During this section we will be considering a canard point. This is when our fold point is shifted along the manifold - Figure 1. To adequately explain the effect that the canard point will have on our system we will need

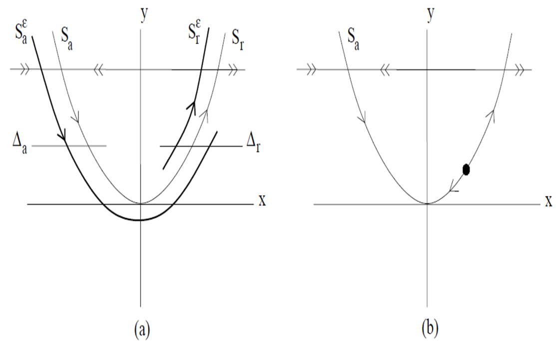

Figure 1: The reduced flow of our system for a) $\lambda = 0$ and b) $\lambda > 0$.

to consider our system,

$$\begin{aligned} x' &= -y + x^2 - \frac{x^3}{3}, \\ y' &= \epsilon(x - 1), \\ \epsilon' &= 0. \end{aligned} \tag{??}$$

Now we need to consider Equation **??** in terms our our cananrd system. To do this we rewrite our system with an extra parameter λ, where λ is our perturbation of our fold point (Krupa & Szmolyan 2001). Krupa & Szmolyan (2001) discusses generally how we should continue with computing our canard system. If we apply his theory to the Van der Pol system we find,

$$\begin{aligned} x' &= -y + x^2 - \frac{x^3}{3}, \\ y' &= \epsilon(x - \lambda), \\ \epsilon' &= 0, \\ \lambda' &= 0, \end{aligned} \tag{1}$$

where the change in ϵ and λ are constant. Now, for the remainder of the section, we follow the method of Krupa & Szmolyan (2001) for the canard system. If we start by rewriting our canard system into the canonical forms we find,

$$x' = -yh_1(x, y, \epsilon, \lambda) + x^2 h_2(x, y, \epsilon, \lambda), \tag{2}$$

$$y' = \epsilon(xh_4(x, y, \epsilon, \lambda) - \lambda h_6(x, y, \epsilon, \lambda)), \tag{3}$$

$$\tag{4}$$

Where we note that $h_j(x, y, \epsilon, \lambda) = 1 + O(x, y, \epsilon, \lambda)$ for $j = 1, 2, 4, 5$ and $h_3(x, y, \epsilon, \lambda) = O(x, y, \epsilon, \lambda)$. However, we should note that for the Van der Pol system our only term that is not solely of leading order is $h_2(x, y, \epsilon, \lambda) = 1 - \frac{x}{3}$. Now we are able to choose such a $\lambda > 0$ that produces an equilibrium on our repelling branch S_r for the reduced flow. By doing this we are then able to define the following conditions for our reduced flow on h_j,

$$a_3 = \frac{\partial}{\partial x} h_2(0, 0, 0, 0) = -\frac{1}{3}, \tag{5}$$

$$A = -a_2 + 3a_3 - (2a_4 + 2a_5) = -1, \tag{6}$$

where we notice that our other solutions for $a_i = 0$ for $i = 1, 2, 4, 5$ are trivial. The reason that we consider the constant A is because we will find that this constant is crucial in our canard point analysis iff $A \neq 0$ (Krupa & Szmolyan 2001). Following this (Krupa & Szmolyan 2001) discusses the existence of a critical value for λ (denoted λ_c), where our two branches S_r and S_a must connect in a smooth fashion. Now from *Theorem 3.1* we nkow that we must have a transition map at our critical point,

$$\lambda_c(\sqrt{\epsilon}) = -\epsilon(\frac{a_1 + a_5}{2} + \frac{A}{8}) + O(\epsilon^{\frac{3}{2}}), \tag{7}$$

which can be written as $\lambda_c(\sqrt{\epsilon}) = \frac{\epsilon}{8} + O(\epsilon^{\frac{3}{2}})$ for the Van der Pol system (Krupa & Szmolyan 2001). Consider Canard cycles and center manifolds / Freddy Dumortier, Robert Roussarie. for more details on canards in Van der Pol .

