ON THE MAGNETO-HYDROSTATIC THEORY OF SUNSPOTS*

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ABSTRACT

Theoretical sunspot models are constructed which are a generalization of those considered by Schlüter and Temesvary (1958). A non-vanishing horizontal component of the pressure gradient, as required by magneto-hydrostatic equilibrium, is obtained by assuming that a magnetic field in the solar hydrogen convection zone (HCZ) inhibits the convective-energy transport. In the framework of Prandtl's mixing-length theory, this is achieved by making the ratio of the mixing length, l, and the pressure scale height, H, smaller in the magnetic-field region than in the surrounding undisturbed HCZ.

The corresponding system of ordinary differential equations was solved numerically, and pressure, temperature, and magnetic-field strength along the axis of symmetry were obtained. When l/H ranges from zero to $(l/H)_{\mathrm{HCZ}}$ a one-parameter family of solutions results; in the one extreme convective-energy transport is completely suppressed; in the other extreme there is no inhibition of convective-energy transport. From this family of models, relations between the coolness of a spot and its magnetic-field strength and between the magnetic-field strength and l/H are derived. The former relation agrees with observations to within 50 per cent; the latter indicates a maximum possible field strength of about 5000 gauss.

I. INTRODUCTION

This investigation is concerned with the magnetic fields and low temperatures observed in sunspots. Its aim is to obtain a quantitative understanding of these two quantities, especially the relation between them. For this purpose, theoretical sunspot models are calculated. As a first step it seems reasonable to ignore dynamical effects; hence, a stationary sunspot is assumed. The visible spot is thought to be the uppermost part of a phenomenon extending through the superficial layers of the Sun. In the following, this whole three-dimensional phenomenon has to be considered; consequently the name "sunspot" will be applied to all of it.

Since the work of Cowling (1946) it has been known that sunspot magnetic fields have decay times of the order of magnitude of 1000 years. Thus this magnetic field must be considered the primary phenomenon. Brought to the surface region by some as-yet unknown mechanism, it produces the observed coolness. Such a magnetic field has two important consequences.

First, it exerts forces on its surroundings. In the stationary case these forces have to be in equilibrium with the pressure gradient and the gravity force. Without a magnetic field the pressure gradient, which then only balances the gravity forces, has a component only in the vertical direction. In the presence of a magnetic field, forces also act in the horizontal direction; hence, the gradient of pressure must have a non-vanishing horizontal component.

Second, the region between the Sun's surface and a depth of about $\frac{1}{10}$ solar radius is known to be thermally unstable. Here turbulent convection rather than radiation is the prevailing energy-transport mechanism. If a magnetic field is brought into this so-called hydrogen convection zone (HCZ), a strong influence on the pattern of motion is to be expected. This, in turn, affects the energy-transport mechanism, and hence the depth dependence of temperature and pressure is changed. Taken together, these remarks lead to the following conclusion about the structure of stationary sunspots: *The magnetic field of a sunspot has to affect the convective-energy transport in such a way that the resulting pres-*

* This paper is based on the author's thesis (1962) done at the Max-Planck-Institut für Physik und Astrophysik, Munich (Germany).

sure gradient together with the gravity force is in turn able to balance the forces exerted by this magnetic field.

If it were possible to formulate this statement quantitatively, the magnetic field would be the self-consistent solution of the problem. Let us see to what extent a formulation can be achieved.

As far as the magneto-hydrostatic part is concerned, there are no essential difficulties. One has to write down the magneto-hydrostatic equation, here a partial differential vector equation. Schlüter and Temesvary (1958) have shown how this three-dimensional problem may be reduced to a one-dimensional one, by assuming that the magnetic field is axially symmetric and untwisted, and that the horizontal dependence of its vertical component obeys a certain similarity law. With these assumptions the magnetic field can be deduced from its field strength at the axis of symmetry; that, in turn, is obtained as the solution of an ordinary differential equation. To solve this equation, one has to know as a function of depth the horizontal difference in pressure between a point at the axis of symmetry and a point far from the axis.

A deductive description of the interaction between a magnetic field and turbulent convection, however, presents a completely unsolved problem. Even for the simpler problem, the description of turbulent convection itself by the hydrodynamical equations, there is no generally accepted solution. Due to this difficulty in solving the full problem, certain tentative suggestions for the case of sunspots have been made. Biermann (1941) suggested that the magnetic field might inhibit convection entirely; to transport all energy by radiation a much steeper temperature gradient is required; consequently, the spot region is cooler than its surroundings. On the other hand, Hoyle (1949) suggested that the only effect of the magnetic field might consist in forcing the convective flow to follow the lines of force. If these funnel out near the surface, that part of the energy transported by convection is then distributed over a greater area; the flux is therefore decreased, and the spot is again cooler than its surroundings.

Numerical calculations based on these assumptions in neither case have led to reasonable sunspot models—at best they gave extreme cases. It seemed plausible, as was in fact already suggested earlier (Cowling 1953), that the truth must lie somewhere in between. In general, there might be a certain reduced but non-negligible amount of energy transported by convection within the spot region.

To arrive at a quantitative formulation of this idea, it is useful to consider first how the structure of the HCZ in the absence of a magnetic field is calculated (see Vitense 1953; Böhm-Vitense 1958). For this purpose, Prandtl's mixing-length theory is applied: "turbulent elements" are formed, rise, and are assumed to mix with their surroundings after they have passed a certain distance l (Prandtl's mixing length). They perform convective-energy transport in communicating their energy excess to the surrounding matter, while they move the distance l. For the quantity l one usually takes the pressure scale height H or some small multiple of it.

For the present application it is important to notice that the ratio l/H governs the efficiency of convection in transporting energy. As l/H decreases, more and more energy is transported by radiation, and in the limit $l/H \to 0$ radiation carries all the energy. This property makes l/H a convenient parameter to describe partial inhibition of convection by the magnetic field.[1] One assigns to the whole sunspot region a ratio $0 \leq l/H \leq (l/H)_{\text{HCZ}}$. Then one can solve the corresponding structure equations to obtain temperature and pressure as functions of depth. Finally, one can use the horizontal pressure difference to solve the simplified magneto-hydrostatic equation. Thus the desired sunspot model is obtained, which one can then compare with observations.

