Fast Slow Dynamics - the van der Pol Oscillator

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Abstract

Abstract

1 Geometric Singular Perturbation Theory

2 Fast-Slow Systems

Fast- Slow systems are systems of differential equations that can be viewed on two different time scales, which are separated by a parameter. These systems are generally of the form

$$\begin{cases} x' &= \frac{dx}{dt} = f(x,y,\lambda,\epsilon) \\ y' &= \frac{dy}{dt} = \epsilon g(x,y,\lambda,\epsilon), \end{cases} \tag{1}$$

which is called the fast system. Using a change of variables, $t = \frac{\tau}{\epsilon}$ this can be rewritten as

$$\begin{cases} \epsilon\dot{x} &= \epsilon\frac{dx}{d\tau} = f(x,y,\lambda,\epsilon) \\ \dot{y} &= \frac{dy}{d\tau} = g(x,y,\lambda,\epsilon), \end{cases} \tag{2}$$

called the slow system.
Here x is called the fast variable, while y is the slow variable. λ is a parameter, ϵ is the time scale separation parameter and satisfies $0 < \epsilon << 1$. The functions f and g are required to be sufficiently smooth (depending on literature $C^1, C^\infty, C^{r+1} for C^r$ invariant manifolds. Choose later, maybe C^r considering fenichel theorem) It is generally possible to have three or more time scales, separated by additional time scale separation parameters parameters, as well as more state-space variables.
In order to analyse systems (1) and (2) using Geometric Singular Pertubation Theory (GSPT), the singular limit $\epsilon \to 0$ is considered:

$$\begin{cases} x' &= \frac{dx}{dt} = f(x,y,\lambda,\epsilon) \\ y' &= 0, \end{cases} \tag{3}$$

which is called the layer problem and

$$\begin{cases} 0 &= \epsilon\frac{dx}{d\tau} = f(x,y,\lambda,0) \\ \dot{y} &= \frac{dy}{d\tau} = g(x,y,\lambda,0), \end{cases} \tag{4}$$

called the reduced system.
Considering (4), the first equation is $f(x,y,\lambda,0) = 0$ and a manifold can be defined as:

$$S = \{(x,y) : f(x,y,\lambda,0) = 0\}, \tag{5}$$

called the critical manifold, where, by definition of S, the points $(x,y) \in S$ are equilibria of (3). Before we continue, it is useful to have a visual interpretation of these flows, where we can see that the flows will travel

towards our fold point, following the relevant branches.

The main idea of GSPT is the following: Under certain conditions it can be concluded that the critical manifold $S = S_0$, where $\epsilon \to 0$ persists as an invariant manifold S_ϵ under a small pertubation $\epsilon > 0$, if ϵ is sufficiently small. (In higher than 2 dimensions the idea of transversality of the flow of the stable and unstable manifolds is essential for analysis, while in 2 dimensions this is rather trivial.) The main contribution to GSPT comes from Fenichel Theory and his three Theorems can be summed up in one, according to (reference MMO Paper or book). However, before stating the Theorem, some formal definitions are needed.

Definition 2.1. Normal Hyperbolicity
A submanifold $M \subseteq S$ is called normally hyperbolic, if the Jacobian $\frac{\partial f}{\partial x}(x, y, \lambda, 0)$, where $(x, y) \in M$, has only eigenvalues with nonzero real part.

(reference paper 1)
Moreover, the points $(x, y) \in M$, M normally hyperbolic, are hyperbolic equilibria of (3). (ref:MMO) A normally hyperbolic submanifold can be classified according to its stability property: If M has only eigenvalues with positive real part it is called repelling, if M has only eigenvalues with negative real part it is called attracting and if M is neither attracting nor repelling it is called a saddle-type submanifold. (ref:MMO paper)
Furthermore, stable and unstable manifolds can be defined as $W^s(M)$ and $W^u(M)$, corresponding to the eigenvalues with negative and positive real part, respectively. (???? pretty sure there are two different concepts in the last two sentences.. check needed) Furthermore, with the following definition it is established which notion of distance is going to be employed throughout this analysis.

Definition 2.2. Hausdorff Distance
The Hausdorff Distance of two nonempty sets $V, W \subset \mathbf{R}^n$, for some $n \in \mathbf{N}$ is defined as

$$d_H(V, W) = \max\{\sup_{v \in V} \inf_{w \in W} ||v - w||, \sup_{w \in W} \inf_{v \in V} ||v - w||\}.$$

(ref: book kuehn)

Now Fenichel's Theorem can be stated:

Theorem 2.3
Fenichel's Theorem
Suppose $M = M_0$ is a compact, normally hyperbolic submanifold (possibly with boundary) of the critical manifold S (5) and that $f, g \in C^r, r < \infty$. Then for $\epsilon > 0$, sufficiently small, the following hold:
(F1) There exists a locally invariant manifold M_ϵ, diffeomorphic to M_0. Local invariance means that M_ϵ can have boundaries through which trajectories enter or leave.
(F2) M_ϵ has a Hausdorff distance of $O(\epsilon)$ from M_0.
(F3) The flow on M_ϵ converges to the slow flow as $\epsilon \to 0$.
(F4) M_ϵ is C^r- smooth.
(F5) M_ϵ is normally hyperbolic and has the same stability properties with respect to the fast variabes as M_0 (attracting, repelling or saddle type).
(F6) M_ϵ is usually not unique. In regions that remain at a fixed distance from the boundary of M_ϵ, all manifolds satisfying (F1)-(F5) lie at a Hausdorff distance $O(e^{-K/\epsilon})$ from each other for some $K > 0$ with $K = O(1)$.
The normally hyperbolic manifold M_0 has associated local stable and unstable manifolds

$$W^s(M_0) = \cup_{p \in M_0} W^s(p) \quad and \quad W^u(M_0) = \cup_{p \in M_0} W^u(p),$$

where $W^s(p)$ and $W^u(p)$ are the local stable and unstable manifolds of p as a hyperbolic equilibrium of the layer equations, respectively. These manifolds also persist for $\epsilon > 0$, sufficiently small: there exist locally stable and unstable manifolds $W^s(M_\epsilon)$ and $W^u(M_\epsilon)$, respectviely, for which conclusions (F1) - (F6) hold if we replace M_ϵ and M_0 by $W^s(M_\epsilon)$ and $W^s(M_0)$ (or similarly by $W^u(M_\epsilon)$ and $W^u(M_0)$).

