

# Fast and Slow Dynamics

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# A Quick Reminder: Dynamical Systems

$$\begin{cases} \frac{dx}{dt} = -y + x^2 - \frac{x^3}{3} \\ \frac{dy}{dt} = \epsilon(-\lambda + x) \end{cases}$$

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$$\begin{cases} \frac{dx}{dt} = -y + x^2 - \frac{x^3}{3} \\ \frac{dy}{dt} = (-1 + x) \end{cases}$$

# The Van der Pol Equations

- Fast System:

$$\begin{cases} \frac{dx}{dt} = -y + x^2 - \frac{x^3}{3} \\ \frac{dy}{dt} = \epsilon(-1 + x) \end{cases}$$

**Figure:** Phase Plane of the Van der Pol Equations

# The Van der Pol Equations

- Fast System:

$$\begin{cases} \frac{dx}{dt} = -y + x^2 - \frac{x^3}{3} \\ \frac{dy}{dt} = \epsilon(-1 + x) \end{cases}$$

- Slow System:

► Let  $t = \frac{\tau}{\epsilon}$ .

$$\begin{cases} \epsilon \frac{dx}{d\tau} = -y + x^2 - \frac{x^3}{3} \\ \frac{dy}{d\tau} = (-1 + x) \end{cases}$$

**Figure:** Phase Plane of the Van der Pol Equations

# The Van der Pol Equations, $\epsilon = 0$

- Layer Problem:

$$\begin{cases} \frac{dx}{dt} = -y + x^2 - \frac{x^3}{3} \\ \frac{dy}{dt} = 0 \end{cases}$$

- Reduced System:

$$\begin{cases} 0 = -y + x^2 - \frac{x^3}{3} \\ \frac{dy}{d\tau} = (-1 + x) \end{cases}$$

**Figure:** Phase Plane of the Van der Pol Equations

# The Reduced System

Reduced System:

$$\begin{cases} 0 & = -y + x^2 - \frac{x^3}{3} := f(x, y) \\ \frac{dy}{d\tau} & = (-1 + x) \end{cases}$$

Reduced flow is restricted to:

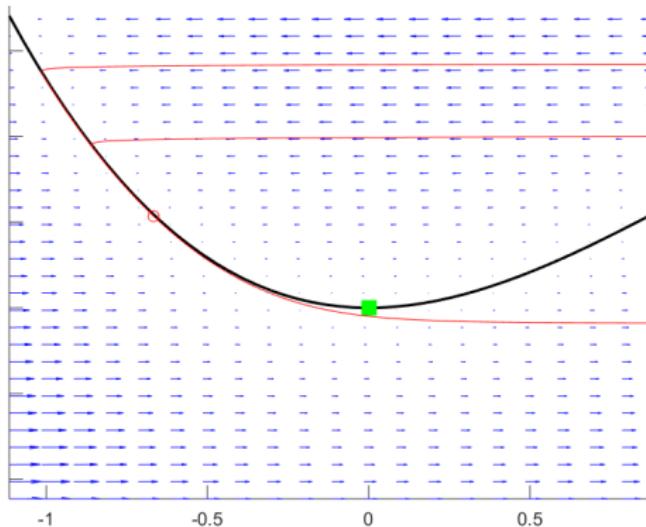
$$f = 0 \Rightarrow y = x^2 - \frac{x^3}{3}$$

Fold Points:

