Econometric Homework 2

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September 25, 2012

Problem 1: Suppose Y_i i.i.d. $N(0, \sigma^2) i = 1, 2, \ldots, n$

a.)

$$\begin{aligned} \sigma^2 &= E[Y_i^2] - E^2[Y_i] \\ &= E[Y_i^2] \end{aligned}$$

since $E[Y_i] = 0$, the square of the expectation is also zero. So,

$$E\left[\frac{Y_i^2}{\sigma^2}\right] = E\left[\frac{\sigma^2}{\sigma^2}\right] = 1$$

b.) Let $W = (1/\sigma^2)\sum_i^n Y_i^2$. Show that W is distributed according to χ_n^2.

Let $X = \frac{1}{\sigma^2} Y_i$. We want to show that $X \sim N(0, 1)$. First note that X is a linear transformation of Y with $a = 0$ and $b = \frac{1}{\sigma^2}$. So $X \sim N(a + b\mu, b^2\sigma^2)$. Simplifying $a + b\mu = 0 + \frac{1}{\sigma^2}0 = 0$, and $b^2\sigma^2 = \frac{1}{\sigma^4}\sigma^2 = \sigma^2$. This gives us the desired result, $X \sim N(0, 1)$. Therefore,

$$W = \sum_i^n X_i^2 = (1/\sigma^2)\sum_i^n Y_i^2 \sim \chi^2(n)$$

c.) Show that $E[W] = n$

$E[W] = E[(1/\sigma^2)\sum_i^n Y_i^2] = (1/\sigma^2)E[\sum_{i=2}^n Y_i^2] = n\frac{\sigma^2}{\sigma^2} = n$

d.) We want to show that $V \equiv Y_1 / \sqrt{\frac{\sum_i^n Y_i^2}{n-1}}$

Let $W \equiv \frac{1}{\sigma^2}\sum_{i=2}^n$. Then $W \sim \chi^2(n-1)$.

Let $Z \equiv (Y_1 - \mu)/\sigma$. Since $E(Y_1) = \mu$, $\mu = 0$, and $V(Y_1) = \sigma^2$, $Z \sim N(0, 1)$. So we have

$$\begin{aligned} V &= Z / \sqrt{1/\sigma^2 \frac{W}{n-1}} \\ &= \frac{1}{\sigma} Y_1 / (\frac{1}{\sigma}\frac{\sum_{i=2}^n Y_i^2}{n-1}) \\ &= Y_1 / \sqrt{\frac{\sum_i^n Y_i^2}{n-1}} \end{aligned}$$

so $V \sim t_{n-1}$.

Problem 2:
a.) Find $E[X], V[X]$,and $Pr(A)$ where A is the event $\{0.3 < X \leq 0.7\}$ for:

- $X \sim$ Bernoulli(p): $E[x] = p = 0.5$ $V(X) = p(1-p) = 0.25$ $Pr(A) = 0$ since $x \in \{0, 1\}$
- $X \sim N(0.5, 0.25)$ $E[X] = \mu = 0.5$ $V[X] = \sigma^2 = 0.25$

$$\begin{aligned} Pr(A) &= Pr(\frac{0.3-0.5}{(1/16)} \leq Z \leq \frac{0.7-0.5}{(1/16)}) \\ &= 1 - 2 * Pr(Z \leq 3.2) \\ &= 1 - 0.26 = 0.974 \end{aligned}$$

- $X \sim$ Exponential(2) $E[X] = \lambda^{-1} = \frac{1}{2}$ $V(X) = \lambda^{-2} = \frac{1}{4}$

$$\begin{aligned} Pr(A) &= (1 - e^{-2(0.7)}) - (1 - e^{2(0.3)}) \\ &= 0.3221 \end{aligned}$$

Problem 3: $Y_i \sim i.i.d.\ N(10, 4)$. Find $Pr(9.6 \leq \overline{Y} \leq 10.4)$. Note that $\overline{Y} \sim N(\mu, \sigma^2/n)$.