+++direct citation needed for theorem (MMO) +++
Fenichel's Theorem establishes that the submanifold M_0 of the critical manifold S_0 persists as slow manifold M_ϵ as $\epsilon > 0$, given it is compact and normally hyperbolic. The theorem furthermore establishes that the stable and unstable manifolds persist as well as the individual fibres of these manifolds, namely $W^s(p)$ and $W^u(p)$, that are associated to each base point $p \in M_0$. Therefore, under the assumptions of the theorem, the flow of the fast-slow system (1)/(2) remains $O(\epsilon)$ close to the flow of the system (3)/(4) in the singular limit $\epsilon \to 0$.
The importance of this result lies in the fact that the behaviour of the full system can be analysed by looking at the system in the singular limit instead, which is often more practical.
++++++++++++also trajectories can be constructed and tested using fenichel... paper 1++++++++++

3 Singularities and Fold Points

(ref : book kuehn) One of the requirements of Fenichel's Theorem is normal hyperbolicity. However, fast-slow systems can display singular points where normal hyperbolicity is no longer given and therefore the conclusions of (2.3) no longer hold at these singularities. Singularities in the setting of fast-slow systems are points (x_0, y_0) on the critical manifold S_0, for which the Jacobian $\frac{\partial f}{\partial x}(x_0, y_0, \lambda, 0)$, has one or more eigenvalues with zero real part. Comparing this with Definition 2.1 shows that this is a negation of normal hyperbolicity. Singularities are points where trajectories can jump between fast and slow flow.
The simplest of those singularities is called a fold point, which is defined as follows:

Definition 3.1. Fold Point
A fold point $(x_0, y_0) \in S_0$ is a point where the Jacobian $\frac{\partial f}{\partial x}(x_0, y_0, \lambda, 0)$ has only one eigenvalue with zero real part.

At the fold point, the system (3) undergoes a saddle-node bifurcation. (+++explain?++++) The fold point is non-degenerate if it satisfies the non-degeneracy assumptions:

$$\begin{cases} \frac{\partial^2 f}{\partial x^2}(x_0, y_0, \lambda, 0) \neq 0 \\ \frac{\partial f}{\partial y}(x_0, y_0, \lambda, 0) \neq 0. \end{cases} \tag{6}$$

Furthermore, if (x_0, y_0) satisfies the transversality condition $g(x_0, y_0, \lambda, 0) \neq 0$, then it is called a generic fold point. For these generic folds there exists a theorem that states that the slow flow on S_ϵ (?) near (x_0, y_0) has either positive or negative sign, implying that no equilibria of the slow flow are close to (x_0, y_0). Therefore, for generic fold points no canards will be observed, which is a relevant observation for Section refCanards. The analysis of fold points is using a method called Blow Up Method, which is discussed in Section 7.
In systems containing generic fold point a certain behaviour of the flow can be observed, called Relaxation Oscillations. These are defined as follows:

Definition 3.2. Relaxation Oscillation
A periodic trajectory γ_ϵ is the relaxation oscillation of the fast-slow system if the following holds: In the singular limit there exists a trajectory γ_0, which alternates between fast and slow bits and describes a closed loop in the system. This trajectory γ_0 persists as γ_ϵ under a small pertubation $\epsilon > 0$.

Systems containing non-generic folds or other types of singularities can display different types of periodic orbits.

3.1 Folded Singularities in a Three Dimensional System

Now that we have considered the two dimensional case for a folded singularity we can extend it to a third dimension in our system. This can be done by considering a system of one fast and two slow variables such that ,

$$\begin{cases} \epsilon\dot{x} &= f(x, y, z, y, \epsilon), \\ \dot{y} &= g_1(x, y, z, y, \epsilon), \\ \dot{z} &= g_2(x, y, z, y, \epsilon), \end{cases} \tag{7}$$

which we can see is an extension of our original form - Equation 2 (**?**). Furthermore, **?** also discusses that the addition of an extra slow variable causes issues with respect to the existence of a canard solution. This is because our existence ranges increases from $O(\epsilon)$ to $O(1)$, noting that $\epsilon \ll 1$ (**?**). Then for this system we are able to make similar assumptions to the previous case, Section **??**, but it is obvious we now must have more than one fold point. We can see that this is the case in Figure 1, as our fold point now can take multiple locations within

Figure 1: Three dimensional folded singularity.

our system. From here we are able to define some non-degeneracy conditions, much like we did in Section 1,

$$\begin{aligned} &f(p_*, \lambda, 0) = 0, \\ &\frac{\partial}{\partial x} f(p_*, \lambda, 0) = 0, \\ &\frac{\partial^2}{\partial x^2} f(p_*, \lambda, 0) \neq 0, \\ &D_{(y,z)} f(p_*, \lambda, 0) \text{ has full rank one,} \end{aligned} \tag{8}$$

where we denote $p_* = (x_*, y_*, z_*) \in F$ as our fold points and $D_{(y,z)}$ as our Jacobian with respect to y and z (**?**). In addition to this we can see from Figure 1 that we have some interesting flows within our system. These flows do not follow the standard pattern as we saw in Figure reffig: vdp flow diagram, instead the slow flow switches its orientation when the flow hits the fold point and continue to flow in that direction, as a desingularised flow - these are called isolated singularities (**?**). As a result we are able to express these flows in the following manner, using Equation 8,

$$\begin{cases} \dot{x} &= g_1 \frac{\partial f}{\partial y} + g_2 \frac{\partial f}{\partial z} \\ \dot{y} &= -g_1 \frac{\partial f}{\partial x}, \\ \dot{z} &= -g_2 \frac{\partial f}{\partial x}, \end{cases} \tag{9}$$

where we can then define a folded singularity if $g_1(p_*, \lambda, 0)\frac{\partial}{\partial y} f(p_*, \lambda, 0) + g_2(p_*, \lambda, 0)\frac{\partial}{\partial z} f(p_*, \lambda, 0) = 0$, for our flow on branches (S) (**?**). Next we need to consider the stability of our fold points. We do this by constructing the Jacobian of our system,

$$J = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial z} & \frac{\partial \dot{x}}{\partial \lambda} & \frac{\partial \dot{x}}{\partial \epsilon} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial z} & \frac{\partial \dot{y}}{\partial \lambda} & \frac{\partial \dot{y}}{\partial \epsilon} \\ \frac{\partial \dot{z}}{\partial x} & \frac{\partial \dot{z}}{\partial y} & \frac{\partial \dot{z}}{\partial z} & \frac{\partial \dot{z}}{\partial \lambda} & \frac{\partial \dot{z}}{\partial \epsilon} \end{bmatrix}, \tag{10}$$

+++++++++++++++Note: either 5x5 matrix J with two last rows zero. however then in the system above need $\lambda' = 0, \epsilon' = 0$. if not, then delete the last two columns in J and make it 3x3++++++++++++++++++++++++++++-

which we can easily find the eigenvalues, by taking the determinant. The result of this analysis gives that we have three eigenvalues, σ_i for $i = 1, 2, 3$ (?). Without loss of generality we can choose $\sigma_3 = 0$ because we know that at least one of our eigenvalues must be zero to account for our fold point in our system, as the **Poincaré diagram describes**. Then we know from standard stability theory that, at our folded singularity we will have three types of phase portrait in the form of,

$$\begin{cases} Saddle\ \sigma_1\sigma_2 < 0 : \sigma_i \in \Re, \\ Node\ \sigma_1\sigma_2 > 0 : \sigma_i \in \Re, \\ Focus\ \sigma_1\sigma_2 > 0 : \Im(\sigma_i) \neq 0, \end{cases} \tag{11}$$

where we can note that only our focus will have imaginary parts (?). **?** illustrates this in the following Figure, where we can see the effect of the varying eigenvalues above. A question which is prudent to consider is, why

