FIR Filtering with Polynomial Coefficients

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Definition. A filter shall be called *polynomial filter* if all of its coefficients can be expressed as a polynomial.

$$w_i = \sum_p a_p i^p$$

An example of such a filter would be the Welch window function, a quadratic polynomial which reaches unity in the middle of the window and zero at the samples just outside the span of the window.

Theorem. Convolution with polynomial filters can be computed recursively.

Proof. We begin by defining $z_{p,t}$ as the convolution of the vector $(1, 2^p, \ldots, n^p)^\top$ with the signal x:

$$\begin{aligned}(w * x)[t] &= \sum_{i=1}^{n} w_i x_{t-(i-1)} = \sum_{i=1}^{n} \sum_p a_p i^p x_{t-(i-1)} \\ &= \sum_p a_p \sum_{i=1}^{n} i^p x_{t-(i-1)} = \sum_p a_p z_{p,t}\end{aligned}$$

Now, by shifting indices we arrive at the recurrence relation:

$$\begin{aligned}z_{p,t} &\equiv \sum_{i=1}^{n} i^p x_{t-(i-1)} = \sum_{i=0}^{n-1} (i+1)^p x_{t-i} = \sum_{i=0}^{n-1} \sum_{k=0}^{p} \binom{p}{k} i^k x_{t-i} \\ &= \sum_{k=0}^{p} \binom{p}{k} \left[\sum_{i=0}^{n-1} i^k x_{t-i} \right] = \sum_{k=0}^{p} \binom{p}{k} \left[\sum_{i=1}^{n} i^k x_{t-i} - n^k x_{t-n} + 0^k x_t \right] \\ &= \sum_{k=0}^{p} \binom{p}{k} \left[z_{k,t-1} - n^k x_{t-n} + 0^k x_t \right] \\ &= z_{p,t-1} - n^p x_{t-n} + \sum_{k=0}^{p-1} \binom{p-1}{k} \left[z_{k,t-1} - n^k x_{t-n} + 0^k x_t \right] \\ &= z_{p,t-1} - n^p x_{t-n} + z_{p-1,t} \quad \text{if} \quad p \geq 1\end{aligned}$$

$\square$

Example. For a quadratic polynomial filter, the recurrence is computed as follows:

$$\begin{aligned}z_{0,t} &= z_{0,t-1} - x_{t-n} + x_t \\ z_{1,t} &= z_{1,t-1} - n x_{t-n} + z_{0,t} \\ z_{2,t} &= z_{2,t-1} - n^2 x_{t-n} + z_{1,t}\end{aligned}$$

Note that due to the lack of divisions this recurrence can be used to implement polynomial filtering in the domain of integers.