-good vertex in $V(C_e) \cap V(C_f)$. By condition (a) applied to e and f , all the ε -edge-good vertices in $V(C_e) \cup V(C_f)$ are in the same component in G , and the claim follows immediately. \square

3.1 Proof of Lemma 1.8

We conclude this section with the proof of Lemma 1.8, which allows us to extract concrete bounds from Theorems 1.7 and 1.9.

Proof of Lemma 1.8. Given a model in $\mathcal{D}_{\geq 0.5847}(Q_6)$, we start by constructing a model in $\mathcal{D}(Q_3, p_v, p_e)$ for some $p_v, p_e \in (0, 1]$ by applying the first step (with $k = 3$) of the renormalisation process used in the proof of Theorem 1.9. That is, we declare a vertex v of Q_3 to be open if the cube $\{v \parallel a : a \in \{0, 1\}^3\}$ is connected under the model on Q_6 , and an edge of Q_3 between two open vertices to be open if at least one of the eight edges between the two cubes corresponding to its endpoints is open. If $P_0 > 0$ is a constant such that in any model in $\mathcal{D}_{\geq 0.5847}(Q_3)$ the probability that the graph is connected is at least P_0 , then by Claim 2.3 the renormalised model on Q_3 is in $\mathcal{D}(Q_3, P_0, 1 - 0.4153^8/P_0^2)$, where $0.4153 = 1 - 0.5847$.

It is clear that this renormalisation has the property that if the random subgraph associated to the renormalised model forms a connected spanning subgraph of Q_3 , then the random subgraph of Q_6 associated to the original model is connected. Moreover, if we condition on all sites being open in the renormalised model, then the bonds follow a model in $\mathcal{D}_{\geq 1 - 0.4153^8/P_0^2}(Q_3)$. Thus if $P_1 > 0$ is a constant such that for any model in this class, the probability that the random subgraph of Q_3 is connected is at least P_1 , then the probability that Q_6 is connected under the original model is at least $P_0^8 P_1$.

In order to obtain suitable values for P_0 and P_1 , we construct a linear program that is satisfied by any 1-independent model on Q_3 (with edge probability exactly p) by removing the non-linear conditions. More specifically, for each subset S of the edges of Q_3 , let x_S denote the probability that the set of open edges is exactly S , and y_S denote the probability that S is a subset of the set of open edges. Let \mathcal{C} be the collection of all

subsets S of $E(Q_3)$ that form a connected spanning subgraph of Q_3 . We consider the following linear programming problem.

$$\begin{aligned}
&\text{Minimise:} && \sum_{S \in \mathcal{C}} x_S \\
&\text{Subject to:} && x_S \geq 0, \\
& && y_S = \sum_{T \supseteq S} x_T, \\
& && y_{S \cup \{e\}} = p \cdot y_S, \\
& && y_\emptyset = 1,
\end{aligned}$$

where S runs over all subsets of $E(Q_3)$ and e runs over all edges that are vertex-disjoint from all edges in S .

It is clear that the above conditions hold in any 1-independent model with edge probability p . Indeed, the only conditions missing are the non-linear constraints $y_{S \cup T} = y_S y_T$ when S and T are sets of edges sharing no vertex and $|S|, |T| \geq 2$. Thus the solution to the linear programming problem gives a lower bound on the minimum probability that the open edges in any 1-independent bond percolation model on Q_3 in which edges are open with probability p form a connected spanning subgraph of Q_3 . As the existence of a spanning connected open subgraph is an increasing event, this bound also holds when the edges are open with probability at least p as we can independently delete edges so as to ensure edges are open with probability exactly p .

Running the above LP problem using the **Gurobi** optimisation package and $p = 0.5847$ gave a lower bound¹ on P_0 of 0.0463. Running the LP problem again with $p = 0.5872 < 1 - 0.4153^8 / 0.0463^2$ gave a lower bound on P_1 of 0.0497. Hence we may take $P = 10^{-12} < 0.0497 \cdot (0.0463)^8$ as a lower bound on the probability of Q_6 being connected under any model in $\mathcal{D}_{\geq 0.5847}(Q_6)$, and $P > 9.93 \times 10^{-13} > 0.4153^{32} \cdot \varphi$ as required. \square

4 Lower bounds on $p_{\max}(\mathbb{Z}^2)$

In this section we detail the proof of Theorem 1.5 and justify Result 1.6, starting with the former.

Proof of Theorem 1.5. Let $p > p_{\text{site}} = p_{\text{site}}(\mathbb{Z}^2)$ and, to each vertex $(i, j) \in \mathbb{Z}^2$ with $i + j \equiv 0 \pmod{2}$, assign independent random variables $X_{i,j} \in \{A, U, D, L, R\}$ which take the value ‘A’ with probability $1 - p$ and each of the other four values with probability $p/4$. We add edges according to the following rules, and leave all other edges closed.

- If $X_{i,j} = U$, we add the edge from (i, j) to $(i, j + 1)$.
- If $X_{i,j} = D$, we add the edge from (i, j) to $(i, j - 1)$.
- If $X_{i,j} = L$, we add the edge from (i, j) to $(i - 1, j)$.
- If $X_{i,j} = R$, we add the edge from (i, j) to $(i + 1, j)$.
- If $X_{i,j} = A$, we add all four edges incident to (i, j) .

