A MODEL OF THE SOLAR CONVECTION ZONE

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Abstract. A model of the convection zone is presented which matches an empirical model atmosphere (HSRA) and an interior model. A mixing length formalism containing four adjustable parameters is used. Thermodynamical considerations provide limits on two of these parameters. The average temperature-pressure relation depends on two or three combinations of the four parameters. Observational information on the structure of the outermost layers of the convection zone, and the value of the solar radius limit the range of possible parameter combinations. It is shown that in spite of the remaining freedom of choice of the parameters, the *mean temperature-pressure relation* is fixed well by these data.

The reality of a small density inversion in the HSRA model is investigated. The discrepancy between the present model and a solar model by Mullan (1971) is discussed briefly.

1. Introduction

This investigation was prompted by the need for an accurate temperature-pressure relation of the solar convection zone, matching an empirical atmosphere model of the Sun. We could not find a published model meeting this requirement.

The pressure-temperature relation in that part of the convection zone, in which the convective flow is adiabatic, is known relatively well since it may be determined completely from the value of the solar radius, and a model of the solar core. For instance, a detailed model is given in Baker and Temesvary (1966). The structure of the superadiabatic outer layers of the Sun, however, is rather uncertain, due to the inadequacy of present methods of treating convection. We show in this paper that agreement with the energy distribution in the spectrum, and the centre-to-limb variation probably is a sufficient condition to fix the structure of the superadiabatic part of the convection zone (i.e. the first 1000 km).

For the calculations the mixing length formalism was used, modified in such a way that a few parameters which usually are kept fixed, were adjustable within certain limits. This was done just for the convenience of finding a model satisfying some observational constraints; the present formulation should not be considered as an attempt to improve upon the standard formalism. We only suppose that the *mean pressure-temperature relation* in the convection zone can be represented by a suitable choice of these parameters.

The constraints used are:

(a) The usual demand that the model should lead to the correct value of the solar radius.

A new constraint which we used is:

(b) The centre-to-limb variation of emergent intensity calculated with the model

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should be consistent with that of the Harvard Smithsonian Reference Atmosphere (HSRA, Gingerich *et al.*, 1971).

At present methods of treating convection exist which are perhaps theoretically better founded and which are numerically feasible (Van der Borght, 1972; Waters, 1971; Waters and Van der Borght, 1972; Spiegel, 1963; Travis and Matsushima, 1973; Ulrich, 1970). We believe, however, that a mixing length approach with parameters fitted to meet the constraints summarized above is sufficient to determine to a large extent the *mean temperature pressure relation* in the convection zone. The justification for this optimism is to be found in the results presented in Section 4. There it is shown that substantial changes in the free parameters result in essentially identical models, *if only* the observational constraints are met. Convective velocities and fluctuations in temperature and density by this approach may be determined only with a very limited accuracy. In the calculations we aimed at consistency with the HSRA by using abundances consistent with those used for the HSRA.

2. The Treatment of Convection

2.1. Free parameters in mixing length convection

The equations used in calculating the structure of the convection zone are the following. (We follow the notation used by Böhm (1966). An account of the mixing length formalism can be found, for instance, in Cox and Giuli (1968)).

The symbols πF, πF_r, πF_c, κ, H, l, v, c_p, ∇, ∇_a, ∇', respectively stand for: total, radiative and convective flux, Rosseland mean opacity, pressure scale height, mixing length, mean convective velocity, specific heat, logarithmic temperature gradient $\mathrm{d}\log T/\mathrm{d}\log P$, the corresponding value for adiabatic flow, and the value of ∇ inside the convective elements.

$$\pi F = \pi F_r + \pi F_c\,, \tag{1}$$

$$\pi F = \frac{16}{3}\frac{\sigma T^4}{\kappa\varrho H}\nabla_r\,, \qquad \pi F_r = \frac{16}{3}\frac{\sigma T^4}{\kappa\varrho H}\nabla\,, \tag{2}$$

$$v^2 = a\,\frac{g}{H}\,l^2\,(\nabla - \nabla')\,. \tag{3}$$

We omitted the factor $Q = 1 - \partial\ln\mu/\partial\ln T$ used by some authors in this expression.

$$\pi F_c = b c_p \varrho T v\,(\nabla - \nabla')\,\frac{l}{H} = b\sqrt{a}\left(\frac{g}{H}\right)^{1/2}\frac{l^2}{H}\,c_p\varrho T\,(\nabla - \nabla')^{3/2}\,, \tag{4}$$

$$\frac{\nabla - \nabla'}{\nabla' - \nabla_a} = \frac{1}{24}\,f\,\frac{c_p\varrho^2 T\kappa H v}{\sigma T^4}\,\frac{l}{H} = \frac{(\nabla - \nabla')^{1/2}}{2u}\,, \tag{5}$$

with

$$u = \frac{1}{\xi}\,\frac{12\sigma T^4}{c_p\varrho^2\kappa T H^2}\left(\frac{H}{g}\right)^{1/2}, \qquad \xi = f\sqrt{a}\left(\frac{l}{H}\right)^2.$$

We neglected turbulent pressure in the calculations.

These equations differ from the usual mixing length equations in that, in addition to the depth dependent parameter l, three adjustable dimensionless factors a, b, and f are included. Equations (3)–(5) may be considered as defining these parameters. The values taken by different authors for these parameters differ by at least a factor of two, owing to slight differences in the derivation of the equations. A usual choice of their values is: $a=\frac{1}{8}$, $b=\frac{1}{2}$, $f=1$. The influence of changes in these parameters on the structure of the convection zone has been investigated by Böhm (1966) and Mizuno and Nishida (1969). The influence on stellar evolution calculations has been investigated by Henyey *et al.* (1965). Their investigation offers no clues for the present problem of how to adjust a convection zone model to an empirical solar atmosphere model. We assume the parameters a, b and f to be *constant throughout the convection zone*. From thermodynamical considerations limits on a and a/b may be derived (cf. Appendix). Equations (1)–(5) can be combined into an equation of third degree in $(u^2+\nabla-\nabla_a)^{1/2}$ (see, e.g., Yun, 1968):

$$\nabla - \nabla_r + \frac{\eta}{u}\left[(u^2 + \nabla - \nabla_a)^{1/2} - u\right]^3 = 0\,, \tag{6}$$

where $\eta = b/f$.

For the case $l=\alpha H$ it follows that $\xi=\alpha^2 f\sqrt{a}=\text{const}$. Then the value of ∇ is determined uniquely by the two constants ξ and η, through (6). The $T-P$ relation then follows through integration of ∇, once the (T, P) values of the starting point have been chosen. However, two sets of parameter values which yield the same values for ξ and η, will in general yield different values of the convective velocity and convective flux. For the case $l=\min(z+z_0, H)$ (see Section 2.2) we can define $\chi = f\sqrt{a}$. Then the $P-T$ relation is fixed by the three constants η, χ, z_0, and the values of T and P at the starting point of the integration. Thus, when the starting point of the integration has been chosen, two or three numerical conditions on the convection zone model are sufficient to determine the $P-T$ relation completely, depending on the type of mixing length chosen.

