A Study Into Bifurcation Theory and Its Applications

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Contents

Abstract

Bifurcation Theory is a rich branch of applied mathematics that studies the behaviour of a dynamical system as the parameters of the system change. The field has applications in many biological and physical topics and is an exciting and active area of research. This essay will discuss the basic theory of bifurcations, as well as, explore their applications.

1 Introduction

Bifurcation theory is an area of applied mathematics that studies a dynamical system's behaviour as its parameters vary. A dynamical system provides a mathematical description of a process - i.e. a pendulum swinging or stock-market prices - and comprises of a function predicting the system's evolution over time and an initial set of conditions which, provided our function is invariant under different initial conditions, uniquely determine the system's behaviour [9]. Often this function takes the form of a differential equation which could result in equilibrium points (where the differential equation is zero) and bifurcation theory studies the creation and destruction of these points as certain parameters vary. To illustrate this, we begin with an example.

Example 1.1 (The Buckling Beam [3])**.** Consider a metal beam with a weight placed on top as shown in Figure 1.

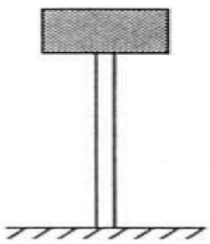


Figure 1: A metal rod with a mass on top [3]

Intuitively, as we increase the mass of the weight, we would expect there to be a minimum mass such that at this weight, or heavier, the beam would buckle. However, it is not clear which way the beam will buckle and this will depend on the minute difference in the position of the mass on the beam, as one can see in Figure 2.

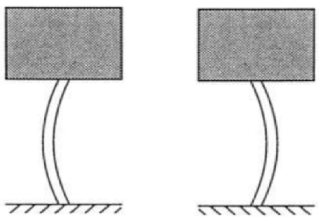


Figure 2: The different cases of the Buckled Beam [3]

Labelling the amount the beam buckles from the centre line as x and calling the mass at which the beam buckles, M, as depicted in Figure 3.

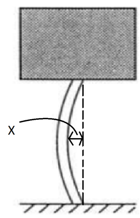


Figure 3: Showing x [3]

We can plot how x varies as our mass is increased. (See Figure 4)

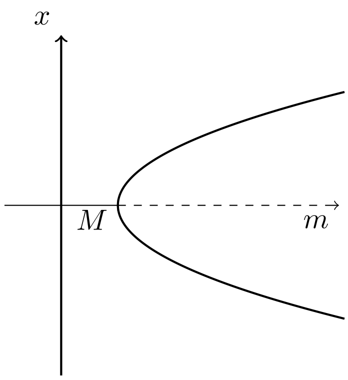


Figure 4: A sketch of what happens to x as m increases

Figure 4 shows that as the mass increases from $m = 0$ to $m = M$, there is no buckling and so $x = 0$. However, from $m = M$, the beam could buckle and so x could either increase left or right, hence the two branches, represented by the positive and negative values of x. There is also the small possibility we place the mass perfectly on the beam resulting in no buckling. This case is very unstable as a minute change in the position of the mass will cause the beam to buckle; we represent this instability by a dotted line. This type of graph is known as the *bifurcation diagram* and is used to show the behaviour of solutions to our equations as our parameter, in this case m, varies.

From the graph, we can see that from the point $(M, 0)$, two curves are created or equivalently for each value of m, we now have two new values of x. The point $(M, 0)$ can therefore be viewed as the creation or destruction of these new values of x. This concept of new solutions appearing from certain points on the graph is key to Bifurcation Theory.

2 Basic Theory

2.1 Definition of a Bifurcation Point

We now begin the formal approach to this topic. Throughout this essay, I will refer to an operator, $F : X \times \mathbb{R} \to Y$, where X, Y are Banach spaces. Further, we will now call our varying parameter λ, the previous example used m to emphasise it was the mass we were changing.

Definition 2.1 (Bifurcation Point [7]). Let $F(x_0, \lambda) = 0$ for any $\lambda \in \mathbb{R}$. The point $\lambda_0 \in \mathbb{R}$ is a *bifurcation point* if for any neighbourhood U of (x_0, λ_0), there exists at least one $(x, \lambda) \in U$ with $x \neq x_0$ such that $F(x, \lambda) = 0$.

We will use Example 1.1, the buckling beam, to demonstrate this notion of a bifurcation point. The dynamics displayed here are known as a *pitchfork bifurcation*, discussed further in Section 4.3, and is modelled by the differential equation:

$$\dot{x} = F(x, \lambda) = \lambda x - x^3,\ x,\ \lambda \in \mathbb{R}$$

Note in Example 1.1, parameter x in the equation above was the displacement from the centre line and λ was the mass. We will now apply Definition 2.1 to show that only $(0, 0)$ is a bifurcation point.

The equilibrium points for this differential equation occur when:

$$F(x, \lambda) = \lambda x - x^3 = 0$$

Namely when:

$$x = 0 \text{ or}$$
$$x = \pm\sqrt{\lambda}$$

Note for $\lambda \leq 0$, we therefore, have $x = 0$ as the only equilibrium point and plotting $\dot{x}$ against x, as in Figure 5, shows this point is stable.

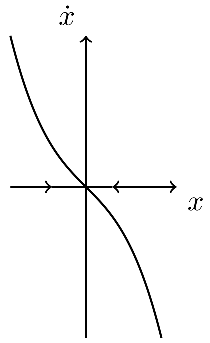


Figure 5: $\dot{x}$ against x for $\lambda = -1$

For $\lambda > 0$, however, we now have three fixed points with $x = 0$ as unstable, as shown in Figure 6.

