Order Of Growth 1

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1 Introduction

The goal of this assignment is to understand how to prove runtimes and to understand the formal definitions of Big-O, Ω, and Θ.

NOTE 1: iff $\equiv$ if and only if.
NOTE 2: s.t. $\equiv$ such that.

2 Pick Your Constants: Big-O

2.1

Prove by selecting the appropriate constants and the definition used in lecture that f(n) is O(n^2)
Given: f(n) = $32n^2 + 17n + 1$
Claim: f(n) is O(n^2)
Formal Definition of Big-O: T(n) = O(f(n)) iff there exists positive constants, c and n_0 s.t. T(n) $\leq$ c x f(n) for all n $\geq n_0$
Proof:
f(n) has a O(n^2) runtime if, and only if, f(n) $\leq$ c*n^2.
1) $32n^2 + 17n + 1 \leq c * n^2$
2) let c = 50 and n_0 = 1.
3) $32n^2 + 17n + 1 \leq 50 * n^2$, plug in 50 for c
4) $\frac{32n^2+17n+1}{n^2} \leq 50$
5) $\frac{32n^2}{n^2} + \frac{17n}{n^2} + \frac{1}{n^2} \leq 50$
6) $32 + \frac{17}{n} + \frac{1}{n^2} \leq 50$
7) $50 \leq 50$, plug in 1 for n
8) If any n is inserted larger than one, the result will become increasingly smaller, and will always be less than or equal to 50. This proves that any n $\geq n_0$ cannot disprove this, and that f(n) has a runtime of O(n^2).

2.2

Conversely: show, using counter-examples, that f(n)is neither O(n) nor O(nlogn)
Given: f(n) = $32n^2$ + 17n + 1
General Claim: f(n) is not O(n) not O(nlogn)
Formal Definition of Big-O: T(n) = O(f(n)) iff there exists positive constants, c and n_0 s.t. T(n) ≤ c x f(n) for all n ≥ n $_0$
Claim 1: f(n) is not O(n).
Proof:
Prove by contradiction.
Assume f(n) is O(n).
If f(n) is O(n) then there is some equation that satisfies f(n) $\leq c * n$
1)$32n^2$ + 17n + 1 $\leq c * n$
2)$\frac{32n^2+17n+1}{n} \leq c$
3)$32n + 17 + \frac{1}{n} \leq$ c
4) Since the first term in the fraction was reduced to 32n, this can never be less than some constant factor. Since n can be any number multiplied by 32, it does not satisfy a number less than c. Namely, n can become infinity, and infinity will always be larger than some constant c. Therefore, this function cannot have a runtime of O(n).
5) Counter Example:
6) Let c = 50 and n_0 = 2
7) 32(2) + 17 + $\frac{1}{2}$ ≤ 50
8) 81.5 ≤ 50 , which is not true. Hence f(n) cannot be O(n).

Claim 2: f(n) is not O(nlogn).
Proof:
Prove by contradiction.
Assume f(n) is O(nlogn).
If f(n) is O(nlogn) then there is some equation that satisfies f(n) $\leq c * nlogn$
1)$32n^2$ + 17n + 1 ≤ c*nlogn
2) $\frac{32n^2+17n+1}{nlogn}$ ≤ c
3) $\frac{32n}{logn}$ + $\frac{17}{logn}$ + $\frac{1}{nlogn}$ ≤ c
4) Since the first term in the fraction was reduced to $\frac{32n}{logn}$, this can never be less than some constant factor. Since n can be any number multiplied by 32, it does not satisfy a number less than c. Namely, n can become infinity, and infinity will always be larger than some constant c. Therefore, this function cannot have a runtime of O(nlogn).
5) Counter Example:
6) $\frac{32n}{logn}$ + $\frac{17}{logn}$ + $\frac{1}{nlogn}$ ≤ c
7) Let c = 50 and n_0 = 2
8) $\frac{32(2)}{log(2)}$ + $\frac{17}{log(2)}$ + $\frac{1}{(2)log(2)}$ ≤ 50
9) 270.73 ≤ 50 , which is not true. Hence f(n) cannot be O(n). Rounded to the nearest second decimal place.*

It has now been proved that f(n) is neither O(n) nor O(nlogn).

3 Pick Your Constants: Ω

3.1

Prove by selecting the appropriate constants and the definition used in lecture that f(n) is both $\Omega(n^2)$ and $\Omega(n)$.
Given: $f(n) = 32n^2 + 17n + 1$
General Claim: f(n) is both $\Omega(n^2)$ and $\Omega(n)$
Formal Definition of Ω runtime: $f(n) = \Omega(g(n))$ iff there exists positive constants c and n_0 s.t. $f(n) \geq c * g(n)$ for all $n \geq n_0$
Proof: f(n) has an $\Omega(n^2)$ and $\Omega(n)$ runtime iff there exist positive consants c and n_0 s.t. $T(n) \geq c \text{ x } f(n)$ for all $n \geq n_0$.