¹We also ran the dual programs and checked that the dual solutions provided by **Gurobi** were feasible.

This clearly gives a 1-independent bond percolation model on \mathbb{Z}^2 and, moreover, any infinite open path must, on alternate internal vertices, pass through vertices in the even sublattice $\{(i, j) : i + j \equiv 0 \pmod{2}\}$ at vertices where $X_{i,j} = A$ (as (i, j) has degree 1 when $X_{i,j} \neq A$). Thus, we must have an infinite component in the ‘diagonally connected’ even sublattice, where each vertex is adjacent to its eight nearest neighbours. It is well known that site percolation on this lattice satisfies a ‘duality’ condition with site percolation on the usual \mathbb{Z}^2 lattice and that the percolation threshold is $1 - p_{\text{site}}$ (see, for example, [6]). As $p > p_{\text{site}}$ and $\mathbb{P}(X_{i,j} = A) = 1 - p$, we almost surely do not have such an infinite component, and hence this model does not percolate.

In this model, each edge is open with probability $\frac{p}{4} + (1 - p) = 1 - \frac{3}{4}p$, so

$$p_{\max}(\mathbb{Z}^2) \geq 1 - \frac{3}{4}p_{\text{site}}.$$

Now, as noted in the introduction, Day, Falgas-Ravry, and Hancock [11] showed that

$$p_{\max}(\mathbb{Z}^2) \geq p_{\text{site}}^2 + \frac{1}{2}(1 - p_{\text{site}}).$$

Hence, *independently* of any assumption on the value of p_{site} , we have

$$p_{\max}(\mathbb{Z}^2) \geq \inf_{x \in [0,1]} \max \left\{ 1 - \frac{3}{4}x, x^2 + \frac{1}{2}(1 - x) \right\} = \frac{1}{32}(35 - 3\sqrt{33}) \approx 0.555197. \quad \square$$

Note that if we use the conjectured value $p_{\text{site}} \approx 0.592746$ we get the slightly better bound $p_{\max}(\mathbb{Z}^2) \geq 0.555440$. However, an even better non-rigorous bound is given by the following model.

Let each site v be given a state $X_v \in \{U, D, L, R\}$ independently at random with $\mathbb{P}(X_v = U) = \mathbb{P}(X_v = R) = \frac{1}{2}\theta$, and $\mathbb{P}(X_v = D) = \mathbb{P}(X_v = L) = \frac{1}{2}(1 - \theta)$, where $\theta \in (0, 1)$. We join neighbouring sites $u = (i, j)$ and v if *any* of the following hold.

- $X_u = X_v$;
- $X_u = U$ and $v = (i, j + 1)$ is above u ;
- $X_u = D$ and $v = (i, j - 1)$ is below u ;
- $X_u = L$ and $v = (i - 1, j)$ is to the left of u ;
- $X_u = R$ and $v = (i + 1, j)$ is to the right of u .

In other words, sites are joined if their states are equal, but a site in state d also forces an edge to its neighbour in direction d regardless of the neighbour’s state.

If v is to the right of u we have

$$\begin{aligned} \mathbb{P}(uv \text{ is open}) &= \mathbb{P}(X_u = X_v = U) + \mathbb{P}(X_u = X_v = D) + \mathbb{P}(X_u = R \text{ or } X_v = L) \\ &= \frac{1}{4}\theta^2 + \frac{1}{4}(1 - \theta)^2 + \frac{1}{2}\theta + \frac{1}{2}(1 - \theta) - \frac{1}{4}\theta(1 - \theta) = \frac{3}{4}(\theta^2 - \theta + 1). \end{aligned}$$

A similar calculation holds for vertical bonds and so every edge is present with probability $p = \frac{3}{4}(\theta^2 - \theta + 1)$. This is clearly a 1-independent model, and numerical simulations suggest the threshold for percolation (above which the model percolates) is at $p \approx 0.592119$.

We provide a high confidence result that this model does not percolate for p just above 0.5921, implying that $p_{\max}(\mathbb{Z}^2) > 0.5921$. See, for example, [3] or [22] for more

examples of this method. We use a renormalisation argument, constructing a new 1-independent model on \mathbb{Z}^2 from the model described above by identifying renormalised sites (x, y) with $N \times N$ blocks of sites $(Nx, Ny) + \{0, \dots, N-1\}^2$ in the original model. A renormalised bond uv , which corresponds to a $2N \times N$ (or $N \times 2N$) rectangle, will be open if some event E_{uv} holds, and this event will depend only on the sites and bonds within the corresponding rectangle to ensure 1-independence of the renormalised model. These events will be chosen so that if there is an open cycle C enclosing a point v in the renormalised model, then in the original model the open component containing the point v is contained within the bounded region enclosed by the blocks corresponding to C .

