Overview

Solving Equations

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A. Make a guess, and try to improve it.

Solving the Quintic

James Farrell & Jure Dobnikar

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At the solution to the quintic, ϕ is equal to x,

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but at other points x_i there is an error, ϵ_i,

$$\phi(x_i) = \xi + \epsilon_i$$

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Problem: how do we know if the sequence $x_0, x_1, \ldots x_k, x_{k+1}$ converges to ξ?

$$\lim_{k \to \infty} \phi(x_k) = \xi$$

Convergence

In order for this sequence to converge, we require the absolute error

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we expand the right-most term as a Taylor series,

$$\begin{aligned} \xi + \epsilon_{k+1} &= \phi\left(\xi\right) + \epsilon_k \phi'\left(\xi\right) + \mathcal{O}\left(\epsilon_k^2\right) \\ \epsilon_{k+1} &= \epsilon_k \phi'\left(\xi\right) + \mathcal{O}\left(\epsilon_k^2\right) \end{aligned}$$

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therefore a necessary (but not sufficient) condition for the sequence to converge to ξ is

$$\left\|\phi'\left(\xi\right)\right\| < 1$$

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we say that the sequence converges linearly to ξ with rate $|\phi'(\xi)|$.

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If ϕ' vanishes at the solution, but ϕ'' is finite and non-zero,

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A heuristic to estimate q is given by,

$$\boxed{q \approx \frac{\log\left|\frac{x_{k+1}-x_k}{x_k-x_{k-1}}\right|}{\log\left|\frac{x_k-x_{k-1}}{x_{k-1}-x_{k-2}}\right|}} \qquad (2)$$

Q-CONVERGENCE RATE ESTIMATE

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Q. Does the quintic sequence converge?

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the sequence converges to the solution for all starting values x_k.

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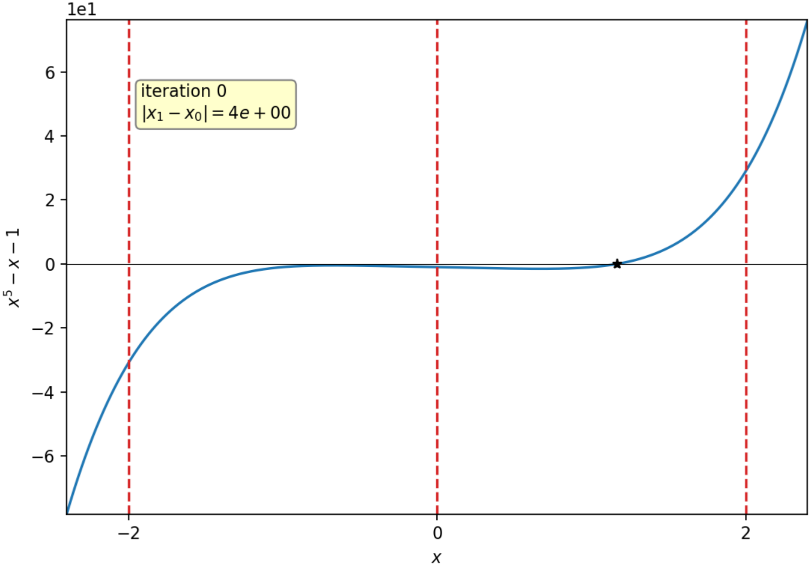
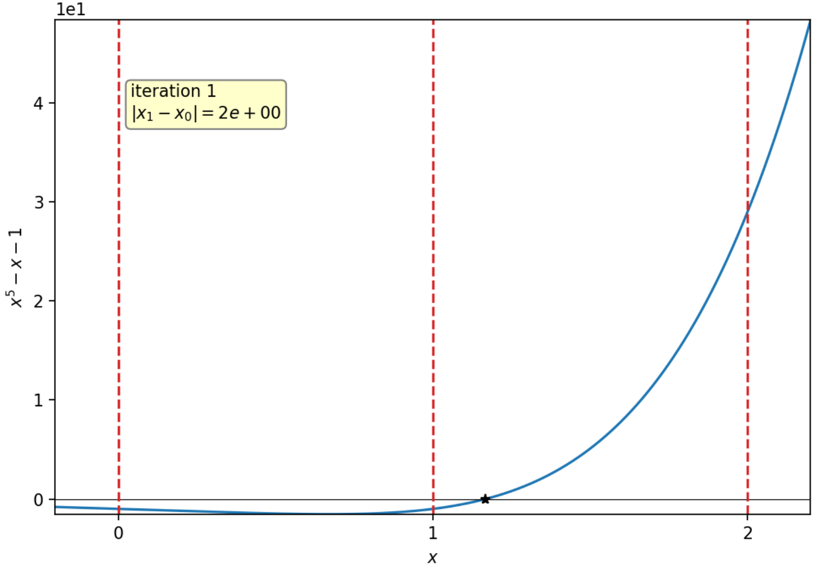
$$\boxed{a_{k+1}, b_{k+1} = \phi\left(a_k, b_k\right) = \begin{cases} \left(a_k, \frac{a_k+b_k}{2}\right) & \text{if } f\left(\frac{a_k+b_k}{2}\right) > 0; \\ \left(\frac{a_k+b_k}{2}, b_k\right) & \text{if } f\left(\frac{a_k+b_k}{2}\right) < 0. \end{cases}} \tag{3}$$