2.2. Choice of the Mixing Length

The types of mixing length considered by us are the classical $l=\alpha H$, and the choice $l=\min(z+z_0, H)$. The second choice is a slight generalisation of the mixing length chosen by Böhm and Stückl (1967). Of the numerous other possibilities, the most essentially different is a mixing length proportional to the density scale height, as used by several authors. We did not consider this mixing length, because models calculated with this assumption cannot contain a density inversion (such an inversion is indicated in the layers with $\tau>2$ in the empirical model). It is seen that l would diverge at the boundaries of the inversion, and would be negative inside the inversion (cf. Kippenhahn, 1962, p. 365).

To check whether the density inversion in the HSRA is significant we calculated the centre-to-limb variation of emergent intensity both for the HSRA and for a model which is equal to the HSRA except for the density inversion (i.e. taking $\nabla=1$ where

$\nabla > 1$ in the HSRA). The difference in the centre-to-limb variation of emergent intensity resulting from this change is given in Table I.

Observations of the limb darkening were not used directly in the construction of the HSRA. Therefore, we compared the HSRA data with the experimental data by Pierce *et al.* (1950) and Peyturaux (1955). The comparison was done at 5500 Å, since at this wavelength the intensities are sensitive to the structure of the deepest layers, while at the same time the sources of opacity are known well enough for theoretical calculations. The Peyturaux data are higher than the HSRA data by about 0.3%, the data by Pierce *et al.* are lower by about 0.3%, in the range $0.9 < \mu < 0.5$. The accuracy of the data, claimed by Peyturaux and Pierce *et. al.*, is of the order of 0.2% in this range. Comparing these numbers with the data from Table I, we see that the model without a density inversion seems rather unlikely. In any case, the actual stratification must be so close to an inversion, that using a density scale height as mixing length would lead to unacceptably high values of l.

3. The Fitting Procedures

3.1. The Depth of the Convection Zone

A complete solar model is obtained by fitting an envelope model to an interior model. The interior model may be determined separately, because the influence of changes in the envelope on the interior is small (e.g. Sears, 1964). Recently several interior models have been computed using modern physical data (e.g. Abraham and Iben, 1971; Bahcall *et al.*, 1969). We adopted the model I by Abraham and Iben. We note that this model, like all recent interior models, does predict a neutrino flux larger than the measured flux.

TABLE I

Centre to limb variations and absolute intensities

μ \ λ	I 4000 Å	II 5000 Å	III 5485 Å	IV 5485 Å	V 5485 Å
0.80	0.009	0.005	0.004	−0.007	−0.001
0.60	0.014	0.008	0.006	−0.013	−0.005
0.50	0.015	0.009	0.007	−0.035	−0.009
0.40	0.014	0.009	0.007	−0.049	−0.013
0.30	0.013	0.008	0.006	−0.062	−0.019
0.25	0.012	0.007	0.005	−0.066	−0.022
0.20	0.010	0.006	0.005	−0.071	−0.025
0.15	0.009	0.006			
1.0				0.19	−0.03

Columns I–III: Difference between HSRA and model without density inversion (model minus HSRA).
Column IV: Difference between HSRA and the model by Baker and Temesvary.
Column V: Idem, model by Mullan.
Last row: Relative difference (with respect to the HSRA) in the absolute intensities for the centre of the disc.

Our envelope models were fitted to the interior model by minimizing the minimum distance between the models, represented as curves in the $\varrho-T-r$ space. This was done by adjusting the free parameters of the convection formalism. An exact match could not be obtained, presumably owing to the use of a somewhat different equation of state. At the point of minimum distance the difference corresponds to a difference of 10% in ϱ. The result of the fitting procedure is that the depth z_t of the convection zone turns out to be:

$$z_t = 198000 \text{ km}.$$

All models with this depth fit equally well to the interior. Accidentally, this value of z_t is almost exactly identical to that of the solar model by Baker and Temesvary (1966)(1.95×10^5 km).

In this context the convection zone recently calculated by Mullan (1971) appears to be inconsistent with existing interior models, because its depth is only 10000 km. Mullan has used a mixing length formalism due to Öpik, which includes effects of turbulent heat transfer between convective elements. If we calculate the radiative extension of Mullan's model, we find that $r/R_\odot = 0.52$ at a temperature of 1.58×10^6 K, while the interior model by Abraham and Iben requires $r/R_\odot = 0.82$ at the same temperature. A model with a convection zone as shallow as Mullan's would require a radically different interior model.

Since the depth of the convection zone is a function of the free mixing length parameters, the value of z_t found fixes a relation between these parameters: $f(\eta, \chi, z_0) = 0$ (in the case $l = \min(z + z_0, H)$). This relation is shown in Figure 1 as curves of η versus χ for various values of z_0.

3.2. The uppermost layers of the convection zone

In a convection zone of depth 2×10^5 km the difference $\nabla - \nabla_a$ decreases to about 0.01 in the first 1000 km (cf. Table II).

Although the observable layers of the convection zone cover no more than 50 km (up to $\tau_{5000} = 5$) they may still provide useful information for fixing the values of the free mixing length parameters (ξ and η, or η, χ and z_0), because the stratification in the uppermost layers is particularly sensitive to the adopted parameter values. Therefore we surmised that observational data on the atmospheric layers might be sufficient to determine the structure of the outer 1000 km of the convection zone, while the remaining part is fixed by the known depth of the zone.

To determine whether a specific convection zone 'fits with' the HSRA, we calculated the centre-to-limb variation of emergent intensity at suitable wavelengths (5000, 16500 Å) for the model. Models which predicted a difference in the centre-to-limb variation of emergent intensity compared with the HSRA of less than 0.5% for both wavelengths were considered to be in accord with the HSRA model.

The starting point for the integration of the convection zone model should satisfy two conflicting conditions:

(i) The point should be chosen at a sufficiently small depth, so that a check on the model can be performed by comparing the continuum intensities with those of the HSRA.

(ii) The point should be located at such a depth that the radiative transfer can be treated with the diffusion approximation, with Rosseland mean opacities (c.f. Equation (2)).

As a compromise we chose the starting point at $\tau_{5000}=2$ on the HSRA model.