The initially assumed l/H is now uniquely related to a certain magnetic-field configuration. The range of values of l/H permitted by our condition $0 \leq l/H \leq (l/H)_{\text{HCZ}}$

[1] Recently E. Spiegel suggested to the author the reduction of the mixing-length perpendicular to the magnetic field as a physical justification for the decrease of l/H.

gives us a one-dimensional family of sunspot models. Letting l/H go to the limits of the interval one should get models corresponding to the ideas of Biermann or of Hoyle.

II. EQUATIONS

In order to get a set of differential equations describing the sunspot model as outlined above, one has to write down the expressions for momentum and energy balance. The former in the present case is the magneto-hydrostatic equation; the latter gives a relation between the energy flux and the temperature gradient.

Let us start with the magneto-hydrostatic equation:

$$\nabla p = -\frac{1}{4\pi} \boldsymbol{B} \times (\nabla \times \boldsymbol{B}) + \rho \boldsymbol{g} , \tag{1}$$

where $\boldsymbol{B}$ is the magnetic-field vector (in Gaussian units), p is the pressure, ρ is the density, and $\boldsymbol{g}$ is the gravitational acceleration. Schlüter and Temesvary (1958) have shown that equation (1) may be simplified considerably in the case where the magnetic field is axially symmetric, untwisted, and its vertical component B_z obeys—in cylindrical coordinates—the similarity law

$$B_z(z, r) = B_z(z, 0) \frac{D(a)}{D(0)} \quad \text{with} \quad a = r \zeta(z) . \tag{2}$$

This last assumption is reasonable only if there are no returning lines of force in the same spot, or, in other words, if the total magnetic flux ϕ has a finite constant value throughout the spot. Since observations show (see Hale and Nicholson 1925) that in general two spots of opposite magnetic polarity belong together, one may assume that the lines of force of the spot under consideration are completed to closed loops by the magnetic field of the accompanying spot.

If one chooses $D(0)$ such that the following normalization condition holds:

$$2\pi \int_0^\infty D(a)\, a\, d a = \phi , \tag{3}$$

the components of the magnetic field according to equation (2) can be written as

$$B_z(z, r) = D(a) \zeta^2(z) , \tag{4a}$$

$$B_r(z, r) = -D(a)\, a \frac{d\zeta}{dz} , \tag{4b}$$

and B_ϕ vanishes, since we assume the field to be untwisted. When equation (4) is substituted into the horizontal component of equation (1) one gets an equation which, integrated over a, leads to the ordinary differential equation:

$$f y'' y - y^4 + 8\pi \Delta p = 0 , \tag{5}$$

where

$$y = [B_z(z, 0)]^{1/2} ,$$

$$f = \frac{\phi}{\pi} \cdot \frac{\int_0^\infty [D(a)/D(0)]^2 a\, d a}{\int_0^\infty [D(a)/D(0)]\, a\, d a} ,$$

and primes denote derivatives with respect to depth z. The horizontal pressure difference, $\Delta p = p(z, 0) - p(z, \infty)$ arises from integrating the pressure gradient over a. If

Δp is known as a function of depth, the corresponding magnetic field can be calculated by solving equation (5).

For the subsequent calculations the function $D(a)/D(0)$ was chosen to be

$$\frac{D(a)}{D(0)}=e^{-a^2}. \quad (6)$$

According to de Jager (1959) this is in fairly good agreement with observations. With this choice for $D(a)/D(0)$ the quantities defined above become

$$D(a)=\frac{\phi}{\pi}e^{-a^2}, \qquad f=\frac{\phi}{2\pi}.$$

Next we consider the vertical component of equation (1). Since in Δp only the pressure on the spot axis enters, we need only consider the vertical component of equation (1) at $r = 0$. There the magnetic term vanishes identically and the equation reduces to the hydrostatic equation:

$$p' = g\rho . \quad (7)$$

For numerical calculations it is more convenient, however, to use log p as the independent variable. Hence, changing the independent variable in equation (7) and using the equation of state for an ideal gas, we obtain

$$\frac{dz}{d\log p}=\frac{RT}{\mu g}\ln 10=H\ln 10, \quad (8)$$

where R is the gas constant, T is the temperature, and μ is the molecular weight. The right-hand side of equation (9) clearly is, apart from the factor ln 10, the pressure scale height, denoted by H.

Now let us consider the equations of energy balance. If the energy is transported by radiation, this leads to

$$\frac{d\log T}{d\log p}=\nabla_R=\frac{3}{16}\frac{\kappa\rho H}{\sigma T^4}, \quad (9)$$

where κ is the Rosseland mean opacity and σ is the Stefan-Boltzmann constant. The atmosphere will only be thermally stable as long as

$$\nabla_R<\nabla_{\text{ad}}=\frac{2+x(1-x)(\frac{5}{2}+\chi/kT)}{5+x(1-x)(\frac{5}{2}+\chi/kT)^2}, \quad (10)$$

where k is the Boltzmann constant, x is the degree of ionization and χ the ionization potential of hydrogen. We consider only hydrogen because this is the only element undergoing ionization in the region under consideration. If ∇_R exceeds ∇_{ad} turbulent convection sets in and one has to include also the convective energy transport. The formalism used is that given by Böhm-Vitense (1958). Briefly summarized this is as follows.