BISECTION UPDATE

Bisection



Bisection



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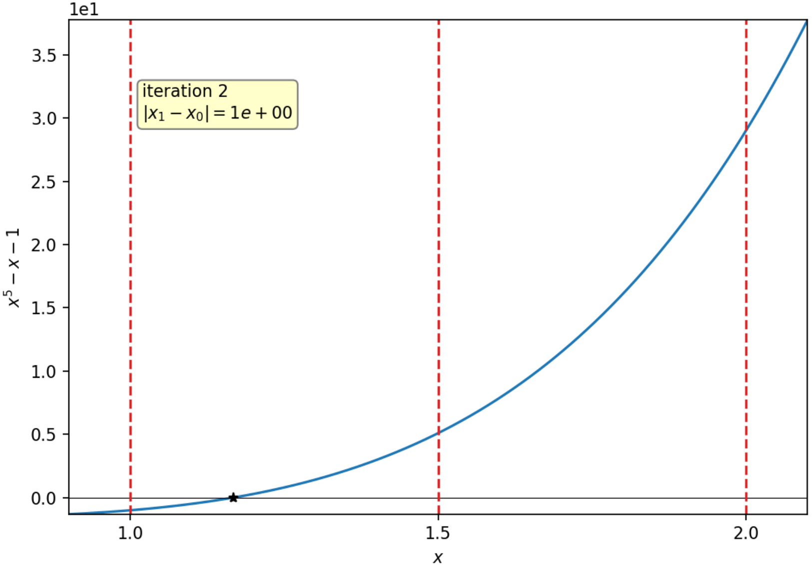
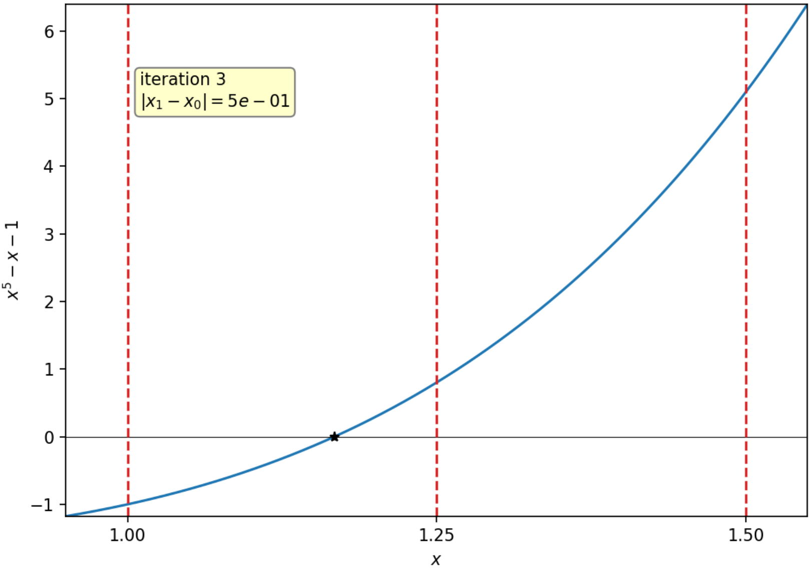
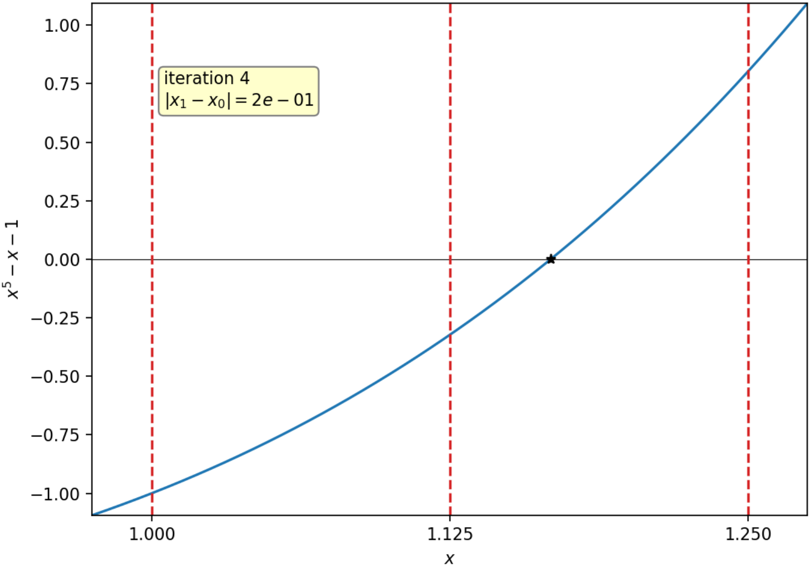


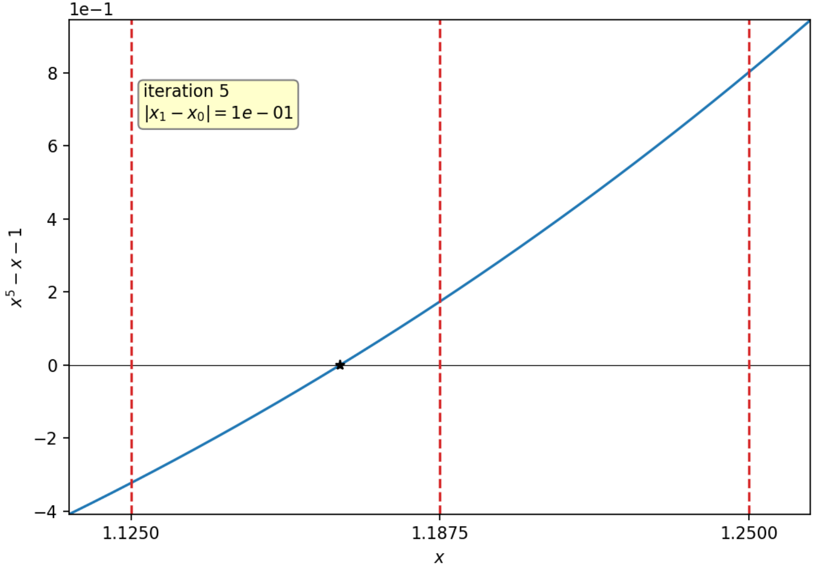
Bisection



Bisection


iteration 4
$|x_1 - x_0| = 2e - 01$
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
$x^5 - x - 1$
1.000
1.125
1.250
x