- $(x_1^*, y_1^*) = (0, 0)$
- $(x_2^*, y_2^*) = (2, \frac{4}{3})$

**Figure:** Phase Plane of the Van der Pol Equations

# The Van der Pol Equations near $(x_1^*, y_1^*)$

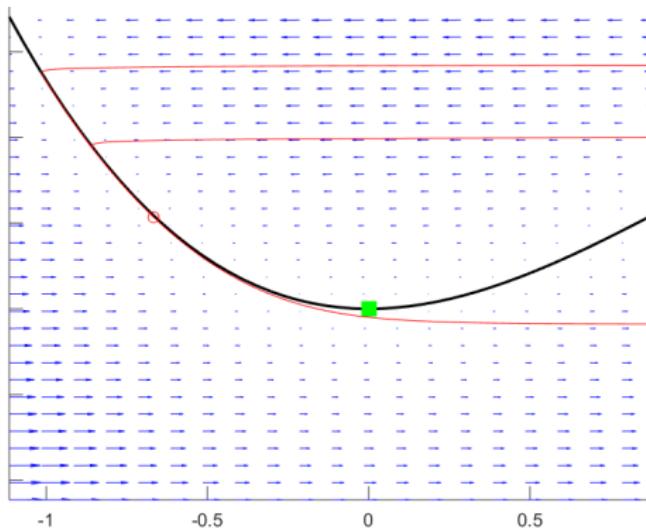


Finding  $\frac{dx}{d\tau}$  for the reduced system:

$$y = x^2 - \frac{x^3}{3}$$
$$\Rightarrow \frac{dy}{d\tau} = \frac{dy}{dx} \frac{dx}{d\tau} = x(2-x) \frac{dx}{d\tau}$$
$$\Rightarrow \frac{dx}{d\tau} = \frac{dy}{d\tau} \frac{1}{(2-x)x}$$

Singularities at  $x = 0, x = 2$ .

# The Van der Pol Equations near $(x_1^*, y_1^*)$



Investigating Hyperbolicity:

$$f(x, y) = -y + x^2 - \frac{x^3}{3}$$
$$\Rightarrow \frac{df}{dx} = 2x - x^2$$
$$= x(2 - x)$$

Fold Point  $(x_1^*, y_1^*) = (0, 0)$  is non-hyperbolic.

# The Blow-up

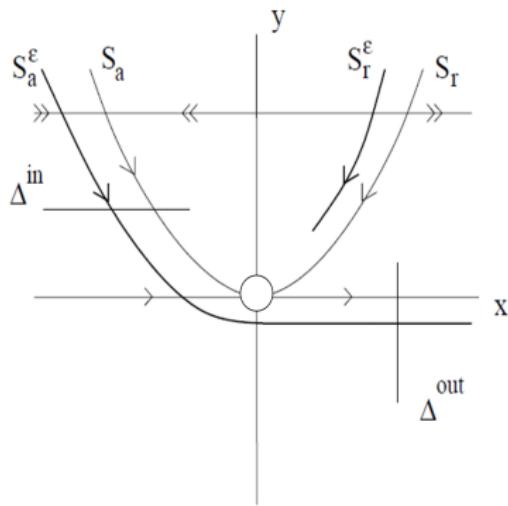
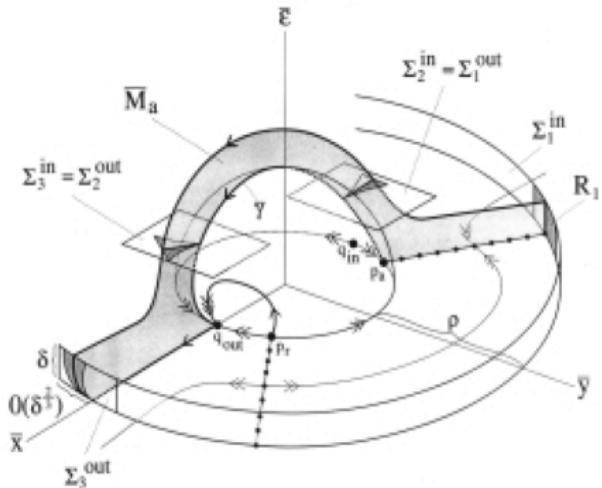


Figure: Blow up of our fold point  
(Kuehn, Multiple Time Scale Dynamics)