If $\mathbb{P}(E_{uv}) > 0.8457$ for all edges uv , then by Corollary 5.1 below there are almost surely cycles enclosing any bounded region in the plane. Hence, no point is in an infinite component in the original model, and so the original model does not percolate. In principle, it is possible to calculate $\mathbb{P}(E_{uv})$ exactly, but this is usually impractical unless N is very small. Instead, we use Monte Carlo simulations to test the hypothesis that $\mathbb{P}(E_{uv}) > 0.8457$. We run T trials, constructing a pseudorandom instance of the states in a $2N \times N$ rectangle then determining whether or not E_{uv} holds. If $\mathbb{P}(E_{uv}) \leq 0.8457$, then with probability at most $p = \mathbb{P}(\text{Bin}(T, 0.8457) \geq k)$ will we have k or more successful trials. Thus, if we obtain k successful trials and p is small, then we can say with high confidence that $\mathbb{P}(E_{uv}) > 0.8457$. Other than a very unfortunate coincidence occurring in the simulation, there are two possible reasons this may fail. One is that the pseudorandom number generator we use may not be sufficiently random, and the other is possible errors in the computer software. To mitigate these errors, we performed two different experiments with different software, different choices of E_{uv} , and different pseudorandom number generators.

Before giving the details of the first experiment, we recall that the dual graph of the \mathbb{Z}^2 lattice has sites corresponding to the square faces of the lattice, and bonds joining these faces are open exactly when the unique bond of the original lattice crossing them is closed. In the first experiment, which closely follows the method in [22], E_{uv} is the event that

- there is a unique largest connected component C_u in the $(N-1) \times (N-1)$ dual graph in the block corresponding to u ;
- there is a unique largest connected component C_v in the $(N-1) \times (N-1)$ dual graph in the block corresponding to v ; and
- C_u and C_v both lie in the same connected component of the $(2N-1) \times (N-1)$ dual graph in the $2N \times N$ rectangle corresponding to the bond uv .

Note that since we want E_{uv} to depend only on the model inside the $2N \times N$ rectangle corresponding to uv , we are restricted to only considering dual bonds in a $(2N-1) \times (N-1)$ rectangle. It is straightforward to see that these events E_{uv} have the required property that the existence of an open cycle enclosing a region in the renormalised model implies the existence of an open cycle enclosing the region in the dual graph of the original model, and hence any open component of the original model meeting this region must be finite.

Apart from the choice of random number generator, the algorithm for finding the component structure in the dual graph is the same as in [22]. In particular, states are revealed column by column, and the component structure is updated accordingly. Thus only states in two or three columns are retained, reducing memory requirements to $O(N)$ (rather than storing all the states which would use $O(N^2)$ memory). We used a 16-bit version of the `arc4` algorithm to supply the random numbers (see, for example, [23] for an overview of the algorithm).² Taking $N = 1,200,000$ and $T = 300$ we obtained 296 successes, giving a p -value of less than 10^{-16} .

The second experiment follows a method suggested (but not used) in [22]. For a horizontal edge uv with u to the left of v , E_{uv} is the event that

- there is no open path from the bottom to the top of the $2N \times N$ rectangle corresponding to the edge uv ; and
- there is no open path from the left to the right of the $N \times N$ square corresponding to u .

Reflecting in the line $x = y$ gives a corresponding definition for vertical edges. By symmetry these have the same probability, $\mathbb{P}(E_{uv})$. Again, it is not difficult to see that these events have the required properties.

To determine if an open path crosses a square or rectangle we use a boundary following algorithm where we explore the boundary of the sites connected to the bottom of the rectangle (or left side of the square). One advantage of this algorithm is that it can be much faster as it only needs to determine the states of a small fraction of the sites. The disadvantage is that we now need a pseudorandom *function* which can generate consistent random data for all sites (x, y) accessed in any order (rather than a sequential list of random numbers as is more usual in a random number generator). For this we used the 20-round `chacha` algorithm introduced in [4] to generate our random states. We again set $N = 1,200,000$ and performed 300 trials. There were 291 successes, giving a p -value of less than 10^{-11} .

The code for both of these experiments is attached to the [arXiv](#) submission.

5 Upper bounds on $p_{\max}(\mathbb{Z}^2)$

We now prove Theorem 1.3 which gives an improved upper bound on $p_{\max}(\mathbb{Z}^2)$. We will apply a renormalisation argument and use linear programs to lower bound the probability that each edge is present.

Proof of Theorem 1.3. We follow the basic renormalisation idea used in [3]. Tile \mathbb{Z}^2 with 2×2 squares $S_{(i,j)} = \{2i, 2i+1\} \times \{2j, 2j+1\}$ and form a new 1-independent model on \mathbb{Z}^2 by identifying $S_{(i,j)}$ with the site $(i, j) \in \mathbb{Z}^2$ and joining neighbouring sites $u, v \in \mathbb{Z}^2$ if a

²While the `arc4` algorithm is known to have some biases (see, for example, [14, 19]), by discarding the initial outputs and using 16-bits rather than 8-bits, all known biases in `arc4` are sufficiently small so as to not affect the simulations.