4. Results and Comparison with Other Work

4.1. The range of possible models

In Figure 1 models calculated with $l=\min(z+z_0, H)$ which start at an optical depth $\tau_{5000}=2$ in the HSRA model and which have a depth $z_t=198000$ km are plotted

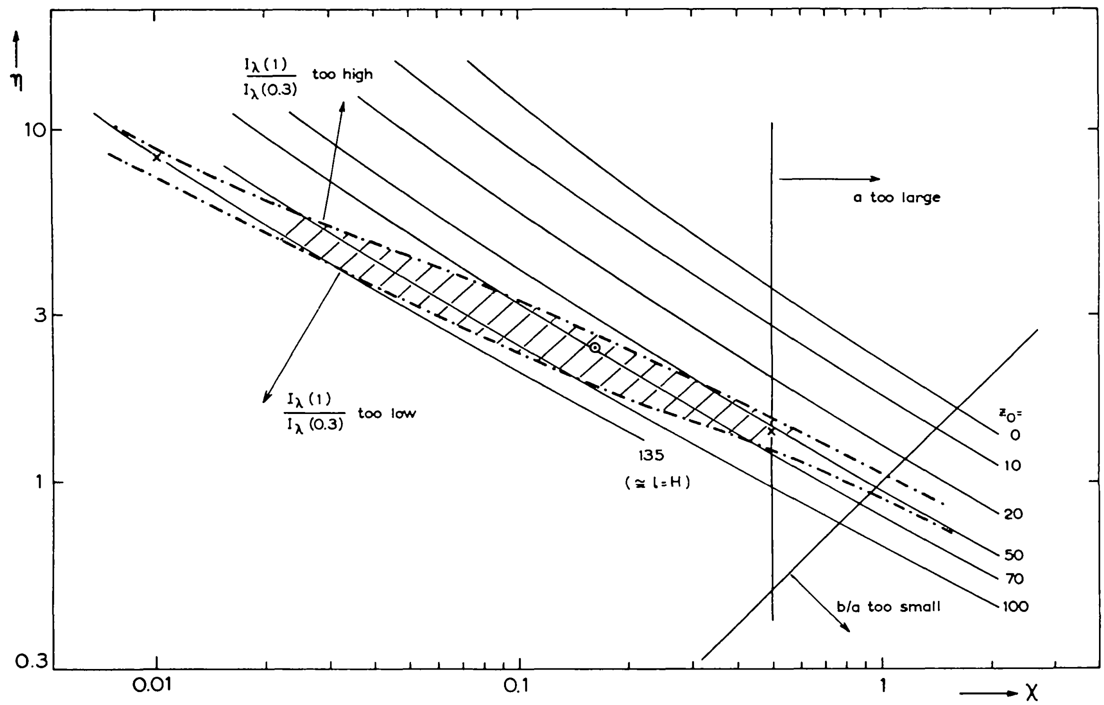

Fig. 1. Convection zone models with $z_t=198000$ km in the $\eta-\chi$ plane. The values of the overshoot parameter z_0 (km) are indicated to the right of the curves. The limits on a and a/b are indicated (solid), as well as the limits set by the continuum intensities (broken). The hatched area indicates the region of the parameter combinations allowed. Circle: finally adopted model (see Table II). Crosses: two extreme models discussed in Section 4.

against η and χ for different values of z_0. The models with $z_0=135$ km are equal to models calculated with $l=H$, since $z(\tau=2)+z_0=33+135$ km$=H(\tau=2)$, The curve for $l=H$ represents models with $l=\alpha H$ as well, if we read the χ-axis as a ξ-axis.

The limits $a<1/\beta$ and $b>\beta a$ derived in the Appendix are indicated for the case that $\beta=2$ and $f=1$, as well as the limits set by the uncertainty of 0.5% in the centre-to-limb variation of emergent intensity. Figure 1 shows that the possible combinations of η, χ and z_0 are restricted to a fairly limited area. The model adopted as the final model (see Section 5) is indicated in Figure 1 by a circle. In Figuie 2 the run of T vs P

is given for two extreme models, within the limits considered here. They are indicated in Figure 1 by crosses; their parameter values are:

$$z_0 = 135, \quad \eta = 8.4, \quad \chi = 0.01 \quad \text{and}$$
$$z_0 = 50, \quad \eta = 1.4, \quad \chi = 0.5.$$

In spite of the large difference in parameter values the temperature-pressure relations of the models are almost identical, and consequently the geometrical depth scales are equal too. Differences between any two models within the shaded area of Figure 1 are of the same order as the difference between the models of Figure 2, or smaller. A change in the value of β (see Appendix) changes the position of the boundary to the right of the shaded area in Figure 1. Since models within the shaded area are practically identical, an uncertainty in the value of β has little influence on the model obtained.

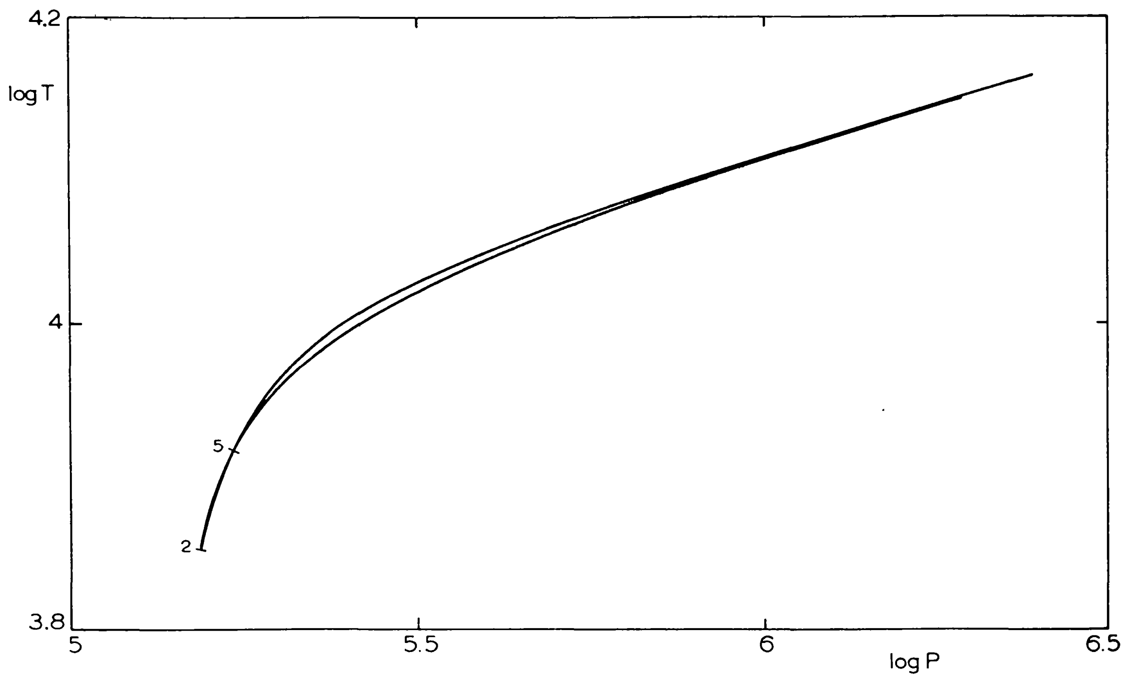

Fig. 2. Log T — log P diagram for the models indicated as crosses in Figure 1. Upper curve: $z_0 = 50$ km, $\chi = 0.5$, $\eta = 1.41$. Lower curve: $z_0 = 135$ km, $\chi = 0.01$, $\eta = 8.4$.

tained. The calculations, therefore, confirm the supposition, raised in the preceding section, that the temperature-pressure relation of the convection zone is well determined by the depth of the convection zone and the conditions at its upper boundary. From Figure 2 we expect that the thermodynamical quantities of the final model are correct within a few per cent, assuming, of course, that the true stratification *can* indeed be represented by some reasonable parameter combination in our formalism. However, the values of the individual mixing length parameters a, b, f, z_0 can be determined with much less certainty. Therefore, there is considerable uncertainty in the velocities as given in the model.

4.2. Comparison with other work, possible improvements

There are three obvious sources of possible errors in the procedure followed in this paper:

(i) The radiative flux was calculated with the diffusion approximation.