The total energy flux F is the sum of a part F_C, transported by convection, and a part F_R, transported by radiation:

$$F = F_R + F_C ; \quad (11)$$

F_C and F_R are related to the actual temperature gradient ∇ by the equations

$$F_R=\frac{3}{16}\frac{\sigma T^4}{\kappa\rho H}\nabla, \quad (12a)$$

$$F_C=\tfrac{1}{2}c_p\rho T\langle v\rangle\frac{l}{H}(\nabla-\nabla'). \quad (12b)$$

where c_p is the specific heat at constant pressure, l is Prandtl's mixing length, ∇' is the temperature gradient governing the change of state of the "turbulent elements," and $\langle v\rangle$ is their mean velocity. The last quantity is given by

$$\langle v\rangle^2 = \tfrac{1}{8}\,\frac{RT}{\mu}\left(\frac{l}{H}\right)^2(\nabla-\nabla'). \tag{13}$$

The rising turbulent elements would undergo adiabatic changes if it were not for radiative exchange of energy between them and the surrounding matter. An approximate treatment of the radiative exchange, which assumes the turbulent elements are optically thick, gives

$$\frac{\nabla-\nabla'}{\nabla'-\nabla_{\rm ad}} = \frac{1}{24}\,\frac{c_p\rho^2 T\kappa H\langle v\rangle}{\sigma T^4}\left(\frac{l}{H}\right). \tag{14}$$

Following Kippenhahn (1962) this system of equations is solved in the following way: equations (13) and (14) combine to the quadratic equation

$$(\nabla-\nabla') + 2U(\nabla-\nabla')^{1/2} - (\nabla-\nabla_{\rm ad}) = 0\,, \tag{15}$$

with

$$U = 12\,\frac{\sigma T^4}{c_p\rho^2 T\kappa H}\left(\frac{H}{l}\right)^2\left(\frac{8\mu}{RT}\right)^{1/2}.$$

Equations (9), (11), (12), and (13) combine to an equation which, after substituting the positive root for $(\nabla-\nabla')^{1/2}$ from expression (15), becomes

$$(\nabla-\nabla_{\rm ad}) - (\nabla_R-\nabla_{\rm ad}) + V([U^2+(\nabla-\nabla_{\rm ad})]^{1/2}-U)^3 = 0\,, \tag{16}$$

with

$$V = \frac{3}{32}\,\frac{c_p\rho T\kappa H}{\sigma T^4}\left(\frac{l}{H}\right)^2\left(\frac{RT}{8\mu}\right)^{1/2}.$$

This is a cubic equation for $[U^2+(\nabla-\nabla_{\rm ad})]^{1/2}$ which is easily shown to have only one real root. Taking the usual formulae for the root of a cubic equation, finally an explicit (and rather lengthy) expression for ∇ is obtained.

Now we have two simultaneous differential equations, equation (8) and the root of equation (16), the solution of which gives the temperature and pressure along the axis of symmetry. Before we can solve this system, however, we must specify κ, c_p, μ, and x as functions of p and T.

A value of κ for each p and T was obtained by interpolating in a table, based on the results of Vitense (1951*a*).

Expressions for c_p, μ, and x were taken over from Kippenhahn, Temesvary, and Biermann (1958):

$$c_p = \frac{R}{\mu}\left[\tfrac{5}{2}+\tfrac{1}{2}x(1-x)\left(\tfrac{5}{2}+\frac{\chi}{kT}\right)^2\right],$$

$$\mu = \frac{\mu_0}{1+x},$$

$$\log\frac{x^2}{1-x^2} = \tfrac{5}{2}\log T - \frac{13.53\times 5040}{T} - 0.48 - \log X - \log p - \log\mu_0\,,$$

where μ_0 is the molecular weight of the neutral mixture of hydrogen and helium with the abundances $X = 0.6$, $Y = 0.4$, respectively.

Above we have described the usual approach to calculate the structure of the undisturbed HCZ. Now we wish to allow for the influence of the magnetic field. We do this by decreasing the quantity l/H. To get insight into the consequences of varying l/H consider equation (16). Letting $l/H \to 0$, the turbulent elements eventually become optically thin and U has to be redefined. For the optically thin case Vitense (1953) gives

$$U = 12 \frac{\sigma T^4 \kappa H}{c_p T} \left(\frac{8\mu}{RT}\right)^{1/2} .$$

This quantity is independent of l, hence, it remains constant, whereas V goes to zero as l^2 and equation (16) gives

$$\lim_{l/H \to 0} \nabla = \nabla_R .$$

Dividing equation (16) by V we see that

$$\lim_{l/H \to \infty} \nabla = \nabla_{\text{ad}} .$$

Thus for any positive value l/H there corresponds a temperature gradient which lies between ∇_R and ∇_{ad}. Furthermore, for any $0 \leq l/H \leq (l/H)_{\text{HCZ}}$ there corresponds a temperature gradient which lies between ∇_R and the actual one in the HCZ. This makes l/H a convenient parameter to interpolate between the two extreme values the temperature gradient can assume.

There remains, however, the question: How crude is the assumption of a constant l/H throughout the spot region? A more accurate discussion of the structure of the HCZ shows that the solution is sensitive to the parameters in the theory only in the non-adiabatic region, that is, where $\nabla - \nabla_{\text{ad}} \gtrsim 10^{-2}$. Deeper down, in the adiabatic region, the solution becomes very insensitive to changes in the parameters. In the solar case the non-adiabatic region extends over the uppermost 500–1000 km, whereas we expect that the magnetic fields under consideration extend to a depth larger by at least a factor 10 and therefore appreciable variation should only occur over this distance. As any variation of l/H would plausibly be related to changes in the magnetic field, the variation will be so small in the critical region that it should be a good approximation to substitute for l/H the value it assumes in this upper region.

The same reasoning can be applied to variations of the total energy flux F. Somehow this flux must be decreased because we observe in sunspots a flux which is smaller than in the surrounding photosphere. Again one might plausibly relate any change in F to variations in the magnetic field. Then they are quite small in the non-adiabatic region; hence it is a good approximation as for l/H, to substitute for F the value occurring in the upper part of the spot.