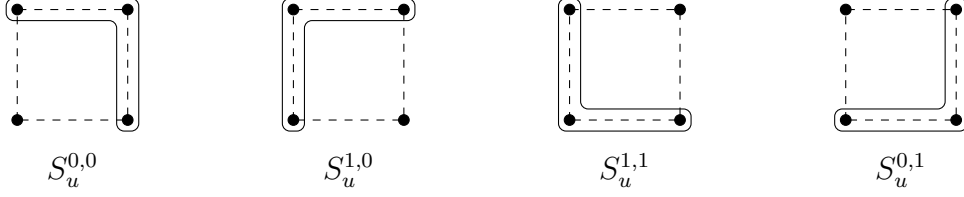


Figure 1: The sets $S_u^{r,s}$

certain ‘good event’ E_{uv} occurs in the corresponding 4×2 or 2×4 rectangle $S_u \cup S_v$ in the original 1-independent model. The good event E_{uv} will be defined to be the existence of an open component in the rectangle $S_u \cup S_v$ that has a ‘large’ intersection with both S_u and S_v . As long as any two ‘large’ subsets of any S_u intersect, these events will have the property that an open path from some v to some w in the renormalised grid will imply the existence of an open path from some site in S_v to some site in S_w in the original model.

Clearly containing at least three out of four sites in S_u would be a sufficient condition to be ‘large’, but we can do better. Fix for each $u \in \mathbb{Z}^2$ a subset $\mathcal{L}_u \subseteq S_u$ of size 3, i.e., one of the subsets shown in Figure 1. We define a large subset of S_u to be one that intersects \mathcal{L}_u in at least two sites. Note that this includes any set containing three or more sites in S_u , but certain 2-element subsets of S_u are now also deemed large and, of course, any two large subsets of S_u intersect.

We now come to the choice of \mathcal{L}_u . These could be chosen to be all (translates of) the same fixed set, but we can improve the bound slightly by having \mathcal{L}_u vary, and indeed vary randomly. Hence, we shall choose \mathcal{L}_u randomly, independently for each u , and also independently of the state of the original model. We note that under these assumptions, the renormalised model is still 1-independent as vertex-disjoint sets of renormalised edges depend on vertex-disjoint subgraphs of the original model, and on the choice of \mathcal{L}_u for disjoint sets of sites u .

Even the probability distributions for the choice of \mathcal{L}_u will vary, depending on u , so our renormalised bonds will not be open with the same probabilities. Hence to inductively renormalise we will need to consider 1-independent models with differing edge probabilities for each edge. However, it will be enough for our purposes to consider models with at most two distinct edge probabilities. More precisely, define $\mathcal{D}_{p,p'}$ as the set of 1-independent bond percolation models on \mathbb{Z}^2 in which each edge uv is open with probability p if u and v both lie in the same 2×2 square S_w , and p' otherwise (see Figure 2). Similarly we define $\mathcal{D}_{\geq p, \geq p'}$ as the set of 1-independent models where the edge probabilities are at least p and p' respectively.

We now define the precise choice of probability distribution for the \mathcal{L}_u . For $r, s \in \{0, 1\}$ let

$$S_{(i,j)}^{r,s} = S_{(i,j)} \setminus \{(2i+r, 2j+s)\},$$

as shown in Figure 1. Now fix $\theta \in [0, 1]$ and assume $u \in \mathbb{Z}^2$ with $u \equiv (r, s) \pmod{2}$. Then

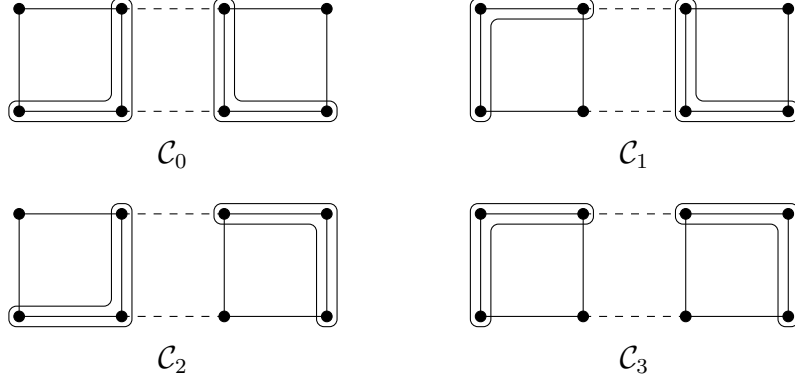


Figure 3: The configurations corresponding to the definitions of \mathcal{C}_0 , \mathcal{C}_1 , \mathcal{C}_2 , and \mathcal{C}_3 .

open edges be $E' \subseteq E$. For fixed $\theta \in [0, 1]$, let p_1 and p'_1 be such that, for all such models on R ,

$$p_1 \leq (1 - \theta)^2 \mathbb{P}(E' \in \mathcal{C}_0) + \theta(1 - \theta) \mathbb{P}(E' \in \mathcal{C}_1) + \theta(1 - \theta) \mathbb{P}(E' \in \mathcal{C}_2) + \theta^2 \mathbb{P}(E' \in \mathcal{C}_3)$$

and

$$p'_1 \leq \theta^2 \mathbb{P}(E' \in \mathcal{C}_0) + \theta(1 - \theta) \mathbb{P}(E' \in \mathcal{C}_1) + \theta(1 - \theta) \mathbb{P}(E' \in \mathcal{C}_2) + (1 - \theta)^2 \mathbb{P}(E' \in \mathcal{C}_3).$$

Then by symmetry, if our original 1-independent model on \mathbb{Z}^2 lies in $\mathcal{D}_{\geq p_0, \geq p'_0}$ then the renormalised model lies in $\mathcal{D}_{\geq p_1, \geq p'_1}$.