(ii) The assumption that four parameters, constant throughout the convection zone, are sufficient to determine the $P-T$ relation.

(iii) The temperatures given in the HSRA may be significantly higher than the horizontally averaged temperatures, since they are derived from average *emergent intensities*. Through the dependence of the source function on temperature the hot elements are represented with greater weight than the cool ones.

The importance of the second error may be estimated from the work of Waters (1971), who has calculated the approximate depth dependence of the mixing length parameters on the basis of Unno's (1969) theory of convection. It appears from this work that for depths larger than 200 km the parameters are constant to within a few percent, while considerable variations occur in the first 60 km. However, since in the first 60 km our model is effectively determined by the empirical atmosphere model, this variation is of little concern in the procedure followed in this paper. To take into account the third error, a more detailed convection theory would be necessary, providing reliable values of the temperature fluctuations. An investigation of this kind has been done by Ulrich (1970a, b) using his non-local mixing length theory. It would be of interest to calculate a standard convection zone with this formalism, using the HSRA (Ulrich's calculations were based upon the BCA).

Finally, detailed *observations* of temperature fluctuations and velocities in the deeper atmospheric layers may be useful in discriminating between more advanced theories of convection, since one of the most important aspects of any stellar convection theory is its description of the outer boundary layer. We feel that the accuracy of mixing length type convection formalisms is insufficient for this purpose.

5. The Model

The model is given in Table II. The values of the convection parameters are

$$\chi = 0.16, \qquad \eta = 2.45, \qquad z_0 = 70 \text{ km}.$$

For calculating the convective velocities we used rather arbitrarily $f=1$, thus $a=0.16$, $b=2.45$ (cf. Equations (5) and (6)). The run of T vs P in the upper layers is shown in Figure 3. The meaning of the symbols used in Table II is as follows:

Z	depth from the level $\tau_{5000}=1$
MZ	the mass above the level z
P	pressure
T	temperature
PE	electron pressure
DEN	density ϱ

