Fast and Slow Dynamics

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A Quick Reminder: Dynamical Systems

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -y + x^2 - \dfrac{x^3}{3} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \epsilon(-\lambda + x) \end{cases}$$

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$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = -y + x^2 - \frac{x^3}{3} \\ \frac{\mathrm{d}y}{\mathrm{d}t} = (-1 + x) \end{cases}$$

The Van der Pol Equations

- Fast System:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} & = -y + x^2 - \frac{x^3}{3} \\ \frac{\mathrm{d}y}{\mathrm{d}t} & = \epsilon(-1 + x) \end{cases}$$

Figure: Phase Plane of the Van der Pol Equations

The Van der Pol Equations

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- Slow System:
  - Let $t = \frac{\tau}{\epsilon}$.

$$\begin{cases} \epsilon\frac{\mathrm{d}x}{\mathrm{d}\tau} &= -y + x^2 - \frac{x^3}{3} \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} &= (-1 + x) \end{cases}$$

Figure: Phase Plane of the Van der Pol Equations

- Layer Problem:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} & = -y + x^2 - \frac{x^3}{3} \\ \frac{\mathrm{d}y}{\mathrm{d}t} & = 0 \end{cases}$$

- Reduced System:

$$\begin{cases} 0 & = -y + x^2 - \frac{x^3}{3} \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} & = (-1 + x) \end{cases}$$

Figure: Phase Plane of the Van der Pol Equations

The Reduced System

Figure: Phase Plane of the Van der Pol Equations

Reduced System:

$$\begin{cases} 0 & = -y + x^2 - \frac{x^3}{3} := f(x, y) \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} & = (-1 + x) \end{cases}$$

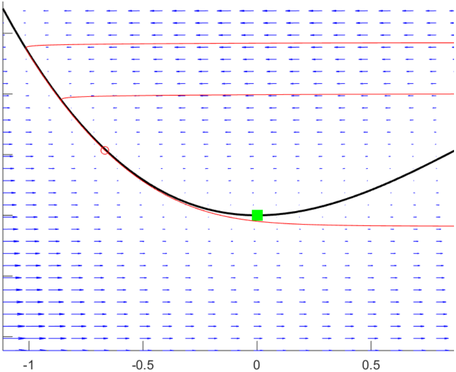
Reduced flow is restricted to:

$$f = 0 \Rightarrow \quad y = x^2 - \frac{x^3}{3}$$

Fold Points:

- $(x_1^*, y_1^*) = (0, 0)$
- $(x_2^*, y_2^*) = (2, \frac{4}{3})$

The Van der Pol Equations near (x_1^*, y_1^*)

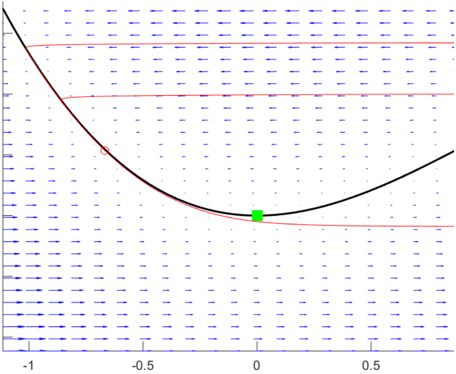


Finding $\frac{dx}{d\tau}$ for the reduced system:

$$y = x^2 - \frac{x^3}{3}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}\tau} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\tau} = x(2-x)\frac{\mathrm{d}x}{\mathrm{d}\tau}$$
$$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\mathrm{d}y}{\mathrm{d}\tau}\frac{1}{(2-x)x}$$

Singularities at $x = 0, x = 2$.

The Van der Pol Equations near (x_1^*, y_1^*)



Investigating Hyperbolicity:

$$f(x,y) = -y + x^2 - \frac{x^3}{3}$$

$$\Rightarrow \frac{\mathrm{d}f}{\mathrm{d}x} = 2x - x^2$$

$$= x(2-x)$$

Fold Point $(x_1^*, y_1^*) = (0,0)$ is non-hyperbolic.

The Blow-up

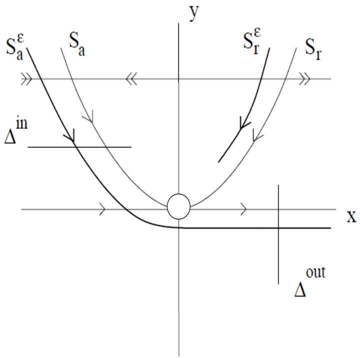


Figure: Blow up of our fold point (Kuehn, Multiple Time Scale Dynamics)

- The radius $r = 0$
- Coordinate transformation
  - $x = r_i x_i, \; y = r_i^2 y_i, \; \epsilon = r_i^3 \epsilon_i$
- Analysing our blow up

Charts Motivation

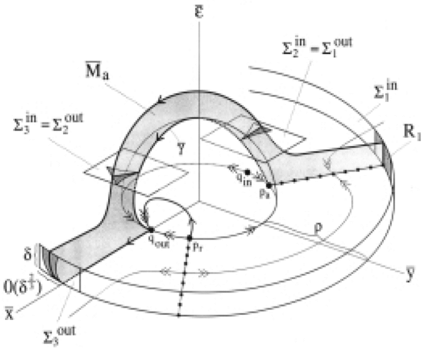


Figure: Sketch of the coordinate charts (Kuehn, Multiple Time Scale Dynamics)

