The rainbow saturation number is linear

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Abstract

Given a graph H, we say that an edge-coloured graph G is H-rainbow saturated if it does not contain a rainbow copy of H, but the addition of any non-edge in any colour creates a rainbow copy of H. The rainbow saturation number $\mathrm{rsat}(n, H)$ is the minimum number of edges among all H-rainbow saturated edge-coloured graphs on n vertices. We prove that for any non-empty graph H, the rainbow saturation number is linear in n, thus proving a conjecture of Girão, Lewis, and Popielarz. In addition, we also give an improved upper bound on the rainbow saturation number of the complete graph, disproving a second conjecture of Girão, Lewis, and Popielarz.

1   Introduction

For a fixed graph H, we say that a graph G is H-*saturated* if it does not contain H as a subgraph, but adding any extra edge creates a copy of H as a subgraph. The *saturation number* of H, denoted by $\mathrm{sat}(n, H)$, is the minimum number of edges in an H-saturated

graph G on n vertices. This can be thought of as a dual to the classical Turán extremal number $ex(n, H)$ that counts the maximum number of edges in an H-free graph on n vertices. Since a maximal H-free graph is H-saturated, $ex(n, H)$ equivalently counts the maximum number of edges in an H-saturated graph on n vertices. The saturation number in contrast counts the minimum number of edges among such graphs.

The saturation number was first studied independently by Zykov [18] and Erdős, Hajnal, and Moon [5], who proved that $\text{sat}(n, K_r) = (r-2)(n-1) - \binom{r-2}{2}$, where as usual K_r denotes the complete r-vertex graph. Later, Kászonyi and Tuza [10] proved that the saturation number of every graph H is linear in n, that is, $\text{sat}(n, H) = O_H(n)$. For more information and many other results related to the saturation number see the survey of Faudree, Faudree, and Schmitt [6]. In this paper we provide analogous results to these for the rainbow saturation number: a variant of the saturation number for edge-coloured graphs.

The generalisation of saturation to edge-coloured graphs was first considered by Hanson and Toft [9]. Following this, Barrus, Ferrara, Vandenbussche, and Wenger [2] considered the particular case of t-rainbow saturation, where there are exactly t colours available. Girão, Lewis, and Popielarz [8] initiated the study of the natural generalisation to the case where the palette of colours available is unlimited, rather than bounded by t. This is the focus of our paper.

A t-*edge-coloured* graph is an ordered pair (G, c), where c is a (not necessarily proper) edge-colouring of the graph G using colours from $[t] = \{1, 2, ..., t\}$. An edge-colouring of a graph is said to be *rainbow* if every edge is assigned a distinct colour. We say that a t-edge-coloured graph (G, c) is (H, t)-*rainbow saturated* if (G, c) does not contain a rainbow copy of H, but the addition of any non-edge in any colour from $[t]$ creates a rainbow copy of H in G. Note that this requires $t \geq |E(H)|$. The t-*rainbow saturation number* of H, denoted by $\text{rsat}_t(n, H)$, is the minimum number of edges in an (H, t)-rainbow-saturated graph on n vertices.

When the number of possible edge colours is infinite (say the set of colours is $\mathbb{N}$), an edge-coloured graph (G, c) is H-*rainbow saturated* if (G, c) does not contain a rainbow copy of H, but the addition of any non-edge in any colour from $\mathbb{N}$ creates a rainbow copy of H. The *rainbow saturation number* of H, denoted by $\text{rsat}(n, H)$, is the minimum number of edges in an H-rainbow saturated graph on n vertices.

In [8], Girão, Lewis, and Popielarz conjectured that, like ordinary saturation numbers, the rainbow saturation number of any non-empty graph H is at most linear in n, and proved this for graphs with some particular properties.

Theorem 1.1 (Girão, Lewis, and Popielarz; Theorem 2 (4) in [8]). *Let H be a graph with a non-pendant edge that is not contained in a triangle. Then* $\mathrm{rsat}(n, H) = O(n)$.

In fact, Theorem 1.1 is a consequence of a stronger result that proves $\mathrm{rsat}_t(n, H) = O(n)$ for any such H and any $t \geq e(H)$.

Our main result proves that the rainbow saturation number of every graph is linear, thus confirming the conjecture of Girão, Lewis, and Popielarz [8].

Theorem 1.2. *Every non-empty graph H satisfies* $\mathrm{rsat}(n, H) = O(n)$.

Interestingly, when t is fixed the t-rainbow saturation number is not in general linear in n. Barrus, Ferrara, Vandenbussche, and Wenger [2] proved in particular that for all $t \geq |E(H)|$, $\mathrm{rsat}_t(n, H) \geq c_1 \frac{n \log n}{\log \log n}$ whenever H is a complete graph, and $\mathrm{rsat}_t(n, H) \geq c_2 n^2$ if H is a star, where c_1 and c_2 are constants depending only on t and the size of H.

The lower bound on the t-rainbow saturation number for complete graphs was improved independently by Ferrara, Johnston, Loeb, Pfender, Schulte, Smith, Sullivan, Tait, and Tompkins [7], by Korándi [11], and by Girão, Lewis, and Popielarz [8] to $\mathrm{rsat}_t(n, K_r) \geq c_1 n \log n$, which is tight up to a constant factor.