TABLE II

The model

Z	MZ	P	T	PE	DEN	CP	AGRAD	MU	K	RGRAD	GR	V	S33
+.3300⏨+ 7	-.0000⏨- 0	+.1540⏨+ 6	+.7140⏨+4	+.2284⏨+ 3	+.3345⏨-6	+.1900⏨+ 9	.3456	1.2897	+.2923⏨+1	+.1323⏨+1	1.2734	+.4575⏨+5	+.1587⏨+13
+.3693⏨+ 7	+.7964⏨+22	+.1576⏨+ 6	+.7370⏨+4	+.3376⏨+ 3	+.3314⏨-6	+.2007⏨+ 9	.3298	1.2889	+.4259⏨+1	+.1738⏨+1	1.4500	+.7424⏨+5	+.2148⏨+13
+.4100⏨+ 7	+.1611⏨+23	+.1613⏨+ 6	+.7620⏨+4	+.5041⏨+ 3	+.3276⏨-6	+.2157⏨+ 9	.3104	1.2876	+.6208⏨+1	+.2268⏨+1	1.4157	+.9575⏨+5	+.2889⏨+13
+.4520⏨+ 7	+.2445⏨+23	+.1650⏨+ 6	+.7860⏨+4	+.7271⏨+ 3	+.3246⏨-6	+.2341⏨+ 9	.2901	1.2860	+.7950⏨+1	+.2625⏨+1	1.2792	+.1039⏨+6	+.3724⏨+13
+.4953⏨+ 7	+.3298⏨+23	+.1688⏨+ 6	+.8080⏨+4	+.1000⏨+ 4	+.3226⏨-6	+.2547⏨+ 9	.2709	1.2840	+.1034⏨+2	+.3128⏨+1	1.1221	+.1094⏨+6	+.4576⏨+13
+.5399⏨+ 7	+.4172⏨+23	+.1728⏨+ 6	+.8277⏨+4	+.1308⏨+ 4	+.3217⏨-6	+.2761⏨+ 9	.2542	1.2819	+.1368⏨+2	+.3848⏨+1	.9701	+.1126⏨+6	+.5400⏨+13
+.5855⏨+ 7	+.5065⏨+23	+.1768⏨+ 6	+.8451⏨+4	+.1648⏨+ 4	+.3219⏨-6	+.2979⏨+ 9	.2397	1.2796	+.1740⏨+2	+.4607⏨+1	.8460	+.1136⏨+6	+.6192⏨+13
+.6321⏨+ 7	+.5979⏨+23	+.1809⏨+ 6	+.8607⏨+4	+.2019⏨+ 4	+.3228⏨-6	+.3201⏨+ 9	.2271	1.2772	+.2141⏨+2	+.5391⏨+1	.7469	+.1135⏨+6	+.6943⏨+13
+.6796⏨+ 7	+.6915⏨+23	+.1851⏨+ 6	+.8748⏨+4	+.2407⏨+ 4	+.3244⏨-6	+.3414⏨+ 9	.2166	1.2749	+.2551⏨+2	+.6160⏨+1	.6696	+.1129⏨+6	+.7631⏨+13
+.7280⏨+ 7	+.7872⏨+23	+.1895⏨+ 6	+.8878⏨+4	+.2823⏨+ 4	+.3265⏨-6	+.3630⏨+ 9	.2073	1.2724	+.2948⏨+2	+.6867⏨+1	.6070	+.1119⏨+6	+.8288⏨+13
+.7771⏨+ 7	+.8852⏨+23	+.1939⏨+ 6	+.8997⏨+4	+.3259⏨+ 4	+.3290⏨-6	+.3841⏨+ 9	.1992	1.2699	+.3372⏨+2	+.7621⏨+1	.5551	+.1109⏨+6	+.8905⏨+13
+.8270⏨+ 7	+.9854⏨+23	+.1984⏨+ 6	+.9108⏨+4	+.3714⏨+ 4	+.3319⏨-6	+.4047⏨+ 9	.1922	1.2675	+.3806⏨+2	+.8381⏨+1	.5117	+.1099⏨+6	+.9485⏨+13
+.8775⏨+ 7	+.1088⏨+24	+.2030⏨+ 6	+.9212⏨+4	+.4195⏨+ 4	+.3352⏨-6	+.4253⏨+ 9	.1859	1.2650	+.4255⏨+2	+.9162⏨+1	.4746	+.1089⏨+6	+.1004⏨+14
+.1139⏨+ 8	+.1638⏨+24	+.2278⏨+ 6	+.9653⏨+4	+.6888⏨+ 4	+.3554⏨-6	+.5212⏨+ 9	.1636	1.2526	+.6890⏨+2	+.1381⏨+2	.3513	+.1044⏨+6	+.1247⏨+14
+.1415⏨+ 8	+.2255⏨+24	+.2556⏨+ 6	+.1001⏨+5	+.1009⏨+ 5	+.3809⏨-6	+.6069⏨+ 9	.1501	1.2406	+.9546⏨+2	+.1857⏨+2	.2830	+.1007⏨+6	+.1449⏨+14
+.1703⏨+ 8	+.2947⏨+24	+.2867⏨+ 6	+.1031⏨+5	+.1388⏨+ 5	+.4109⏨-6	+.6847⏨+ 9	.1412	1.2291	+.1322⏨+3	+.2560⏨+2	.2396	+.9761⏨+5	+.1628⏨+14
+.2001⏨+ 8	+.3723⏨+24	+.3217⏨+ 6	+.1058⏨+5	+.1823⏨+ 5	+.4454⏨-6	+.7523⏨+ 9	.1352	1.2184	+.1722⏨+3	+.3375⏨+2	.2133	+.9424⏨+5	+.1786⏨+14
+.2309⏨+ 8	+.4594⏨+24	+.3610⏨+ 6	+.1084⏨+5	+.2345⏨+ 5	+.4838⏨-6	+.8169⏨+ 9	.1306	1.2078	+.2268⏨+3	+.4536⏨+2	.1985	+.8953⏨+5	+.1938⏨+14
+.2628⏨+ 8	+.5572⏨+24	+.4050⏨+ 6	+.1108⏨+5	+.2971⏨+ 5	+.5262⏨-6	+.8797⏨+ 9	.1270	1.1969	+.2732⏨+3	+.5611⏨+2	.1864	+.8522⏨+5	+.2089⏨+14
+.2956⏨+ 8	+.6668⏨+24	+.4545⏨+ 6	+.1131⏨+5	+.3696⏨+ 5	+.5732⏨-6	+.9356⏨+ 9	.1244	1.1866	+.3219⏨+3	+.6823⏨+2	.1769	+.8139⏨+5	+.2232⏨+14
+.3293⏨+ 8	+.7898⏨+24	+.5099⏨+ 6	+.1154⏨+5	+.4558⏨+ 5	+.6249⏨-6	+.9901⏨+ 9	.1222	1.1762	+.3759⏨+3	+.8256⏨+2	.1689	+.7785⏨+5	+.2376⏨+14
+.3641⏨+ 8	+.9277⏨+24	+.5721⏨+ 6	+.1176⏨+5	+.5569⏨+ 5	+.6820⏨-6	+.1041⏨+10	.1206	1.1659	+.4373⏨+3	+.9987⏨+2	.1622	+.7459⏨+5	+.2518⏨+14
+.3998⏨+ 8	+.1083⏨+25	+.6419⏨+ 6	+.1198⏨+5	+.6751⏨+ 5	+.7447⏨-6	+.1089⏨+10	.1194	1.1558	+.5376⏨+3	+.1280⏨+3	.1566	+.7158⏨+5	+.2660⏨+14
+.4364⏨+ 8	+.1256⏨+25	+.7203⏨+ 6	+.1219⏨+5	+.8127⏨+ 5	+.8139⏨-6	+.1133⏨+10	.1185	1.1459	+.6235⏨+3	+.1552⏨+3	.1519	+.6877⏨+5	+.2802⏨+14
+.4741⏨+ 8	+.1451⏨+25	+.8082⏨+ 6	+.1241⏨+5	+.9727⏨+ 5	+.8899⏨-6	+.1175⏨+10	.1178	1.1362	+.7212⏨+3	+.1880⏨+3	.1480	+.6616⏨+5	+.2943⏨+14
+.5127⏨+ 8	+.1669⏨+25	+.9068⏨+ 6	+.1262⏨+5	+.1159⏨+ 6	+.9735⏨-6	+.1213⏨+10	.1174	1.1266	+.8204⏨+3	+.2243⏨+3	.1446	+.6370⏨+5	+.3086⏨+14
+.5522⏨+ 8	+.1914⏨+25	+.1017⏨+ 7	+.1283⏨+5	+.1370⏨+ 6	+.1066⏨-5	+.1245⏨+10	.1172	1.1177	+.9208⏨+3	+.2644⏨+3	.1420	+.6144⏨+5	+.3225⏨+14
+.5928⏨+ 8	+.2189⏨+25	+.1142⏨+ 7	+.1304⏨+5	+.1619⏨+ 6	+.1167⏨-5	+.1278⏨+10	.1172	1.1085	+.1028⏨+4	+.3105⏨+3	.1396	+.5927⏨+5	+.3370⏨+14
+.6343⏨+ 8	+.2498⏨+25	+.1281⏨+ 7	+.1325⏨+5	+.1906⏨+ 6	+.1278⏨-5	+.1308⏨+10	.1172	1.0995	+.1144⏨+4	+.3635⏨+3	.1376	+.5721⏨+5	+.3517⏨+14
+.6769⏨+ 8	+.2844⏨+25	+.1437⏨+ 7	+.1346⏨+5	+.2237⏨+ 6	+.1400⏨-5	+.1335⏨+10	.1174	1.0906	+.1342⏨+4	+.4493⏨+3	.1360	+.5527⏨+5	+.3666⏨+14
+.7204⏨+ 8	+.3232⏨+25	+.1613⏨+ 7	+.1367⏨+5	+.2620⏨+ 6	+.1535⏨-5	+.1360⏨+10	.1177	1.0818	+.1484⏨+4	+.5237⏨+3	.1347	+.5343⏨+5	+.3818⏨+14
+.7650⏨+ 8	+.3667⏨+25	+.1809⏨+ 7	+.1388⏨+5	+.3060⏨+ 6	+.1682⏨-5	+.1383⏨+10	.1181	1.0732	+.1632⏨+4	+.6079⏨+3	.1336	+.5167⏨+5	+.3972⏨+14
+.8106⏨+ 8	+.4155⏨+25	+.2030⏨+ 7	+.1409⏨+5	+.3566⏨+ 6	+.1844⏨-5	+.1403⏨+10	.1185	1.0648	+.1791⏨+4	+.7037⏨+3	.1327	+.5001⏨+5	+.4129⏨+14
+.8573⏨+ 8	+.4703⏨+25	+.2278⏨+ 7	+.1431⏨+5	+.4132⏨+ 6	+.2023⏨-5	+.1416⏨+10	.1192	1.0573	+.1957⏨+4	+.8118⏨+3	.1323	+.4845⏨+5	+.4281⏨+14
+.9050⏨+ 8	+.5318⏨+25	+.2556⏨+ 7	+.1453⏨+5	+.4796⏨+ 6	+.2219⏨-5	+.1431⏨+10	.1199	1.0492	+.2142⏨+4	+.9381⏨+3	.1318	+.4694⏨+5	+.4436⏨+14
+.9539⏨+ 8	+.6007⏨+25	+.2867⏨+ 7	+.1475⏨+5	+.5558⏨+ 6	+.2434⏨-5	+.1444⏨+10	.1206	1.0413	+.2346⏨+4	+.1085⏨+4	.1316	+.4549⏨+5	+.4594⏨+14
+.1004⏨+ 9	+.6780⏨+25	+.3217⏨+ 7	+.1498⏨+5	+.6432⏨+ 6	+.2669⏨-5	+.1456⏨+10	.1214	1.0334	+.2575⏨+4	+.1258⏨+4	.1315	+.4411⏨+5	+.4757⏨+14
+.1055⏨+ 9	+.7647⏨+25	+.3610⏨+ 7	+.1521⏨+5	+.7433⏨+ 6	+.2928⏨-5	+.1465⏨+10	.1223	1.0257	+.2834⏨+4	+.1462⏨+4	.1315	+.4279⏨+5	+.4924⏨+14
+.1107⏨+ 9	+.8620⏨+25	+.4050⏨+ 7	+.1544⏨+5	+.8579⏨+ 6	+.3212⏨-5	+.1473⏨+10	.1232	1.0181	+.3131⏨+4	+.1705⏨+4	.1317	+.4153⏨+5	+.5096⏨+14
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+.1215⏨+ 9	+.1093⏨+26	+.5099⏨+ 7	+.1591⏨+5	+.1139⏨+ 7	+.3864⏨-5	+.1483⏨+10	.1252	1.0030	+.3898⏨+4	+.2367⏨+4	.1325	+.3915⏨+5	+.5457⏨+14
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+.1328⏨+ 9	+.1385⏨+26	+.6419⏨+ 7	+.1641⏨+5	+.1507⏨+ 7	+.4649⏨-5	+.1486⏨+10	.1275	.9883	+.4488⏨+4	+.3035⏨+4	.1336	+.3697⏨+5	+.5841⏨+14
+.1508⏨+ 9	+.1968⏨+26	+.9068⏨+ 7	+.1719⏨+5	+.2281⏨+ 7	+.6131⏨-5	+.1480⏨+10	.1313	.9668	+.5520⏨+4	+.4377⏨+4	.1361	+.3401⏨+5	+.6470⏨+14
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+.3198⏨+ 9	+.2226⏨+27	+.1017⏨+ 9	+.2487⏨+5	+.3644⏨+ 8	+.4078⏨-4	+.1153⏨+10	.1742	.8290	+.2882⏨+5	+.5859⏨+5	.1753	+.2072⏨+5	+.1372⏨+15
+.4683⏨+ 9	+.8809⏨+27	+.4050⏨+ 9	+.3246⏨+5	+.1663⏨+ 9	+.1142⏨-3	+.8914⏨+ 9	.2113	.7615	+.7177⏨+5	+.1999⏨+6	.2118	+.1650⏨+5	+.2220⏨+15
+.6837⏨+ 9	+.3474⏨+28	+.1613⏨+10	+.4496⏨+5	+.7312⏨+ 9	+.3045⏨-3	+.6398⏨+ 9	.2651	.7060	+.7462⏨+5	+.2250⏨+6	.2653	+.1366⏨+5	+.3756⏨+15
+.1015⏨+10	+.1363⏨+29	+.6419⏨+10	+.6852⏨+5	+.3090⏨+10	+.7546⏨-3	+.4300⏨+ 9	.3403	.6699	+.3816⏨+5	+.8489⏨+5	.3404	+.1176⏨+5	+.6817⏨+15
+.1554⏨+10	+.5303⏨+29	+.2556⏨+11	+.1102⏨+6	+.1269⏨+11	+.1812⏨-2	+.4305⏨+ 9	.3390	.6500	+.1115⏨+5	+.1476⏨+5	.3390	+.8917⏨+4	+.1276⏨+16
+.2427⏨+10	+.2033⏨+30	+.1017⏨+12	+.1789⏨+6	+.5188⏨+11	+.4329⏨-2	+.3893⏨+ 9	.3694	.6331	+.2880⏨+4	+.2184⏨+4	.3694	+.7019⏨+4	+.2187⏨+16
+.3861⏨+10	+.7612⏨+30	+.4050⏨+12	+.3039⏨+6	+.2088⏨+12	+.1003⏨-1	+.3534⏨+ 9	.3927	.6257	+.8624⏨+3	+.3128⏨+3	.3927	+.5579⏨+4	+.3522⏨+16
+.6216⏨+10	+.2735⏨+31	+.1613⏨+13	+.5262⏨+6	+.8381⏨+12	+.2286⏨-1	+.3409⏨+ 9	.3988	.6203	+.3305⏨+3	+.5316⏨+2	.3988	+.4399⏨+4	+.7104⏨+16
+.9942⏨+10	+.9198⏨+31	+.6419⏨+13	+.9133⏨+6	+.3339⏨+13	+.5238⏨-1	+.3399⏨+ 9	.3993	.6198	+.1244⏨+3	+.8803⏨+1	.3993	+.3433⏨+4	+.9616⏨+16
+.1551⏨+11	+.2796⏨+32	+.2556⏨+14	+.1586⏨+7	+.1330⏨+14	+.1201⏨-0	+.3404⏨+ 9	.3992	.6196	+.4408⏨+2	+.1380⏨+1	.3992	+.2519⏨+4	+.1072⏨+17
+.1982⏨+11	+.5083⏨+32	+.5810⏨+14	+.2201⏨+7	+.3023⏨+14	+.1966⏨-0	+.3413⏨+ 9	.3989	.6195	+.2066⏨+2	+.4006⏨-0	.3989	+.4104⏨+3	+.1049⏨+17