Finally, the question should be discussed: Is there any significant radiative-energy transport in horizontal direction due to the temperature difference ΔT between the spot axis and the surroundings? The existence of energy transport in horizontal direction requires a horizontal component F_r of the flux vector $\boldsymbol{F}$, approximately given by

$$F_r = -\left(\frac{16\sigma T^3}{3\kappa\rho}\right) \frac{\Delta T}{\langle r \rangle} ,$$

where $\langle r \rangle$ is the mean radius of the spot. The two components F_z and F_r are related to each other by the requirement that $\boldsymbol{F}$ is a source-free field:

$$\frac{\partial F_z}{\partial z} = -\frac{2}{\langle r \rangle} F_r .$$

Together these equations yield a differential equation for F_z which can be used to make an order of magnitude estimate for the change ΔF of F_z through the spot (with depth h):

$$\Delta F = \left(\frac{16\sigma T^3}{3\kappa\rho}\right) h \frac{\Delta T}{\langle r \rangle^2} .$$

An average value for the "conductivity" is 10^{13} erg cm^{-3} sec^{-1} $^\circ$ K^{-1}; h and the mean radius $\langle r \rangle$ were taken to be 10^9 cm. The temperature difference ΔT was in all models found to be positive and of order 10^3 $^\circ$ K with little variation. Hence we finally have

$$\Delta F \approx 10^7 \text{ erg cm}^{-2} \text{ sec}^{-1} ,$$

which is less than 0.1 per cent of the total energy flux and is therefore unimportant.

In a previous investigation (Deinzer 1960) only energy transport by radiation was considered within the spot region. To get a self-consistent, singularity-free model for spots of the usual effective temperatures of about 4000° K a significant horizontal energy flux was needed. Hence, consistent with the above formulae, models resulted with drastically reduced mean radii.

III. BOUNDARY CONDITIONS

It is now necessary to specify the boundary conditions of the problem. Let us first consider the conditions to be satisfied at the upper boundary. There we want to fit our interior solutions for pressure and temperature to those of the solar atmosphere. If we choose the level where we want to make this fit as the one with an optical depth $\tau = \frac{2}{3}$, the upper boundary coincides (more or less) with the center of the visible sunspot. Vitense (1951*b*) has carried out calculations for model atmospheres and obtained the photospheric pressure as a function of effective temperature and gravitational acceleration. Since we are, in the actual calculation of our model, only concerned with the axis of symmetry, the magnetic forces vanish identically and Vitense's results can be used as the fitting condition. This means that the photospheric pressure of a sunspot is assumed to be the same as for an atmosphere of correspondingly lower effective temperature and the same g as for the Sun.

Next we should specify the position of the upper boundary compared to the outside geometrical scale, because later we need the horizontal pressure difference. If we choose the zero-point of the geometrical scale to coincide with the undisturbed photospheric level of the Sun, then we have to specify the depth z_D of the photospheric level at the center of the spot. As pressure varies very rapidly with depth, z_D enters the pressure difference very sensitively and therefore greatly affects the calculations. Unfortunately this quantity, known as the geometrical depression of a sunspot, is poorly determined by observation. Therefore the reference of the two geometrical scales to each other was established by choosing an appropriate condition at the lower boundary of the spot region; z_D is then a prediction of our theory.

The magnetic field above the photospheric level was required to be that of a monopole; in other words, the effect of an accompanying spot was neglected. In the adopted description of the magnetic field this means y and y' have to vanish identically at infinity. A necessary and sufficient condition for that is obtained from direct inspection of equation (5). As Δp probably decreases very rapidly above the photosphere, $\Delta p = 0$ can be assumed. Integration of equation (5) then gives

$$\tfrac{1}{2} f y'^2 - \tfrac{1}{4} y^4 = c_1 .$$

From this expression it is seen that vanishing of y and y' at infinity requires that c_1, the constant of integration, be zero, i.e.,

$$f y'^2 = \tfrac{1}{2} y^4 .$$

To carry out a further integration we have to take the square root. The resulting ambiguity in sign is removed by the requirement that the magnetic field has to decrease in the outward direction; hence we have

$$y' = (2f)^{-1/2} y^2 . \tag{17}$$

This is also a sufficient condition for y and y' vanishing at $z \to -\infty$. Integration of (17) leads to

$$y = -\frac{(2f)^{1/2}}{z + c_2},$$

which clearly vanishes at infinity with all its derivatives. Thus equation (17) serves as an upper boundary condition for the magnetic field.

As in the calculation of stellar models, we have to solve a system of non-linear differential equations. To avoid running into singularities, one knows from stellar structure calculations (see, e.g., Schwarzschild 1958) that it is necessary to get hold of the behavior of the solutions at both ends of the integration interval. Any statement about the solutions at the lower boundary, however, should in the present case be based on ideas concerning the continuation of the magnetic field into the interior. Since there is no generally accepted theory on this continuation, any condition may be imposed. This condition, however, must be such that our results at the upper boundary are not appreciably influenced. A condition of that kind turned out to be that the magnetic field should become independent of depth at the lower boundary of the spot region, z_0. Quantitatively this is achieved by requiring

$$y' = 0 \quad \text{at} \quad z = z_0 , \tag{18}$$

$$y'' = 0 \quad \text{at} \quad z = z_0 . \tag{19}$$

With the help of equation (5), the latter condition is transformed into the relation between y and Δp:

$$y^4 = 8\pi\Delta p \quad \text{at} \quad z = z_0 . \tag{20}$$

Differentiation of equation (20), with respect to depth, and equation (18) finally give as a requirement on Δp

$$(\Delta p)' = 0 \quad \text{at} \quad z = z_0 . \tag{21}$$

The effect of these conditions upon the results at the upper boundary is studied by posing them at different depths z_0. This depth was shifted downward until finally the results at the upper part of the spot region did not change anymore.

Conditions (18) and (20) can be applied as they stand. Equation (21) is transformed by virtue of the hydrostatic equation and the equation of state (since ionization is almost complete in the region of interest, the molecular weight is assumed to be constant) into

$$\frac{\Delta p}{p} = \frac{\Delta T}{T} \quad \text{at} \quad z = z_0 . \tag{22}$$

Conditions (21) and (22) refer to horizontal differences in p and T, thus providing the desired relation between functions at the spot axis and those outside.