We will iterate this, giving a $\theta \in [0, 1]$ and a sequence of pairs (p_i, p'_i) , $i = 0, \dots, k$ such that, for every model in $\mathcal{D}_{\geq p_i, \geq p'_i}$, the renormalised model lies in $\mathcal{D}_{\geq p_{i+1}, \geq p'_{i+1}}$. The theorem will follow if we can exhibit such a sequence with $p_0 = p'_0 = 0.8457$ and $\min\{p_k, p'_k\} \geq 0.8639$, as there is then almost surely an infinite open component in the k times renormalised model. Note that at each stage we may assume that in the i times renormalised model each edge is open with probability *exactly* p_i or p'_i as appropriate. Indeed, independently deleting edges with the appropriate probabilities only makes the events \mathcal{C}_i less likely.

After renormalising i times, we wish to minimise the right-hand sides of the two inequalities above over all models in \mathcal{D}_{p_i, p'_i} . As in the proof of Lemma 1.8, we relax the condition that we have a 1-independent model on R to obtain the following two linear programming problems.

$$\begin{aligned} \text{Minimise: } & (1 - \theta)^2 \sum_{S \in \mathcal{C}_0} x_S + \theta(1 - \theta) \sum_{S \in \mathcal{C}_1} x_S + \theta(1 - \theta) \sum_{S \in \mathcal{C}_2} x_S + \theta^2 \sum_{S \in \mathcal{C}_3} x_S \\ \text{or: } & \theta^2 \sum_{S \in \mathcal{C}_0} x_S + \theta(1 - \theta) \sum_{S \in \mathcal{C}_1} x_S + \theta(1 - \theta) \sum_{S \in \mathcal{C}_2} x_S + (1 - \theta)^2 \sum_{S \in \mathcal{C}_3} x_S \end{aligned}$$

$$\begin{aligned} \text{Subject to: } & x_S \geq 0, \\ & y_S = \sum_{T \supseteq S} x_T, \\ & y_{S \cup \{e\}} = p_i \cdot y_S, \\ & y_{S \cup \{f\}} = p'_i \cdot y_S, \\ & y_\emptyset = 1, \end{aligned}$$

where S runs over all subsets of E , e runs over all edges induced by $S_{(0,0)}$ or $S_{(1,0)}$ that are vertex-disjoint from all edges in S , and f runs over the edges $\{(1,0), (2,0)\}$ and

i	p_i	p'_i
0	0.845700	0.845700
1	0.859167	0.829055
2	0.856981	0.831846
3	0.857370	0.831456
4	0.857391	0.831616
5	0.857546	0.831779
6	0.857826	0.832114
7	0.858365	0.832753
8	0.859396	0.833976
9	0.861358	0.836303
10	0.865058	0.840691
11	0.871911	0.848815
12	0.884171	0.863343
13	0.904695	0.887637
14	0.934851	0.923277

Table 1: Suitable values for p_i and p'_i (with $\theta = 0.18$).

$\{(1, 1), (2, 1)\}$ that are vertex-disjoint from all edges in S . The first of these optimisation problems yields a suitable value for p_{i+1} and the second yields a suitable value for p'_{i+1} .

Running the above LP problems with $\theta = 0.18$ using the **Gurobi** optimisation package we can find suitable values for p_i and p'_i , which we checked by confirming that the dual LP problem was feasible. The results are listed in Table 1. For $k = 13$ we see that $p_k, p'_k \geq 0.8639$, and hence the model percolates by the results of [3]. \square

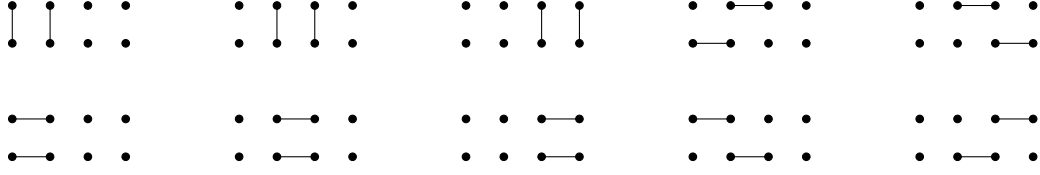
We remark that there is no need to use the same value of θ for every renormalisation in the proof of Theorem 1.3. Indeed, it seems rather unlikely that the optimal sequence of θ values is constant. We have, however, been unable to find a sequence that gives $p_{\max}(\mathbb{Z}^2) \leq 0.8456$. Note that taking $\theta = 0$ in the above, so that the \mathcal{L}_u are as illustrated in Figure 2, yields only $p_{\max}(\mathbb{Z}^2) \leq 0.8463$. Moreover, the rather simpler homogeneous renormalisation where we take $\mathcal{L}_u = S_u^{0,0}$ for all u , so that in the renormalised model we assume all edges are open with the same probability, gives $p_{\max}(\mathbb{Z}^2) \leq 0.8493$.

The renormalisation used in the proof of Theorem 1.3 can be combined with some ideas from [3] to give the following corollary.