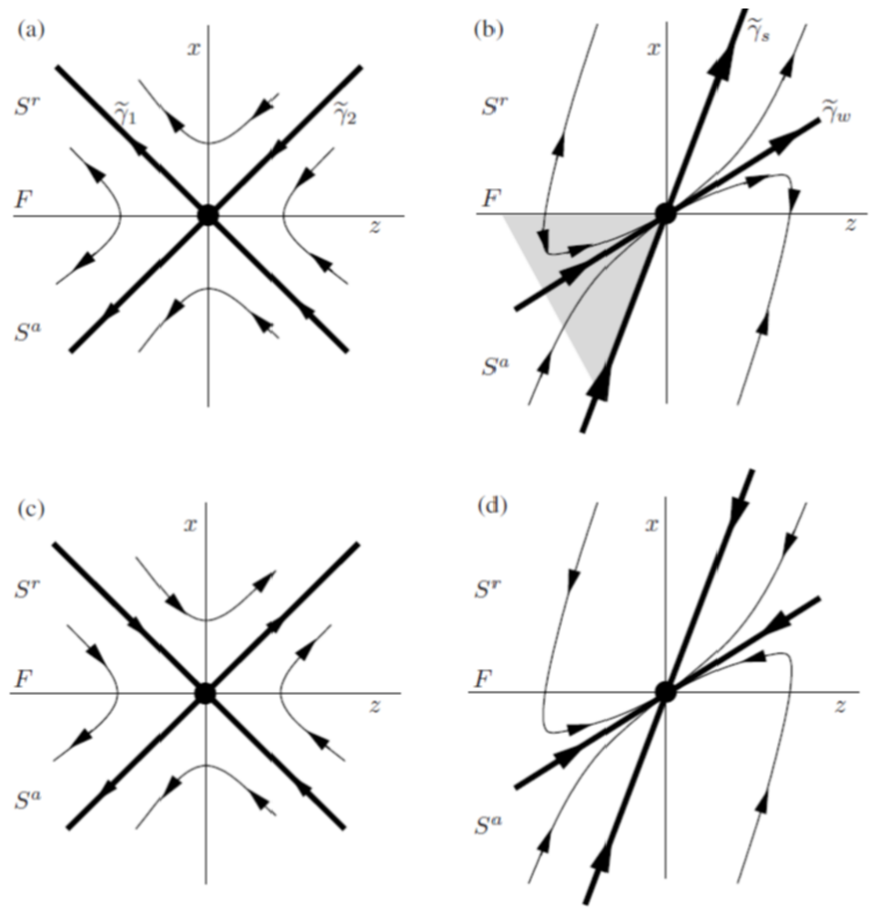

Figure 2: Phase portraits of our three dimensional system where a) is a folded saddle, b) folded node, c) and d) are desingularised flows (?).

hasn't (?) illustrated the singular canard case for a folded focus. This is easily answered as we know that a singular canard is only present if the node or saddle connect our two connecting branches (S^r and S^a) whereas we find that for the focus **this is not the case due to its imaginary parts?** (?).

Theorem 3.3 (Canards in $\Re^3$ (?))
For slow-fast systems (Equation 7) with $\epsilon > 0$ sufficiently small the following holds:

— *There are no maximal canards generated by a folded focus. For a folded saddle the two singular canards* $\bar{\gamma_{1,2}}$ *perturb to maximaal canards* $\gamma_{1,2}$.

— *For a folded node let* $\mu = \frac{\sigma_w}{\sigma_s} < 1$. *the singular canard* $\bar{\gamma_s}$ *("the strong canard") always perturbs to a maximal canard* γ_s. *If* $\mu^{-1} \notin \mathbb{N}$, *then the singualr canard* $\bar{\gamma_w}$ *("weak canard") also perturbs to a maximal canard. We call* γ_s *and* γ_w *primary canards.*

— *For a folded node suppose k¿0 is an integer such that* $2k+1 < \mu^{-1} < 2k+3$ *and* $\mu^{-}1 \neq 2(k+1)$. *Then, in addition to* $\gamma_{s,w}$ *there are k other maximal canards, which we call secondary canard.*

— *The primary weak canard of a node undegoes a trancritical bifurcation for odd* $\mu^{-1} \in \mathbb{N}$ *and a pitchfork bifurcation for even* $\mu^{-1} \in \mathbb{N}$

From Theorem 3.3 we have now defined the existence of a strong and weak eigenvalue such that $|\sigma_1| > |\sigma_2| \iff |\sigma_s| > |\sigma_w|$. From this theorem we are then able to carry out explicit investigations, as we will see in Section **Not done yet**

4 The Van Der Pol Equation

One fast-slow system that contains generic fold points and therefore displays relaxation oscillations is called the Van der Pol System. This can be derived from the Van der Pol Oscillator, which is a well-studied second order ODE that is used to model a variety of physical and biological phenomena. It was developed by the dutch physicist and electrical engineer Balthasar Van der Pol, who conducted research on electrical circuits, in which he observed stable oscillations, later named relaxation oscillations. The derivation of the Van der Pol fast-slow system of the form (1) is presented in the following section.

4.1 Derivation of the Van der Pol Fast-Slow System

The Van der Pol Oscillator describes the evolution of the position coordinate $x(t)$ according to the following the ODE:

$$\ddot{x}(t) - \mu\left(1 - x^2(t)\right)\dot{x}(t) + x(t) = 0, \tag{12}$$

where $\mu \gg 1$ is a scalar constant.
A new variable$w = \dot{x} + \mu F(x)$ is introduced, where $F(x) = \frac{x^3}{3} - x$. F is chosen such that $F'(x) = -(1-x^2)$ is the nonlinear term in Equation 12. Differentiating w we obtain

$$\begin{aligned} \dot{w} &= \ddot{x} + \mu\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^3}{3} - x\right)\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \ddot{x} + \mu(x^2-1)\dot{x} \\ &= -x \end{aligned}$$

Here, the last equality follows from rearranging (12). We now have a two dimensional system:

$$\begin{cases} \dot{x} = w - \mu F(x) \\ \dot{w} = -x \end{cases}$$

and letting $y = \frac{w}{\mu}$ results in

$$\begin{cases} \dot{x} = \mu\left(y - F(x)\right) \\ \dot{y} = -\frac{x}{\mu}. \end{cases}$$