Girão, Lewis, and Popielarz also conjectured that for any complete graph K_r where $r \geq 3$, the rainbow saturation number satisfies $\mathrm{rsat}(n, K_r) = 2(r-1)n + O(1)$ when $n \geq 2(r-2)$. We prove that $\mathrm{rsat}(n, K_r)$ is in fact significantly less than this conjectured value.

Theorem 1.3. *Let $r \geq 3$. Then* $\mathrm{rsat}(n, K_r) \leq (r + 2\sqrt{2r})n + c_r$, *where c_r is a constant depending only on r.*

In Section 2, we establish some useful basic results regarding rainbow saturation, including Propositions 2.2 and 2.3 which allow us to assume that any counterexample to Theorem 1.2 is connected and has no pendant edges. We describe the construction used to prove Theorem 1.2 in Section 3: given a graph H and an edge $e \in E(H)$, we construct an edge-coloured graph on n vertices with $O(n)$ edges such that the addition of (almost) any non-edge creates a rainbow copy of H. In Section 4, we show that if e is an edge contained in a triangle then the graph produced by this construction does not contain a rainbow copy of H. Together with Theorem 1.1, this completes the proof of Theorem 1.2. Finally, we prove Theorem 1.3 in Section 5.

2 Preliminaries

In this section we provide preliminary results that deal with some easy classes of graphs. We begin by establishing that the existence of an edge-coloured graph G on n vertices with $O(n)$ edges that is "almost" H-rainbow-saturated (save for at most linearly many problematic non-edges) is enough to prove that $\mathrm{rsat}(n, H) = O(n)$. For a colour c, we say that a non-edge $e \in E(\overline{G})$ is *c-bad* for G if the addition of e in colour c to G does not create a rainbow copy of H. Say that a non-edge is *bad* if there exists a colour c such that e is c-bad for G.

Proposition 2.1. *Let H be a graph, and let G be an n-vertex edge-coloured graph. Suppose that G has m bad non-edges and does not contain a rainbow copy of H. Then $\mathrm{rsat}(n, H) \leq |E(G)| + m$.*

Proof. Let $\{e_1, \ldots, e_m\}$ be the set of bad non-edges of G. We construct an n-vertex graph G_m and colouring χ of $E(G_m)$ as follows: set $G_0 := G$, where each edge of G_0 is coloured as it is in G. Consider each $i \in [m]$ in turn. If there is a colour c such that e_i is c-bad for G_{i-1}, then we define G_i to be $G_{i-1} \cup \{e_i\}$, set $\chi(e_i) = c$, and keep the colours of the other edges as in G_{i-1}. Note that there may be multiple such c, in which case we pick one arbitrarily. Otherwise, if e_i is not bad for G_{i-1}, set $G_i := G_{i-1}$. Observe that G_m is H-rainbow-saturated, and since we added at most m edges during this process, G_m has at most $|E(G)| + m$ edges. Hence, $\mathrm{rsat}(n, H) \leq |E(G)| + m$ as required. $\qquad\square$

A direct result of Proposition 2.1 shows that in order to prove Theorem 1.2 it suffices to consider connected graphs H.

Proposition 2.2. *Let H be a disconnected graph and let H' be a component of H that has the most vertices, and subject to this, the most edges. If $\mathrm{rsat}(n, H') = O(n)$, then $\mathrm{rsat}(n, H) = O(n)$.*

Proof. Let s be the number of components of H isomorphic to H'. As $\mathrm{rsat}(n, H') = O(n)$, we have $\mathrm{rsat}(n - m, H') = O(n)$, where $m := 2\left(|V(H)| - |V(H')|\right) - (s - 1)$. Let (G, χ) be an H'-rainbow-saturated graph on $n - m$ vertices with as few edges as possible.

Define a graph G^* on n vertices as follows: let H_2 be obtained from H' by gluing two disjoint copies of H' together at an arbitrary vertex. Let G^* be the disjoint union of G with $s - 1$ disjoint copies of H_2 and two disjoint copies of each component of H that is not isomorphic to H'. See Figure 1 for an example. Let χ^* be an edge-colouring of $E(G^*)$ where $\chi^*(e) = \chi(e)$ for every edge $e \in G$, and $G^* \setminus V(G)$ is rainbow with colours not used in χ.
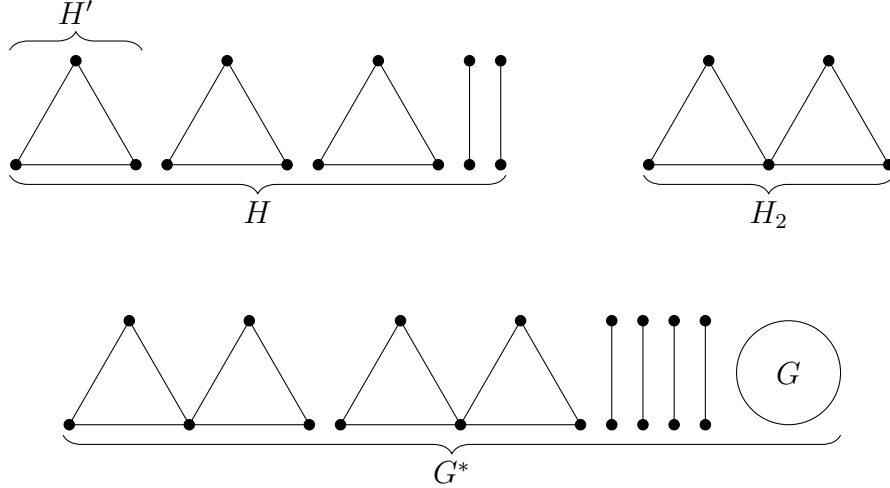
Figure 1: An example of the graph G^* created via the construction described in Proposition 2.2. Here $s = 3$.