DEPTH OF CONVECTIONZONE:+.1984⏨+11

CP	specific heat per gram
AGRAD	adiabatic gradient
MU	mean molecular weight μ
K	Rosseland mean opacity
RGRAD	radiative gradient ∇_r
GR	mean gradient $\nabla = \mathrm{d}\log T/\mathrm{d}\log P$
V	convective velocity
S33	electrical conductivity in esu

The ionisation equilibrium, ∇_a and C_p were calculated with the formulae compiled by Mihalas (1967). The elements included were H, He, C, O, in the ratios according to the abundances assumed in the HSRA. Two stages of ionisation were taken into account, paıtition functions ware assumed to be constant. The Ohmic conductivity σ was calculated according to Oster (1968) except that collisions with molecules were neglected.

Rosseland opacities for the HSRA element mixture were calculated by inteıpolation in two tables provided by Dr A. N. Cox and his collaborators.

6. Comparison with Two Other Models

In Figure 3 the model of Table II is compared with the model by Baker and Temesvary (1966), which is 195000 km deep, and Mullan's (1971) model which is 10000 km deep. The centre-to-limb variations and the absolute intensities of these models are compared with the HSRA in columns IV and V of Table I. It is seen that the model by Baker and Temesvary differs considerably from the HSRA. However, Baker and Temesvary did not explicitly intend to match an empirical atmosphere model.

Mullan's model does much better, though the differences with HSRA are certainly significant. In fact the model by Mullan provides the reasonable fit to the BCA model which he aimed at in his investigation. However, as stated in Section 3.1, Mullan's model cannot be reconciled with the type of interior model commonly used today. Apparently Mullan considers the uncertainty in the interior models to be large enough to accommodate a convection zone with a depth of 10000 km. Concluding, we remark that an interior model, together with the solar radius, fixes the particular adiabat that must be followed by the deeper layers of the convection zone. The mixing length formalism, in this picture, is essentially a way of estimating the behaviour of convection in the superadiabatic transition layer. We have illustrated in this article how observations of the solar atmosphere provide an important check on this estimate.

Appendix: Limits to the Parameters a and a/b

An upper limit to the ratio a/b (cf. Equations (3) and (4)) can be estimated by comparing the efficiency of conversion of thermal into kinetic energy with the maximum possible efficiency (a Carnot cycle). We will consider the following cycle of a unit mass in an idealized convective medium (see Figure 4).

