Orthonormal Matrices

A square matrix Q is orthonormal if $Q'Q = I$. Let $\mathbf{v}_j$ be the jth column of Q, then, the definition implies that $\mathbf{v}_i'\mathbf{v}_i = 1, i = 1, \ldots, n$ and $\mathbf{v}_i'\mathbf{v}_j = 0, i \neq j$, that is that the columns of Q are all orthogonal to each other, i.e. the inner product of any two columns is 0.

Say we have data (X, y). Rotating the data by Q does not affect our least squares estimator $\widehat{\beta}$. Note that

$$\widehat{\beta} = (X'X)^{-1}X'y = (X'Q'QX)^{-1}X'Q'Qy = (\widetilde{X}'\widetilde{X})^{-1}\widetilde{X}'\widetilde{y},$$

where $\widetilde{X} = QX$ and $\widetilde{y} = Qy$. More is true. Note that

$$Q'Q = I \implies Q'QQ^{-1} = Q^{-1} \implies Q' = Q^{-1}$$

thus

$$\widehat{\beta} = (X'X)^{-1}X'y = (X'QQ^{-1}X)^{-1}X'QQ^{-1}y = (X'QQ''X)^{-1}X'QQ'y = (\widehat{X}'\widehat{X})^{-1}\widehat{X}'\widehat{y},$$

Let's see a practical example now.

```
data <- cars
Q <- randortho(length(cars$speed))
data$speedQ <- Q %*% data$speed
data$distQ <- Q %*% data$dist
data$speedQtr <- t(Q) %*% data$speed
data$distQtr <- t(Q) %*% data$dist
```

Regression with raw data.

```
lm(dist ~ speed - 1 , data = cars)
```

```
##
## Call:
## lm(formula = dist ~ speed - 1, data = cars)
##
## Coefficients:
## speed
## 2.909
```

Regression with data pre-multiplied by Q.

```
lm(distQ ~ speedQ - 1 , data = data)
```

```
##
## Call:
## lm(formula = distQ ~ speedQ - 1, data = data)
##
## Coefficients:
## speedQ
##  2.909
```

Regression with data pre-multiplied by Q'.

```
lm(distQtr ~ speedQtr - 1 , data = data)
```

```
##
## Call:
## lm(formula = distQtr ~ speedQtr - 1, data = data)
##
## Coefficients:
## speedQtr
##    2.909
```