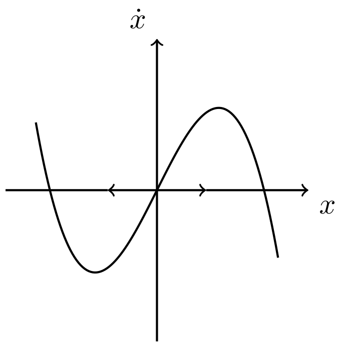


Figure 6: $\dot{x}$ against x for $\lambda = 2$

Now plotting the *bifurcation diagram* of this system, i.e the fixed points, x_*, as a function of λ, we obtain Figure 7:

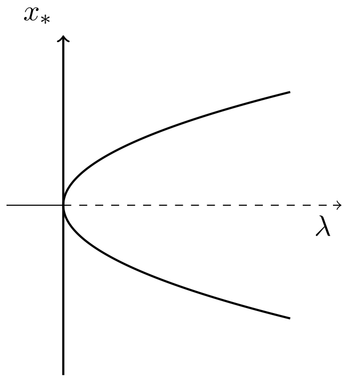


Figure 7: x_* as a function of λ

Inside any neighbourhood of (0,0), see U in Figure 8, there exists at least one point, (x_1, λ_1), away from the origin but on the curve $x_* = \sqrt{\lambda}$. Since $F(0, \lambda) = 0\ \forall \lambda \in \mathbb{R}$, we have that $F(x_1, \lambda_1) = F(0, \lambda_1) = 0$ which results in two possible values of x_* for the same value of λ. As a result, $(0,0)$ satisfies the definition for being a bifurcation point. Intuitively, we see that as λ increases past 0, the number of fixed points of the system changes. For $\lambda < 0$, we have no fixed points; for $\lambda = 0$, we have one and then for $\lambda > 0$ we have three (including the $-\sqrt{\lambda}$ branch). The point $(0,0)$ is then seen as a *bifurcation point* for this system as from here new equilibrium points are created. However, if we take any neighbourhood of a point away from the bifurcation point, see W in Figure 8, we see that although there exists at least one point $(x_2, \lambda_2) \in W$ on the curve $x_* = \sqrt{\lambda}$, x_2 is the only value of x for this value of λ and so no new fixed points are created nor destroyed near this point and so it is not a bifurcation point.

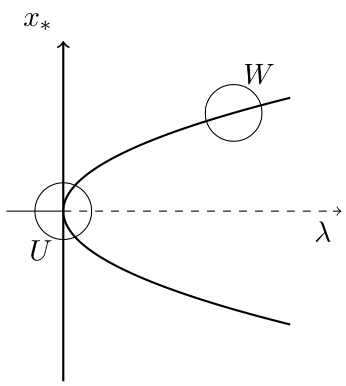


Figure 8: x_* as a function of λ

It is not surprising therefore that at the bifurcation point, the Implicit Function Theorem (I.F.T) breaks down as in any neighbourhood of that point we can not write x_* as a function of λ. We see this with the buckling example where $\frac{\partial F}{\partial x}(x, \lambda) = \lambda - 3x^2$. Consequently at the bifurcation point, $(0,0)$, $\frac{\partial F}{\partial x}(0,0) = 0$ thus breaking the I.F.T meaning we cannot write x_* as a function of λ. As a result, in any neighbourhood of the bifurcation point, there must be multiple values of x_* for the same value of λ. However, as we can take our neighbourhoods arbitrarily small, these values of x_* and λ must converge to the bifurcation point resulting in the creation or destruction (depending on the direction λ changes) of equilibrium points near this point.

This breaking of the I.F.T poses a problem when wanting to study solutions to the function at the bifurcation point. Thankfully, mathematicians have developed a technique for circumventing this issue called the *Lyapunov-Schmidt Reduction*.

3 Lyapunov-Schmidt Reduction

3.1 Introduction

We now move away from the notion of fixed points of differential equations when dealing with bifurcations and instead consider the bifurcation problem as looking for solutions to the equation:

$$F(x, \lambda) = 0 \tag{1}$$

with $F : X \times \mathbb{R} \to Y$, then this could be viewed as solving a system of n non-linear equations with m unknowns, where $n = \dim(Y)$ and $m = \dim(X)$. However, $\dim(X)$ and $\dim(Y)$ could be infinite possibly resulting in an infinite system of non-linear equations with an infinite number of unknowns, for which it is hard to find solutions. The whole idea of the Lyapunov-Schmidt Reduction is to reduce the dimensionality of this problem, thus making it easier to solve.

For the reduction, we assume that $\frac{\partial F}{\partial x}(x, \lambda)$ is a *Fredholm Operator*, defined below, and that $\mathrm{Im}(\frac{\partial F}{\partial x}(x, \lambda))$ has finite co-dimension (i.e letting the $\mathrm{Im}(\frac{\partial F}{\partial x}(x, \lambda)) = A$, then the subspace, B, such that $Y = A \oplus B$, has finite dimension).

Definition 3.1 (Fredholm Operator [2])**.** Let X, Y be Banach spaces and let $T : X \to Y$ be a bounded linear operator. T is said to be *Fredholm* if the following hold:

1. $\mathrm{Ker}(T)$ is finite dimensional,
2. $\mathrm{Im}(T)$ is closed,
3. The quotient set $Y/\mathrm{Im}(T)$ (also known as the *Cokernel*) is finite dimensional.

Noteworthy here is that given a Fredholm operator, $T : X \to Y$ with X, Y Banach spaces, the $\dim(\mathrm{Ker(T)})$ is finite.

3.2 Lyapunov-Schmidt Reduction

Without loss of generality, we assume equation (1) has a solution set and that $(0, 0)$ is in it and is a bifurcation point of the equation.

Then taking the derivative of F with respect to x and evaluating it at $(0, 0)$ we obtain the operator, L, defined as

$$L = \frac{\partial F}{\partial x}(0, 0).$$

We assume that L is not invertible because if it were, we could apply the Implicit Function Theorem to reduce the dimensionality of (1), and as discussed before, the I.F.T is not valid at a bifurcation point.