Bisection



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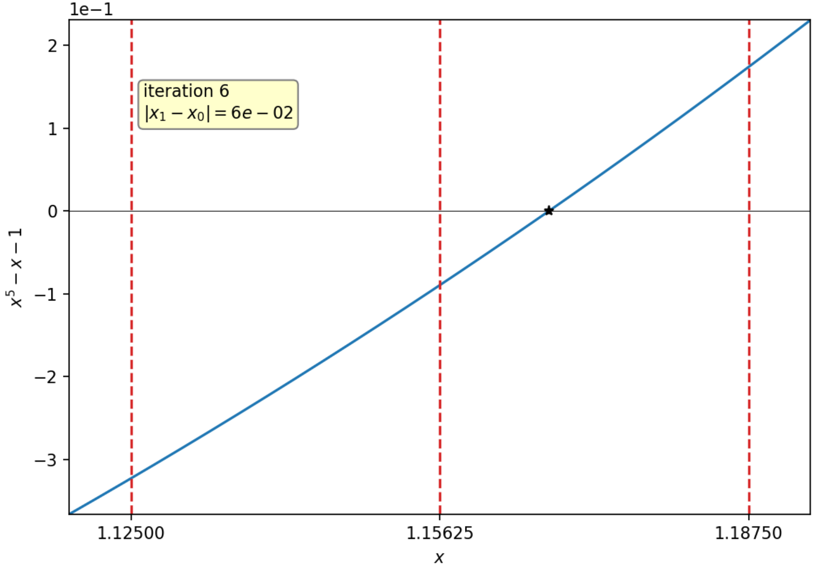
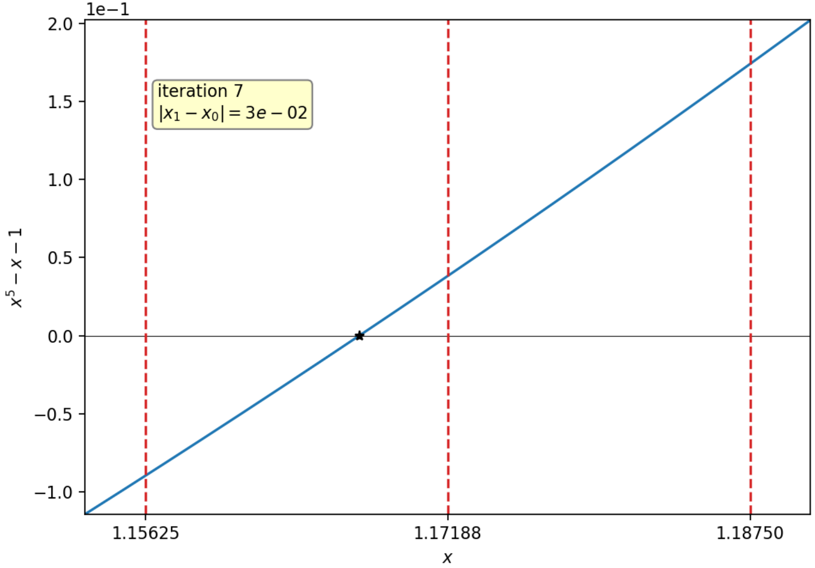
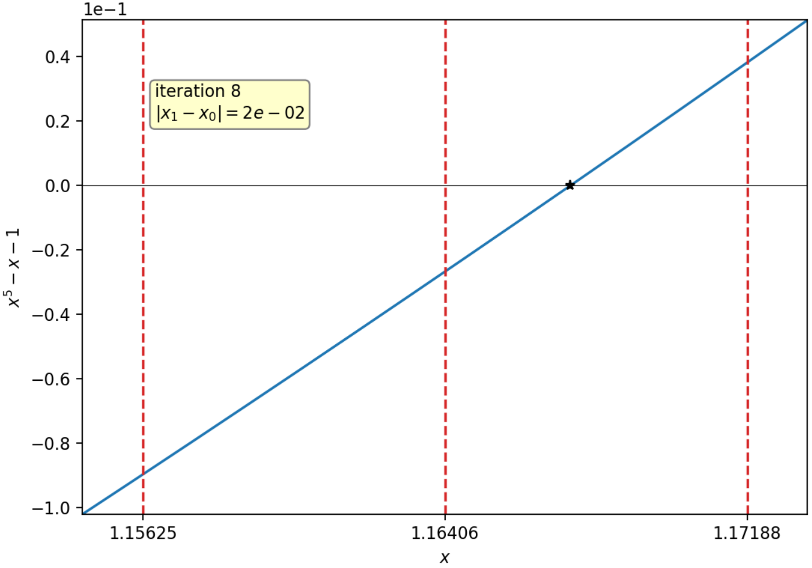

1e−1
iteration 6
$|x_1 - x_0| = 6e-02$
2
1
0
−1
−2
−3
$x^5 - x - 1$
1.12500
1.15625
1.18750
x

Bisection


1e−1
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
iteration 7
$|x_1 - x_0| = 3e-02$
$x^5 - x - 1$
1.15625
1.17188
1.18750
x

Bisection


1e−1
iteration 8
$|x_1 - x_0| = 2e - 02$
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
$x^5 - x - 1$
1.15625
1.16406
1.17188
x

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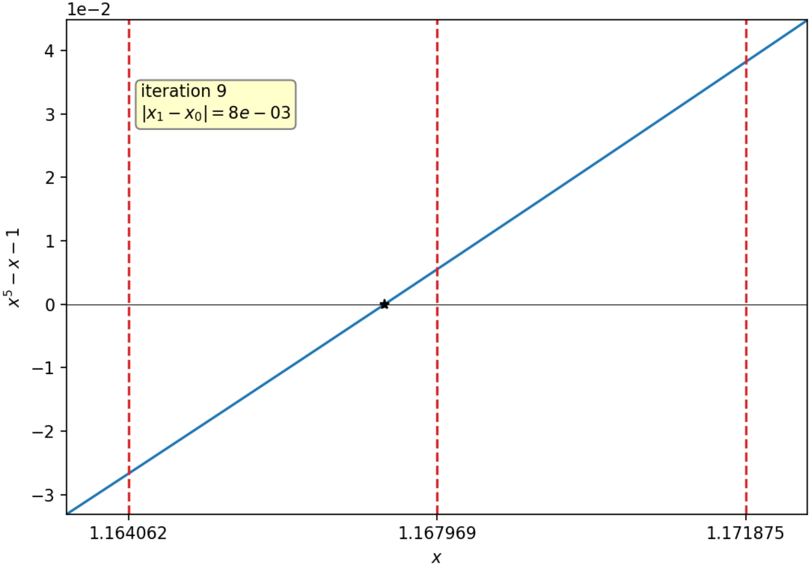
1e−2
iteration 9
$|x_1 - x_0| = 8e-03$
4
3
2
1
0
−1
−2
−3
$x^5 - x - 1$
1.164062
1.167969
1.171875
x

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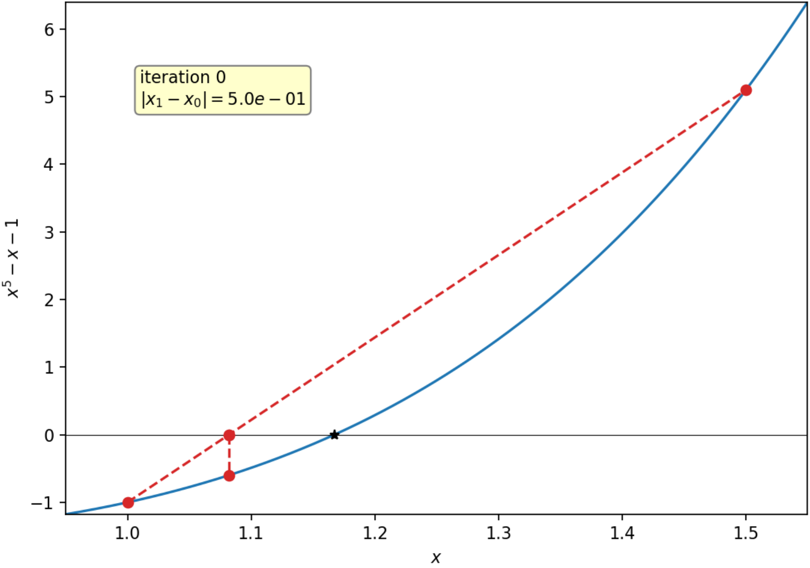
SECANT UPDATE