Observe that G^* contains no rainbow copy of H (by our choice of H', there are at most $s - 1$ disjoint copies of H' in $G^* \setminus G$ and G contains no rainbow copy of H'). Moreover, the addition of any non-edge (in any colour) within G creates a rainbow copy of H in G^*, since $G^* \setminus V(G)$ contains a rainbow copy of $H \setminus V(H')$ avoiding any given colour. As there are at most $m(n - m) + \binom{m}{2} = O(n)$ other non-edges in G^*, by Proposition 2.1 we have $\mathrm{rsat}(n, H) = O(n)$, as required. $\qquad\square$

Therefore, in what follows we may (and will) assume that H is a connected graph. Our next result shows that, if H contains a vertex of degree 1, then $\mathrm{rsat}(n, H) = O(n)$.

Proposition 2.3. *Let H be a connected graph with $\delta(H) = 1$. Then $\mathrm{rsat}(n, H) = O(n)$.*

Proof. Write $k = |V(H)|$ and $n = q(k - 1) + r$, where $0 \le r < k - 1$. Define an edge-coloured graph G on n vertices as follows: take q disjoint copies of K_{k-1} and let χ be a rainbow colouring of them. Finally, add r isolated vertices. Note that G has n vertices and $q\binom{k-1}{2} = (n - r)\left(\frac{k-2}{2}\right) = O(n)$ edges. Clearly G does not contain a rainbow copy of H. Moreover, since $\delta(H) = 1$, if any non-edge in any colour is added between two distinct copies of K_{k-1} in G, then a rainbow copy of H is created. Therefore, since G has at most $q(k - 1)r + \binom{r}{2} = (n - r)r + \binom{r}{2} = O(n)$ other non-edges, Proposition 2.1 implies that $\mathrm{rsat}(n, H) = O(n)$. $\qquad\square$

3 The Construction

In this section, we give the construction that lies at the heart of our proof of Theorem 1.2. Given a graph H and an edge $e \in E(H)$, we create an edge-coloured graph on n vertices with $O(n)$ edges such that the addition of (almost) any edge creates a rainbow copy of H. The main work in the proof of Theorem 1.2 will be to show that this graph does not contain a rainbow copy of H, which will be done in Section 4.

Construction 3.1. Let H be a connected graph on at least three vertices, and let $xy \in E(H)$. Write $S := N_H(x) \cap N_H(y)$ and let T be the set of edges in H with one endpoint in S and the other in $\{x, y\}$. Define H' to be the graph obtained from H by contracting the edge xy and replacing any multi-edges by single edges. We label the vertices of H' as in H, with the single vertex obtained from contracting xy labelled by $x|y$. Let T' be the set of edges in H' between $x|y$ and S, and write $H'' := H' \setminus T'$.

For an integer $m \geq 2$, define $F = F(m)$ to be the graph obtained from H' by replacing the vertex $x|y$ with m duplicates of itself, denoted $v_1, v_2, \ldots, v_m$. Write $M := \{v_1, v_2, \ldots, v_m\}$ and label the vertices in M by $x|y$. Label the remaining vertices as in H. This means F has m vertices labelled $x|y$ and one vertex labelled v for every $v \in V(H) \setminus \{x, y\}$.

Given any integers $m \geq 2$ and $r \geq |E(H'')| + 1$, define the graph $G := G_{xy}^H(m, r)$ as follows. Start with $|E(H'')|$ disjoint copies of $F(m)$, indexed by the edges in H'' as $F_{e_1}, \ldots, F_{e_{|E(H'')|}}$. To these we add $r - |E(H'')|$ copies of $F(m)$, which we index as $F_1, \ldots, F_{r-|E(H'')|}$. Finally, we obtain G by identifying the vertices corresponding to v_i for each i in turn. That is, for each $i = 1, \ldots, m$, we replace the r vertices $u_1, \ldots, u_r$ corresponding to v_i with one vertex and connect this vertex to the vertices in $\bigcup_{j=1}^{r} N(u_j)$. Note that the graph G has r copies of each vertex in $V(H') \setminus \{x|y\}$ and m copies of the vertex $x|y$, and that the labelling of the vertices naturally defines a labelling of the edges, where an edge in G whose vertices are labelled u and v is labelled uv. By construction, uv is an edge in H'.