Letting $\mathrm{Ker}(L) = X_1$ and $\mathrm{Im}(L) = Y_1$, we split our spaces X and Y as such:

$$X = X_1 \oplus X_2$$

$$Y = Y_1 \oplus Y_2$$

Remark. We can safely decompose our spaces X and Y if we assume the *Axiom of Choice* [1]. If X and Y are finite dimensional, then we can take the basis for X_1 and Y_2, which we have assumed to be finite dimensional, and extend these to form the bases for X and Y. The extension bases form the basis of X_2 and Y_1 respectively. In the case that X and Y are infinite dimensional and under the assumption that the *Axiom of Choice* holds, we can form an explicit bases for each space thus allowing us to decompose our X and Y into their respective subspaces.

We then take projection maps into X_1 and Y_1 from X and Y respectively.

$$P : X \to X_1$$

$$Q : Y \to Y_1$$

[1] See Appendix.

From the above, we can now write for each $x \in X$, $x = Px + (I-P)x = x_1 + x_2$, letting $x_1 = Px$ and $(I-P)x = x_2$. Using equation (1), we can also write $QF(x,\lambda) = 0$ and $(I-Q)F(x,\lambda) = 0$. Combining these and remarking that $Y_1 \cap Y_2 = 0$, we get that:

$$QF(x_1 + x_2, \lambda) = 0 \tag{2}$$

$$(I-Q)F(x_1 + x_2, \lambda) = 0 \tag{3}$$

We now define the function $G : X_1 \times X_2 \times \mathbb{R} \to Y_1$ as

$$G(x_1, x_2, \lambda) = QF(x_1 + x_2, \lambda)$$

Since $G(0,0,0) = 0, QF(0,0) = 0$. Our aim now is to apply the I.F.T on G to show that in a neighbourhood of $(0,0,0)$, we can write $x_2 = \psi(x_1)$ for a C^1 function ψ and so $G(x_1, \psi(x_1), \lambda) = 0$. We then substitute this function $\psi(x_1)$ into equation (3) to obtain:

$$(I-Q)F(x_1 + \psi(x_1), \lambda) = 0 \tag{4}$$

Since we assumed $\frac{\partial F}{\partial x}$ to be a Fredholm Operator, its kernel and consequently X_1 is finite dimensional. This also means that Y_2 is finite dimensional and since $I-Q$ maps into Y_2, we obtain that (4) is a finite dimensional equation. Consequently, we will have reduced our function:

$$F(x, \lambda) = 0,$$

which could be possibly an infinite system of equations with an infinite number of unknowns, to

$$(I-Q)F(x_1 + \psi(x_1), \lambda) = 0,$$

a finite system of equations with a finite number of unknowns.

In order to be able to apply the I.F.T to G at (0,0,0), we require $\frac{\partial G}{\partial x_2}(0,0,0)$ to be invertible. We see this by first rewriting $\frac{\partial G}{\partial x_2}(x_1, x_2, \lambda)$ in terms of F.

$$\frac{\partial G}{\partial x_2}(x_1, x_2, \lambda) = \lim_{h \to 0} \frac{Q(F(x_1 + x_2 + h, \lambda)) - Q(F(x_1 + x_2, \lambda))}{h}.$$

Since Q is a linear operator, we see that

$$\begin{aligned}\frac{\partial G}{\partial x_2}(x_1, x_2, \lambda) &= \lim_{h \to 0} Q\left(\frac{F(x_1 + x_2 + h, \lambda) - F(x_1 + x_2, \lambda)}{h}\right) \\ &= Q\left(\lim_{h \to 0} \frac{F(x_1 + x_2 + h, \lambda) - F(x_1 + x_2, \lambda)}{h}\right).\end{aligned}$$

We note that:

$$\lim_{h \to 0} \frac{F(x_1 + x_2 + h, \lambda) - F(x_1 + x_2, \lambda)}{h} = \frac{\partial}{\partial x_2} F(x_1 + x_2, \lambda),$$

and as a result we have:

$$\frac{\partial G}{\partial x_2}(x_1, x_2, \lambda) = Q\left(\frac{\partial}{\partial x_2} F(x_1 + x_2, \lambda)\right).$$

Therefore,

$$\frac{\partial G}{\partial x_2}(x_1, x_2, \lambda) = Q\left(\frac{\partial}{\partial x} F(x, \lambda)\right).$$

Evaluating this at $(0,0,0)$, we see that

$$\frac{\partial G}{\partial x_2}(0,0,0) = Q\left(\frac{\partial}{\partial x} F(0,0)\right).$$

We recall that $\frac{\partial}{\partial x} F(0,0) = L$ and therefore:

$$\frac{\partial G}{\partial x_2}(0,0,0) = Q(L)$$

which, if we restrict to X_2, means that $\frac{\partial G}{\partial x_2}(0,0,0)$ is an isomorphism and is therefore invertible as required. We now define:

$$\frac{\partial G}{\partial x_2}(0,0,0) = L$$

If we restrict this to X_2, we see that this is an isomorphism.
Define $\phi : X_2 \to Y_1$ by $\phi(x) = Lx$.

Proposition 1. $\phi : X_2 \to Y_1$ given by $\phi(x) = Lx$ is an isomorphism.