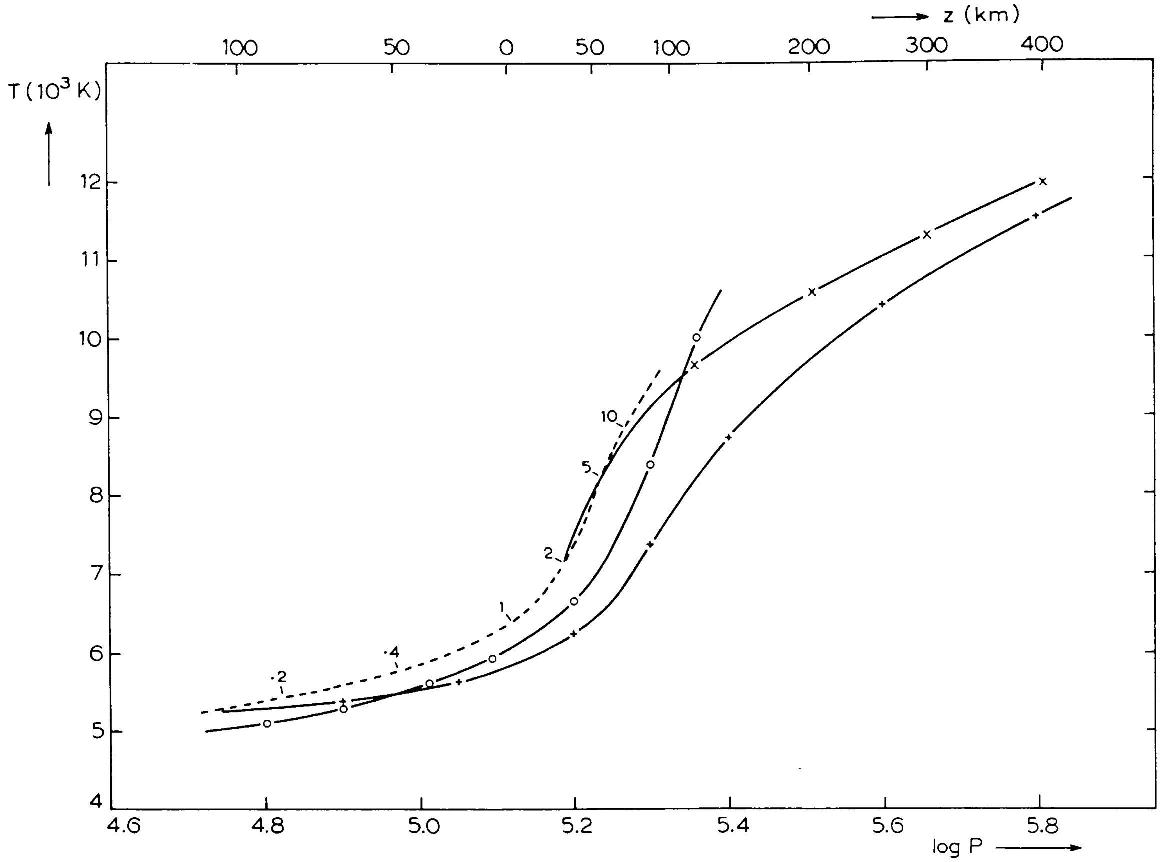

Fig. 3. HSRA and convection zone models. Broken: HSRA. Indicated along the curve are the values of τ_{5000}. $\times$: Model given in Table II. $+$: Baker and Temesvary (1966). $\bigcirc$: Mullan (1971). On top is the depth scale relative to the level $\tau_{5000} = 1$ for the HSRA (up to $\tau = 2$) and the model of Table II (from $\tau = 2$).

Starting at point Q the temperature of the mass is equal to the temperature T_2^0 of its surroundings. It rises and expands adiabatically over a distance l and it arrives at point R with a temperature T_1. Then the mass is cooled by mixing at constant pressure until it has attained the temperature T_1^0 of its surroundings. Next the mass is compressed adiabatically during downward motion from S to P and finally its mixes again to temperature T_2^0. The thermodynamic efficiency of this cycle is:

$$e = \frac{\text{kinetic energy gained during the cycle}}{\text{heat absorbed during the phase } PQ}.$$

The kinetic energy gained during the phases QR and SP is lost by mixing during PQ and RS. We define the quantity β as the ratio between the maximum kinetic energy during each of the phases QR and SP, and the average kinetic energy during these phases. It is clear from this definition that β is of the order of 2. Equation (3) gives the *mixing length* estimate of this average kinetic energy. Now we write (4) as

$$\pi F_c = \varrho v Q, \qquad Q = bc_p T(\nabla - \nabla')l/H,$$

where Q is the mixing length estimate of the heat absorbed during PQ, per unit mass.

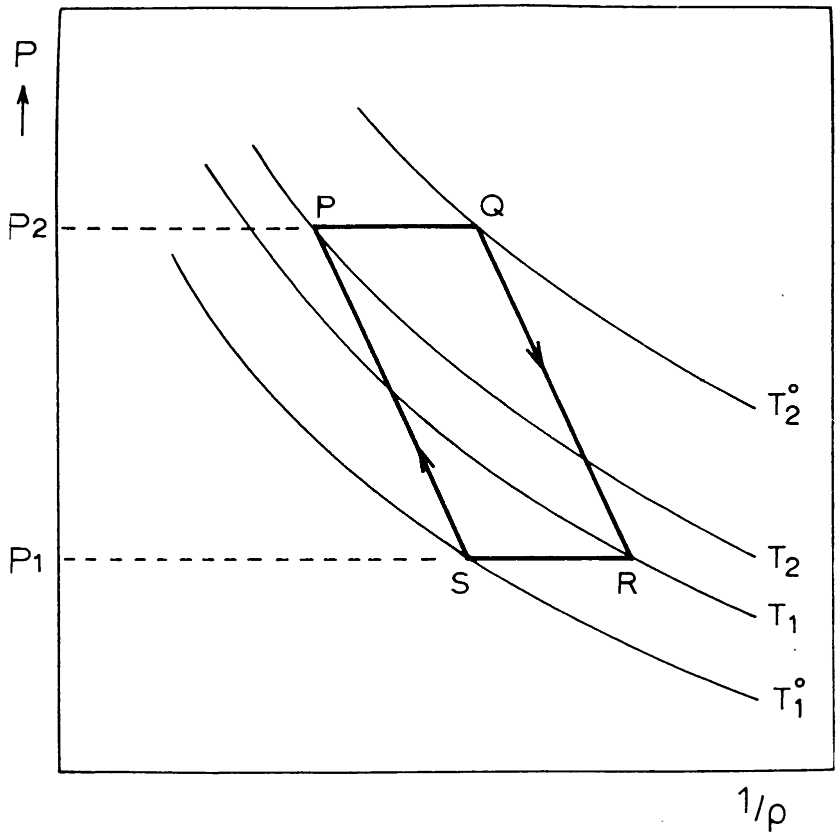

Fig. 4. Idealised convection cycle in the pressure-specific volume diagram.