Now, using a rescaling of time $\tilde{t} = \mu\tau$ and setting $\frac{1}{\mu^2} = \epsilon$ results in the system:
+++$\tilde{t}$ is the original variable, we transform into the slow system but state the fast system first because thats the order we always have them in. slightly confusing. ideas? Also. Need to define λ as either zero or 1 depending on where to mention it...++++++++

$$\begin{cases} x' = y - \frac{x^3}{3} + x \\ y' = -\epsilon x, \end{cases} \tag{13}$$

which is of the form (1), the fast system, and the rescaling of time $t = \epsilon\tau$ results in

$$\begin{cases} \epsilon\dot{x} = y - \frac{x^3}{3} + x \\ \dot{y} = -x, \end{cases} \tag{14}$$

which is in the form of (2), the slow system.
As in Sechtion 2 the fast and slow system can be analysed by considering the limiting case $\epsilon \to 0$. The two systems then become

$$\begin{cases} x' = y - \frac{x^3}{3} + x \\ y' = 0, \end{cases} \tag{15}$$

which is of the form (3), the layer problem, and the reduced problem

$$\begin{cases} 0 = y - \frac{x^3}{3} + x := f \\ \dot{y} = -x. \end{cases} \tag{16}$$

4.2 Phase Plane Analysis (is it?? some part 'singularity analysis')

Considering (15), it can be observed that the flow is dominated by the dynamics in x which is cubically depending on x. Furthermore, it is clear that in the layer problem the dynamics in y are constant and therefore the flow is horizontal and is only influenced by y as a constant parameter. Then x is called the fast variable. This is immediately obvious when comparing this to the reduced problem (16), where the flow is restricted to $f = 0$, which is in the form of a cubic function. This defines a critical manifold. Restricted to this manifold, the flow is dominated by the dynamics in y, which linearly depends on x, which is much slower than the cubic dependence in the layer problem. Therefore, this is called the slow flow and y is the slow variable.
The aim of this analysis is to be able to analyse the system in the singular limit $\epsilon \to 0$ and apply appropriate theory to conclude the persistence of the dynamic for $\epsilon > 0$. Section 1 introduced one instance where this persistence can be concluded. The main requirement for the theory in Section 1 is normal hyperbolicity of the critical manifold. Considering the manifold $C_0 = \{(x, y) : 0 = y - \frac{x^3}{3} + x := f\}$,the Jakobian $\frac{\partial f}{\partial x}(x, y, 0) = -x^2 + 1$, which has a zero real part at $x_0 = \pm 1$. Together with the corresponding y_0 are singularities of the system. Further analysis has to be done below in order to conclude that they are generic fold points. The points of interest are $(x_0^+, y_0^+) = (1, -\dfrac{2}{3})$ and $(x_0^-, y_0^-) = (-1, \dfrac{2}{3})$.
By Definition 3.1, there is only one eigenvalue with zero real part at (x_0, y_0). Evaluating the Jakobian at each of the points in turn shows:

$$\begin{cases} \frac{\partial f}{\partial x}(x_0^+, y_0^+, 0) = -1^2 + 1 = 0 \\ \frac{\partial f}{\partial x}(x_0^-, y_0^-, 0) = -(-1)^2 + 1 = 0, \end{cases}$$

where each of the zeros are simple. Therefore (x_0^+, y_0^+) and (x_0^-, y_0^-) are fold points. These points are nondegenerate if the non-degeneracy assumptions (6) hold:

$$\begin{cases} \frac{\partial^2 f}{\partial x^2}(x_0^+, y_0^+, \lambda, 0) = -2x_0^+ = -2 \neq 0 \\ \frac{\partial f}{\partial y}(x_0^+, y_0^+, \lambda, 0) = 1 \neq 0, \end{cases}$$

and equivalently for the other fold point

$$\begin{cases} \frac{\partial^2 f}{\partial x^2}(x_0^-, y_0^-, \lambda, 0) = -2x_0^- = 2 \neq 0 \\ \frac{\partial f}{\partial y}(x_0^-, y_0^-, \lambda, 0) = 1 \neq 0. \end{cases}$$

Therefore, the two fold points are non-degenerate. Futhermore, it can be checked if a fold point is generic. It then has to satisfy the transversality condition $g(x_0, y_0, 0) \neq 0$. The two fold points considered here are generic, since

$$g(x_0^+, y_0^+, 0) = -1 \neq 0$$
$$g(x_0^-, y_0^-, 0) = 1 \neq 0.$$

Now we know that the Van der Pol System displays Relaxation Oscillations and that normal hyperbolicity of the system breaks down at the fold points. Fenichel Theory can be applied for regions that are not in the neighbourhood of the fold points. However, a different approach has to be employed for the analysis of the dynamics around the folds.

In order to analyse a fold point it is convenient to transform the Van der Pol system using a coordinate transformation that satisfies the following:

$$\begin{cases} (x_0, y_0) = (0,0) \text{ is a fold point,} \\ \frac{\partial^2 f}{\partial x^2}(0,0,0) > 0 \\ \frac{\partial f}{\partial y}(0,0,0) < 0 \\ g(0,0,0) < 0. \end{cases} \tag{17}$$

4.3 Transformation of the Van der Pol System

In order to analyse the system at the fold points, one fold point at $(x_0^+, y_0^+) = (1, -\dfrac{2}{3})$ is considered, and the further analysis is identical for the second fold point (x_0^-, y_0^-) with a slightly different coordinate transformation. The aim is to find a coordinate transformation that satisfies the conditions in (17). The proposed transformation is $(x, y) \to (1 - \tilde{x}, \tilde{y} - \frac{2}{3})$, which represents a reflection and a translation of the system such that (x_0^+, y_0^+) is mapped to $(0,0)$ - Figure 4.

Now using the proposed mapping $(x, y) \to (1 - \tilde{x}, \tilde{y} - \frac{2}{3})$ we are able to redefine the fast system (13) in the following way,

$$\begin{cases} x' = -y + x^2 - \dfrac{(x)^3}{3} \\ y' = \epsilon(x - 1), \end{cases} \tag{18}$$

where the tilde has been dropped on x and y for convenience. The slow system (14) is redefined as

$$\begin{cases} \epsilon x' = -y + x^2 - \dfrac{(x)^3}{3} \\ y' = (x - 1), \end{cases} \tag{19}$$

using the normal rescaling of time. These two systems will be used throughout the following analysis of the generic fold point.
It is readily checked that the coordinate transfomation is correct by evaluating (17) for the transformed system. It is clear to see that $(x_0, y_0) = (0, 0)$, and differentiation of f yields $\frac{\partial^2 f}{\partial x^2}(0, 0, 0) = 2 > 0$ and $\frac{\partial^1 f}{\partial y^1}(0, 0, 0) = -1 < 0$. Furthermore, $g(0, 0, 0) = -1 < 0$. Therefore, the new system of equations posesses the required qualities.

