6E:221
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Problem Set on Eigenvalues and Eigenvectors

1. Let

$$Q = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{-5}{\sqrt{30}} \end{bmatrix}.$$

Show that Q is orthgonal.

2. Consider Matrices

$$A_1 = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ 1 & -1 & 4 \end{pmatrix}$$

Find the eigenvalues and the associated eigenvectors of A_1 and A_2.

3. Use the answers obtained in question 2 to verify the following statements:

   a. In a symmetric matrix the eigenvectors corresponding to distinct eigenvalues are pairwise orthogonal.

   b. The rank of a symmetric matrix is equal to the number of its nonzero eigenvalues.

   c. The determinant of a matrix equals the product of its eigenvalues.

   d. The sum of eigenvalues is equal to the trace.

   e. For any symmetric matrix A, the eigenvalues of A^2 are the square of those of A and the eigenvectors are the same.

f. If A^{-1} exists, then the eigenvalues of A^{-1} are the reciprocal of those of A and the eigenvalues are the same.

4. Let

$$A_1 = \begin{pmatrix} 2 & -2 & 2 & 1 \\ -1 & 3 & 0 & 3 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 2 & -1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$$

a. Find the characteristics roots and the associated characteristic vectors of A_1 and A_2 .

b. Obtain the matrix which diagonalizes A_1 and the **orthogonal** matrix which diagonalizes A_2.

5. Consider the following Quadratic form:

$$2x_1^2 + 8x_2^2 + 5x_3^2 + 8x_1x_2 + 6x_3x_1 + 12x_3x_2$$

a. Find the symmetric matrix that represents the form.

b. Find the roots of the matrix.

c. Is the matrix positive definite?