The set of boundary conditions is now as follows:

At the upper boundary: $p = p(T_{\text{eff}})$ $\qquad y' = (2f)^{-1/2} y^2$,

At the lower boundary: $\Delta p = \dfrac{y^4}{8\pi}$ $\qquad y' = 0$,

$$\Delta T = \Delta p \, \frac{T}{p} .$$

When we compare the number of boundary conditions with the order of the problem and the number of free parameters, we see that in general a model is uniquely determined by specifying the three quantities: z_0, the position of the lower boundary, $T_{\rm eff}$, the effective temperature, and ϕ, the total magnetic flux. For convenience in the later calculations, $T_{\rm eff}$ is preferred as a free parameter to the quantity l/H.

To show how a model is obtained with this formulation of the problem, let us outline the course of the actual calculations.

First pressure and temperature for the undisturbed HCZ had to be obtained. For this purpose, a solution of the structure equations was calculated, determined by the initial value at $z = 0$: $\log p = 4.85$, $T_{\rm eff} = 5800°$ K, and the quantity $l/H = 1$.

Now a value z_0 was chosen and a guess $\Delta p(z_0)$ was made. This number and the values for pressure and temperature of the undisturbed HCZ at the depth z_0 were substituted into equation (22) to get $\Delta T(z_0)$. Then we obtained pressure and temperature at the spot axis by adding $\Delta p(z_0)$ and $\Delta T(z_0)$ to pressure and temperature of the HCZ at z_0.

Now an effective temperature $T_{\rm eff}$ was chosen for the model. From it a value for the photospheric pressure at the spot axis was obtained by making use of Vitense's (1951*b*) results.

With knowledge of the pressure and temperature on the spot axis at $z = 0$ and at $\tau = \frac{2}{3}$, it was possible to solve the structure equations only for a particular l/H; in other words this quantity is an eigenvalue of the problem. This was done by trial and error, and pressure and temperature along the spot axis were obtained. These solutions, in particular, gave the geometrical depression z_D as the depth where the photospheric conditions are actually assumed.

Now everything was known to set up the horizontal pressure difference Δp as a function of depth. With $\Delta p(z)$ and after a total magnetic flux ϕ was chosen, the solution $y = y(z)$ of equation (5) was calculated in the interval $z_D \leq z \leq z_0$, satisfying the boundary conditions (17) and (18). Then we took $y(z_0)$ from the solution and substituted it into equation (20) to get a new value for $\Delta p(z_0)$. If this agreed with the initially guessed value, the calculation was finished. Otherwise the whole procedure was started again with the new value for $\Delta p(z_0)$ and was repeated until finally two successive values $\Delta p(z_0)$ agreed reasonably well.

The numerical calculations were carried out by the step-wise method of Adams and Stormer, i.e., a virtual initial value problem was solved. The necessary adjustment of the parameters or undetermined values at one boundary to satisfy the conditions at the other boundary was obtained by a rapidly converging automatic fitting procedure. All the numerical work was done on the electronic computers G2 and G3 at the Max-Planck-Institut für Physik und Astrophysik, Munich (Germany).

IV. SOLUTIONS

It is of advantage for the following to introduce first an alternative description of the magnetic field. So far it has been characterized by y, the square root of the field strength at the axis of symmetry. Let us now consider the lines of force, which, according to Schlüter and Temesvary (1958), are given in the r, z-plane by

$$\alpha(z, r) = \alpha = \text{const}\,.$$

From equations (2), (3), and (4) a more explicit form may be obtained:

$$r = \left[\frac{\phi}{\pi}\frac{1}{B_z(z, 0)}\right]^{1/2} \alpha \left[2\int_0^\infty \frac{D(\alpha)}{D(0)}\,\alpha\, d\alpha\right]^{-1/2}. \tag{23}$$

Furthermore, the mean radius $\langle r\rangle$ of the magnetic flux tube was defined by

$$\phi = \pi\langle r\rangle^2 B_z(z, 0)\,. \tag{24}$$

From equation (23) it is seen that $\langle r\rangle(z)$ is the line of force with

$$a=\left[2\int_0^{\infty}\frac{D(a)}{D(0)}\,a\,d\,a\right]^{1/2}.$$

Substituting for $D(a)/D(0)$ from equation (6) the lines of force are

$$r=\left(\frac{\phi}{\pi}\right)^{1/2}\frac{a}{y(z)}, \tag{23'}$$

and the mean radius $\langle r\rangle(z)$ is now the line of force given by $a = 1$.

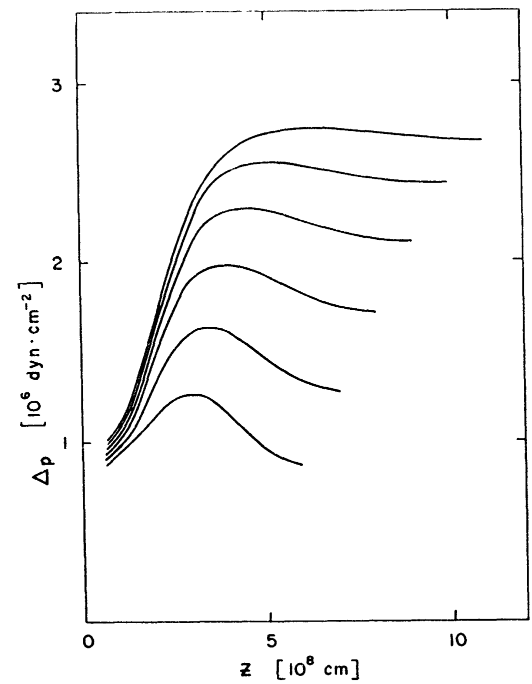

FIG. 1.—Horizontal pressure difference Δp as function of depth z The curves correspond to $T_{\rm eff} = 4000°$ K, $\phi = 5 \times 10^{21}$ gauss cm² and varying total depth z_0 Notice the geometrical depression $z_D \approx 0.7 \times 10^8$ cm at the left.

Let us now discuss the solutions. The problem formulated in the last two sections contains three free parameters: the total depth of the sunspot region z_0, its total magnetic flux ϕ and its effective temperature $T_{\rm eff}$. Hence we are concerned with a three-parameter family of solutions.