We now define a colouring χ of G, which we will later show does not contain a rainbow copy of H. Let χ_0 be a rainbow colouring of H'' and fix a colour that is not in χ_0, say black. We extend χ_0 to a colouring of H' by colouring the edges in T' with a colour $\star$. Let an edge uv in G be coloured by $\chi_0(e)$, where e is the colour of the edge in H' matching the label of uv. For each edge e in H'', recolour the edge corresponding to e in F_e with the colour black, and finally recolour each edge coloured with $\star$ with unique colours that are not used elsewhere in G.

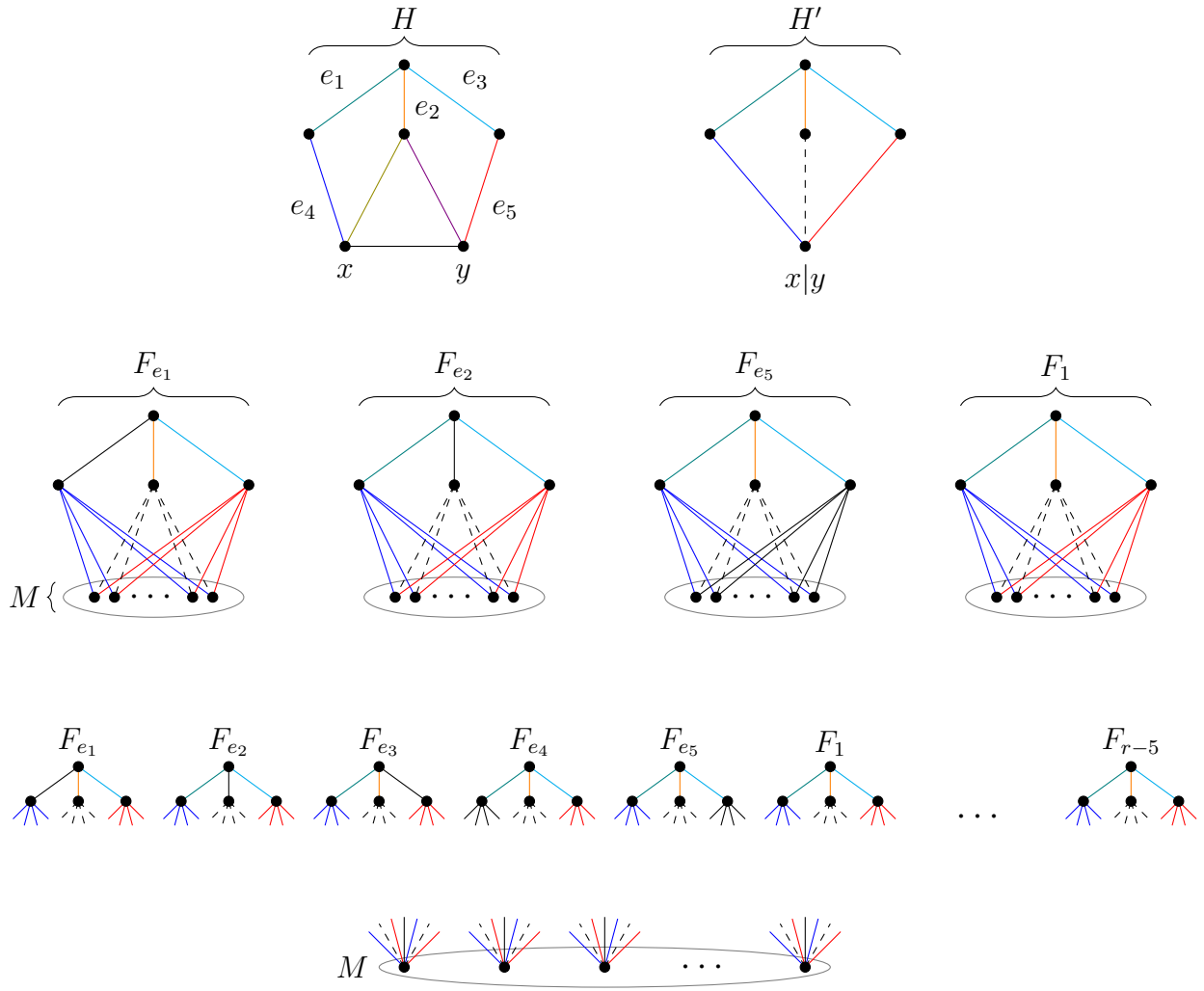Construction 3.1 was inspired by the construction used in [8] to prove Theorem 1.1. In fact,

Figure 2: An example of the construction of the graph $G_{xy}^H(m, r)$. The dashed edge represents edges which are originally coloured $\star$ and will be coloured with unique colours by χ. The colour black can be seen replacing other colours according to the index of the copy of F.

when the edge xy is not contained in a triangle, $N_H(x) \cap N_H(y) = \emptyset$, and the graph described in Construction 3.1 is equivalent to the construction in [8].

