Fast Slow Dynamics - the van der Pol Oscillator

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1 Setup

We wish to study the behaviour of the van der Pol oscillator using blow up. The van der Pol oscillator is a well-studied second order ODE used to model a variety of physical and biological phenomena. A position coordinate $x(t)$ evolves according to the following equation.

$$\ddot{x}(t) - \mu\left(1 - x^2(t)\right)\dot{x}(t) + x(t) = 0 \tag{1}$$

Here, $\mu \gg 1$ is a scalar constant.
Currently, it doesn't resemble a anything like what we've seen in Krupa & Szmolyan (2001). To make it resemble a fast/slow system, we introduce a new variable. Let $w = \dot{x} + \mu F(x)$ where $F(x) = \frac{x^3}{3} - x$. Why have we chosen this function F? Notice that $F'(x) = -(1 - x^2)$, our nonlinear term in Equation 1. Differentiating w we obtain

$$\begin{aligned} \dot{w} &= \ddot{x} + \mu \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^3}{3} - x\right)\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \ddot{x} + \mu(x^2 - 1)\dot{x} \\ &= -x \end{aligned}$$

Here, the last equality follows from the van der Pol equation. We now have a two dimensional system.

$$\begin{cases} \dot{x} = w - \mu F(x) \\ \dot{w} = -x \end{cases}$$

Let $y = \frac{w}{\mu}$. Then

$$\begin{cases} \dot{x} = \mu\left(y - F(x)\right) \\ \dot{y} = -\frac{x}{\mu} \end{cases}$$

We will pause here although it doesn't look quite right and do a phase plane analysis to better understand the behaviour of the system. Setting each equation equal to zero in turn gives nullclines of $x = 0$ and $y = F(x)$.
+++ PHASE PLANE ANALYSIS +++
+++ New transformation to fast/slow system +++
Fast System:

$$\begin{cases} x' = y - F(x) \\ y' = -\epsilon x \end{cases} \tag{2}$$

Slow system:

$$\begin{cases} \epsilon\dot{x} = y - F(x) \\ \dot{y} = -x \end{cases} \tag{3}$$

1.1 Fold Points

The fold points are $(x_0^+, y_0^+) = (1, -\frac{2}{3})$ and $(x_0^-, y_0^-) = (-1, \frac{2}{3})$

Figure 1: Our Manifold $f(x, y, \epsilon)$

1.2 Non-degeneracy

Now that we have established our fast-slow systems (Equation 2 and 3) we need to check our non-degeneration conditions (Krupa & Szmolyan 2001). Now we first check that $\frac{\partial^2 f}{\partial x^2}(x_0, y_0, 0) \neq 0$ then we have,

$$\frac{\partial^2}{\partial x^2}(y - \frac{x^3}{3} - x) = -2x, \tag{4}$$

which we can evaluate at our fold points (Section 1.1) to give,

$$\begin{cases} \frac{\partial^2 f}{\partial x^2}(x_0^+, y_0^+, 0) = 2 < 0 \\ \frac{\partial^2 f}{\partial x^2}(x_0^-, y_0^-, 0) = 2 > 0. \end{cases} \tag{5}$$

Then we can consider that $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$. We show this by the following,

$$\frac{\partial f}{\partial y}(x_0^+, y_0^+, 0) = 1 \tag{6}$$

$$\frac{\partial f}{\partial y}(x_0^-, y_0^-, 0) = 1. \tag{7}$$

Lastly we need to consider that $g(x_0, y_0, 0) \neq 0$. This is easily seen as we find that $g(x_0, y_0, 0) = \pm 1$ for our two fold points. From here we are can now consider our transformation.

2 Charts

include picture from the book

Figure 2: Charts

3 Transformation

Krupa & Szmolyan (2001) discusses why we should make a transformation for our system. We find that if we map $(x_0, y_0) = (0, 0)$ we are then able to simplify our non-degeneracy conditions - as shown in Equation 8 (Krupa & Szmolyan 2001).

$$\begin{cases} (x_0, y_0) = (0, 0) \\ \frac{\partial^2 f}{\partial x^2}(0, 0, 0) > 0 \\ \frac{\partial f}{\partial y}(0, 0, 0) < 0 \end{cases} \tag{8}$$

3.1 Mapping Transformation

To be able to continue with our analysis we should consider the transformations. We will first only consider the case for our fold point at $(x_0^+, y_0^+) = (1, -\dfrac{2}{3})$. We wish to map $(x, y) \to (1 - \tilde{x}, \tilde{y} - \frac{2}{3})$ which reflects and translates our system such that our fold points are now mapped to $(0, 0)$ - Figure 3.