Proof. ϕ is clearly a homomorphism. Furthermore, since Y_1 is the image of L and X_2 contains all the elements of X not in the kernel of L, besides the zero element, we see that ϕ is surjective. We also have that ϕ is injective: if we take two different elements of X_2, say $x, y \in X_2$ with $x - y \neq 0$ and assume $\phi(x) = \phi(y)$, we see this implies $Lx = Ly$ and so $L(x-y) = 0$. As a result, $(x-y) \in Ker(L)$ or equivalently $(x-y) \in X_1$. However since $x \neq y$, $(x-y)$ is a non-zero element of the kernel and so is not in X_2 but since X_2 is a subspace of X, it is closed and so this leads to a contradiction. Therefore, $\phi(x) \neq \phi(y)$ and ϕ is injective. Combining these results, we have that ϕ is a bijective homomorphism and so is an isomorphism. $\square$

We have just shown that L, and consequently, $\frac{\partial G}{\partial x_2}(0,0,0)$ is an isomorphism and so is invertible. As discussed earlier, this means we can apply the I.F.T on G in a neighbourhood of $(0,0,0)$ which allows us to write x_2 as a function of x_1 and substitute this into (3), thus reducing the dimensionality of our problem.

Example 3.1. Consider $F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ with:

$$F(x,y,\lambda) = (\lambda^2 + x - x^2 + y^2, \lambda + x^2 - xy) = (f_1(x,y,\lambda), f_2(x,y,\lambda)), \tag{5}$$

and the system of ODES given by - i.e $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (f_1, f_2)^T$. We consider the solutions to the equation F $= 0$.

Note that F(0,0,0) $= 0$ and $DF_{x,y}(x,y) = \begin{pmatrix} 1-2x & 2x-y \\ 2y & -x \end{pmatrix}$ so $DF_{x,y}(0,0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ which is not invertible. The kernel of L is spanned by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and its image is spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We see, however, that $D_x f_1(0,0,0) = 1$ which is invertible and so by the Implicit Function Theorem, there exists a function ϕ such that $\phi(y,\lambda) = x$. As a result, we have the equation $f_1(\phi(y,\lambda), y, \lambda) = 0$.

However, note that for $(\phi(y,\lambda), y, \lambda)$ to be a solution of F, we need $f_2(\phi(y,\lambda), y, \lambda) = 0$.

Let $\psi(y,\lambda) = f_2(\phi(y,\lambda), y, \lambda) = \lambda + (\phi(y,\lambda))^2 - y\phi(y,\lambda)$. Then

$$D_\lambda \psi(y,\lambda) = 1 + 2\phi(y,\lambda) D_\lambda \phi(y,\lambda) - y D_\lambda(y,\lambda) \tag{6}$$

so $D_\lambda \psi(0,0) = 1$ and $\phi(0,0) = 0$.

So by the Implicit Function Theorem, there exists a function τ such that $\tau(y) = \lambda$. This means we can write f_2 as $f_2(\phi(y,\tau(y)), y, \tau(y))$ and so we have reduced our problem to a lower dimensional problem.

4 The Different Norms of Local Bifurcation

Dynamical systems containing bifurcations generally show similar patterns of behaviour. Mathematicians have modelled the types of behaviour of these bifurcations and often dynamical systems can be reduced to look like these models (called the *normal forms* of the bifurcation). We have already encountered this when discussing the buckling beam example, where the system could be modelled by the differential equation $\dot{x} = \lambda x - x^3$. Below, we will introduce the most commonly found types of bifurcations.

4.1 Saddle Node or Fold Bifurcation

The Saddle-Node bifurcation models systems where two equilibrium points collide and disappear. Figure 6 shows the two equilibrium curves which as λ decreases, collide at $(0,0)$ before disappearing for $\lambda \leq 0$. The normal form of this type of bifurcation in the 1D case takes the form:

$$\dot{x} = \lambda + x^2$$

This type of behaviour is found in higher dimensions where the Jacobian of the linearised system around the bifurcation point has one simple eigenvalue, n_s eigenvalues with $\mathrm{Re}(\lambda) < 0$ and n_u with $\mathrm{Re}(\lambda) > 0$ (where $n_s + n_u + 1 =$ the dimension of the system). See [4] for more details.

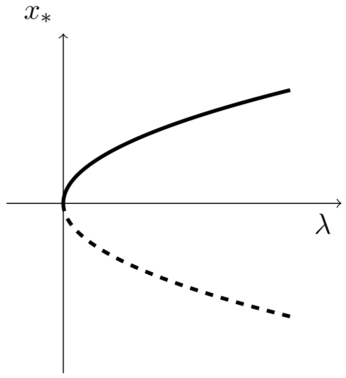
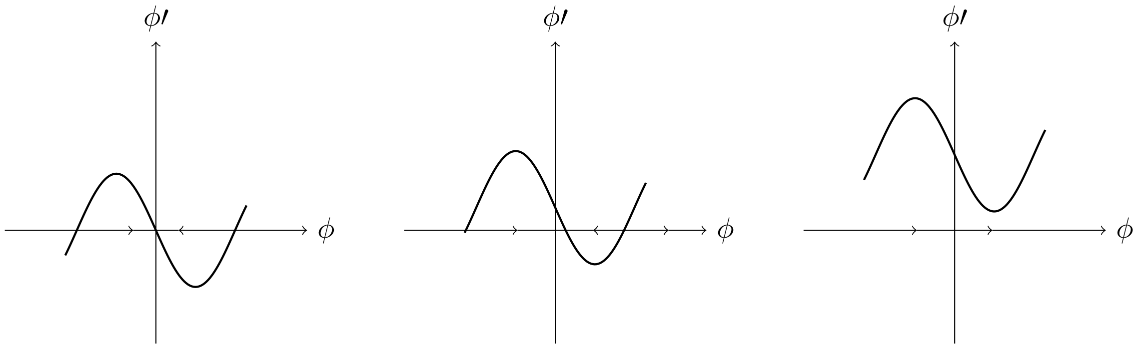


Figure 9: x_* as a function of λ

An incredible example of this type of bifurcation occurring in the natural world can be found in the synchronisation of firefly flashes whilst trying to attract a mate. Although initially unsynchronized, the male fireflies adapt the period of their flashes until eventually all the flashes occur together.
A simple model for this flashing rhythm was developed in 1984 [14] and in short is:

$$\dot{\phi} = \lambda - \sin(\phi),$$

where ϕ is the phase difference between the firefly and the rest of the males and λ is the frequency difference.