Secant Method



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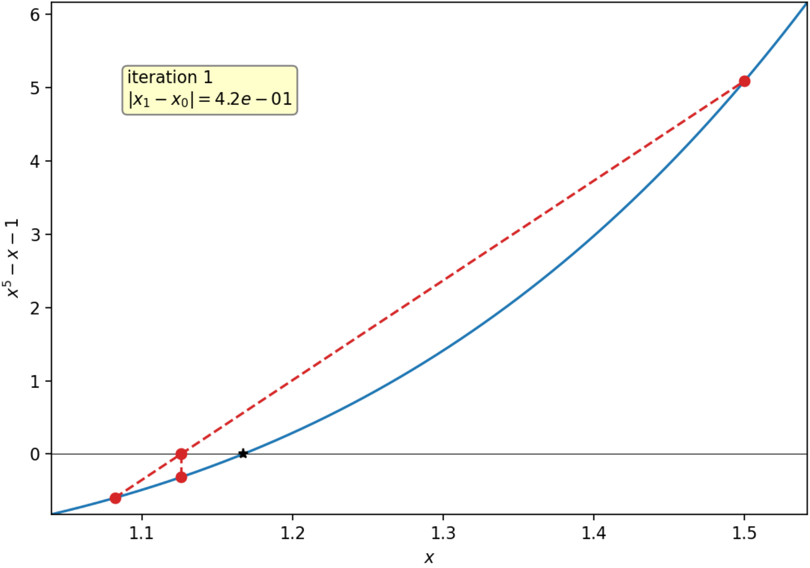

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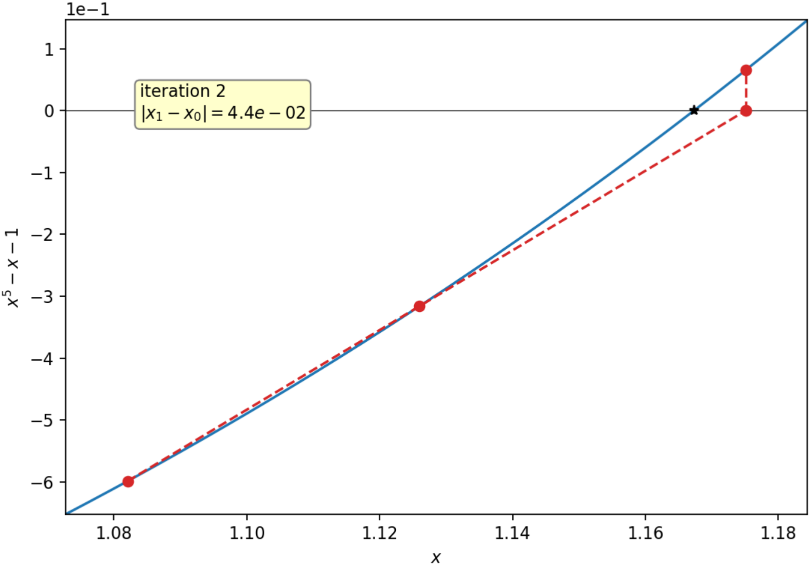

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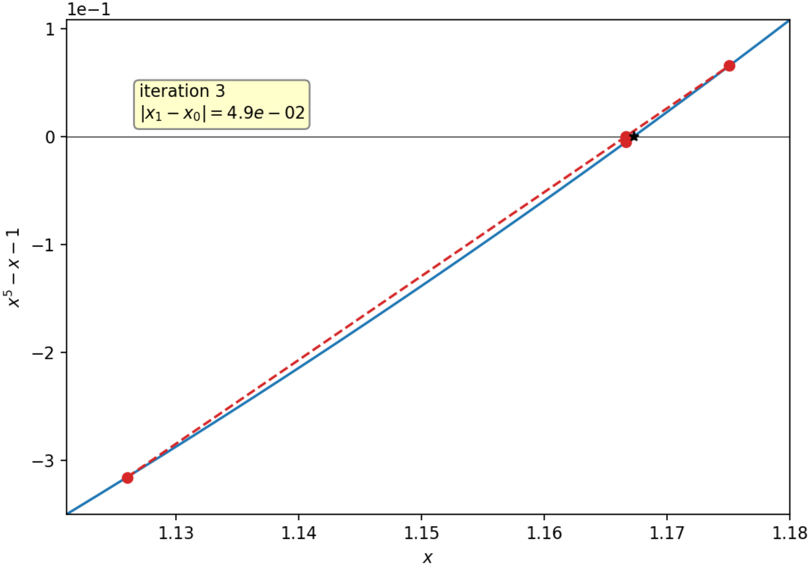

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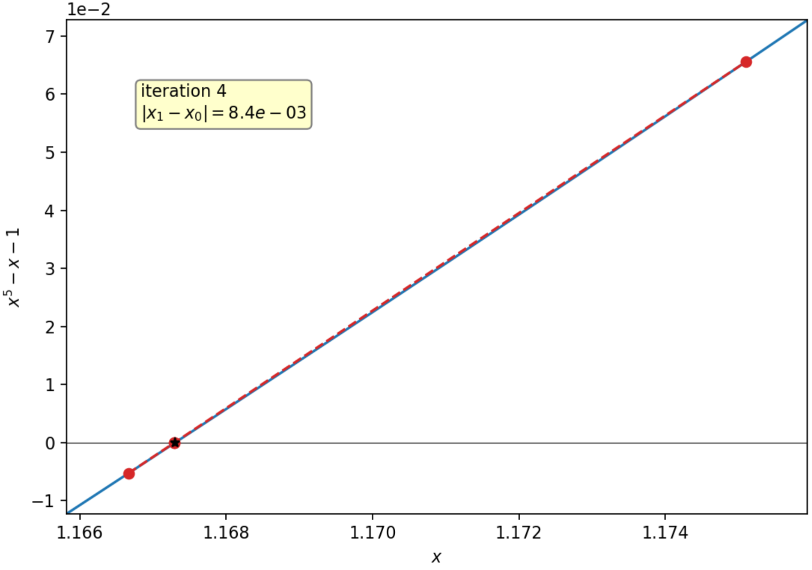