0.1 Canard Blow-up

Now similarly to Section **??** we consider various transformations of our coordinate system to be able to be able to consider the non-hyperbolic equilibrium induced by our canard point. However, as we would expect with our new system we should consider a new set of transformations (Krupa & Szmolyan 2001).

$$x = \bar{r}\bar{x},\ y = \bar{r}^2 y,\ \epsilon = \bar{r}^2\bar{\epsilon},\ \lambda = \bar{r}\bar{\lambda} \tag{8}$$

Now that we have established the transformation we can then define our transformations for K_1 and K_2 but it is not necessary to consider the third chart (K_3). This is because we find that the attracting slow manifold connects to the repelling slow manifold. As a result of this we find that our flow will 'bend back' from K_2 into K_1 instead of flowing out into the fast flow, which is described by K_3. This concept can be described by the Figure 2.

Figure 2: Figure describing canard flow in manifold

Since we have established why we need only consider two charts we can our transformations,

$$x = r_1 x_1,\ y = r_1^2,\ \epsilon = r_1^2\epsilon_1,\ \lambda = r_1\lambda_1 \tag{9a}$$

$$x = r_2 x_2,\ y = r_2^2 y_2,\ \epsilon = r_2^2,\ \lambda = r_2\lambda_2 \tag{9b}$$

Since these transformations have been defined we should consider our charts. We will first consider chart 2, for analogous reasoning to Section **??**.

0.1.1 Dynamics in K_2

We start by noting that we are considering our invariant plane at $r_2 = 0$ which will significantly simplify our system for K_2. Further we should note that we are taking a transformation in time, $\frac{\mathrm{d}r}{\mathrm{d}t_2} = \frac{\mathrm{d}t}{\mathrm{d}t_2}\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{r_2}\frac{\mathrm{d}r_2}{\mathrm{d}t}$, as well as in our coordinates. Then if we substitute our time transformation and Equation 9b into our system of Equations 1 we find,

$$\begin{aligned} r_2^2 x_2' - r_2 x_2 r_2' &= -r_2^2 y_2 h_1 + r_2^2 x_2^2 h_2, \\ \implies x_2 &= -y_2 + x_2^2 - r_2 G_2(x_2, y_2), \end{aligned} \tag{10a}$$

$$\begin{aligned} r_2^3 y_2' - 3r_2^2 y_2 r_2' &= r_2^2(r_2 x_2 h_4 - r_2\lambda_2 h_5), \\ \implies y_2' &= x_2 - \lambda_2 + r_2 G_2(x_2, y_2), \end{aligned} \tag{10b}$$

where we note that $h_j = h_j(x, y, \epsilon, \lambda)$ for $j = 1, 2, 3, 4, 5$. We should also recall that $r_2' = \lambda_2' = 0$. Notice that we have included an additional term in Equation 10b - we define $G_2(x_2, y_2)$ in the following way, $G(x_2, y_2) = (G_1(x_1, y_1), G_2(x_2, y_2))^T = (-\frac{x_2^2}{3}, 0)^T$. The reason we also define this vector is to aide in the Melnikov computations which we will see later. Krupa & Szmolyan (2001) discusses that for this chart we have an interesting result. They note that at $r_2 = \lambda_2 = 0$ our system is integrable which allows us to define a constant of motion $H(x_2, y_2) = \frac{1}{2}\exp(-2y_2)\left(y_2 - x_2^2 + \frac{1}{2}\right)$ which we can easily verify (Krupa & Szmolyan 2001) using the following equations,

$$\begin{aligned} x_2' &= e^{2y_2}\frac{\partial H}{\partial y_2}(x_2, y_2), \\ y_2' &= -e^{2y_2}\frac{\partial H}{\partial x_2}(x_2, y_2). \end{aligned}$$

Further to this we can see, when we consider our reduced system, that we have an equilibrium at the origin, implying that $H(x_2, y_2) = h$. This then allows us to define a trajectory for the orbit by WHY??????????

$$\gamma_{c,2}(t_2) = (x_{c,2}(t_2), y_{c,2}(t_2)) = \left(\frac{t_2}{2}, \frac{t_2^2}{4} - \frac{1}{2}\right) \tag{11}$$

Next we will be continuing our analysis onto K_1.