(a) $\lambda = 0$ (b) $0 < \lambda < 1$ (c) $1 < \lambda$

Figure 10: Example of a Saddle-Node Bifurcation [5]

Plotting $\phi\prime$ against ϕ and looking at a single period, we see that for $0 < \lambda < 1$ we have one stable and one unstable fixed point. However, as λ increase past 1, these two fixed points collide and annihilate each other. This type of behaviour, as seen in Figure 10, means we have a saddle node bifurcation. This is a highly simplified model and we refer the reader to [14] which contains a much more detailed explanation.

4.2 Transcritical Bifurcation

Unlike the saddle-node case in Section 4.1, certain dynamical systems will contain at least one fixed point for all values of the changing parameter. Although this fixed point is never destroyed, it is common for it to change its stability at the bifurcation point. Dynamical systems showing this behaviour are known as *transcritical bifurcations* and have the normal form

$$\dot{x} = F(x, \lambda) = \lambda x - x^2. \tag{7}$$

We can plot the fixed points against λ to form the bifurcation diagram in Figure 11.

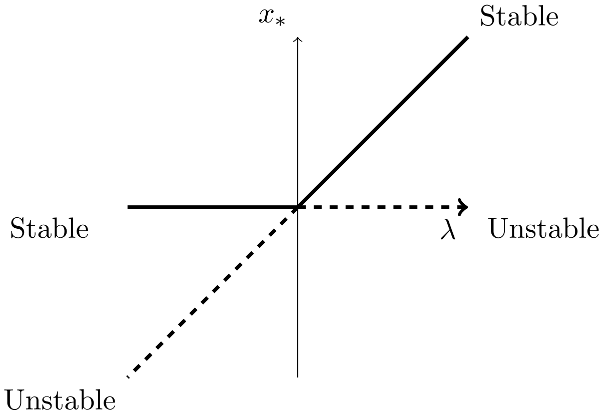


Figure 11: Transcritical Bifurcation [3]

Note how we have $F(0, \lambda) = 0$, $\forall \lambda \in \mathbb{R}$ but changes stability at $(0, 0)$, which is the defining feature of a transcritical bifurcation. This type of bifurcation frequently occurs in modelling situations such as the logistic equation [3] and, surprisingly lasers [14].

Lasers are produced by taking a material with "laser-active" atoms embedded inside and pumping external energy into this material, thus exciting the atoms and making them oscillate. As the energy pumped in increases past a certain threshold, the oscillations of these "laser-active" atoms suddenly begin to sync and in a short time, you have all your atoms oscillating in phase producing a large beam of radiation, the laser. We can model the rate of change of the number of photons, $n(t)$, being produced by:

$$\dot{n}(t) = \text{ gain - loss} = GnN - kn$$

The *gain* term arises due to photons stimulating already excited atoms causing them to produce yet more photons. This process therefore depends on the random interactions between photons and atoms and so occurs at a rate proportional to the number of photons, $n(t)$, and the number of excited atoms, $N(t)$, with $G > 0$ as the constant of proportionality. The *loss* term is due to photons escaping from the material which has a constant rate $k > 0$.

However, when an atom attains a certain level of "excitedness" , it will produce a photon and become unexcited. As a result, the number of excited atoms, N, will decrease as the number of photons, n, increases. We suppose that in the event no laser is produced, i.e. no photons are emitted, the energy pumped in can maintain a constant number of excited atoms, N_0. Then the actual number of excited atoms is given by:

$$N(t) = N_0 - \alpha n,$$

where α is the rate at which the excited atoms become "unexcited". Combining these two equations we attain

$$\dot{n}(t) = Gn(N_0 - \alpha n) - kn = (GN_0 - k)n - (\alpha G)n^2.$$

Which if we plot the fixed points against N_0, we obtain Figure 12.

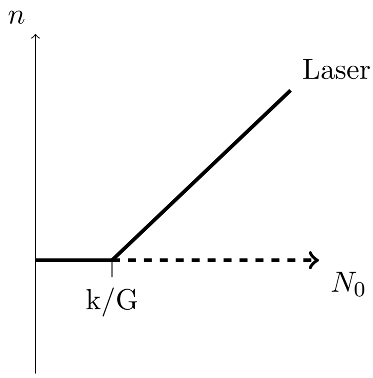


Figure 12: Laser Model Showing Transcritical Bifurcation [14]

We can see that this model has a transcritical bifurcation at $N_0 = k/G$, which is known as the laser threshold. Once the energy pumped into the material attains or exceeds the energy needed to produce this N_0, when no photons are allowed out, a laser will be emitted from the material! This is an incredibly exciting application of bifurcation theory and is discussed more thoroughly in [14].

4.3 Pitchfork Bifurcation

This type of bifurcation often arises in situations involving symmetry; for example, many physical problems contain spatial symmetry between the left and right [14]. We have already encountered a problem containing a pitchfork bifurcation in Example 1.1. Note how here the system is invariant to buckling to the left or right, with both cases forming stable fixed points which are destroyed at $(0,0)$.There are two different types of pitchfork bifurcation:

4.3.1 Supercritical Pitchfork Bifurcation

The Supercritical Pitchfork Bifurcation has the normal form

$$\dot{x} = \lambda x - x^3.$$

Note the similarity with the buckling beam example. We also remark that the system is invariant under the sign of x. Plotting the fixed points against λ, we see an identical graph to the one discussed earlier. Note how the graph looks like a pitchfork, hence the bifurcation's name.