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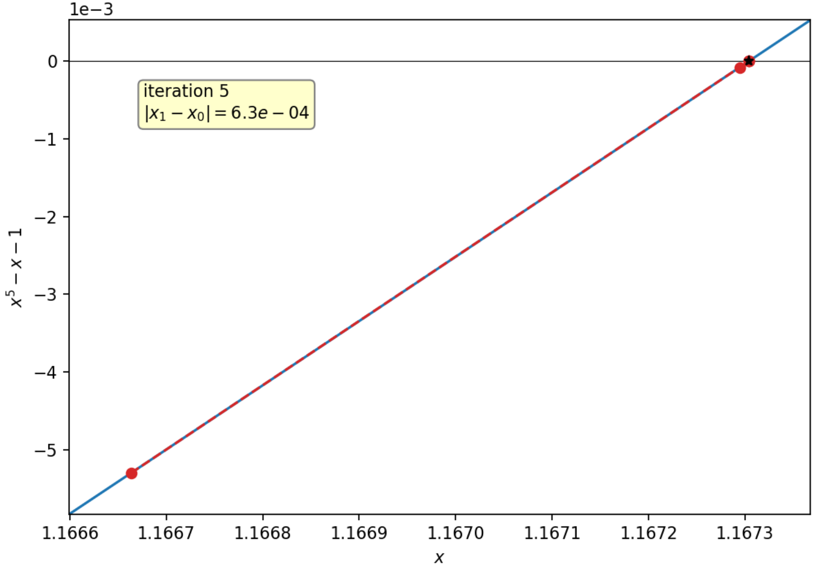

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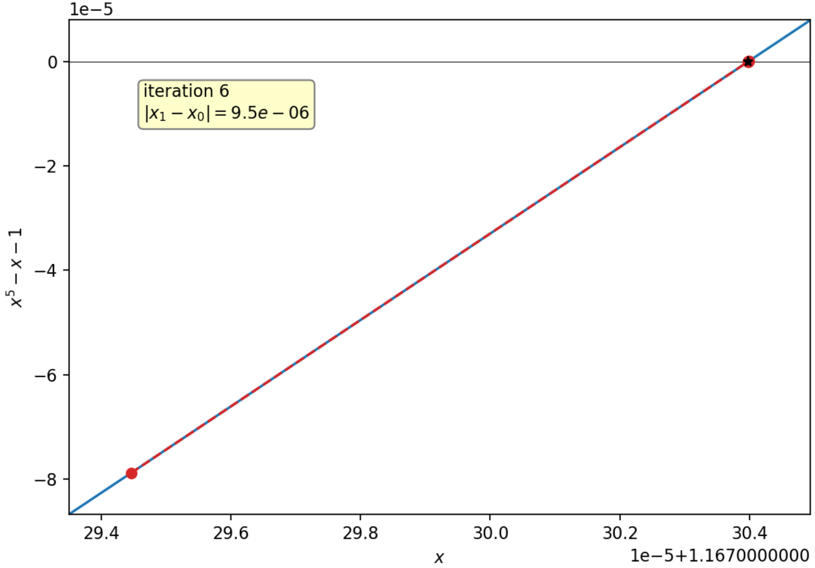

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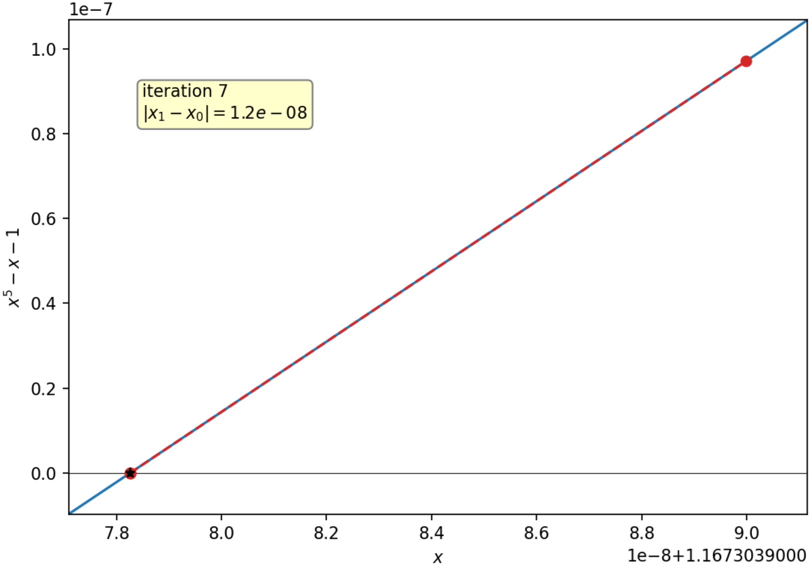

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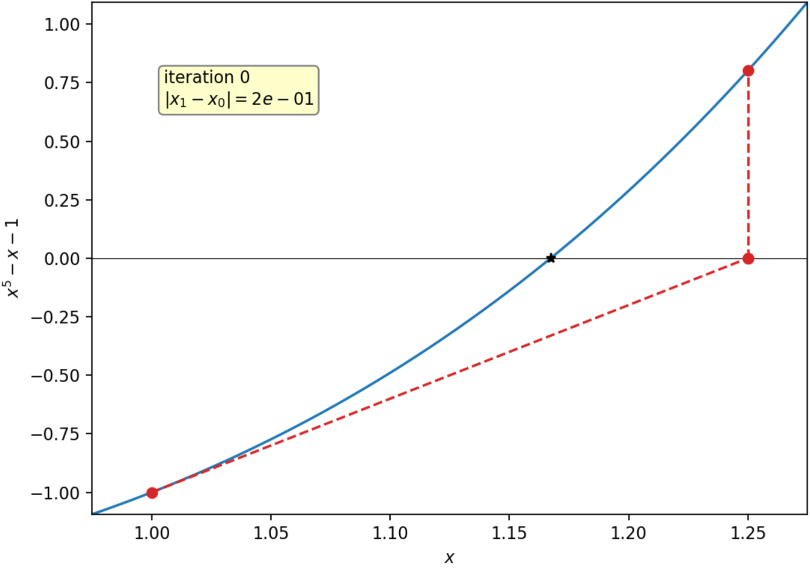
$$\boxed{x_{k+1} = x_k - \frac{f_k}{f_k'}} \tag{5}$$