Let us first consider the family of solutions in which only z_0 is varied, ϕ and $T_{\rm eff}$ being kept constant. In Figure 1 we show the function $\Delta p(z)$ and in Figure 2 we show $\langle r\rangle(z)$ and $y(z)$. After z_0 has assumed values $\gtrsim 10000$ km, the solutions converge reasonably well. The limiting functions vary strongly only in the upper 500 km, say, and remain almost constant further down. They are monotonic with $\Delta p(z)$ and $y(z)$ increasing and $\langle r\rangle(z)$ decreasing; the latter shows the expected funnel shape of the magnetic field. The independence of our solutions to the position of the lower boundary is gratifying in view of the somewhat arbitrary choice of the lower boundary condition (see Sec. III).

The example displayed in Figures 1 and 2 represents the case $\phi = 5 \times 10^{21}$ gauss/cm², $T_{\rm eff} = 4000°$ K. The essential features of this solution are, however, quite general; the other cases differ only quantitatively. Once we know the general behavior of the solutions, the desired convergence in the other cases may be inferred from a simpler graph. Figure 3 shows the quantity at the upper and lower boundaries as a function of increasing z_0 for different effective temperatures and with $\phi = 5 \times 10^{21}$ gauss/cm^2 ("T_{eff}-family"). Figure 4 is the corresponding diagram for the ϕ-family of solutions with $T_{\text{eff}} = 4000°$ K. Unfortunately, it was not possible to get solutions for $\phi < 5 \times 10^{21}$ by solving equation (5) as an initial-value problem. For these cases, due to its nonlinear character, equation (5) would have to be solved as a boundary-value problem. It seems, however, that this is a purely numerical difficulty and there is no objection to extrapolating the results of Figure 4 to lower magnetic fluxes.

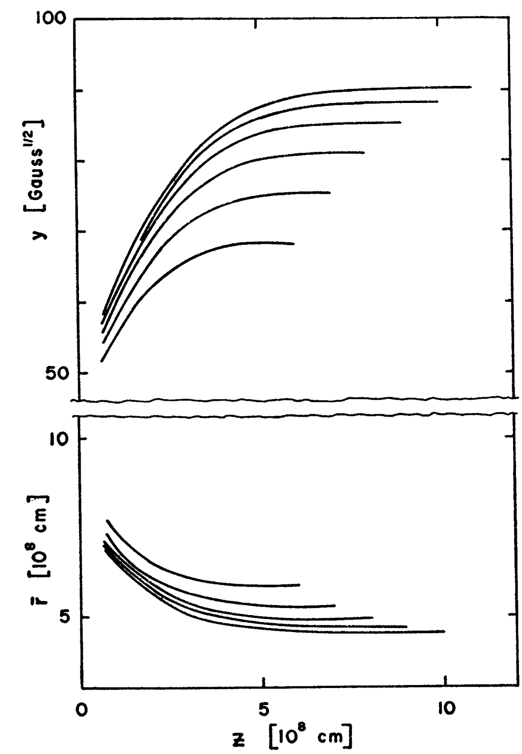

FIG. 2.—Square root of the magnetic-field strength at the axis of symmetry y and mean radius $\langle r \rangle$ as functions of depth. The curves correspond to $T_{\text{eff}} = 4000°$ K, $\phi = 5 \times 10^{21}$ gauss cm^2 and varying total depth z_0. (The curve $\langle r \rangle(z)$ for $z_0 = 11 \times 10^8$ cm was omitted because it almost coincides with the curve $\langle r \rangle(z)$ for $z_0 = 10 \times 10^8$ cm.)

Both figures show the desired convergence for values greater than 10000 km. At the upper boundary the solutions converge faster than at the lower boundary. The main difference between the two families of solutions is that at z_D the curves of the T_{eff}-family converge to different values of y, whereas the curves of the ϕ-family converge to the same values of y. This means that our sunspot models are independent of ϕ in the observable part. In other words, the central magnetic field strength, B_c, is independent of the total magnetic flux, ϕ, in our sunspots. When equation (24) is used, this means that B_c is independent of the mean area, $\pi\langle r \rangle^2(z_D)$, in our sunspot models, which is in obvious contradiction to the observational results of Houtgast and van Sluiters (1948) or Nicholson (1938). This failure of our models might be due to the special assumptions for the magnetic field, although there is no definite answer at present.

For the following, this simply means that we are now only concerned with a one-dimensional family of solutions; namely, the T_{eff}-family. In Table 1 the characteristic quantities of these models are presented. In the first column the parameter T_{eff} is given.

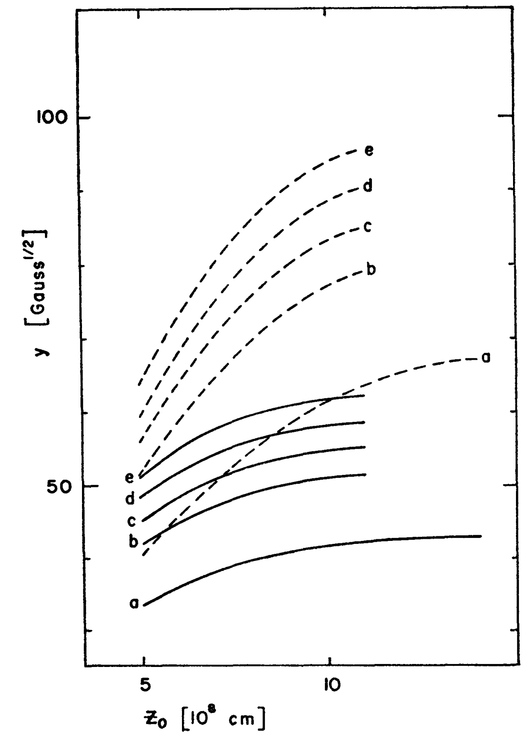

FIG. 3.—Square root of the magnetic-field strength on the axis of symmetry at the upper boundary $y(z_D)$ (*solid curve*) and at the lower boundary $y(z_0)$ (*dashed curve*) as functions of total depth z_0. The curves correspond to $\phi = 5 \times 10^{21}$ gauss cm^2 and varying $T_{\rm eff}$: curves *a*, *b*, *c*, *d*, *e*, correspond to $T_{\rm eff} = 5000$, 4400, 4200, 4000, 3800° K.