4.4 Reduced Dynamics

In order to determine the reduced dynamics on the critical manifold S, equation (19) in the limit $\epsilon \to 0$ is considered which yields the following system,

$$\begin{cases} 0 = f(x, y, 0) = -y + x^2 - \dfrac{x^3}{3} \\ \dot{y} = g(x, y, 0) = 0 \end{cases} \tag{20}$$

which is the reduced problem (**?**). The critical manifold is then defined as

$$S = \{(x, y) : f(x, y, 0) = 0\} = \left\{(x, y) : y = x^2 - \frac{x^3}{3}\right\}, \tag{21}$$

which is an S-shaped curve. Since the flow on S is determined by $\dot{y}$, it can be seen that since the sign of g is negative in the neighbourhood of the fold point $(0, 0)$, the slow flow on S is directed towards the fold point. The two fold points $(x_0^{\pm}, y_0^{\pm})$ coincide with the extrema of the cubic function $\phi(x) = y = x^2 - \frac{x^3}{3}$. Then using the chain rule, the second equation of (28b) is (**?**),

$$\phi_x(x)\dot{x} = g(x, \phi(x), 0). \tag{22}$$

Rearranging this gives an expression for the dynamics in x on S. We find that $\phi(x) = x^2 - \frac{x^3}{3}$, where the derivative with respect to x gives $\phi_x(x) = 2x - x^2$. Therefore (29) becomes

$$\dot{x} = \frac{g(x, \phi(x), 0)}{\phi_x(x)} = \frac{x - 1}{2x - x^2} = \frac{x - 1}{x(2 - x)}.$$

This calculation confirms that the fold points at $x = 0$ and $x = 2$ are singularities of the reduced system. Therefore, no conclusions about the dynamics of x can be made at the fold points. Therefore, alternative methods have to be employed to describe the dynamics on the fold points in the singular limit and furthermore to be able to conclude the dynamics of the full system at the fold points from this analysis. The method considered for analysis is called the Blow-Up Method and is considered in the following section.

4.5 Extended System

Now that we have established the above theorems then we need to consider the extended system such that $\epsilon' = 0$, thus $\epsilon = const$. By considering this system we will be able to consider a blow up (magnification) around our fold point in three dimensions.
From Figure 5 we can consider the stability of our fold point. To do this we are able to establish the following determinant,

$$A = \begin{vmatrix} 2x - x^2 - \lambda & -1 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = \lambda^2(2x - x^2 - \lambda). \tag{23}$$

However, a far easier approach is to note that our matrix is an upper triangular matrix which we can take the eigenvalues directly from Equation 30 such that $(\lambda_1.\lambda_2, \lambda_3) = tr(A)$. Then we can clearly see that for our fold points $\lambda_i = 0$ for $i = 1, 2, 3$ and for any $x \neq 0$ we have $\lambda_1 = x(2 - x)$ and $\lambda_2 = \lambda_3 \equiv 0$. As a result we can see that we are forced to blow up our system around our fold point as our steady states are non-hyperbolic whereas outside of this we find that we have one hyperbolic steady state.

4.5.1 Canonical Form

Now that we have established our reduced system we are able to rewrite it in canonical form - Equation 31.

$$\begin{aligned} x' &= -y + x^2 - \frac{x^3}{3} = -y + x^2 + h(x) \\ y' &= \epsilon(x - 1) \end{aligned} \tag{24}$$

There is ample reasoning for doing this. This is because we find that the canonical form has been studied in great detail allowing us to make comparisons and to avoid excess computation. This then allows us to follow an analogous approach to the many papers associated with this topic - as seen in **?** paper on Extending Geometric Singular Perturbation Theory.

5 Charts

include picture from the book

Figure 3: Charts

6 Transformation

? discusses why we should make a transformation for our system. We find that if we map $(x_0, y_0) = (0, 0)$ we are then able to simplify our non-degeneracy conditions - as shown in Equation 25 (**?**).

$$\begin{cases} (x_0, y_0) = (0, 0) \\ \frac{\partial^2 f}{\partial x^2}(0, 0, 0) > 0 \\ \frac{\partial f}{\partial y}(0, 0, 0) < 0 \end{cases} \tag{25}$$

6.1 Mapping Transformation

To be able to continue with our analysis we should consider the transformations. We will first only consider the case for our fold point at $(x_0^+, y_0^+) = (1, -\frac{2}{3})$. We wish to map $(x, y) \to (1 - \tilde{x}, \tilde{y} - \frac{2}{3})$ which reflects and translates our system such that our fold points are now mapped to $(0, 0)$ - Figure 4.

Figure 4: Our transformed system.

Now that we have made our transformation we can check our non-degeneracy conditions. However, before we continue we should check that our the sign of the derivative is conserved through the transformation. We can do this by using the chain rule as follows,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\tilde{x}}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\mathrm{d}\tilde{x}}{\mathrm{d}t}. \tag{26}$$

Now using Equation 26 and our new mapping ($(x, y) \to (1 - \tilde{x}, \tilde{y} - \frac{2}{3})$) we are able to define our new Fast System in the following way,

$$\begin{cases} x' = -y + x^2 - \dfrac{(x)^3}{3} \\ y' = \epsilon(x - 1) \end{cases} \tag{27}$$

where we note that we have dropped the tilde on x and y for convenience.

6.2 Non-degeneracy Conditions

$$\begin{cases} (x_0, y_0) = (0, 0) \\ \frac{\partial^2 f}{\partial x^2}(0, 0, 0) > 0 \\ \frac{\partial f}{\partial y}(0, 0, 0) < 0 \end{cases} \tag{25}$$

Now that we have constructed our transformed system, we are now able to check our non-degeneracy conditions - Equation 25. It is clear to see that $(x_0, y_0) = (0, 0)$ by the mapping we defined in Section 6.1. Following this we are able to check our other non-degeneracy conditions conform to our new mapping. The differentiation is easily seen from Equation 27 which yields that $\frac{\partial^2 f}{\partial x^2}(0, 0, 0) = 2 > 0$ and $\frac{\partial^1 f}{\partial y^1}(0, 0, 0) = -1 < 0$, confirming our assumptions.

6.3 Reduced Dynamics

The next progression for our system is to consider the reduced dynamics within our system. To do this we consider Equation 14 and take the $\epsilon \to 0$ which yields the following system,

$$0 = f(x, y, 0) = -y + x^2 - \frac{x^3}{3} \tag{28a}$$

$$\dot{y} = g(x, y, 0) = 0 \tag{28b}$$

which is known as the slow subsystem (**?**). We are then able to compute the reduced flow by computing (**?**),

$$\phi_x(x)\dot{x} = g(x, \phi(x), 0), \text{ for } y = \phi(x). \tag{29}$$

We find that $\phi(x) = x^2 - \dfrac{x^3}{3}$ where the derivative with respect to x gives $\phi_x(x) = 2x - x^2$. Now it is clear that we will have a singularity at $x = 0$, by Equation 29, thus we will find that our system blows up ($\dot{x} \to \infty$) which motivates the process that will follow.