Corollary 5.1. *In any model in $\mathcal{D}_{\geq 0.8457}(\mathbb{Z}^2)$ and for any $C > 0$, the probability that there is an open path crossing the rectangle $[0, Cn] \times [0, n]$ from left to right tends to 1 as $n \rightarrow \infty$. In particular, there are almost surely open cycles enclosing any bounded region of the plane.*

Proof. We recall from Theorem 2 from [3] that a much simpler 2×2 renormalisation argument gives that if the original model is in $\mathcal{D}_{\geq 1-q}(\mathbb{Z}^2)$ then the renormalised model lies in $\mathcal{D}_{\geq 1-10q^2}(\mathbb{Z}^2)$. Indeed, suppose each edge is open with probability at least $1 - q$

and take the good event E_{uv} to be the event that there is a component meeting at least three out of the four vertices in each of S_u and S_v . If this fails, we must have at least one of the following ten sets of edges closed³



Hence, $\mathbb{P}(E_{uv}) \geq 1 - 10q^2$. Note that this lower bound on edge probabilities also applies to the renormalisation used in the proof of Theorem 1.3, since the good events there contain the good events here. Thus, once $p_k, p'_k > 0.9$ in the proof of Theorem 1.3 (see Table 1), we can inductively choose $p_{i+1} = p'_{i+1} = 1 - 10q_i^2$ where $q_i = 1 - \min\{p_i, p'_i\}$ for all $i \geq k$. It is clear that then $p_i \rightarrow 1$ rapidly as $i \rightarrow \infty$.

Now given any $\varepsilon > 0$ we can find some i_0 such that $p_i, p'_i > 1 - \varepsilon / \lceil 2C + 1 \rceil$ for all $i \geq i_0$. Suppose $n \geq 2^{i_0}$ and pick i so that $2^i \leq n < 2^{i+1}$. Then in the i times renormalised model we have an open path from $(-1, 0)$ to $(\lceil 2C \rceil, 0)$ with probability at least $1 - \varepsilon$. It follows that in the original model there is an open path in $\mathbb{Z} \times \{0, \dots, n\}$ starting before $x = 0$ and ending at or after $x = \lceil 2C \rceil 2^i \geq Cn$ with probability at least $1 - \varepsilon$. Thus we have a path crossing $[0, Cn] \times [0, n]$ from left to right with probability at least $1 - \varepsilon$ as desired.

For the last part, fix a bounded set and let m_0 be large enough that $(-m_0, m_0)^2$ contains it. Then for all $m \geq m_0$, if there are open paths crossing the rectangles $[-2m, 2m] \times [m, 2m]$ and $[-2m, 2m] \times [-2m, -m]$ from left to right and the rectangles $[-2m, -m] \times [-2m, 2m]$ and $[m, 2m] \times [-2m, 2m]$ from top to bottom, then there is an open cycle enclosing the bounded set. By the above, in each case such a path is present with probability $1 - o(1)$ as $m \rightarrow \infty$, so such an open cycle exists with probability 1. It follows by a union bound that, almost surely, for every bounded set there is an open cycle enclosing it. \square

As noted in [3], it follows from Corollary 5.1 that for every model in $\mathcal{D}_{\geq 0.8457}(\mathbb{Z}^2)$ the infinite component is almost surely unique: otherwise pick two vertices in different infinite components, then there is an open cycle enclosing both of them and this connects the two infinite components. It also follows that there is no infinite component in the dual graph associated to such models.

In Theorem 2 of [3] it was claimed that under every model in $\mathcal{D}_{\geq 0.8639}$ the origin is in an infinite component with positive probability, but this result does not seem to follow from the proof given. However, we can now adapt the renormalisation used to prove Theorem 1.3 to show this claim holds for all models in $\mathcal{D}_{\geq 0.8459}(\mathbb{Z}^2)$.

Theorem 5.2. *In any model in $\mathcal{D}_{\geq 0.8459}(\mathbb{Z}^2)$ the origin is in an infinite open component with positive probability.*

³For each of the first three columns, having at least one of the two edges present in both upper and lower sets implies the corresponding 2×2 square has at least three vertices in the same component. The last two columns ensure that these components connect up.

Proof. We shall inductively define nested events F_k that imply that the origin is in a component of size at least $k + 2$. Let F_0 be the event that the edge from $(0, 0)$ to $(1, 0)$ is open in the original model. Now given F_k in the k times renormalised model (which will imply the event that the edge from $(0, 0)$ to $(1, 0)$ in this model is open), we first reflect the model in the line $x + y = 1$ (i.e., map (x, y) to $(1 - y, 1 - x)$) and then renormalise as in the proof of Theorem 1.3. Note that this reflection preserves $\mathcal{D}_{\geq p, \geq p'}$ as the p -edges and p' -edges are mapped onto edges of the same type. Also the event F_k now implies that the edge from $(1, 0)$ to $(1, 1)$ is open. Let F_{k+1} be the event that F_k holds and that the horizontal renormalised edge from $S_{(0,0)}$ to $S_{(1,0)}$ is open, i.e., that the event $E_{(0,0)(1,0)}$ (in the sense of the proof of Theorem 1.3) holds.