Proving f(n) has an $\Omega(n^2)$ runtime:
1) $f(n) \geq c \text{ x } n^2$
2) $32n^2 + 17n + 1 \geq c \text{ x } n^2$
3) let $c = 32$ and $n_0 = 1$
4) $32n^2 + 17n + 1 \geq 32 \text{ x } n^2$, plug in 32 for c
5) $\frac{32(n)^2+17(n)+1}{(n)^2} \geq 32$
6) $\frac{32(n)^2}{(n)^2} + \frac{17n}{(n)^2} + \frac{1}{(n)^2} \geq 32$
7) $32+\frac{17}{n} + \frac{1}{(n)^2} \geq 32$
8) $32+\frac{17}{(1)} + \frac{1}{(1)^2} \geq 32$, plug in 1 for n
9) $50 \geq 32$
10) If any n is inserted larger than one, become increasing smaller, but will always be greater than or equal to 32. In fact, if n is substituted by infinity then the function will equal 32, as the second and third term would just equal zero, and then there would only be a constant of 32 leftover. This proves that any $n \geq n_0$ cannot disprove this, and that f(n) has a runtime of $\Omega(n^2)$.

Proving f(n) has an $\Omega(n)$ runtime:
1) $f(n) \geq c \text{ x } n$
2) $32n^2 + 17n + 1 \geq c \text{ x } n$
3) let $c = 50$ and $n_0 = 1$
4) $32n^2 + 17n + 1 \geq 50 \text{ x } n$, plug in 50 for c
5) $\frac{32(n)^2+17(n)+1}{n} \geq 50$
6) $\frac{32(n)^2}{n} + \frac{17n}{n} + \frac{1}{n} \geq 50$
7) $32n + 17 + \frac{1}{n} \geq 50$
8) $50 \geq 50$, plug in 1 for n
9) If any n is inserted larger than one, the result will become increasingly larger, and will always be greater than or equal to 50. This proves that any $n \geq n_0$ cannot disprove this, and that f(n) has a runtime of $\Omega(n)$.

Since f(n) has both a Ω(n) and Ω(n^2) runtime, the original claim above has now been proved. The original claim being that f(n) is both Ω(n) and Ω(n^2).

3.2

Conversely, show using counter-examples, that f(n) is not Ω(n^3).
Given: f(n) = $32n^2$ + 17n + 1
Claim: f(n) is not Ω(n^3).
Formal Definition of Ω runtime: f(n) = Ω(g(n)) iff there exists positive constants c and n_0 s.t. f(n) $\geq$ c * g(n) for all n $\geq$ n_0
Proof:
Prove through contradiction.
Assume f(n) has a Ω(n^3) runtime.
If this is true then there is some equation that satisfies f(n) $\geq c * n^3$
1) f(n) $\geq$ c x n^3
2) $32n^2$ + 17n + 1 $\geq$ c x n^3
3) $\frac{32(n)^2+17(n)+1}{n^3} \geq$ c
4) $\frac{32}{n} + \frac{17}{n^2} + \frac{1}{n^3} \geq$ c
5) Since there are terms that have n as a denominator, f(n) can never be greater than some constant factor. Since n can be any number, namely infinity, it does not satisfy a number greater than c, as any number divided by infinity is zero. Therefore, this function cannot have a runtime of Ω(n^3).
6) Counter Example:
7) $32n^2$ + 17n + 1 $\geq$ c x n^3
8) Let c = 50 and n_0 = 2
9) $\frac{32}{(2)} + \frac{17}{(2)^2} + \frac{1}{(2)^3} \geq 50$
10) 20.375 $\geq$ 50 , this is false. Therefore f(n) cannot have an Ω(n^3) runtime.

4 Pick Your Constants: Θ

4.1

Claim: f(n) is Θ(n^2).
Formal Definition of Θ runtime: f(n) = Θ(g(n)) iff there exist positive constants c_1, c_2, and n_0 s.t. c_1 x g(n) $\leq$ f(n) $\leq$ c_2 x g(n) for all n $\geq$ n_0
Given: f(n) = $32n^2$ + 17n + 1
Proof:
Claim: f(n) $\leq$ c_2*n^2.
1) $32n^2$ + 17n + 1 $\leq$ c_2*n^2
2) let c_2 = 50 and n_0 = 1.
3) $32n^2$ + 17n + 1 $\leq 50 * n^2$, plug in 50 for c
4) $\frac{32n^2+17n+1}{n^2} \leq 50$
5)$\frac{32n^2}{n^2} + \frac{17n}{n^2} + \frac{1}{n^2} \leq 50$
6)$32 + \frac{17}{n} + \frac{1}{n^2} \leq 50$
7) 50 $\leq$ 50, plug in 1 for n

8) If any n is inserted larger than one, the result will become increasingly smaller, and will always be less than 50. This proves that any n $\geq n_0$ cannot disprove this, and that f(n) $\leq$ c*n^2.