After finding our reduced system we are able to determine the way in which it flows by considering the sign of $g(0, 0, 0)$. We see from Equation 28b that $g < 0$ then we have that our flow is directed towards the fold points $(0, 0)$. To continue with our analysis we first need to define Fenichel's theorems.

6.4 Fenichel and Standard Theory

THEOREM

6.5 Extended System

Now that we have established the above theorems then we need to consider the extended system such that $\epsilon' = 0$, thus $\epsilon = const$. By considering this system we will be able to consider a blow up (magnification) around our fold point in three dimensions. From Figure 5 we can consider the stability of our fold point. To do this we are

Figure 5: Blown up system

able to establish the following determinant,

$$A = \begin{vmatrix} 2x - x^2 - \lambda & -1 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = \lambda^2(2x - x^2 - \lambda). \tag{30}$$

However, a far easier approach is to note that our matrix is an upper triangular matrix which we can take the eigenvalues directly from Equation 30 such that $(\lambda_1.\lambda_2, \lambda_3) = tr(A)$. Then we can clearly see that for our fold points $\lambda_i = 0$ for $i = 1, 2, 3$ and for any $x \neq 0$ we have $\lambda_1 = x(2 - x)$ and $\lambda_2 = \lambda_3 \equiv 0$. As a result we can see that we are forced to blow up our system around our fold point as our steady states are non-hyperbolic whereas outside of this we find that we have one hyperbolic steady state.

6.5.1 Canonical Form

Now that we have established our reduced system we are able to rewrite it in canonical form - Equation 31.

$$\begin{aligned} x' &= -y + x^2 - \frac{x^3}{3} = -y + x^2 + h(x) \\ y' &= \epsilon(x - 1) \end{aligned} \tag{31}$$

There is ample reasoning for doing this. This is because we find that the canonical form has been studied in great detail allowing us to make comparisons and to avoid excess computation. This then allows us to follow an analogous approach to the many papers associated with this topic - as seen in **?** paper on Extending Geometric Singular Perturbation Theory.

7 Blow-ups in our System

We first need to transform our system in a new coordinate and time system. This is because we are considering our point as a circle with radius zero. By doing this we are then able to consider a localised flow within our system which is a most on the boundary of our point ($r = 0$). To do this we need to consider varying powers of r in each of our variables such that we have the system shown in Equation 32.

$$x = \bar{r}\bar{x} \tag{32a}$$

$$y = \bar{r}^2\bar{y} \tag{32b}$$

$$\epsilon = \bar{r}^3\bar{\epsilon} \tag{32c}$$

Now that we have this transformation we are then able to consider our charts (see Section 5) in our system.

7.1 Charts

We note that our system will have three charts K_1, K_2, K_3, where we have $\bar{y}=1,\ \bar{\epsilon}=1,\ \bar{x}=1$. By inserting these into Equations 32a, 32b and 32c respectively to give,

$$x = r_1 x_1,\ y = r_1^2,\ \epsilon = r_1^3\epsilon_1, \tag{33a}$$
$$x = r_2 x_2,\ y = r_2^2 y_2,\ \epsilon = r_2^3 \tag{33b}$$
$$x = r_3,\ y = r_3^2 y_3,\ \epsilon = r_3^3\epsilon_3 \tag{33c}$$

where $(x_i, r_i, \epsilon_i) \in \Re^3$ for $i = 1, 2, 3$ (?). Now that we have done this we can consider the individual charts explicitly. We start with chart two (K_2) for a simple reason. This is because we area able to glean the most information out of this chart. We will see that we then find that we can define the mappings from chart one to two and two to three, further simplifying our analysis in the future.

7.2 Dynamics in K_2

To be able to consider chart K_2 we will use the transformation - shown in Equation 33b - in our canonical system. We also need to use a time rescaling ($t_2 = r_2 t$) to be able to desingularise the system. Now substituting this into Equation 31 yields,

$$\frac{\mathrm{d}}{\mathrm{d}t}(r_2 x_2) = r_2^2\frac{\mathrm{d}x_2}{\mathrm{d}t} = -y_2 + x_2^2 - \frac{x_2^3 r_2}{3}, \tag{34}$$
$$r_2^3 y_2' = r_2^3(-1 + r_2 x), \tag{35}$$
$$r_2' = 0, \tag{36}$$

noting that $\frac{\mathrm{d}r}{\mathrm{d}t_2} = \frac{\mathrm{d}t}{\mathrm{d}t_2}\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{r_2}\frac{\mathrm{d}r_2}{\mathrm{d}t}$ and we are taking the equations to $O(r_2)$. Now dividing through by r_2^2 and r_2^3 respectively for each equation we get,

$$\begin{aligned} x_2' &= x_2^2 - y_2 + O(r_2), \\ y_2' &= -1 + O(r_2), \\ r_2' &= 0, \end{aligned} \tag{37}$$

which are then able to evaluate as a layer problem. Now we know that this is the Riccati Equation - see ?.

7.3 Dynamics in K_1

7.4 Dynamics in K_3

Similarly to K_1 and K_2, the system can be transformed using Equation 33c.

$$\frac{\mathrm{d}r_3}{\mathrm{d}t_3} = r_3 F(r_3, y_3, \epsilon_3)$$
$$\frac{\mathrm{d}y_3}{\mathrm{d}t_3} = \epsilon_3(r_3 - 1) - 2y_3 F(r_3, y_3, \epsilon_3)$$
$$\frac{\mathrm{d}\epsilon_3}{\mathrm{d}t_3} = -3\epsilon_3 F(r_3, y_3, \epsilon_3)$$

where $F(r_3, y_3, \epsilon_3) = \left(1 - y_3 - \frac{r_3}{3}\right)$

8 Canard Points

During this section we will be considering a canard point. This is when our fold point is shifted along the manifold - Figure 6. To adequately explain the effect that the canard point will have on our system we will need

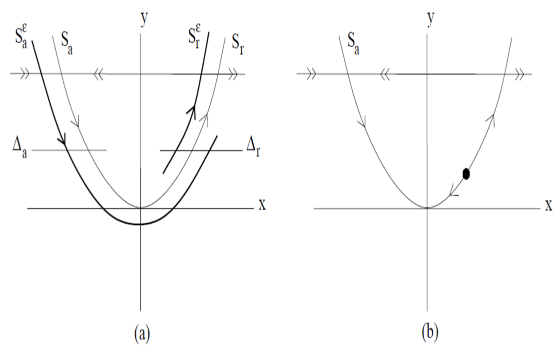

Figure 6: The reduced flow of our system for a) $\lambda = 0$ and b) $\lambda > 0$.

to consider our system,

$$\begin{aligned} x' &= -y + x^2 - \frac{x^3}{3}, \\ y' &= \epsilon(x-1), \\ \epsilon' &= 0. \end{aligned} \tag{27}$$