Figure 3: Our transformed system.

Now that we have made our transformation we can check our non-degeneracy conditions. However, before we continue we should check that our the sign of the derivative is conserved through the transformation. We can do this by using the chain rule as follows,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\tilde{x}}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\mathrm{d}\tilde{x}}{\mathrm{d}t}. \tag{9}$$

Now using Equation 9 and our new mapping ($(x, y) \to (1 - \tilde{x}, \tilde{y} - \frac{2}{3})$) we are able to define our new Fast System in the following way,

$$\begin{cases} x' = -y + x^2 - \dfrac{(x)^3}{3} \\ y' = \epsilon(x - 1) \end{cases} \tag{10}$$

where we note that we have dropped the tilde on x and y for convenience.

3.2 Non-degeneracy Conditions

$$\begin{cases} (x_0, y_0) = (0, 0) \\ \frac{\partial^2 f}{\partial x^2}(0, 0, 0) > 0 \\ \frac{\partial f}{\partial y}(0, 0, 0) < 0 \end{cases} \tag{8}$$

Now that we have constructed our transformed system, we are now able to check our non-degeneracy conditions - Equation 8. It is clear to see that $(x_0, y_0) = (0, 0)$ by the mapping we defined in Section 3.1. Following this we are able to check our other non-degeneracy conditions conform to our new mapping. The differentiation is easily seen from Equation 10 which yields that $\frac{\partial^2 f}{\partial x^2}(0, 0, 0) = 2 > 0$ and $\frac{\partial^1 f}{\partial y^1}(0, 0, 0) = -1 < 0$, confirming our assumptions.

3.3 Reduced Dynamics

The next progression for our system is to consider the reduced dynamics within our system. To do this we consider Equation 3 and take the $\epsilon \to 0$ which yields the following system,

$$0 = f(x, y, 0) = -y + x^2 - \frac{x^3}{3} \tag{11a}$$

$$\dot{y} = g(x, y, 0) = 0 \tag{11b}$$

which is known as the slow subsystem (Kuehn 2015). We are then able to compute the reduced flow by computing (Krupa & Szmolyan 2001),

$$\phi_x(x)\dot{x} = g(x, \phi(x), 0), \text{ for } y = \phi(x). \tag{12}$$

We find that $\phi(x) = x^2 - \dfrac{x^3}{3}$ where the derivative with respect to x gives $\phi_x(x) = 2x - x^2$. Now it is clear that we will have a singularity at $x = 0$, by Equation 12, thus we will find that our system blows up ($\dot{x} \to \infty$) which motivates the process that will follow.

After finding our reduced system we are able to determine the way in which it flows by considering the sign of $g(0, 0, 0)$. We see from Equation 11b that $g < 0$ then we have that our flow is directed towards the fold points $(0, 0)$. To continue with our analysis we first need to define Fenichel's theorems.

3.4 Fenichel and Standard Theory

THEOREM

3.5 Extended System

Now that we have established the above theorems then we need to consider the extended system such that $\epsilon' = 0$, thus $\epsilon = const$. By considering this system we will be able to consider a blow up (magnification) around our fold point in three dimensions. From Figure 4 we can consider the stability of our fold point. To do this we are

Figure 4: Blown up system

able to establish the following determinant,

$$A = \begin{vmatrix} 2x - x^2 - \lambda & -1 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = \lambda^2(2x - x^2 - \lambda). \tag{13}$$

However, a far easier approach is to note that our matrix is an upper triangular matrix which we can take the eigenvalues directly from Equation 13 such that $(\lambda_1.\lambda_2, \lambda_3) = tr(A)$. Then we can clearly see that for our fold points $\lambda_i = 0$ for $i = 1, 2, 3$ and for any $x \neq 0$ we have $\lambda_1 = x(2 - x)$ and $\lambda_2 = \lambda_3 \equiv 0$. As a result we can see that we are forced to blow up our system around our fold point as our steady states are non-hyperbolic whereas outside of this we find that we have one hyperbolic steady state.