Claim from definition: c_1 x g(n) $\leq$ f(n):
Proving:
1) f(n) $\geq c_1$ x n^2
2) $32n^2 + 17n + 1 \geq c_1$ x n^2
3) let $c_1 = 32$ and $n_0 = 1$
4) $32n^2 + 17n + 1 \geq 32$ x n^2, plug in 32 for c
5) $\frac{32(n)^2+17(n)+1}{(n)^2} \geq 32$
6) $\frac{32(n)^2}{(n)^2} + \frac{17n}{(n)^2} + \frac{1}{(n)^2} \geq 32$
7) $32+\frac{17}{n} + \frac{1}{(n)^2} \geq 32$
8) $32+\frac{17}{(1)} + \frac{1}{(1)^2} \geq 32$, plug in 1 for n
9) $50 \geq 32$
10) If any n is inserted larger than one, become increasing smaller, but will always be greater than or equal to 32. In fact, if n is substituted by infinity then the function will equal 32, as the second and third term would just equal zero, and then there would only be a constant of 32 leftover. This proves that any n $\geq n_0$ cannot disprove this, and that c_1 x g(n) $\leq$ f(n).

Since we have determined that both c_1 x g(n) $\leq$ f(n) and f(n) $\leq c_1$ x g(n), this means that c_1 x g(n) $\leq$ f(n) $\leq c_2$ x g(n) is true which satisfies the original definition of Θ runtime, which means that this function is $\Theta(n^2)$.

4.2

Conversely, show using counter-examples, that f(n) is neither Θ(n) nor $\Theta(n^3)$.
Given: f(n) = $32n^2 + 17n + 1$
General Claim: f(n) is neither Θ(n) nor $\Theta(n^3)$.
Formal Definition of Θ runtime: f(n) = Θ(g(n)) iff there exist positive constants c_1, c_2, and n_0 s.t. c_1 x g(n) $\leq$ f(n) $\leq c_2$ x g(n) for all n $\geq n_0$
Proof:
Prove through contradiction:

Proving f(n) is not Θ(n).
Assume f(n) is Θ(n) s.t. c_1 x g(n) $\leq$ f(n) $\leq c_2$ x g(n) for all n $\geq n_0$
Proving f(n) $\leq c_2$ x g(n):
1) f(n) $\leq c_2$*n, let g(n) be n as this is the claim.
2) $32n^2 + 17n + 1 \leq c_2$*n
3) $\frac{32n^2+17n+1}{n} \leq c_2$
4) 32n + 17 $\frac{1}{n} \leq c_2$
5) Since the first term is still in terms of n, this equation can never be true. If n were to be substituted with infinity it would surely be larger than c_2, meaning

that it is impossible for f(n) to be Θ(n).
6) Counter Example:
7) Let c = 50 and n_0 = 2
8) 32(2) + 17 + $\frac{1}{2} \leq 50$
9) 81.5 $\leq$ 50 , which is not true. Hence f(n) cannot be Θ(n).

Proving f(n) is not $\Theta(n^3)$.
Assume f(n) is Θ(n) s.t. c_1 x g(n) $\leq$ f(n) $\leq$ c_2 x g(n) for all n $\geq$ n_0
Proving f(n) $\geq$ c_2 x g(n):
1) f(n) $\geq c_2$*n^3, let g(n) be n^3 as this is the claim.
2) $32n^2 + 17n + 1 \geq c_2$*n^3
3) $\frac{32n^2+17n+1}{n^3} \geq c_2$
4) $\frac{32}{n} + \frac{7}{n^2} + \frac{1}{n^3} \geq c_2$
5) Since the first term, and the other terms, are still in terms of n, this equation can never be true. If n were to be substituted with infinity it would surely be smaller than c_2. It would be smaller because anything divided by infinity is zero. This means that it is impossible for f(n) to be $\Theta(n^3)$.
6) Counter Example:
7) $32n^2 + 17n + 1 \geq$ c x n^3
8) Let c = 50 and n_0 = 2
9) $\frac{32}{(2)} + \frac{17}{(2)^2} + \frac{1}{(2)^3} \geq 50$
10) 20.375 $\geq$ 50 , this is false. Therefore f(n) cannot have an $\Theta(n^3)$ runtime.