Now we need to consider Equation 27 in terms our our cananrd system. To do this we rewrite our system with an extra parameter λ, where λ is our perturbation of our fold point (**?**). **?** discusses generally how we should continue with computing our canard system. If we apply his theory to the Van der Pol system we find,

$$\begin{aligned} x' &= -y + x^2 - \frac{x^3}{3}, \\ y' &= \epsilon(x-\lambda), \\ \epsilon' &= 0, \\ \lambda' &= 0, \end{aligned} \tag{38}$$

where the change in ϵ and λ are constant. Now, for the remainder of the section, we follow the method of **?** for the canard system. If we start by rewriting our canard system into the canonical forms we find,

$$x' = -yh_1(x, y, \epsilon, \lambda) + x^2 h_2(x, y, \epsilon, \lambda), \tag{39}$$

$$y' = \epsilon(xh_4(x, y, \epsilon, \lambda) - \lambda h_6(x, y, \epsilon, \lambda)), \tag{40}$$

$$\tag{41}$$

Where we note that $h_j(x, y, \epsilon, \lambda) = 1 + O(x, y, \epsilon, \lambda)$ for $j = 1, 2, 4, 5$ and $h_3(x, y, \epsilon, \lambda) = O(x, y, \epsilon, \lambda)$. However, we should note that for the Van der Pol system our only term that is not solely of leading order is $h_2(x, y, \epsilon, \lambda) = 1 - \frac{x}{3}$. Now we are able to choose such a $\lambda > 0$ that produces an equilibrium on our repelling branch S_r for the reduced flow. By doing this we are then able to define the following conditions for our reduced flow on h_j,

$$a_3 = \frac{\partial}{\partial x} h_2(0, 0, 0, 0) = -\frac{1}{3}, \tag{42}$$

$$A = -a_2 + 3a_3 - (2a_4 + 2a_5) = -1, \tag{43}$$

where we notice that our other solutions for $a_i = 0$ for $i = 1, 2, 4, 5$ are trivial. The reason that we consider the constant A is because we will find that this constant is crucial in our canard point analysis iff $A \neq 0$ (**?**). Following this (**?**) discusses the existence of a critical value for λ (denoted λ_c), where our two branches S_r and S_a must connect in a smooth fashion. Now from *Theorem 3.1* we nkow that we must have a transition map at our critical point,

$$\lambda_c(\sqrt{\epsilon}) = -\epsilon(\frac{a_1 + a_5}{2} + \frac{A}{8}) + O(\epsilon^{\frac{3}{2}}), \tag{44}$$

which can be written as $\lambda_c(\sqrt{\epsilon}) = \frac{\epsilon}{8} + O(\epsilon^{\frac{3}{2}})$ for the Van der Pol system (**?**). Consider Canard cycles and center manifolds / Freddy Dumortier, Robert Roussarie. for more details on canards in Van der Pol .

8.1 Canard Blow-up

Now similarly to Section 7 we consider various transformations of our coordinate system to be able to be able to consider the non-hyperbolic equilibrium induced by our canard point. However, as we would expect with our new system we should consider a new set of transformations (**?**).

$$x = \bar{r}\bar{x},\ y = \bar{r}^2 y,\ \epsilon = \bar{r}^2\bar{\epsilon},\ \lambda = \bar{r}\bar{\lambda} \tag{45}$$

Now that we have established the transformation we can then define our transformations for K_1 and K_2 but it is not necessary to consider the third chart (K_3). This is because we find that the attracting slow manifold connects to the repelling slow manifold. As a result of this we find that our flow will 'bend back' from K_2 into K_1 instead of flowing out into the fast flow, which is described by K_3. This concept can be described by the Figure 7.

Figure 7: Figure describing canard flow in manifold

Since we have established why we need only consider two charts we can our transformations,

$$x = r_1 x_1,\ y = r_1^2,\ \epsilon = r_1^2\epsilon_1,\ \lambda = r_1\lambda_1 \tag{46a}$$
$$x = r_2 x_2,\ y = r_2^2 y_2,\ \epsilon = r_2^2,\ \lambda = r_2\lambda_2 \tag{46b}$$

Since these transformations have been defined we should consider our charts. We will first consider chart 2, for analogous reasoning to Section 7.2.

8.1.1 Dynamics in K_2

We start by noting that we are considering our invariant plane at $r_2 = 0$ which will significantly simplify our system for K_2. Further we should note that we are taking a transformation in time, $\frac{\mathrm{d}r}{\mathrm{d}t_2} = \frac{\mathrm{d}t}{\mathrm{d}t_2}\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{r_2}\frac{\mathrm{d}r_2}{\mathrm{d}t}$, as well as in our coordinates. Then if we substitute our time transformation and Equation 46b into our system of Equations 38 we find,

$$\begin{aligned} r_2^2 x_2' - r_2 x_2 r_2' &= -r_2^2 y_2 h_1 + r_2^2 x_2^2 h_2, \\ \implies x_2' &= -y_2 + x_2^2 - r_2 G_2(x_2, y_2), \end{aligned} \tag{47a}$$

$$\begin{aligned} r_2^3 y_2' - 3r_2^2 y_2 r_2' &= r_2^2(r_2 x_2 h_4 - r_2\lambda_2 h_5), \\ \implies y_2' &= x_2 - \lambda_2 + r_2 G_2(x_2, y_2), \end{aligned} \tag{47b}$$

where we note that $h_j = h_j(x, y, \epsilon, \lambda)$ for $j = 1, 2, 3, 4, 5$. We should also recall that $r_2' = \lambda_2' = 0$. Notice that we have included an additional term in Equation 47b - we define $G_2(x_2, y_2)$ in the following way, $G(x_2, y_2) = (G_1(x_1, y_1), G_2(x_2, y_2))^T = (-\frac{x_2^2}{3}, 0)^T$. The reason we also define this vector is to aide in the Melnikov computations which we will see later. **?** discusses that for this chart we have an interesting result. They note that at $r_2 = \lambda_2 = 0$ our system is integrable which allows us to define a constant of motion $H(x_2, y_2) = \frac{1}{2}\exp(-2y_2)\left(y_2 - x_2^2 + \frac{1}{2}\right)$ which we can easily verify (**?**) using the following equations,

$$x_2' = e^{2y_2}\frac{\partial H}{\partial y_2}(x_2, y_2),$$
$$y_2' = -e^{2y_2}\frac{\partial H}{\partial x_2}(x_2, y_2).$$

Further to this we can see, when we consider our reduced system, that we have an equilibrium at the origin, implying that $H(x_2, y_2) = h$. considering the reduced system (Equation 47) we find from $H(x_2, y_2) = 0$ that,

$$x_2' = \frac{1}{2} \implies x_2 = \frac{t_2}{2} + A, \tag{48a}$$
$$y_2' = \frac{t_2}{2} \implies y_2 = \frac{t_2^2}{4} - \frac{1}{2}, \tag{48b}$$

where we have directly integrated Equation 48a with respect to our time (t_2). However, we can note that we are able to choose $A = 0$ as we are considering an autonomous (time-invariant) system. Then for Equation 48b we are able to rearrange constant of motion at zero to give, $y_2 = x_2^2 - \frac{1}{2}$. Clearly from this analysis we are then able to define our trajectories in terms of $\gamma_{c,2}$,

$$\gamma_{c,2}(t_2) = (x_{c,2}(t_2), y_{c,2}(t_2)) = \left(\frac{t_2}{2}, \frac{t_2^2}{4} - \frac{1}{2}\right). \tag{49}$$

Now that we have established that we must have a flow on our second chart, then there must also exist transition maps. Therefore this now enables us to consider the first chart in the following section.