Note that any component meeting both $S_{(0,0)}$ and $S_{(1,0)}$ in large sets must also meet the edge from $(1, 0)$ to $(1, 1)$. Consider the component containing the origin in the restriction of the original model to the vertices corresponding to $\{(1, 0), (1, 1)\}$, and the component containing the origin in the restriction of the original model to the vertices corresponding to $S_{(0,0)} \cup S_{(1,0)}$. If F_{k+1} holds, then the latter component strictly contains the former (as it contains some vertices from $S_{(1,0)}$) so by induction F_{k+1} implies that the origin is in a component of size at least $k + 3$ in the original model.

Now define $G_0 = F_0$ and the event G_{k+1} to be the event that, if the edge from $(1, 0)$ to $(1, 1)$ in the (reflected) k -fold renormalised model were open, then $E_{(0,0)(1,0)}$ would hold. In other words, G_{k+1} is the event that adding the edge from $(1, 0)$ to $(1, 1)$ gives a suitable connected component in $S_{(0,0)} \cup S_{(1,0)}$. Note that G_{k+1} depends only on the other nine edges in $S_{(0,0)} \cup S_{(1,0)}$ and is clearly an increasing event as a function of these edges. Hence, if we lower bound $\mathbb{P}(G_{k+1})$ in $\mathcal{D}_{p,p'}$ we also have the same bound in $\mathcal{D}_{\geq p, \geq p'}$. Now $F_{k+1} = F_k \cap G_{k+1}$, so by induction

$$\mathbb{P}(F_k) \geq 1 - \sum_{i=0}^k \mathbb{P}(G_i^c).$$

It is straightforward to modify the linear programming problem from the proof of Theorem 1.3 so as to minimise the probability of G_k and hence find a lower bound on F_k for small k . For large k we note that clearly $\mathbb{P}(G_k^c) \leq q_k = 1 - p_k$, where the k -fold renormalised model is in $\mathcal{D}_{\geq p_k, \geq p'_k}$, and we have the inequality $q_{k+1} \leq 10q_k^2$ as in the proof of Corollary 5.1.

To show that the origin is in an infinite component with positive probability, it is sufficient to show that $\sum_{k=0}^{\infty} \mathbb{P}(G_k^c) < 1$. Unfortunately, we cannot show this all the way down to $p = 0.8457$, but we show it does hold for $p = 0.8459$. Using the results from Table 2 we have

$$\begin{aligned} \sum_{k=0}^{\infty} \mathbb{P}(G_k^c) &\leq \sum_{k=0}^{13} \mathbb{P}(G_k^c) + \sum_{k=14}^{\infty} q_k \\ &\leq 0.998359 + \sum_{i=1}^{\infty} 10^{2^i-1} q_{13}^{2^i} \\ &\leq 0.99836, \end{aligned}$$

i	p_i	p'_i	$\mathbb{P}(G_k^c) \leq$
0	0.845900	0.845900	0.154100
1	0.859515	0.829480	0.096201
2	0.857661	0.832648	0.097540
3	0.858670	0.832999	0.096787
4	0.859879	0.834568	0.095945
5	0.862289	0.837404	0.094255
6	0.866795	0.842751	0.091100
7	0.875072	0.852561	0.085314
8	0.889637	0.869816	0.075168
9	0.913248	0.897752	0.058828
10	0.945814	0.936217	0.036503
11	0.978577	0.974824	0.014314
12	0.996611	0.996024	0.002247
13	0.999914	0.999899	0.000057

Table 2: Bounds used in Theorem 5.2 (with $\theta = 0.18$).

where we have used that $q_{13} \leq 0.0002$. Hence, $\mathbb{P}(\cap_{k=0}^{\infty} F_k) > 0$ and so the origin is in an infinite component with positive probability. \square

6 Open problems

In this section we discuss some interesting related problems on 1-independent percolation models on hypercubes and lattices. We begin by restating a question first posed by Day, Falgas-Ravry, and Hancock in [11].

Problem 6.1 ([11]). *For $n \geq 3$, what is the largest $p = p(n)$ for which there is a model in $\mathcal{D}_{\geq p}(Q_n)$ under which the hypercube is always disconnected?*

The simple construction used in the proof of Theorem 1.4 in [2] shows that this p satisfies $p \geq 1/2$, but is this best possible? The case $n = 2$ of Problem 6.1, where $Q_n = C_4$, was answered in [11] where they showed that the maximum is indeed $p = 1/2$ (this follows immediately from considering the expected number of edges in a model with edge probability greater than $1/2$).

It would also be interesting to determine the models above the threshold which minimise the probability that Q_n is connected. When $n = 2$, it is not difficult to show that these models are exactly those supported on subgraphs with either two or four edges, with

$$\mathbb{P}(\sqcup) = 2p - 1, \quad \mathbb{P}(\updownarrow) = \mathbb{P}(\leftrightarrow) = q^2, \quad \mathbb{P}(\lrcorner) = \mathbb{P}(\ulcorner) = \alpha, \quad \mathbb{P}(\lfloor) = \mathbb{P}(\rceil) = pq - \alpha,$$

where $q = 1 - p$ and $\alpha \in [0, pq]$.