See Figure 2 for an example of how Construction 3.1 works. In what follows, we will use $G_{xy}^H(n)$ to refer to the edge-coloured graph $(G_{xy}^H(m,r), \chi)$, where $m = n - r(|V(H)| - 2)$ and $r = |E(H'')| + 1$; so $G_{xy}^H(n)$ has n vertices. Observe that the addition of any non-edge in any colour within M creates a rainbow copy of H. Since $r = O(1)$ and $m = n - O(1)$, there are at most $O(n)$ bad non-edges in $G_{xy}^H(n)$. Therefore, if $G_{xy}^H(n)$ contains no rainbow copy of H, then Proposition 2.1 will yield $\mathrm{rsat}(n, H) = O(n)$, as required.

4   Completing the proof of Theorem 1.2

The goal of this section is to show that G_{xy}^H contains no rainbow copy of H.

Proposition 4.1. *Let H be a non-empty connected graph. For any edge $xy \in E(H)$ that is contained in a triangle and all integers m and r, the graph $G_{xy}^H(m,r)$ does not contain a rainbow copy of H.*

Before proving this, we show why it implies Theorem 1.2 (restated here for convenience).

Theorem 1.2. *Every non-empty graph H satisfies $\mathrm{rsat}(n, H) = O(n)$.*

Proof of Theorem 1.2. First, note that by Proposition 2.2, it suffices to prove that every connected graph H satisfies $\mathrm{rsat}(n, H) = O(n)$. Let H be a non-empty connected graph. If H has a triangle, let xy be an edge contained in a triangle, and write $G := G_{xy}^H(n)$. Then $|E(G)| = O(n)$, and G has no rainbow copies of H by Proposition 4.1. Moreover, as described at the end of Section 3, for all but at most $O(n)$ non-edges e, the addition of e in any colour creates a rainbow copy of H. In particular, $\mathrm{rsat}(n, H) = O(n)$, as required. Otherwise, if H is triangle-free, then $\mathrm{rsat}(n, H) = O(n)$, using either Proposition 2.3 if H has a pendant edge or Theorem 1.1 if H has a non-pendant edge. $\qquad\square$

Proof of Proposition 4.1. Suppose for contradiction that there exists a connected graph H and an edge $xy \in E(H)$ contained in a triangle such that $G_{xy}^H(m,r)$ contains a rainbow copy $H_{\mathfrak{R}}$ of H, for some integers m, r. Take H to have the minimal number of vertices with respect to these properties, and fix some m and r such that $G := G_{xy}^H(m,r)$ has a rainbow copy of H. Recall that the vertices of G are labelled by vertices in H' (the graph obtained

from H by contracting xy), where M is the set of vertices labelled $x|y$, and that the edges of G are labelled by edges in H'.

Since $H_\mathfrak{R}$ is connected it must contain at least one vertex in M, else it would be contained entirely within a copy of $H \setminus \{x, y\}$, which is impossible as $|V(H_\mathfrak{R})| = |V(H)|$.

Claim 4.2. For every $v \in V(H) \setminus \{x, y\}$, the rainbow copy $H_\mathfrak{R}$ uses at least one vertex labelled v.

Proof of Claim 4.2. Suppose for contradiction that v is a vertex in $V(H) \setminus \{x, y\}$ such that $H_\mathfrak{R}$ uses no vertex labelled v.

Suppose first that v is the unique common neighbour of x and y in H. Then, since $H_\mathfrak{R}$ uses no vertex labelled v, it uses at most $|E(H)| - 2$ colours: at most $|E(H'')| = |E(H)| - |T| - 1$ colours used by χ_0, and possibly also black. This is a contradiction, as a rainbow copy of H requires $|E(H)|$ colours.

Next, suppose that the graph $H^v := H \setminus \{v\}$ is connected. By the previous paragraph, we may (and will) assume that v is not the unique common neighbour of x and y. Consider the edge-coloured graph G^* obtained from $G_{xy}^H(m, r)$ by the deletion of every vertex labelled v. As $H_\mathfrak{R}$ contains no vertex labelled v, it is present in G^*, and, in particular, G^* contains a rainbow copy of H^v. Observe that $G^* = (G_{xy}^{H^v}(m, r), \chi_0')$, where χ_0' is the restriction of χ_0 to $(H^v)''$. Since v is not the unique common neighbour of x and y, the edge xy is in a triangle in H^v. Therefore, since H^v is connected, by minimality of H we know that $G_{xy}^{H^v}(m, r)$ contains no rainbow copy of H^v. Hence, we obtain a contradiction.

Therefore, we may (and will) assume that, for every non-cut vertex u in H, the rainbow copy $H_\mathfrak{R}$ contains a vertex labelled u. In particular, v is a cut vertex in H. Note that x and y are in the same component of H^v since $xy \in E(H)$ and $v \in V(H) \setminus \{x, y\}$. Let C be a component of H^v not containing $\{x, y\}$. Take $u \in C$ of maximum distance from v. Note that u is not a cut vertex of H, and thus, by assumption, there is a vertex u' in $H_\mathfrak{R}$ labelled u. Since $H_\mathfrak{R}$ is connected and contains at least one vertex in M, there is a path in $H_\mathfrak{R}$ from M to u'. However, by our choice of C, any such path passes through a vertex labelled v, which is a contradiction. □

Recall that T is the set of edges in H with one endpoint in $S = N_H(x) \cap N_H(y)$ and the other in $\{x, y\}$, and that T' is the set of edges in H' between $x|y$ and S. Note that $|S| = |T'| = \frac{|T|}{2} \geq 1$ since xy is contained in a triangle. By Claim 4.2, we see that $|H_\mathfrak{R} \cap M| \leq 2$. We will obtain a contradiction by counting the edges in $H_\mathfrak{R}$ (via two cases).