- The radius  $r = 0$
- Coordinate transformation
  - ▶  $x = r_i x_i, y = r_i^2 y_i, \epsilon = r_i^3 \epsilon_i$
- Analysing our blow up

# Charts Motivation



**Figure:** Sketch of the coordinate charts  
(Kuehn, Multiple Time Scale Dynamics)

- What are charts?
- Why use charts?
  - ▶ Simplifies our singularity
  - ▶ Magnifies our flow
- Transition Maps  $\Pi_1 \rightarrow \Pi_2 \rightarrow \Pi_3$

# Dynamics on $K_2$

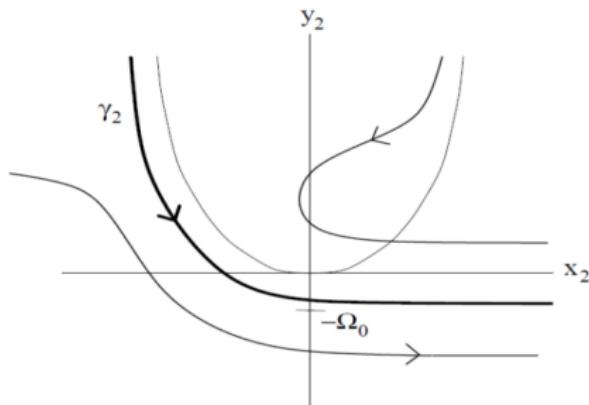


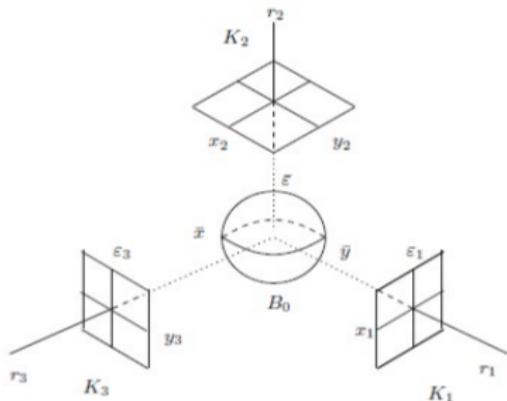
Figure: Phase portrait for  $\bar{\epsilon} = 0$  (Kuehn, Multiple Time Scale Dynamics)

- Transformed Van der Pol equations on  $\epsilon = 1$ :

$$\begin{cases} x'_2 &= x_2^2 - y_2 + O(r_2) \\ y'_2 &= -1 + O(r_2) \\ r'_2 &= 0 \end{cases}$$

- Riccati Equations.
  - ▶ Asymptotic solution.
- Transition map,  $\Pi_2$ , exists

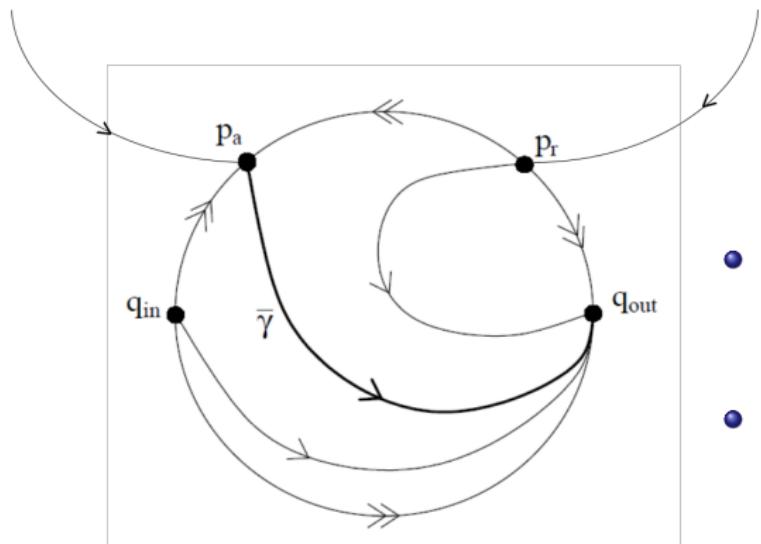
# Dynamics on $K_1$ and $K_3$



- Dynamics on  $K_1$ :
  - ▶ Centre Manifold Theorem
- Dynamics in  $K_3$ 
  - ▶ Resonance

Figure: The attracting branch connecting to the blow up.