- n = 20. $|Z_{9.6}| = Z_{10.4} = \sqrt{20}\frac{9.6-10}{(2)} = .894427$ so $Pr(9.6 \leq \overline{Y} \leq 10.4)) \approx 1 - 2(.1867) = .6266$
- n = 100 $Z_{9.6} \approx -2.00$ so $Pr(9.6 \leq \overline{Y} \leq 10.4) \approx 1 - 2(.0228) = .9544$
- n = 1000 $Z_{9.6} \approx -6.325$ so $Pr(9.6 \leq \overline{Y} \leq 10.4) \approx 0$

b.) Let $c > 0$. $E[\overline{Y_n}] = 10$ and $V(\overline{Y_n}) = \sigma^2/n$ so lim $E[\overline{Y_n}] = 10$ and lim $V(\overline{Y_n}) = 0$. This is an application of the LLN.

c.) This means that $\overline{Y_n}$ converges in mean square to 10. That is lim $E(\overline{Y_n} - c^2) = 0$ so if $A_n = \{|\overline{Y_n} - c| \geq \epsilon\}$, $0 \leq Pr(A_n) \leq E(\overline{Y_n} - c)^2/\epsilon^2$. In the limit we have $0 \leq \lim Pr(\overline{Y_n}) \leq 0$. That is $\overline{Y} \overset{p}{\sim} 10$.

Problem 4 Let $\overline{X} \sim$ exponential(λ). Let $\theta = E[X] = 1/\lambda$. Let $U = 1/\overline{X}$.

a.) We want to show that plim $U = \lambda$ Using S1, plim $\overline{X} = (1/\lambda)$ and $E[\overline{X}] = \theta\ \forall\ n$ so plim $U = 1/\overline{X} = 1/\theta = \lambda$.

b.) Find the limiting distribution of $\sqrt{n}(U - \lambda)$. Since $X_n \sim$ exponential(λ), $\sqrt{n}(\overline{X} - \theta)$ converges in distribution to $N(0, \theta^2)$. And since $U = 1/\overline{X}$ is continuously differentiable at θ,

$\sqrt{n}(U - \lambda) \overset{D}{\sim} N(0, [U'(\theta)]^2\theta^2)$, or alternativly

$$\overline{X} \overset{A}{\sim} N(\theta, \theta^{-2}/n) \equiv \overline{X} \overset{A}{\sim} N(1/\lambda, \lambda^2/n)$$

c.) Approximate $Pr(U \leq 5/2)$ for $n = 16, \lambda = 2$.

$Pr(U \leq 5/2)$ normalizes to the probability that Z is less than $c^* = \sqrt{n}(\frac{5/2-\mu}{\sigma}) = 4(\frac{(5/2-2)}{(1/2)}) = 1$. From a standard normal table (or integration) we find that this is 0.8413.

d.) For the exact $Pr(U \leq 5/2)$ first note that if $X \sim exp(\lambda)$ then $W \sim \chi^2(2n)$ where $W = 2n\lambda\overline{X}$. So here we have that $W \sim \chi^2(32)$ and $W = 64\overline{X}$ or $\overline{X} = (1/64)W$

We want $Pr(U \leq 5/2) = Pr(1/\overline{X} \leq 5/2) = Pr(\overline{X} \geq 2/5)$. Now substituting in W: $Pr(\overline{X} \geq 2/5) = Pr(W \geq 25.6) = .7810$

Problem 5: The following table contains the results from a Monte Carlo simulation. X is a sample of 100 draws from a uniform distribution over $[0, 10]$. Y is draw from a conditional distribution given by $E[Y_i|x_i] = 10 - x_i$, with a conditional variance of 25 for every x_i (homoskedasticity). $\overline{X}$ is the sample mean for the independent variable, $\overline{Y}$ is the mean of the sample means for the dependent variable. That is the mean of the unconditional expectation of Y over the 1000 simulations. β_0 is the coefficient on the intercept and $beta_1$ is the coefficient on X. Finally, we have s^2, the conditional variance.

	$\overline{X}$	$\overline{Y}$	β_0	β_1	s^2
Expected	5	5	10	-1	25
Observed	5.083	4.917	10.009	-1.002	24.912

So we can see that the Monte Carlo simulation approximates the population distribution well.