The efficiency of conversion is thus given by

$$e = \frac{2\beta\frac{1}{2}v^2}{Q} = \frac{\beta a l^2/Hg\,(\nabla - \nabla')}{bc_pT\,(\nabla - \nabla')\,l/H} = \frac{gl\beta a}{c_pTb}.$$

The efficiency is smaller than the efficiency of a Carnot cycle between T_2^0 and T_1^0:

$$e_{\max} = \frac{T_2^0 - T_1^0}{T_2^0} \approx \nabla l/H\,.$$

Thus

$$\frac{gl}{c_pT}\frac{\beta a}{b} < \frac{l}{H}\nabla \quad \text{or} \quad \frac{\beta a}{b} < \frac{\mu c_p \nabla}{\mathscr{R}} = \frac{\nabla}{\nabla_a}.$$

Since we assume the parameters a and b to be constant throughout the convection zone,

$$\frac{\beta a}{b} < \min\,(\nabla/\nabla_a) = 1 \quad \text{or} \quad b > \beta a\ .$$

An upper limit to the value of a can be found by comparing the kinetic energy in the convective medium with the work done by the gravitational force during one cycle of upward and downward motion. We can calculate the work done by the gravitational force by thermodynamical means from the cycle in Figure 4. (It can also be calculated directly from the gravitational force. See, e.g. Mihalas, 1970). Assuming that the ratio of specific heats γ, and the mean molecular weight μ during the cycle may be approximated by constant mean values, the work done on the mass turns out to be

$$-\oint P \,\mathrm{d}\frac{1}{\varrho} = \frac{\gamma \mathscr{R}}{\mu(\gamma-1)}\left[(T_2^0 - T_1) - (T_2 - T_1^0)\right]. \tag{7}$$

Now

$$T_2 - T_1^0 \approx l\left(\frac{\mathrm{d}T}{\mathrm{d}z}\right)_{ad}^{d}, \qquad T_2^0 - T_1^0 \approx l\left(\frac{\mathrm{d}T}{\mathrm{d}z}\right)_{ad}^{u},$$

where the superscripts u and d indicate average values during upward and downward motion, respectively. From

$$\frac{\mathrm{d}T}{\mathrm{d}z} = \frac{T}{H}\nabla \tag{8}$$

follows

$$(T_2^0 - T_1) - (T_2 - T_1^0) = \frac{l}{H}\nabla_a (T^u - T^d).$$

Putting the average temperatures during upward and downward motion equal to $T^u = \frac{1}{2}(T_2^0 + T_1)$ and $T^d = \frac{1}{2}(T_2 + T_1^0)$, and approximating

$$T_2^0 - T_2 \approx T_1 - T_1^0 \approx l\left[\frac{\mathrm{d}T}{\mathrm{d}z} - \left(\frac{\mathrm{d}T}{\mathrm{d}z}\right)_{ad}\right]$$

we obtain, with $(\gamma-1)/\gamma = \nabla_a$:

$$-\oint P \,\mathrm{d}V = \frac{\mathscr{R}T}{\mu}\frac{l^2}{H^2}(\nabla - \nabla_a) = g\frac{l^2}{H}(\nabla - \nabla_a). \tag{9}$$

Since the convective element is brought to rest after half a cycle, half the energy given by (9) is the upper limit of the kinetic energy (per unit mass) of the element. Thus for the maximum velocity v_m of the element *during its life*, we obtain

$$\tfrac{1}{2}v_m^2 < \tfrac{1}{2}g\frac{l^2}{H}(\nabla - \nabla_a). \tag{10}$$

Again assuming that the mean energy $\frac{1}{2}v^2$ during the life of the element is $1/\beta$ times the maximum value and putting $\nabla' = \nabla_a$ in Equation (3):

$$v^2 = ag\frac{l^2}{H}(\nabla - \nabla_a) < 1/\beta g\frac{l^2}{H}(\nabla - \nabla_a)$$

or

$$\boxed{a < 1/\beta}.$$

This maximum holds a fortiori if the elements do not move adiabatically, i.e. if $\nabla' > \nabla_a$.

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References

Abraham, Z. and Iben, I., Jr.: 1971, *Astrophys. J.* **170**, 157.
Bahcall, J. N., Bahcall, N. A., and Ulrich, R. K.: 1969, *Astrophys. J.* **156**, 559.
Baker, N. and Temesvary, S.: 1966, *Tables of Convective Stellar Atmospheres*, 2nd edition, NASA Institute for Space Studies, New York.
Böhm, K. H.: 1966, *Z. Naturforsch.* **219**, 1107.
Böhm, K. H. and Stückl, E.: 1967, *Z. Astrophys.* **66**, 487.
Cox, J. P. and Giuli, R. T.: 1968, *Principles of Stellar Structure*, Gordon and Breach, New York.
Cox, A. N. and Stewart, J. N.: 1970, *Astrophys. J. Suppl.* **19**, 243.
Gingerich, O., Noyes, R. W., Kalkofen, W., and Cuny, Y.: 1971, *Solar Phys.* **18**, 347.
Henyey, P., Vardya, M. S., and Bodenheimer, L.: 1965, *Astrophys. J.* **142**, 841.
Kippenhahn, R.: 1963, in L. Gratton (ed.), *Proc. Intern. School of Physics 'Enrico Fermi'*, Course 28, Acad. Press, New York, p. 330.
Mihalas, D.: 1967, in B. Alder (ed.), *Methods in Computational Physics* **7**, Acad. Press, New York, p. 15.
Mihalas, D.: 1970, *Stellar Atmospheres*, Freeman and Co., San Francisco, p. 203.
Mizuno, S. and Nishida, M.: 1969, *Publ. Astron. Soc. Japan* **21**, 121.
Mullan, D. J.: 1971, *Monthly Notices Roy. Astron. Soc.* **154**, 467.
Oster, L.: 1968, *Solar Phys.* **3**, 543.
Peyturaux, R.: 1955, *Ann. Astrophys.* **18**, 34.
Pierce, A. K., McMath, R. R., Goldberg, L., and Mohler, O. C.: 1950, *Astrophys. J.* **112**, 289.
Sears, R. L.: 1964, *Astrophys. J.* **140**, 477.
Spiegel, E. A.: 1963, *Astrophys. J.* **138**, 216.
Travis, L. D. and Matsushima, S.: 1973, *Astrophys. J.* **180**, 975.
Ulrich, R. K.: 1970a, *Astrophys. Space Sci.* **7**, 183.
Ulrich, R. K.: 1970b, *Astrophys. Space Sci.* **9**, 80.
Unno, W.: 1969, *Publ. Astron. Soc. Japan* **21**, 240.
Van der Borght, R.: 1971, *Proc. Astron. Soc. Australia* **2**, 46.
Waters, B. E.: 1971, *Proc. Astron. Soc. Australia* **2**, 48.
Waters, B. E. and Van der Borght, R.: 1972, *Proc. Astron. Soc. Australia* **2**, 92.
Watson, W. P.: 1970, *Astrophys. J.* **161**, 139.
Yun, H. S.: 1968, Ph.D. Thesis, Indiana University, p. 42.