The answer is already unknown for $n = 3$, although computer experiments suggest an answer for large p . Let the *signs model* on a graph G be the 1-independent percolation model where each vertex of G is independently assigned the sign ‘+’ with probability θ and the sign ‘−’ otherwise. An edge is open if both its endpoints have the same sign (so each edge is open with probability $p = \theta^2 + (1 - \theta)^2$). Computational evidence strongly suggests that this model minimises the probability that Q_3 is connected when $p > 0.55$. However, for smaller p there are models where connectivity is less likely, and indeed for $p < 0.516$ one can find models that are disconnected almost surely.

Conjecture 6.2. *For $p > 0.55$, among models in $\mathcal{D}_{\geq p}(Q_3)$ the signs model minimises the probability that Q_3 is connected.*

We also hypothesise that for general $n \geq 3$, the signs model minimises the probability that Q_n is connected when p is large enough. It should be noted that sadly none of the models listed above which minimise the probability of connectivity when $n = 2$ are signs models (unless $p = 1$).

Problem 6.3. *Is it true that for $n \geq 3$, if p is sufficiently close to 1, then among models in $\mathcal{D}_{\geq p}(Q_n)$ the signs model minimises the probability that Q_n is connected? If so, how close to 1 does p need to be?*

If the signs model is indeed optimal on Q_3 , say for $p \geq 0.5720$, then this leads to improved bounds on p_{giant} and $\lim_{n \rightarrow \infty} p_{\text{max}}(\mathbb{Z}^n)$ as follows. Start by applying the first step of the renormalisation in the proof of Theorem 1.9, with $k = 3$, to a model in $\mathcal{D}_{\geq p}(Q_n)$ to obtain a model on Q_{n-3} in which the probability that an edge is open given both its endpoints are open is at least

$$1 - \frac{2^{16}(1-p)^8}{\left((1 + \sqrt{2p-1})^8 + (1 - \sqrt{2p-1})^8\right)^2}.$$

This expression is greater than p when $p \geq 0.5720$ so for such p we can apply this renormalisation repeatedly until we obtain a constant $I = I(p)$ and a model in $\mathcal{D}(Q_{n-I}, p_v, p_e)$ where $p_v > 0$ and $p_e > 18/19$ such that the existence of a giant component in this model implies the existence of a giant component in the original. It follows from Proposition 2.2 that there exists $C = C(p) > 0$ such that with high probability this model, and hence the original model, has a component containing at least a C proportion of its vertices. Thus, if the signs model is optimal on Q_3 for $p \geq 0.5720$, then $p_{\text{giant}} \leq 0.5720$. Similarly, $\lim_{n \rightarrow \infty} p_{\text{max}}(\mathbb{Z}^n) \leq 0.5720$ under the same assumption.

We have shown that when each edge of the hypercube is open with probability at least 0.5847 (and n is large), there is a component containing a constant fraction of the vertices with high probability, so in particular $p_{\text{giant}} \leq 0.5847$. The best lower bound on this threshold is $1/2$ (using the model from the proof of Theorem 1.4 in [2] again). It was conjectured in [13] that this is tight, that is, $p_{\text{giant}} = 1/2$: they conjectured that for all $p > 1/2$, under every model in $\mathcal{D}_{\geq p}(Q_n)$ the hypercube contains a component of size at least $\left(\frac{1+\sqrt{2p-1}}{2} - o(1)\right) \cdot 2^n$ with probability $1 - o(1)$.

Problem 6.4. *What is the value of p_{giant} ?*

It is not clear to us whether the threshold probabilities for always disconnecting Q_n (as in Problem 6.1) have a limit as $n \rightarrow \infty$, but if they do it would be interesting to know whether this limit differs from p_{giant} .

We have improved the bounds on $p_{\text{max}}(\mathbb{Z}^2)$ and $\lim_{n \rightarrow \infty} p_{\text{max}}(\mathbb{Z}^n)$ so that they now stand at

$$\begin{aligned} 0.555197 \dots &\leq p_{\text{max}}(\mathbb{Z}^2) \leq 0.8457, \\ 0.535898 \dots &\leq \lim_{n \rightarrow \infty} p_{\text{max}}(\mathbb{Z}^n) \leq 0.5847. \end{aligned}$$

There remains a large gap between the upper and lower bounds in both cases and it would be interesting to reduce either of these gaps. We note that Result 1.6 shows that the lower bound for $p_{\text{max}}(\mathbb{Z}^2)$ is unlikely to be correct, and suggests that the correct value of $p_{\text{max}}(\mathbb{Z}^2)$ should be higher than $\lim_{n \rightarrow \infty} p_{\text{max}}(\mathbb{Z}^n)$. We close by restating the fundamental Questions 1.1 and 1.2 from above, recalling that the authors of [13] conjecture that $p_{\text{max}}(\mathbb{Z}^n) = 4 - 2\sqrt{3}$ for some $n \geq 3$.

Question 1.1 ([2]). *What is the value of $p_{\text{max}}(\mathbb{Z}^2)$?*

Question 1.2 ([2]). *What is the value of $\lim_{n \rightarrow \infty} p_{\text{max}}(\mathbb{Z}^n)$?*

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