First, consider the case where $|H_\mathfrak{R} \cap M| = 2$. By Claim 4.2, $H_\mathfrak{R}$ contains exactly one vertex labelled v, for each $v \in V(H) \setminus \{x, y\}$. Hence there are at most $2|T'|$ edges in $H_\mathfrak{R}$ with a label in T'. Now, consider an edge $uv \in E(H') \setminus T'$, where $u \neq x|y$ without loss of generality. Since there is exactly one vertex labelled u, all edges labelled uv belong to the same copy of F, and thus have the same colour. It follows that the edges of $H_\mathfrak{R}$ are coloured using at most $|E(H')| - |T'| + 2|T'| \leq |E(H)| - 1$ colours, which contradicts the assumption that $H_\mathfrak{R}$ is a rainbow copy of H.

Now, consider the case where $|H_\mathfrak{R} \cap M| = 1$. In this case, there is exactly one $v \in V(H) \setminus \{x, y\}$ that labels two vertices in $H_\mathfrak{R}$, and every other vertex in $V(H')$ appears once as a label. We claim that for every edge $e \in E(H')$, except possibly for $vx|y$ (if this is an edge), all edges labelled e have the same colour. Indeed, if e does not contain v then the claim clearly follows, as there is at most one edge labelled e. Otherwise, $e = uv$ with $u \neq x|y$, and thus, since $e \in E(H') \setminus T'$ and all edges labelled e are in the same copy of F, they all receive the same colour. Note that, if $vx|y$ is an edge in H', then there are at most two edges in $H_\mathfrak{R}$ with this label. Altogether, the edges of $H_\mathfrak{R}$ see at most $|E(H')| + 1 = |E(H)| - |T'| < |E(H)|$ colours, which contradicts $H_\mathfrak{R}$ being a rainbow copy of H. $\qquad\square$

5 Rainbow Saturation for Cliques

The purpose of this section is to prove Theorem 1.3, restated here for convenience.

Theorem 1.3. *Let $r \geq 3$. Then $\mathrm{rsat}(n, K_r) \leq (r + 2\sqrt{2r})n + c_r$, where c_r is a constant depending only on r.*

We begin by proving the following lemma.

Lemma 5.1. *For every $s \geq 3$, there is an edge-coloured graph (H, χ_H) on $\binom{s}{2} + s$ vertices with the following properties:*

(1) *the largest rainbow clique in H has $\binom{s}{2} + 1$ vertices,*

(2) *for every $v \in V(H)$, there is a rainbow clique in $H \setminus \{v\}$ on $\binom{s}{2} + 1$ vertices,*

(3) *for every colour c, there is a rainbow clique with $\binom{s}{2} + 1$ vertices containing no edge coloured c.*

Proof. Let S be a set of size s. Let H be a complete graph on vertices $S \cup S^{(2)}$, where $S^{(2)} = \{\{x, y\} \subseteq S : x \neq y\}$ is the set of pairs of distinct elements from S. For each pair $\{x, y\} \in S^{(2)}$, colour the edges $(x, \{x, y\})$ and $(y, \{x, y\})$ with the same colour, picking a different colour for each pair. The remaining edges receive a rainbow colouring, with colours distinct from those used so far.

We first show that the largest rainbow clique in H has size $\binom{s}{2} + 1$. Suppose that T is a rainbow clique, and write $t := |T \cap S|$. For every two distinct vertices $x, y \in T \cap S$, the pair $\{x, y\}$ is not in T, since the edges $(x, \{x, y\})$ and $(y, \{x, y\})$ have the same colour. It follows that $|T| \leq t + \binom{s}{2} - \binom{t}{2} \leq \binom{s}{2} + 1$, using the fact that $\binom{t}{2} \geq t - 1$ for $t \geq 0$. This shows that H has no rainbow cliques of size at least $\binom{s}{2} + 2$. Given distinct vertices $x, y \in S$, let $T(x, y) := \{x, y\} \cup (S^{(2)} \setminus \{\{x, y\}\})$. It is not difficult to see that $T(x, y)$ is a rainbow clique of size $\binom{s}{2} + 1$, and hence a maximum rainbow clique. This proves (1).

For (2), let $v \in V(H)$. If $v \in S$, then for any $x, y \in S \setminus \{v\}$, the clique $T(x, y)$ avoids v (using $s \geq 3$). If $v \in S^{(2)}$, then $v = \{x, y\}$ for some $x \neq y \in S$, and thus the clique $T(x, y)$ avoids v. This proves (2).