# Global Solution



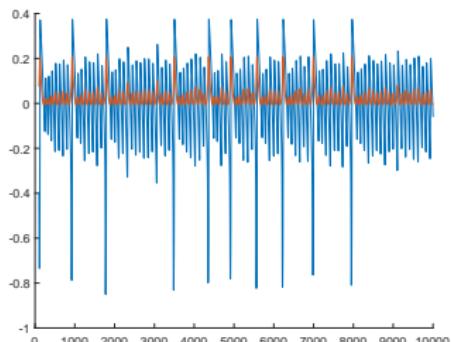
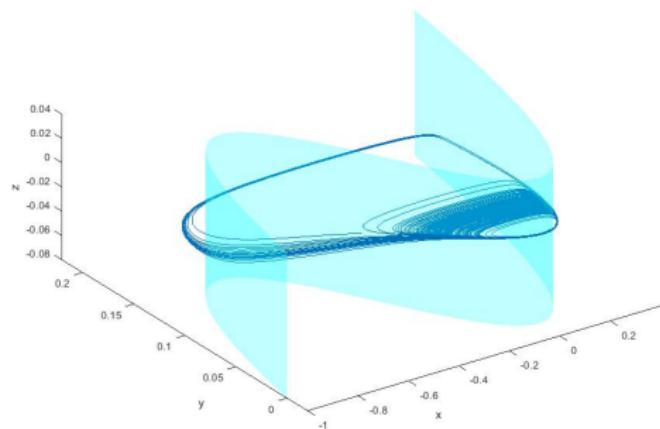
- $\Pi$  describes the global transition of trajectories.
  - ▶  $\Pi = \Pi_3 \circ \kappa_{23} \circ \Pi_2 \circ \kappa_{12} \circ \Pi_1$
- What does  $q_{out}$  do?

**Figure:** Global dynamics of the blown up system (Kuehn, Multiple Time Scale Dynamics)

# Canard Points in the Van der Pol

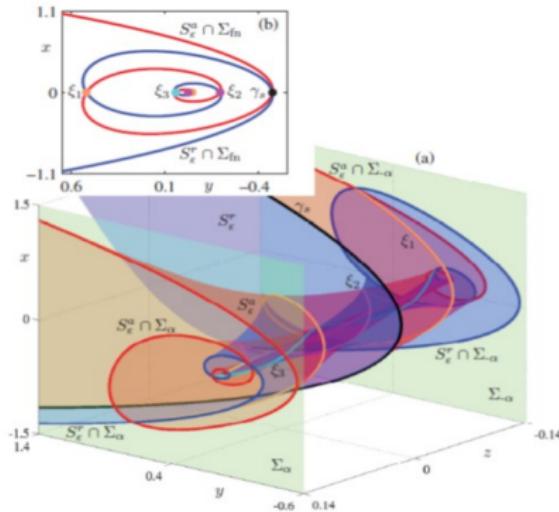
$$\begin{cases} \epsilon \frac{dx}{d\tau} = -y + x^2 - \frac{x^3}{3} \\ \frac{dy}{d\tau} = (-\lambda + x) \end{cases}$$

# Mixed Mode Oscillations (MMO's)



Phase Portrait and Time Series of chaotic MMO

# Mixed Mode Oscillations (MMO's)



**Figure:** Canards in the normal form system (Desroches et al., 2015)

- Normal Form:

$$\begin{cases} \dot{x} &= y - x^2 \\ \dot{y} &= z - x \\ \dot{z} &= -\nu \end{cases}$$

- MMO due to a Folded Node

# Conclusion

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