NEWTON–RAPHSON UPDATE

Newton–Raphson



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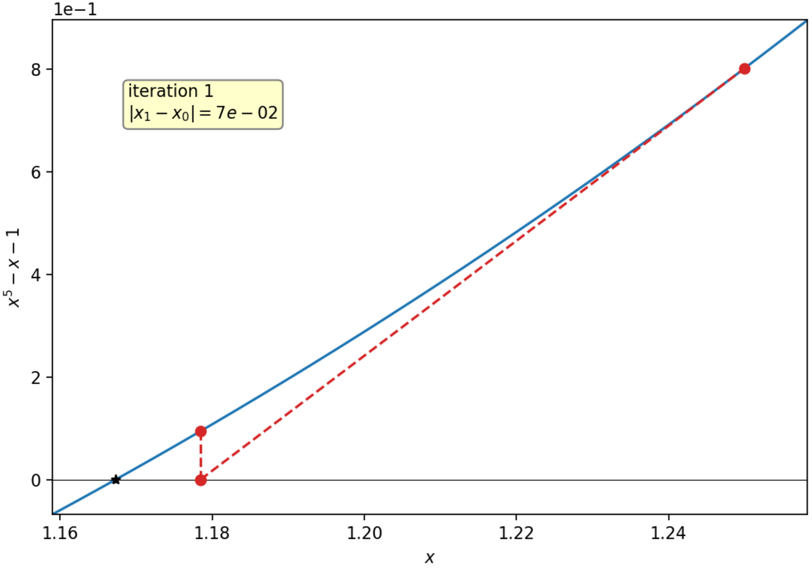

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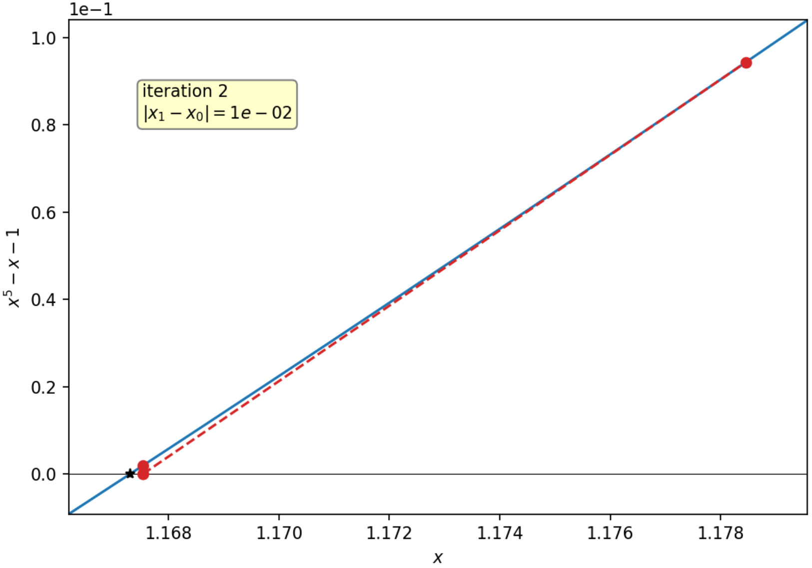

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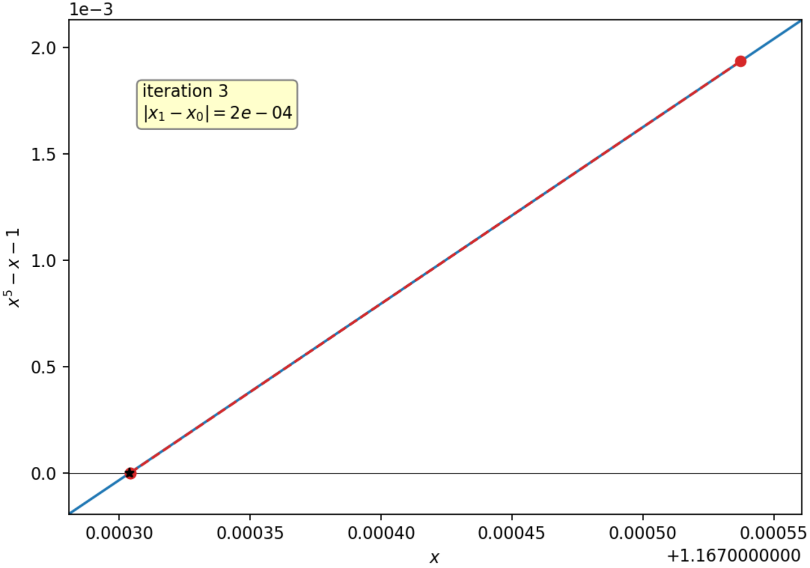

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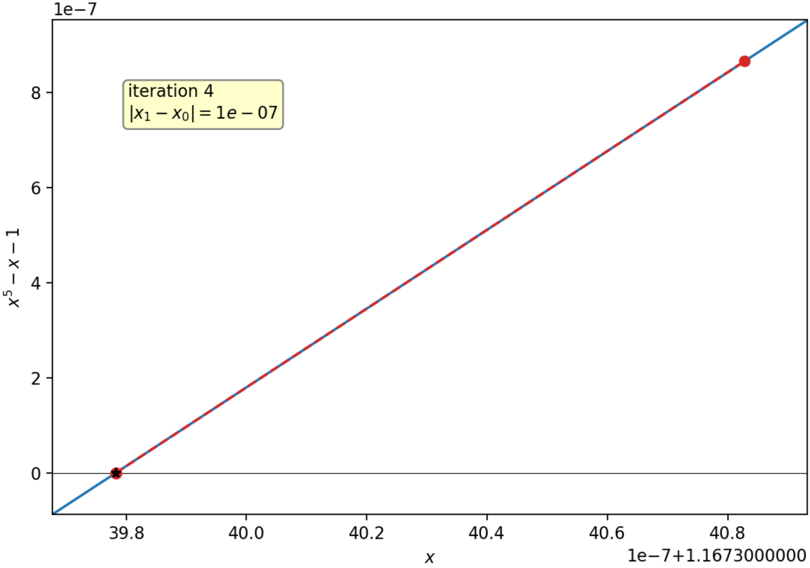

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$$\boxed{\begin{aligned} x_{k+1} = & \frac{f_{k-1} f_k}{(f_{k-2} - f_{k-1})(f_{k-2} - f_k)} x_{k-2} \\ & + \frac{f_{k-2} f_k}{(f_{k-1} - f_{k-2})(f_{k-1} - f_k)} x_{k-1} \\ & + \frac{f_{k-2} f_{k-1}}{(f_k - f_{k-2})(f_k - f_{k-1})} x_k \end{aligned}} \tag{6}$$