8.2 Dynamics in K_1

For K_1 we follow a similar approach to the above. We will use the transformations,

$$x = r_1 x_1,\ y = r_1^2,\ \epsilon = r_1^2\epsilon_1,\ \lambda = r_1\lambda_1, \tag{46a}$$

to find the relevant pathways of our flows. Now if we first consider the r_1 component,

$$2r_1^2 r_1' = r_1^2\epsilon(r_1 x_1 - r_1\lambda_1), \tag{50}$$

where we can call $F = F(x, y, \epsilon, \lambda) = x_1 - \lambda_1 + O(r_1(r_1 + \lambda_1)$. Now we will see the motivation with starting with $y = r_1$ when we transform our other coordinates. Now if we consider $x = r_1 x_1$,

$$r_1 r_1' x_1 + r_1^2 x_1' = -r_1^2 + r_1^2 x_1^2,$$
$$x_1' = -1 + x_1^2 - \frac{x_1 r_1'}{r_1},$$

where we can use Equation 50 to simplify this further - Equation 51.

$$x_1' = -1 + x_1^2 - \frac{x_1}{r_1}\left(\frac{r_1\epsilon_1 F}{2}\right) \tag{51}$$

We now consider our $\epsilon = \epsilon_1 r_1^2$ and noting $\epsilon' = 0$. Then we have, $r_1^3\epsilon' = -2r_1^2\epsilon_1 r_1'$, where we can use Equation 50 to simplify to,

$$\epsilon' = -\epsilon_1^2 F. \tag{52}$$

Our last transformation is for our new coordinate $\lambda = r_1\lambda$, noting that $\lambda' = 0$. Similarly to the above we find $r_1^2\lambda_1' + r_1\lambda_1 r_1' = 0$ then,

$$\lambda_1' = -\frac{\lambda_1\epsilon_1 F}{2}, \tag{53}$$

which is a trivial rearrangement as seen in Equation 52. Now if we combine the above we find that our transformed system is of the following form,

$$r_1' = \frac{\epsilon}{2}(r_1x_1 - r_1\lambda_1), \tag{54a}$$

$$x_1' = -1 + x_1^2 - \frac{x_1\epsilon_1 F}{2}, \tag{54b}$$

$$\epsilon' = -\epsilon_1^2 F, \tag{54c}$$

$$\lambda_1' = -\frac{\lambda_1\epsilon_1 F}{2}. \tag{54d}$$

From this system we are now able to make some deductions. We first can observe that the hyperplanes are along the $r_1 = \epsilon_1 = \lambda_1 = 0$ with an invariant line at $l_1 = \{(x_1, 0, 0, 0) : x_1 \in \Re\}$ (**?**). As **?** discusses the equilibria present at the end of both of our branches - Figure 6 - which are found at $p_a = (-1, 0, 0, 0)$ and $p_r = (1, 0, 0, 0)$ (**?**). Now we can go one step further, we can consider Equation 54 and find the eigenvalues of the system for the invariant planes. We find that,

$$J - \lambda I = \begin{bmatrix} 2x - \lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix}, \tag{55}$$

which clearly has three zero eigenvalues and one non-zero eigenvalue $\lambda = \pm 2$. Which further empahsises that our equilibrium point is non-hyperbolic. As a result we intuitively expect that something interesting occurs at this point. In the section following we will be considering what effect these mappings and eigenvalues will have on our system.

8.2.1 Separation of the Manifolds

Discuss splitting on the manifold

8.3 Effect of the Canard Point

Now that we have shown that there must exist a flow around our fold point we should now consider the global effect of the canard point. We can see by considering the system of Equations 54 that our equilibriums are at $(x, y) = (\lambda, \lambda^2[\frac{1-\lambda}{3}])$ and find the eigenvalues from the matrix,

$$A - \mu I = \begin{bmatrix} 2x - x^2 - \mu & -1 & 0 & 0 \\ \epsilon & -\mu & x - \lambda & -\epsilon \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & -\mu \end{bmatrix}. \tag{56}$$

Then our eigenvalues are, $\mu = (2-x)x$ **and** $\mu = 0$**, noting that we have an upper triangular matrix. Then we can note that we have a complex eignevalue which causes a Hopf bifurcation, as shown below:**
From the following Figures we can see the pregression of our flow over the system, , which we can see that

Figure 8: Development of the Hopf Bifurcation.

we have an unstable periodic solution within our canard system. **We can also further deduce from this calculation that we have an amplitude of** $O()$ (**?**).

8.3.1 Singular Hopf Bifurcation

In this section we will further expand on our Hopf Bifurcation of the previous section (Section 8.3). We note that we get a singular bifurcation iff our system is equivalent to Equation 27, such that $\lambda = 1$. **?** discusses that our bifuraction will only exist within a small range of $O(\epsilon)$. Then to model this behaviour we need to consider a small perturbation along the slow flow where we will have, from Equation 38,

$$\dot{y} = \lambda - x + \nu y, \tag{57}$$

where ν is of order $O(\epsilon)$, thus small. We can immediately see that when $\nu = 0$ that we have our orignal flow at our equilbrium but we are now able to perturb our flow over a small domain, which are described in Figures 8. We can also see how our system behaves when our ν is of larger order than $O(\epsilon)$,

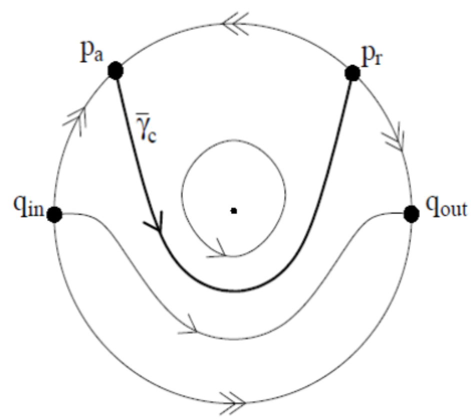

Figure 9: The flow within our canard system (**?**).

where it is clear that our Hopf bifurcation is the periodic solution in the centre of Figure 9 but we can see that below our special flow $\bar{\gamma}_c$, our solution traverses through our equilbrium into our fast flow as we would expect in our original system.

A Dynamics in K_2

A Log