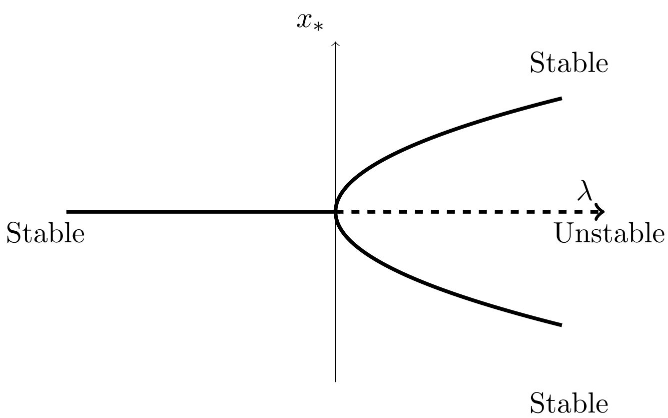


Figure 13: Supercritical Pitchfork Bifurcation

4.3.2 Subcritical Pitchfork Bifurcation

In the supercritical case seen before, the cubic term in the normal equation acts as a restoring force, pulling x back to 0. The subcritical case deals with the opposite situations, where the cubic term destabilises the system. It has normal form

$$\dot{x} = \lambda x + x^3.$$

As with the supercritical case, plotting the fixed points as a function of λ gives Figure 14. Note how

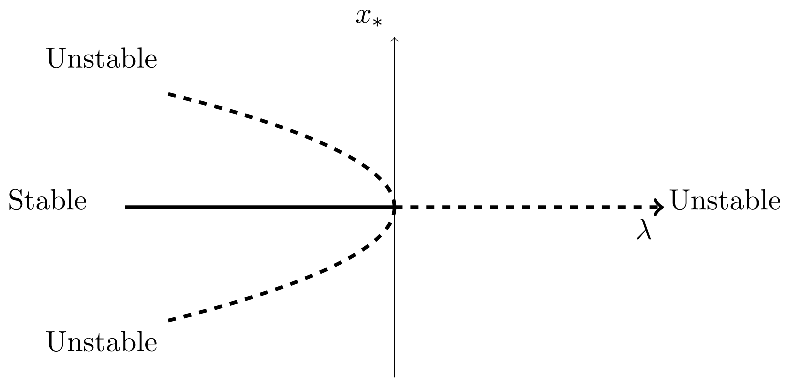


Figure 14: Subcritical Pitchfork Bifurcation

the non-zero fixed points are now unstable and only occur for $\lambda < 0$. As with the supercritical case, we see that the origin is stable for $\lambda < 0$ and unstable for $\lambda > 0$. However, unlike the supercritical case, we have no cubic term stabilising the system, pulling $x(t)$ back to 0 so we find that trajectories *blow up*. Thankfully, real world models usually contain a higher power term acting as the stabilising force (an example would be $\dot{x} = \lambda x + x^3 - x^5$).

Example 4.1. An interesting example of a pitchfork bifurcation is found in the relationship between the temperature and the magnetism of an object. The electrons of the atoms which form our object hold an intrinsic property known as *spin* which collectively give our object a magnetic field. A simple model, known as the Ising Model [14], for our electrons is that electrons spin either up or down, with each scenario being assigned $S_i = \pm 1$ for $i = 1, 2, ..., N$, where $1 << N$ is the number of electrons in the object. Electrons generally prefer to spin in the same direction as their neighbours but this can be disrupted by temperature. An important quantity in Quantum Mechanics is known as *magnetisation* and can be viewed as the average spin of the object. We define magnetisation as

$$m = \frac{1}{N} | \sum_{i=1}^{N} S_i |.$$

High temperatures cause the electrons to spin in random directions and so $m \approx 0$. Our value, m, then remains close to 0 as we cool the object until we reach a critical temperature , T_c. Here m quickly moves away from 0 and our object is suddenly magnetized. A pitchfork bifurcation occurs at this critical temperature, known as the Curie Temperature, and although the theory is too detailed and complicated to discuss here, the symmetry of the model, with the up and down spins, lends us a high-level intuitive understanding as to why this is the case. We refer the reader to [8] where this topic, known as the Curie-Weisse Model, is discussed further.

4.4 Hopf Bifurcation

The Hopf Bifurcation exists only in dimensions two or higher and deals with the changing of periodic or limit cycle solutions as parameters within the system vary [2]. As a result, the Hopf bifurcation is particularly useful when modelling situations in the natural world, such as chemical reactions.
The key concept of a Hopf Bifurcation is that as our changing parameters, cause the eigenvalues of the linearised system about the fixed point to vary resulting in changes to the system's stability. Consequently, to reduce confusion, our bifurcation parameter will now be denoted μ, leaving λ to be used in the context of eigenvalues.
We consider a 2D system which has a stable fixed point, such that the eigenvalues of the Jacobian at this fixed point $\lambda_1(\mu), \lambda_2(\mu)$, and we note that they both depend on μ. Since we have a stable fixed point for certain values of μ, if $\lambda_1(\mu), \lambda_2(\mu) \in \mathbb{R}$, then $\lambda_1(\mu)$, $\lambda_2(\mu) < 0$, $\forall \mu in \mathbb{R}$; alternatively if $\lambda_1, \lambda_2 \in \mathbb{C}$, we have that $\mathrm{Re}(\lambda_1)$, $\mathrm{Re}(\lambda_2) < 0$. So our eigenvalues in the Complex Plane could look as depicted in Figure 16.