- What are charts?
- Why use charts?
  - Simplifies our singularity
  - Magnifies our flow
- Transition Maps $\Pi_1 \to \Pi_2 \to \Pi_3$

Dynamics on K_2

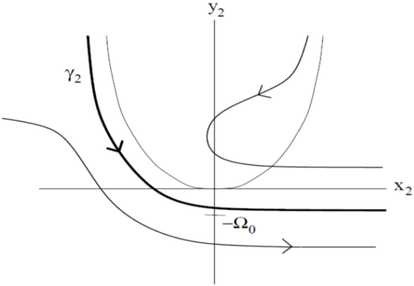


Figure: Phase portrait for $\bar{\epsilon} = 0$ (Kuehn, Multiple Time Scale Dynamics)

- Transformed Van der Pol equations on $\epsilon = 1$:

$$\begin{cases} x_2' &= x_2^2 - y_2 + O(r_2) \\ y_2' &= -1 + O(r_2) \\ r_2' &= 0 \end{cases}$$

- Riccati Equations.
  - Asymptotic solution.
- Transition map, Π_2, exists

Dynamics on K_1 and K_3

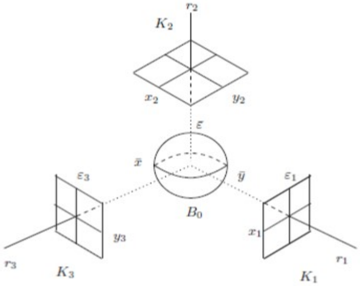


Figure: The attracting branch connecting to the blow up.

- Dynamics on K_1:
  - Centre Manifold Theorem
- Dynamics in K_3
  - Resonance

Global Solution

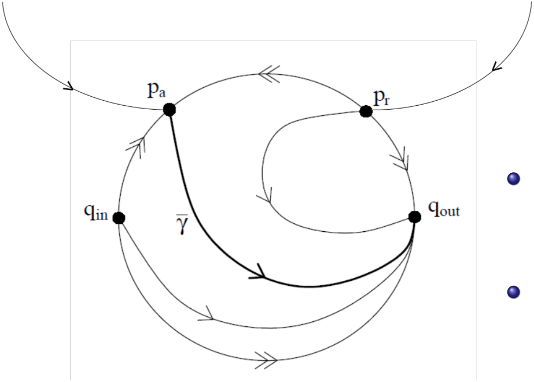


Figure: Global dynamics of the blown up system (Kuehn, Multiple Time Scale Dynamics)

- Π describes the global transition of trajectories.
  - $\Pi = \Pi_3 \circ \kappa_{23} \circ \Pi_2 \circ \kappa_{12} \circ \Pi_1$
- What does q_{out} do?

Canard Points in the Van der Pol

$$\begin{cases} \epsilon \frac{\mathrm{d}x}{\mathrm{d}\tau} & = -y + x^2 - \frac{x^3}{3} \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} & = (-\lambda + x) \end{cases}$$

Mixed Mode Oscillations (MMO's)

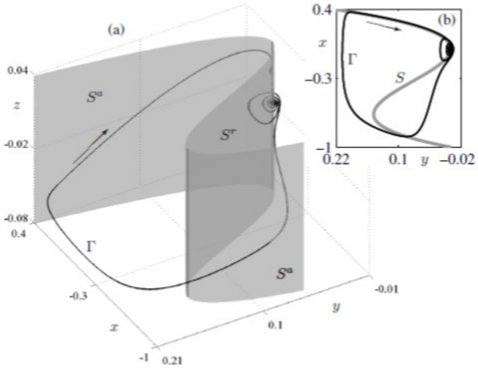


Figure: Phase portrait of an MMO (Desroches et al - Mixed Mode Oscillation with Multiple Time Scales)

- What are MMO's?
- Why are they useful?
  - Applications?
  -

Mixed Mode Oscillations (MMO's)

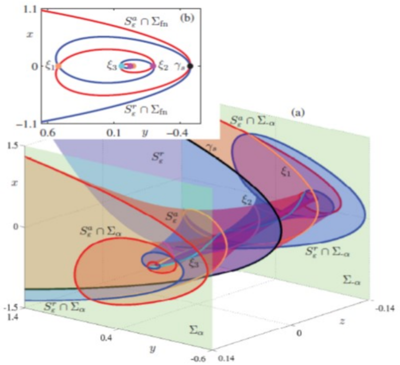


Figure: Oscillations in the canonical form system (Desroches et al - Mixed Mode Oscillation with Multiple Time Scales)

- Canonical System:

$$\begin{cases} \dot{x} &= y - x^2 \\ \dot{y} &= z - x \\ \dot{z} &= -\nu \end{cases}$$

- Small Amplitude Oscillations (SAO's)
- Large Amplitude Oscillations (LAO's)

Conclusion

- Van der Pol
- Canard Point
- MMO's