For (3), let c be a colour that is used on H (if c does not appear on H, then (3) holds trivially). Since every colour class is either an edge or a path of length 2, there is a vertex v which is incident with all edges of colour c. By (2), there is a maximum rainbow clique avoiding v and thus c. This proves (3). $\qquad\square$

We now complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Given $r \geq 10$, let s be the largest integer such that $r \geq \binom{s}{2} + 1$ (so $r \leq \binom{s+1}{2}$). Write $t := r - \binom{s}{2} - 1$. Notice that we may take n to be sufficiently large.

Let (H, χ_H) be an edge coloured graph as in Lemma 5.1 (applied with s), and let T_1, T_2 and U be pairwise disjoint sets of sizes t, t, and $n - 2t - |V(H)|$ respectively which are disjoint from $V(H)$. Let G be a graph on vertex set $V(H) \cup U \cup T_1 \cup T_2$ with edge set $E(G) := \{uv : \{u, v\} \nsubseteq U \text{ and } uv \notin T_1 \times T_2\}$. We define a colouring χ on $E(G)$ such that $\chi(e) = \chi_H(e)$ for each $e \in E(H)$, and the edges of $E(G) \setminus E(H)$ receive a rainbow colouring with colours distinct from those used by χ_H. Note that the vertices of each T_i induce a rainbow clique of size t.

We claim that G contains no rainbow copy of K_{r+2}. Indeed, suppose W is a rainbow clique in G. Then W contains at most one vertex from U (as U is an independent set), at most t vertices from $T_1 \cup T_2$ (as no edges exist between T_1 and T_2), and at most $\binom{s}{2} + 1$ vertices from H (by the choice of χ_H). Thus, $|W| \leq \binom{s}{2} + 1 + t + 1 = r + 1$, as claimed.

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Next, we show that the addition of any non-edge xy in U in any colour creates a rainbow copy of K_{r+2}. Suppose that xy is coloured red, and denote by G' the graph obtained from G by adding xy in red. Then exactly one of the following holds: no edge in $G[V(H) \cup \{x,y\}]$ is red; exactly one of the edges from $\{x,y\}$ to $V(H)$ is red; red is used in H. Hence, by the properties of H given by Lemma 5.1 and by definition of G, there is a maximum rainbow clique W in H such that $G[W \cup \{x,y\}]$ does not contain any red edges. Thus $W \cup \{x,y\}$ is a rainbow clique of size $\binom{s}{2} + 3 = r - t + 2$ in G'. Since either T_1 or T_2 is not incident to any red edge, there is $i \in \{1,2\}$ for which $W \cup T_i \cup \{x,y\}$ is a rainbow clique of size $r + 2$, as required.

Now, as $r \leq \binom{s+1}{2}$, we have $t \leq \binom{s+1}{2} - \binom{s}{2} - 1 = s - 1$. Also, since $r \geq \binom{s}{2}$, we have $s - 1 \leq \sqrt{2r}$. Thus, thinking of r as fixed, we have

$$
\begin{aligned}
|E(G)| &= (|V(H)| + 2t)(n - |V(H)| - 2t) + |E(H)| + 2\binom{t}{2} + 2t|V(H)| \\
&= (|V(H)| + 2t)(n - |V(H)| - 2t) + O(1) \\
&= \left(\binom{s}{2} + s + 2t \right) \cdot n + O(1) \\
&= (r + s + t - 1) \cdot n + O(1) \\
&\leq (r + 2s - 2) \cdot n + O(1) \\
&\leq (r + 2\sqrt{2r}) \cdot n + O(1).
\end{aligned}
$$

Observe that G has at most $t^2 = O(1)$ bad non-edges, namely the edges uv such that $u \in T_1$ and $v \in T_2$. So applying Proposition 2.1 yields that $\mathrm{rsat}(n, K_{r+2}) \leq (r + 2\sqrt{2r})n + O(1)$. $\quad \square$

6   Conclusion and open problems

Our main theorem shows that for any graph H we have that $\mathrm{rsat}(n, H)$ is linear in n. Given this, it is natural to ask the following.

Question 6.1. For every graph H, does there exist a constant $c = c(H)$ such that

$$
\mathrm{rsat}(n, H) = (c(H) + o(1))n?
$$

Analogous questions have been considered for saturation and the related notion of weak saturation. The limit $\lim_{n \to \infty} \frac{\mathrm{sat}(n, H)}{n}$ was conjectured to exist by Tuza, see [16, 17]. Although some progress has been made towards Tuza's conjecture, (see [13, 15]), it remains open.

Another natural direction would be to generalise the notion of weak saturation to edge-coloured graphs. A subgraph G of F is said to be *weakly (F, H)-saturated* if the edges of $E(F) \setminus E(G)$ can be added to G, one edge at a time, in such a way that every added edge creates a new copy of H. The minimum number of edges in a weakly (F, H)-saturated graph is known as the *weak saturation number*, and is denoted $\mathrm{wsat}(F, H)$. When $F = K_n$, we write $\mathrm{wsat}(n, H)$ instead of $\mathrm{wsat}(K_n, H)$.