Figure 15: Subcritical Pitchfork Bifurcation

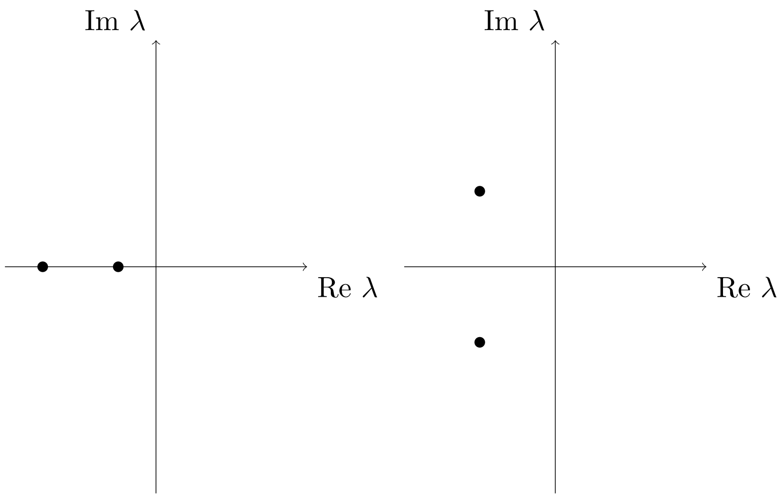


Figure 16: The different cases of our eigenvalues [14]

As μ varies, we assume that our eigenvalues will cross the Imaginary Axis which would represent a change of stability of the fixed point. For the eigenvalues on the real line, this means they will pass through $\lambda = 0$. This situation is known as a *zero bifurcation* and the behaviour is split identical to the cases above (Saddle-Node, Transcritical, Pitchfork). However, in the case of non-real complex eigenvalues, we see that both eigenvalues cross the Imaginary Axis away from 0 which leads to three cases of interesting behaviour, discussed below. Although there are ways one can discern the type of Hopf bifurcation analytically, it is difficult to use and so mathematicians often apply computer simulations to observe the behaviour of the system and deduce the bifurcation type from there.

4.4.1 Supercritical Hopf Bifurcation

This occurs when, as μ increases past a certain value μ_c, the flow in the phase space changes from a stable to an unstable spiral which is now surrounded by a small limit cycle. Figure 17 shows an example in 2D, with $\mu_c = 0$.

[2]See Appendix 1 for an explanation of limit cycles.

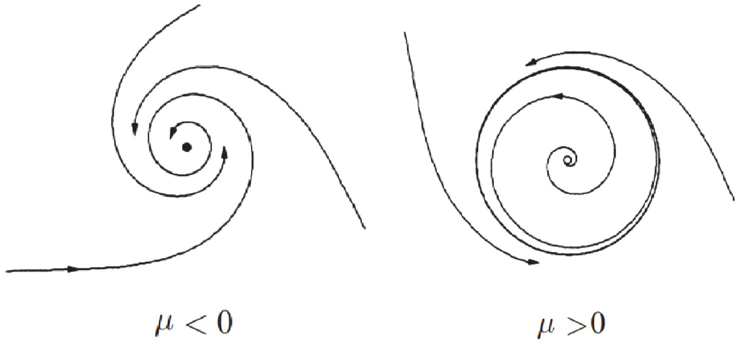


Figure 17: Showing the Different Spirals for μ either side of μ_c [14]

For μ near μ_c, the size of this limit cycle grows proportional to $\sqrt{\mu - \mu_c}$ with frequency equal to the imaginary part of λ evaluated at $\mu = \mu_c$.[14]
This is a highly idealised model because we are assuming that $\text{Im}(\lambda)$ is independent of μ whereas in reality the path taken by the eigenvalue across the Imaginary Axis may have a non-zero slope. [14]

4.4.2 Subcritical Hopf Bifurcation

This is a much more dramatic and problematic type of bifurcation, similar to the Pitchfork bifurcation. We assume the eigenvalues cross the Imaginary Axis at $\mu = 0$, i.e. $Re(\lambda_1(0)) = 0$ and $Re(\lambda_2(0)) = 0$ and that we cross the axis from left to right as μ increases.

For $\mu < 0$, we find that the origin is a stable fixed point surrounded by a stable limit cycle and that between them, there is an unstable limit cycle that tightens around the origin as μ increases to $\mu = 0$. This unstable limit cycle eventually consumes the origin making it unstable. However for $\mu > 0$, we only have the original outer stable limit cycle and so solutions that used to stay near the origin are now suddenly forced out to this limit cycle as shown in Figure 18.

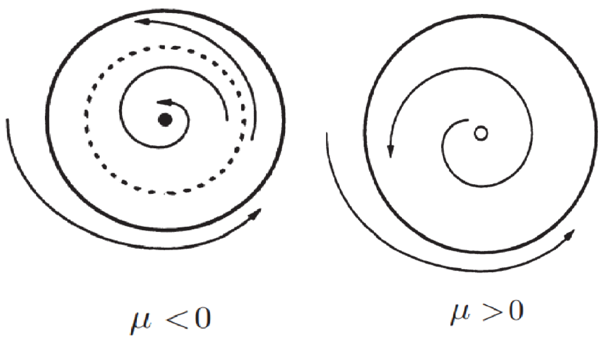


Figure 18: Subcritical Hopf Bifurcation Spirals [14]

Subcritical hopf bifurcations are found in many areas of the natural world, from nerve cells to aero-elastic flutter. [14]

4.4.3 Degenerate Hopf Bifurcation

Before discussing this case, we first need to define *conservative systems.*

Definition 4.1 (Conservative Systems[11])**.** A system $\dot{x} = f(x)$ is said to be conservative if there exists a non-trivial C^1 function $H : X \to \mathbb{R}$ that is constant along all orbits, i.e.

$$\frac{d}{dt}H(\phi_t(x))|_{t=0} = 0 \ \forall x \in X,$$

where $\phi_t(x)$, known as a *flow* of the system, is the solution at time t generated by having x as the system's initial condition.

Remark. A system is *non-conservative* if no such function exists.