0.2 Dynamics in K_1

For K_1 we follow a similar approach to the above. We will use the transformations,

$$x = r_1 x_1,\ y = r_1^2,\ \epsilon = r_1^2\epsilon_1,\ \lambda = r_1\lambda_1, \tag{9a}$$

to find the relevant pathways of our flows. Now if we first consider the r_1 component,

$$2r_1^2 r_1' = r_1^2\epsilon(r_1 x_1 - r_1\lambda_1), \tag{12}$$

where we can call $F = F(x, y, \epsilon, \lambda) = x_1 - \lambda_1 + O(r_1(r_1 + \lambda_1)$. Now we will see the motivation with starting with $y = r_1$ when we transform our other coordinates. Now if we consider $x = r_1 x_1$,

$$\begin{aligned} r_1 r_1' x_1 + r_1^2 x_1' &= -r_1^2 + r_1^2 x_1^2, \\ x_1' &= -1 + x_1^2 - \frac{x_1 r_1'}{r_1}, \end{aligned}$$

where we can use Equation 12 to simplify this further - Equation 13.

$$x_1' = -1 + x_1^2 - \frac{x_1}{r_1}\left(\frac{r_1\epsilon_1 F}{2}\right) \tag{13}$$

We now consider our $\epsilon = \epsilon_1 r_1^2$ and noting $\epsilon' = 0$. Then we have, $r_1^3\epsilon' = -2r_1^2\epsilon_1 r_1'$, where we can use Equation 12 to simplify to,

$$\epsilon' = -\epsilon_1^2 F. \tag{14}$$

Our last transformation is for our new coordinate $\lambda = r_1\lambda$, noting that $\lambda' = 0$. Similarly to the above we find $r_1^2\lambda_1' + r_1\lambda_1 r_1' = 0$ then,

$$\lambda_1' = -\frac{\lambda_1\epsilon_1 F}{2}, \tag{15}$$

which is a trivial rearrangement as seen in Equation 14. Now if we combine the above we find that our transformed system is of the following form,

$$r_1' = \frac{\epsilon}{2}(r_1 x_1 - r_1 \lambda_1), \tag{16a}$$

$$x_1' = -1 + x_1^2 - \frac{x_1 \epsilon_1 F}{2}, \tag{16b}$$

$$\epsilon' = -\epsilon_1^2 F, \tag{16c}$$

$$\lambda_1' = -\frac{\lambda_1 \epsilon_1 F}{2}. \tag{16d}$$

From this system we are now able to make some deductions. We first can observe that the hyperplanes are along the $r_1 = \epsilon_1 = \lambda_1 = 0$ with an invariant line at $l_1 = \{(x_1, 0, 0, 0) : x_1 \in \Re\}$ (Krupa & Szmolyan 2001). As Krupa & Szmolyan (2001) discusses the equilibria present at the end of both of our branches - Figure 1 - which are found at $p_a = (-1, 0, 0, 0)$ and $p_r = (1, 0, 0, 0)$ (Krupa & Szmolyan 2001). Now we can go one step further, we can consider Equation 16 and find the eigenvalues of the system for the invariant planes. We find that,

$$J - \lambda I = \begin{bmatrix} 2x - \lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix}, \tag{17}$$

which clearly has three zero eigenvalues and one non-zero eigenvalue $\lambda = \pm 2$. Which further empahsises that our equilibrium point is non-hyperbolic.

References

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