IQR UPDATE

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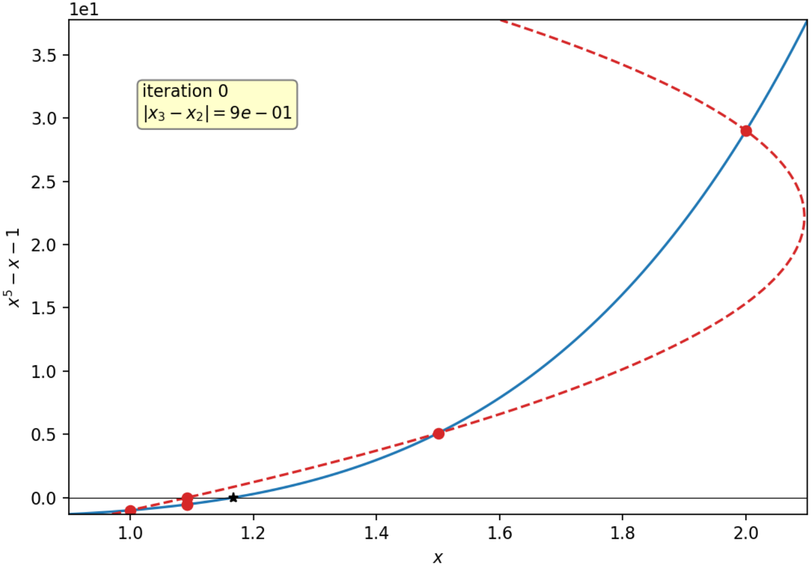

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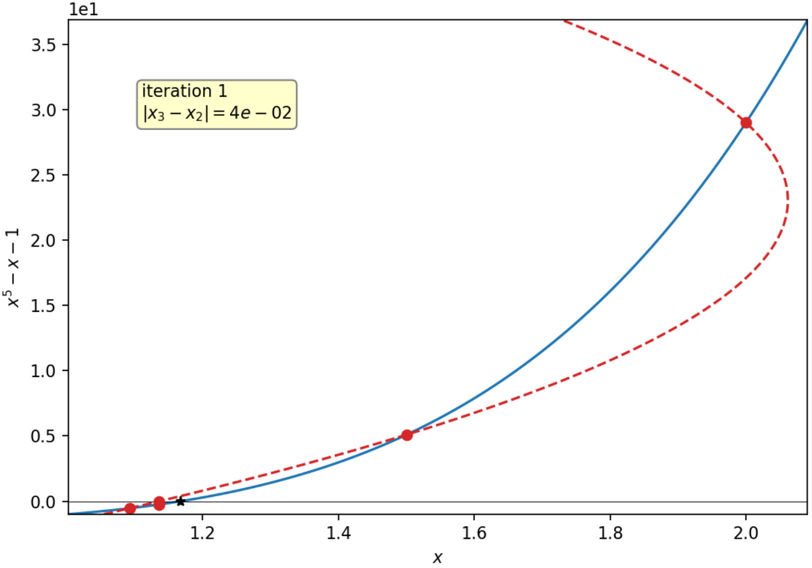

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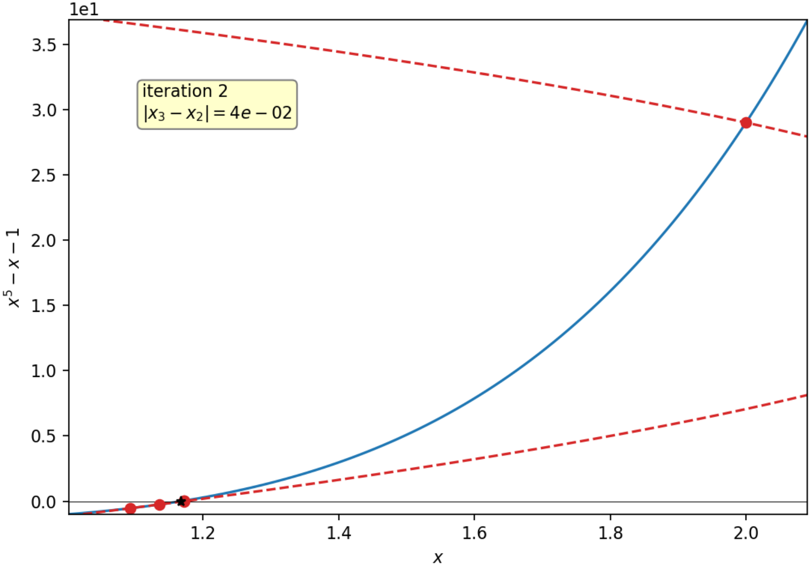

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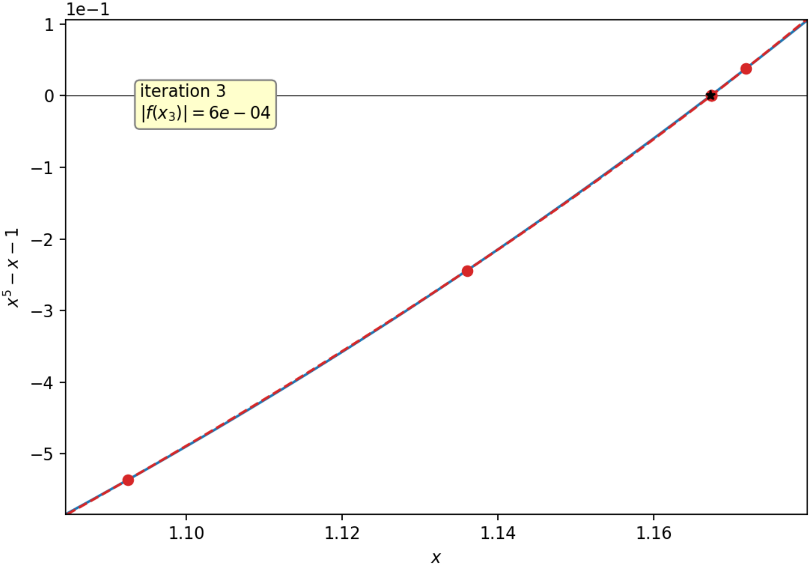