We say an edge-coloured subgraph (G, χ) of F is *weakly H-rainbow saturated* if there exists an ordering $e_1, ..., e_m$ of $E(F) \setminus E(G)$ such that, for any list $c_1, ..., c_m$ of distinct colours from $\mathbb{N}$, the non-edges e_i in colour c_i can be added to G, one at a time, so that every added edge creates a new rainbow copy of H. The *weak rainbow saturation number* of H, denoted by $\mathrm{rwsat}(F, H)$, is the minimum number of edges in a weakly H-rainbow saturated graph. When $F = K_n$, we write $\mathrm{rwsat}(n, H)$ instead of $\mathrm{rwsat}(K_n, H)$.

Note that we require the collection of added edges to receive distinct colours. In particular, we wish to exclude the possibility that all added edges have the same colour, in which case the previously added edges do not contribute to making new rainbow copies and the problem reduces to the standard rainbow saturation number.

The study of weak saturation numbers was introduced by Bollobás [3] in 1968, where he proved that $\mathrm{sat}(n, K_m) = \mathrm{wsat}(n, K_m)$ for all $3 \le m \le 7$ and conjectured that this equality holds for all m. This conjecture was first proved by Lovász [12] using a beautiful generalisation of the Bollobás two families theorem. It would be interesting to determine whether a similar phenomenon holds for the rainbow saturation number.

Question 6.2. Let $r \ge 3$. Is $\mathrm{rwsat}(n, K_r) = \mathrm{rsat}(n, K_r)$?

It would be interesting to see if upper bounds on $\mathrm{rwsat}(n, K_r)$ can be found that are much stronger than those provided by Theorem 1.3 (note $\mathrm{rsat}(n, K_r) \ge \mathrm{rwsat}(n, K_r)$).

Question 6.3. Let $r \ge 3$. Is it the case that $\mathrm{rwsat}(n, K_r) \le rn + O(1)$, for n sufficiently large?

Alon [1] proved in 1985 that $\mathrm{wsat}(n, H) = (c(H) + o(1))n$, for all graphs H. The natural generalisation of this to hypergraphs was recently proved by Shapira and Tyomkin [14]. An analogous question can be considered regarding the weak rainbow saturation number.

Question 6.4. For every graph H, does there exist a constant $c = c(H)$ such that

$$\mathrm{rwsat}(n, H) = (c(H) + o(1))n?$$

6.1 Proper rainbow saturation

It is worth noting that the phrase 'rainbow saturation' has appeared in the literature in a different context. Recently, Bushaw, Johnston, and Rombach [4] defined a different form of rainbow saturation which requires the colouring to be proper. We will refer to this concept as *proper rainbow saturation* to distinguish it from the rainbow saturation that we considered in this paper. A properly edge-coloured graph (G, c) is *properly H-rainbow saturated* if (G, c) does not contain a rainbow copy of H, but the addition of any non-edge, in any colour from $\mathbb{N}$ which preserves the proper colouring, creates a rainbow copy of H. The *proper rainbow saturation number* of H, denoted by $\mathrm{prsat}(n, H)$, is the minimum number of edges in a properly H-rainbow saturated graph on n vertices.

Bushaw, Johnston, and Rombach [4] proved that, for any graph H that does not include an induced even cycle, the proper rainbow saturation number of H is linear in n, that is, $\mathrm{prsat}(n, H) = O(n)$. Note that an analogous result to Proposition 2.1 holds for $\mathrm{prsat}(n, H)$, via a similar argument. Therefore, using Construction 3.1, we obtain the following result, which confirms that proper rainbow saturation is linear in n for a further class of graphs.

Theorem 6.5. *Let H be a connected graph, and suppose there exists $xy \in E(H)$ where $N[x] \cap N[y] = N(x) \cup N(y)$ (here $N[u] := N(u) \cup \{u\}$). Then $\mathrm{prsat}(n, H) = O(n)$.*

Proof. Note that the edge-colouring of the graph G_{xy}^H described in Construction 3.1 is a proper edge-colouring when $N[x] \cap N[y] = N(x) \cup N(y)$, that is, when the only vertices that are in exactly one of $N(x)$ and $N(y)$ are x and y. By Proposition 4.1, G_{xy}^H contains no rainbow copy of H. Since any non-edge added in *any* colour between two vertices in L creates a rainbow copy of H, clearly the addition of any non-edge in any colour that preserves the proper colouring creates a rainbow copy of H. This leaves at most $O(n)$ other non-edges in G_{xy}^H, which, by an argument analogous to the one used to prove Proposition 2.1, implies that $\mathrm{prsat}(n, H) = O(n)$. $\qquad\square$

Bushaw, Johnston and Rombach [4] conjectured that, analogously to the classical saturation numbers, the proper rainbow saturation number is linear in n.

Conjecture 6.6 (Bushaw, Johnston and Rombach [4]). *For every graph H, there exists a constant $c = c(H)$ such that $\mathrm{prsat}(n, H) \leq cn$.*

Regarding graphs that do contain induced even cycles, Bushaw, Johnston and Rombach [4] show that $\mathrm{prsat}(n, C_4)$ is linear. It would be interesting to see if this is also true for other even

cycles. Many other interesting problems concerning the proper rainbow saturation number and its connections to other saturation numbers can be found in [4].

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