This type of Hopf bifurcation occurs when a non-conservative system suddenly becomes a conservative as the system's parameters vary. This results in the fixed point being a non-linear centre, rather than a spiral. The theory behind this will not be discussed here but below demonstrates a classic example of this type of bifurcation.

Example 4.2 (Damped Pendulum [14])**.** The damped pendulum is modelled by:

$$\ddot{x} + \mu x + \sin(x) = 0$$

As μ increases from negative to positive, the origin, which is a fixed point, goes from having an unstable to a stable spiral. However, at $\mu = 0$, we have continuous bands of closed orbits around the origin, which are not limit cycles and so we do not have a proper Hopf bifurcation. Note that at $\mu = 0$, the system has no damping and so is conservative, hence the degenerate Hopf bifurcation.

Hopf bifurcations are particularly useful when modelling situations in the real world, such as oscillations of substances in chemical reactions as shown below.

Example 4.3. Here we will give an example of a model for the chlorine dioxide-iodine-malonic acid (C_1O_2-I_2-MA) reaction [14], which is used when studying non-linear thermodynamics [6].
This reaction can be simplified into three separate chemical reactions with corresponding differential equations.

$$MA + I_2 \to IMA + I^- + H^+ \tag{8}$$
$$\frac{d[I_2]}{dt} = -\frac{k_{1a}[MA][I_2]}{k_{1b} + [I_2]}$$

$$ClO_2 + I^- \to ClO_2^- + 0.5I_2 \tag{9}$$
$$\frac{d[ClO_2]}{dt} = -k_2[ClO_2][I^-]$$

$$ClO_2^- + 4I^- + 4H^+ \to Cl^- + 2I_2 + 2H_2O \tag{10}$$
$$\frac{d[ClO_2^-]}{dt} = -k_3[ClO_2^-][I^-][H^+] - k_{3b}[ClO_2^-][I_2]\frac{[I_2^-]}{u + [I^-]^2}$$

Although computational work can give us a rough idea of this system's behaviour, it is still far too complicated to handle analytically. During the reaction, however, MA, I_2 and ClO_2 change much slower in comparison to I^- and ClO_2^- (multiple orders of magnitude slower) and so we can view these three variables as constants in our system, reducing our problem to only two variables. We can then find a model for just these two variables [14]:

$$\dot{x} = a - x - \frac{4xy}{1 + x^2}$$

$$\dot{y} = bx\left(1 - \frac{y}{1 + x^2}\right)$$

where x and y are dimensionless quantities representing the concentrations of I^- and ClO_2^-, respectively, and $a, b > 0$ are constants dependent on the chemicals we made constant and the rates of reactions (the k_i's above). It can be shown via various methods such as the Poincaré-Bendixson Theorem [3] that this system displays the behaviour of a supercritical Hopf bifurcation. It should be noted that the simplification of this model means equilibrium of chemicals will never be reached, however it is still useful for determining the behaviour of the oscillations.

5 Conclusion

As we have seen, bifurcations occur in many areas of the natural world: from insects attracting mates through to complicated chemical reactions in thermo-dynamics and as such are, and will continue to be, an influential part of applied mathematics. Hopefully this essay will have provided a theoretical grounding in this subject whilst also motivating further study. For those interested, there is a third year module called *Bifurcations, Catastrophes and Symmetry* which explores some of the topics discussed here as well as delving into new and equally exciting areas of applied mathematics.

[3] See Appendix for Poincaré-Bendixson Theorem.

A Appendix

A.1 Limit Cycles

A limit cycle is a cyclic, closed trajectory in the phase space that is defined as an asymptotic limit of other oscillatory trajectories nearby [12]. Its behaviour is depicted in Figure 19.

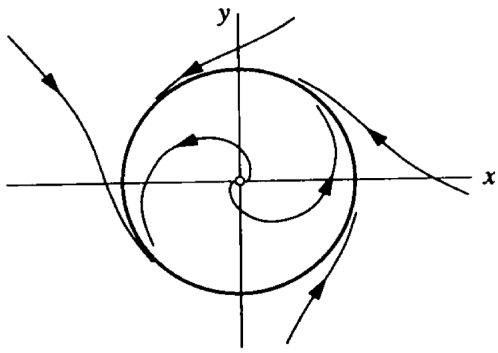


Figure 19: Example of a Limit Cycle Solution [1]

A.2 Axiom of Choice

The Axiom of Choice is an important and widely accepted axiom in Set Theory that states that given a collection of sets, one can form a new set by taking an element of each of the sets.

Definition A.1 (Axiom of Choice [13])**.** If X_α, $\alpha \in A$, is a collection of non-empty sets indexed by the set A — that is, for each element $\alpha \in A$, there is a set X_α in the collection — then there is a map from the index set A to $\cup_\alpha X_\alpha$ such that $f(\alpha) \in X_\alpha$ for each $\alpha \in A$.

Assuming this axiom to be true, one can prove that every Banach space has a certain type of basis, (known as a Schauder Basis). This topic begins to delve into *Functional Analysis* and so will not be discussed here, however for those interested, Chapter 5 of [10] discusses this topic much more thoroughly.

A.3 Poincaré-Bendixson Theorem

Theorem 1 (Poincaré-Bendixson Theorem [11])**.** Suppose that $\dot{x} = f(x)$ on $\mathbb{R}^2$, with f continuously differentiable. Denote the associated flow by ϕ. If $\phi_t(x) \in K$ for all t ≥ 0, where K is compact, then either $\omega(x)$ contains a fixed point or $\omega(x)$ is a periodic orbit.

Definition A.2 (Omega Limit Set [11])**.** $\omega(x)$ is the ω- limit set: $y \in X : \phi_{t_k}(x) \to y$ as $k \to \infty$ for some sequence $(t_k) \to \infty$.

This theorem allows us to prove the existence of limit cycles given certain criteria and is a very useful tool when studying Hopf bifurcations.

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