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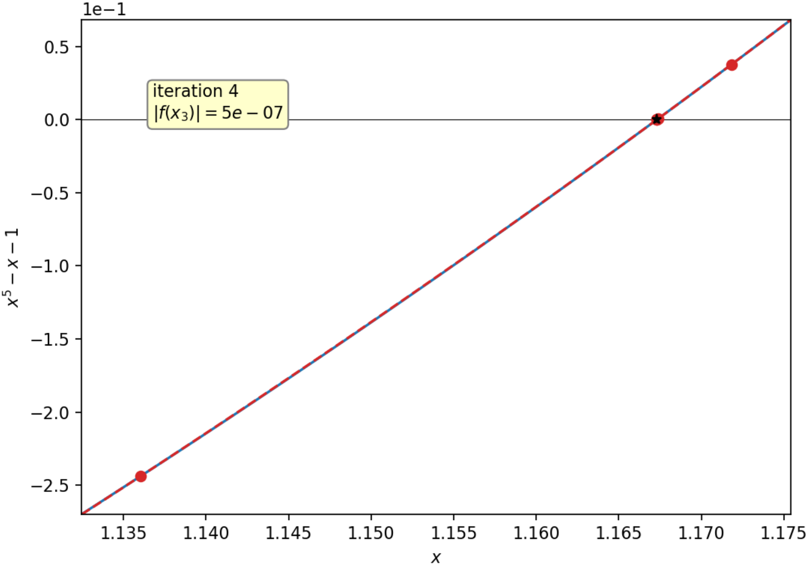

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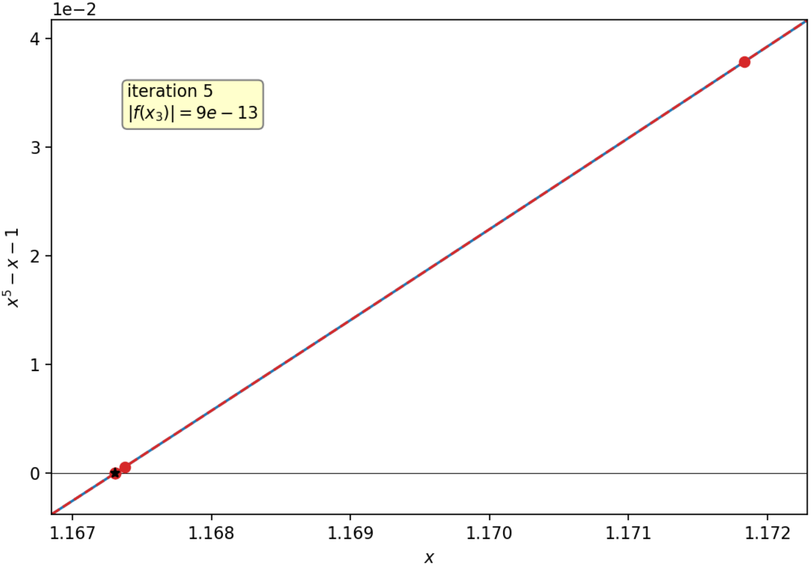

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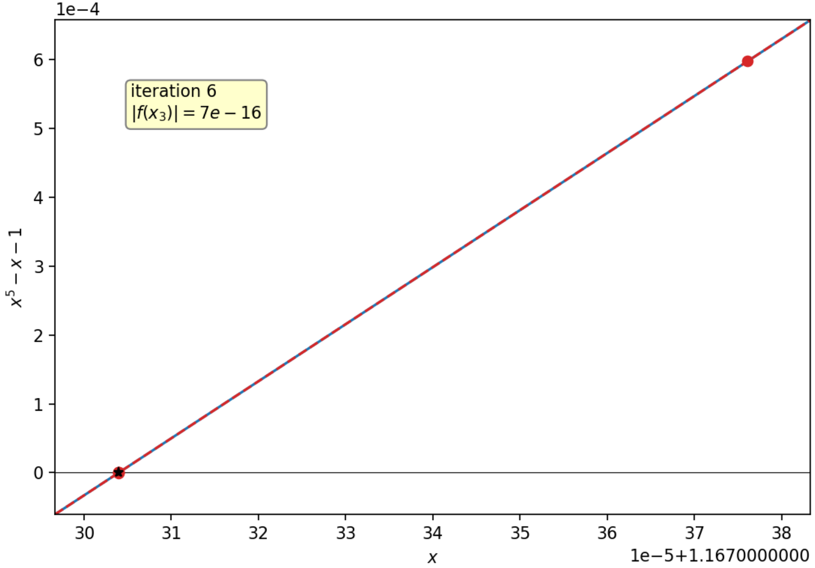

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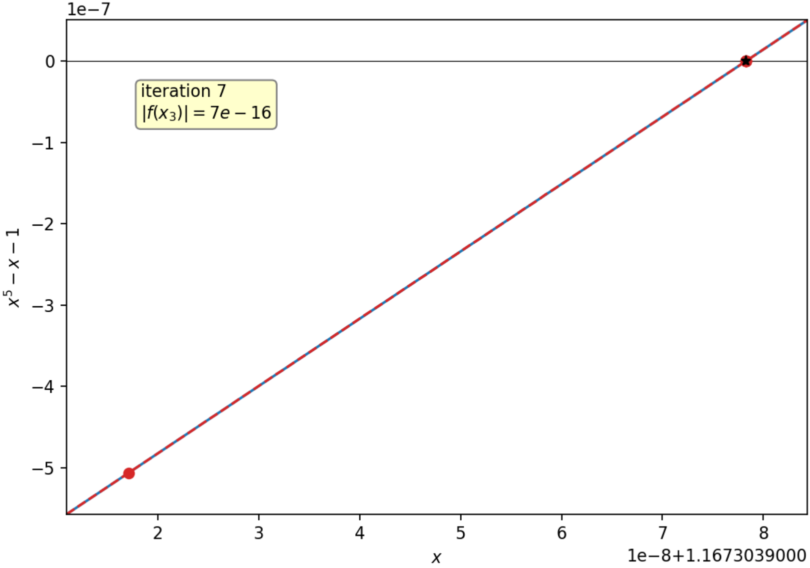

Homework

Pathological Cases

Implement the five root-finding methods we have studied, and use them to find the roots of,

1. $x^5 - x - 1 = 0$;
2. $16x^4 - 8x + 3 = 0$;
3. $x^3 - 2x^2 - 11x + 12 = (x-4)(x-1)(x+3) = 0$,

for a range of starting values (plot the curves to get an idea of what a sensible range of values might be). What do you notice? Do all of the methods find all of the roots? Estimate the order of convergence using the formula given above. Is the order of convergence always the theoretical maximum?