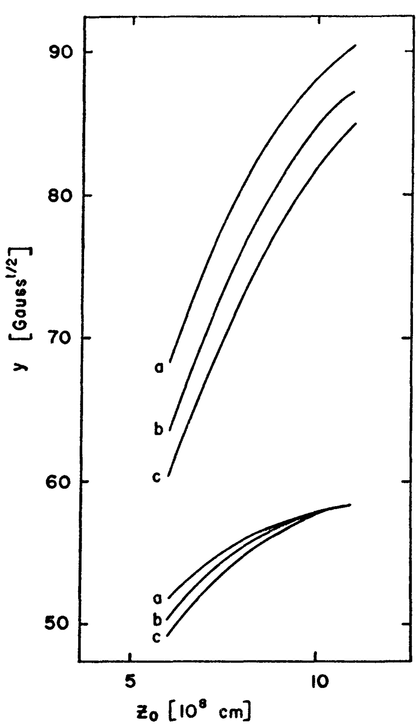

FIG. 4.—Square root of the magnetic-field strength on the axis of symmetry at the upper boundary $y(z_D)$ (*lower part of the diagram*) and at the lower boundary $y(z_0)$ (*upper part of the diagram*) as functions of total depth z_0. The curves correspond to $T_{\rm eff} = 4000°$ K and varying ϕ: curves *a*, *b*, *c* correspond to $\phi = 5 \times 10^{21}$, 7.5×10^{21}, 10×10^{21} gauss cm^2.

In the second and fourth columns the logarithm of the corresponding photospheric pressure, as obtained from Vitense's (1951*b*) work, and the corresponding energy flux F are given. The remaining columns give the geometrical depression z_D, the quantities l/H, $y(z_D)$ at the surface, and $y(\infty)$, taken at a depth where it has converged to its constant value. In the last row of Table 1 corresponding data for the undisturbed photosphere of the Sun are given for comparison.

TABLE 1

PROPERTIES OF THE SUNSPOT MODELS

T_{eff} (° K)	log p	z_D (10^7 cm)	F (10^{10} erg/cm^2 sec)	l/H	$y(z_D)$ (gauss$^{1/2}$)	$y(\infty)$ (gauss$^{1/2}$)
3800	5 17	8 15	1 170	0 185	62 0	98
4000	5 15	7 00	1 425	0 226	58 5	93
4200	5 12	5 90	1 750	0 277	55 5	87
4400	5 08	4 80	2 100	0 345	51 5	82
5000	4 98	2 40	3 500	0 556	42 6	68
5800	4 85	0 0	6 418	1 0	0 0	0

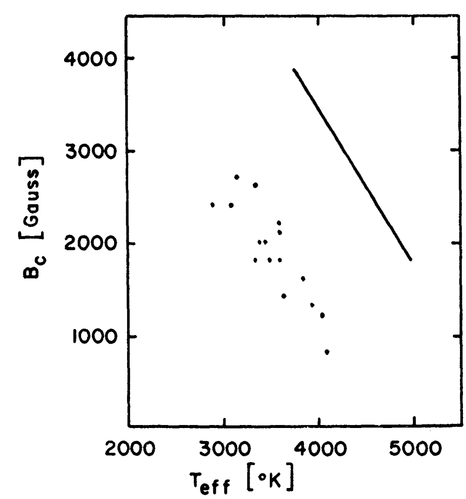

FIG. 5.—Central magnetic-field strength B_c as a function of effective temperature T_{eff} in sunspots. The line represents the theoretical results; the dots are derived from Stumpff (1961, Fig. 17), by means of a relation of Houtgast and van Sluiters (1948).

From a comparison of photospheric pressures one sees that in the sunspots the photospheric pressure is always larger than in the surrounding undisturbed photosphere. As there must, however, be a positive pressure difference Δp in order to balance the magnetic forces, the visible level in sunspots cannot be the same as in the photosphere. It must lie at a depth where the pressure in the surrounding undisturbed atmosphere has increased sufficiently to make $\Delta p > 0$. In fact, the numerical results show geometrical depressions of several hundred kilometers. This agrees in order of magnitude with the observed "Wilson phenomenon."

The first and the sixth columns give the desired relation between T_{eff} and $B_c = y^2(z_D)$. This relation is plotted in Figure 5, and we see that the relation is approximately linear, going from small effective temperatures and high magnetic fields to higher effective temperatures and small magnetic fields. This means that an effective temperature much smaller than that of the undisturbed photosphere indicates a sunspot structure, which deviates very much from that of the undisturbed photosphere; hence, it is possible to balance considerable magnetic fields.

We shall now compare this result with observations. Stumpff (1961) has made measurements of the effective temperature and the umbral area for several sunspots. To compare these results with the predictions of our model, a relation between the central magnetic field and the umbral area is needed. Since no theoretical relation of that kind resulted from the present theory, the observational results of Houtgast and van Sluiters (1948) or Nicholson (1938) were used. Then Stumpff's results yield a relation between T_{eff} and B_c; this relation is also plotted in Figure 5. Observations show the same slope as the theoretical models but the theoretical field strengths are higher by about 50 per