3.5.1 Canonical Form

Now that we have established our reduced system we are able to rewrite it in canonical form - Equation 14.

$$\begin{aligned} x' &= -y + x^2 - \frac{x^3}{3} = -y + x^2 + h(x) \\ y' &= \epsilon(x - 1) \end{aligned} \tag{14}$$

There is ample reasoning for doing this. This is because we find that the canonical form has been studied in great detail allowing us to make comparisons and to avoid excess computation. This then allows us to follow an analogous approach to the many papers associated with this topic - as seen in Krupa & Szmolyan (2001) paper on Extending Geometric Singular Perturbation Theory.

4 Blow-ups in our System

We first need to transform our system in a new coordinate and time system. This is because we are considering our point as a circle with radius zero. By doing this we are then able to consider a localised flow within our system which is a most on the boundary of our point ($r = 0$). To do this we need to consider varying powers of r in each of our variables such that we have the system shown in Equation 15.

$$x = \bar{r}\bar{x} \tag{15a}$$

$$y = \bar{r}^2\bar{y} \tag{15b}$$

$$\epsilon = \bar{r}^3\bar{\epsilon} \tag{15c}$$

Now that we have this transformation we are then able to consider our charts (see Section 2) in our system.

4.1 Charts

We note that our system will have three charts K_1, K_2, K_3, where we have $\bar{y} = 1,\ \bar{\epsilon} = 1,\ \bar{x} = 1$. By inserting these into Equations 15a, 15b and 15c respectively to give,

$$x = r_1x_1,\ y = r_1^2,\ \epsilon = r_1^3\epsilon_1, \tag{16a}$$

$$x = r_2x_2,\ y = r_2^2y_2,\ \epsilon = r_2^3 \tag{16b}$$

$$x = r_3,\ y = r_3^2y_3,\ \epsilon = r_3^3\epsilon_3 \tag{16c}$$

where $(x_i, r_i, \epsilon_i) \in \Re^3$ for $i = 1, 2, 3$ (Krupa & Szmolyan 2001). Now that we have done this we can consider the individual charts explicitly. We start with chart two (K_2) for a simple reason. This is because we area able to glean the most information out of this chart. We will see that we then find that we can define the mappings from chart one to two and two to three, further simplifying our analysis in the future.

4.2 Dynamics in K_2

To be able to consider chart K_2 we will use the transformation - shown in Equation 16b - in our canonical system. We also need to use a time rescaling ($t_2 = r_2t$) to be able to desingularise the system. Now substituting this into Equation 14 yields,

$$\frac{\mathrm{d}}{\mathrm{d}t}(r_2x_2) = r_2^2\frac{\mathrm{d}x_2}{\mathrm{d}t} = -y_2 + x_2^2 - \frac{x_2^3r_2}{3}, \tag{17}$$

$$r_2^3y_2' = r_2^3(-1 + r_2x), \tag{18}$$

$$r_2' = 0, \tag{19}$$

noting that $\frac{\mathrm{d}r}{\mathrm{d}t_2} = \frac{\mathrm{d}t}{\mathrm{d}t_2}\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{r_2}\frac{\mathrm{d}r_2}{\mathrm{d}t}$ and we are taking the equations to $O(r_2)$. Now dividing through by r_2^2 and r_2^3 respectively for each equation we get,

$$\begin{aligned} x_2' &= x_2^2 - y_2 + O(r_2), \\ y_2' &= -1 + O(r_2), \\ r_2' &= 0, \end{aligned} \tag{20}$$

which are then able to evaluate as a layer problem. Now we know that this is the Riccati Equation - see Mishchenko (2012).

4.3 Dynamics in K_1

4.4 Dynamics in K_3

Similarly to K_1 and K_2, the system can be transformed using Equation 16c.

$$\begin{aligned}
\frac{\mathrm{d}r_3}{\mathrm{d}t_3} &= r_3 F(r_3, y_3, \epsilon_3) \\
\frac{\mathrm{d}y_3}{\mathrm{d}t_3} &= \epsilon_3 (r_3 - 1) - 2y_3 F(r_3, y_3, \epsilon_3) \\
\frac{\mathrm{d}\epsilon_3}{\mathrm{d}t_3} &= -3\epsilon_3 F(r_3, y_3, \epsilon_3)
\end{aligned}$$

where $F(r_3, y_3, \epsilon_3) = \left(1 - y_3 - \frac{r_3}{3}\right)$

5 Canard Points

A Dynamics in K_2

References

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