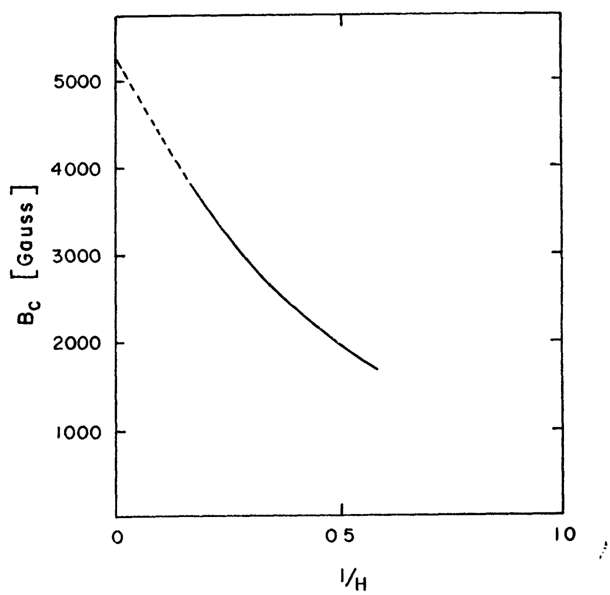

FIG. 6—Central magnetic-field strength B_c as a function of l/H. The dashed part is obtained by extrapolation.

cent. Considering the crude treatment of convection in the presence of a magnetic field and the observational uncertainties, this result is still encouraging.[2]

The fifth and sixth columns give a relation between B_c and l/H. This is plotted in Figure 6. In the framework of the present phenomenological formulation, the relation shows quantitatively the inhibition of convective-energy transport by a magnetic field. From Figure 6 it is seen that strong magnetic fields correspond to l/H-values close to 0, while weaker fields correspond to l/H-values close to 1, the value assumed for the undisturbed HCZ.

The present one-dimensional family of models corresponds to a range in l/H of $0 \leq l/H \leq (l/H)_{\text{HCZ}}$. Figure 6 shows the corresponding range in B_c. By extrapolation it follows that the magnetic-field strength has an upper bound of about 5000 gauss. This occurs in the model where convective-energy transport is suppressed entirely. Thus we get in the limit a model of the kind suggested by Biermann (1941). If the convective-energy transport is suppressed entirely by a certain magnetic field, the structure of sun-

[2] *Added in proof:* Dr P. Stumpff kindly drew my attention to the fact that the temperatures given in his paper are surface temperatures, not effective temperatures. Accordingly, the abscissae of the dots in Figure 5 must be multiplied by the factor 1 19. This shifts the dots toward the theoretical curve and makes the agreement between observations and theory more favorable.

spots can change no more; consequently, stronger magnetic fields cannot be balanced and hence no equilibrium exists for them. The existence of an upper bound for the magnetic-field strengths in sunspots is also an observational fact, although the observed value is about 20 per cent smaller than the theoretical one. This is another positive check with observations.

Finally, let us consider Hoyle's suggestion concerning the reduction of energy flux F in the spot. With the data from Table 1 we can check whether it is possible to explain this reduction. Following Hoyle we assume the convective flow follows the lines of force. With the funnel-shaped magnetic field of the present models this means that the energy flowing through a certain cross-section further down is distributed over a larger cross-section in the upper part of the sunspot. If the total flow of energy through the spot is constant we get

$$\pi\langle r\rangle^2 F = \text{const}\,.$$

Hence we obtain for the flux at different depths the ratio, remembering equation (23′),

$$\frac{F(z_1)}{F(z_2)} = \left[\frac{y(z_1)}{y(z_2)}\right]^2 .$$

TABLE 2

TEST OF UNCERTAINTIES IN THE MODEL WITH $T_{\text{eff}} = 4000°$ K

$\log p$	$(l/H)_{\text{HCZ}}$	z_D (10^7 cm)	l/H	$y(z_D)$ (gauss$^{1/2}$)	$y(\infty)$ (gauss$^{1/2}$)
5 15	1	7 00	0 226	58 5	93
4 25	1	4 70	238	56 5	91
5 15 .	2	3 70	0 404	53 5	97 5

Letting $y(z_1)$ go to $y(z_D)$ and $y(z_2)$ go to $y(\infty)$, we get an expression for the maximum possible reduction. Its value varies only slightly and is about 0.4. Assuming the full energy flux transported through the HCZ, $F = 6.32 \times 10^{10}$ erg cm^{-2} sec^{-1}, enters the spot region and is by Hoyle's mechanism reduced to the observed values, the above ratio must vary from 0.5 to 0.19. Hence for the present models, Hoyle's mechanism is, in general, not able to account for the observed flux values in sunspots. It would give a reduction almost independent of B_c, whereas observations require a reduction which is the greater the stronger B_c is.

There are two major uncertainties in the results presented above, for which we shall now try and make numerical estimates. There is first the question: To what extent is Vitense's (1951*b*) relation between photospheric pressure and effective temperature applicable to sunspots? Observational results of Elsässer (1960) give, for a sunspot with $T_{\text{eff}} = 4000°$ K, a photospheric pressure $\log p = 4.25$. This is smaller than the pressure in the undisturbed solar photosphere, in contrast to Vitense's theory, which gives larger values. For this reason the model with $T_{\text{eff}} = 4000°$ K was repeated with this empirical pressure value.

Before we discuss the results of this calculation, let us consider the second uncertainty. In applying Prandtl's mixing-length theory to convection in stellar atmospheres, l/H has to be fixed *ad hoc;* usually $l/H = 1$ is assumed. To see how sensitively the results depend on this assumption, calculations are usually repeated for $l/H = 2$. In our case, this check had to be done with $(l/H)_{\text{HCZ}}$; l/H for the sunspot adjusted itself according to the boundary conditions. Again the model with $T_{\text{eff}} = 4000°$ K was repeated.

The changes in the models caused by these two alterations can be seen from Table 2.

There are in both cases only minor changes in the magnetic field; they did not exceed 10 per cent. In both cases the geometrical depression was decreased considerably; the change, however, is still within the observational uncertainty. In the case $(l/H)_{\mathrm{HCZ}} = 2$, this ratio for the sunspot is increased by somewhat less than a factor of 2. This might indicate that a number characterizing the inhibition of convection by the magnetic field independent of the particular choice for $(l/H)_{\mathrm{HCZ}}$ is $(l/H)/(l/H)_{\mathrm{HCZ}}$. No important changes in the results are caused by the discussed uncertainties. Neither the main features of the presented sunspot models nor their main deviation from observations are influenced appreciably.

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