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Balanced home–away assignments

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Abstract

In recent years several approaches for generating sports league schedules have been proposed. In this paper we consider foundations for a two-stage approach to construct schedules for a single round robin tournament (or the first half series of a double round robin tournament). In the first stage for each game a mode (home or away) has to be determined and in the second stage the games have to be scheduled in their assigned modes. We study a problem of the first stage where balanced home–away assignments have to be constructed such that for each team the numbers of home and away games differ by at most one. After showing that it is easy to construct balanced home–away assignments we propose repairing mechanisms for unbalanced home–away assignments. Then, neighborhoods on the set of balanced home–away assignments are defined which are shown to be connected. Finally, situations with preassignments are studied.

MSC: 90B35

Keywords: Sports scheduling; Home–away assignment; Neighborhood; Preassignments

1. Introduction

The basic problem for scheduling a sports league may be formulated as follows. Usually, the league consists of an even number n of teams, each team has to play against each other team exactly $l \geq 1$ times. In order to schedule these $\binom{n}{2} l = \frac{n(n-1)}{2} l$ games, $(n-1)l$ rounds are available, i.e. each team has to play one game in each round. Thus, for each round $t = 1, \ldots, (n-1)l$ one has to determine which teams $i, j \in \{1, \ldots, n\}$ play against each other in this round and for each of these $\frac{n}{2}$ pairings $i - j$, whether it is played in the home stadium of team i (home game for i) or in the home stadium of team j (away game for i). In most cases we have $l = 1$ (**single round robin tournament**) or $l = 2$ (double round robin tournament).

If the league contains an odd number n of teams, nl rounds are needed since in each round one team has a ‘bye’, i.e. does not play. This situation may be reduced to the case with an even number of teams by adding a dummy team $n + 1$. Then in each round the team playing against $n + 1$ has a bye.

Several approaches for constructing sports league schedules are based on a graph model introduced by de Werra [13]. For single round robin tournaments (i.e. $l = 1$), the complete graph K_n on n nodes is considered. The

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nodes $i = 1, \ldots, n$ represent the teams, the edges $i - j$ the games between teams $i, j \in \{1, \ldots, n\}$. An edge coloring with $n - 1$ colors, i.e. a partitioning of the edge set into 1-factors $F_1, \ldots, F_{n-1}$ (each consisting of $\frac{n}{2}$ non-adjacent edges), corresponds to the games scheduled in the rounds $t = 1, \ldots, n-1$. If additionally home and away games have to be distinguished, the edges are directed into (i, j) for a game in the home stadium of j or (j, i) for a game in the stadium of i. In this situation the so-called **home–away pattern** (HAP) is defined as $n \times (n-1)$-matrix $H = (h_{it})$, where h_{it} equals 'H' (respectively 'A') when team i has a home (away) game in round t. If in a row i two consecutive entries $h_{i,t-1}$ and h_{it} are equal, team i has a so-called **break** in round t (i.e. the alternating sequence of home and away games is broken).

Often a sports scheduling problem is decomposed into subproblems which are solved sequentially by exact or heuristic algorithms. Two important approaches can be distinguished (for examples cf. the recent survey by Easton et al. [5]):

1. '**First-schedule-then-break**': At first only the $\frac{n}{2}$ pairings are determined for each round (i.e. which teams play against each other in this round). Afterwards for these pairings a corresponding home–away pattern is calculated (with a minimum number of breaks).
2. '**First-break-then-schedule**': At first a feasible home–away pattern (with a minimum number of breaks) is determined, afterwards the pairings for the corresponding pattern are fixed. In this case often at first keys are used to determine the pairings and specific teams are assigned to the keys afterwards.

While in professional leagues often schedules with a minimum number of breaks are required, in non-professional leagues minimizing breaks is not an important issue (spectators for the games are rare). Moreover, in such leagues the number of periods in the planning horizon is often much larger than the minimum number of days needed for scheduling all games (e.g. 100 days for a league with $n = 10$ teams). In this situation the games may be scheduled in different periods and teams have many periods without any game. Since the players and the venues (stadium or gymnasium) are not available each day, these unavailabilities must seriously be taken into account in the scheduling process. An example for such a scheduling problem in practice can be found in Knust [8] where a table tennis league with several constraints is considered. This problem can be modeled as a multi-mode resource-constrained scheduling problem similar to a general sports league scheduling problem in Drexl and Knust [4].

The **multi-mode resource-constrained project scheduling problem (MRCPSP)** with time-dependent resource profiles (cf. e.g. Brucker et al. [3]) may be formulated as follows. We are given N activities $i = 1, \ldots, N$, where each activity may be processed in different modes $\mathcal{M}_i$. Furthermore, renewable and non-renewable resources are given. While the availability R_k of a so-called non-renewable resource k is limited over the whole time horizon $[0, T]$, the availability of a so-called renewable resource k in the time period $[t-1, t[$ is given by a function $R_k(t)$ for $t = 1, \ldots, T$. If activity i is processed in mode $m \in \mathcal{M}_i$, it has to be processed for p_{im} time units and needs r_{imk} units of resource k. A non-renewable resource k is consumed, i.e. when i is processed in mode m, the availability R_k is decreased by r_{imk}. On the other hand, for a renewable resource k in each time period t in which activity i is processed, the availability $R_k(t)$ is decreased by r_{imk} and may be formulated as follows. Additionally, precedence relations between some activities may be given. The objective is to find a feasible mode $m_i \in \mathcal{M}_i$ for each activity i and a starting time $0 \leq S_i \leq T$ such that all resource and precedence constraints are respected and a certain objective function $F = \sum_{i=1}^{N} f_i(S_i, m_i)$ is minimized.

In this paper we lay foundations for a two-stage approach which is based on such a multi-mode resource-based model. We consider a single round robin tournament or the first half series of a double round robin tournament where the games correspond to activities with unit processing times and constraints are covered by the introduction of resources. The $\frac{n(n-1)}{2}$ games may be represented by pairs $[i, j]$ with $i, j \in \{1, \ldots, n\}$ and $i < j$. Each game $[i, j]$ may be processed in two different modes: in mode H scheduling the game at team i, or in mode A scheduling the game at team j. A possible solution approach to finding a schedule for this multi-mode RCPSP may proceed in two stages:

1. Determine a so-called **home–away assignment** (HAA) for the games, i.e. for each game $[i, j]$ a mode H or A.
2. Determine a schedule for all games in the corresponding modes, i.e. solve the resulting RCPSP with fixed modes taking into account additional constraints.

Note that in our approach the term 'home–away assignment' refers to assigning modes H and A only to games and not to schedules (where additionally for each game the round is fixed) like in Miyashiro and Matsui [10] or Post

	1	2	3	4	h_i	a_i	Δ_i
1	–	H	A	H	2	1	+1
2	A	–	H	A	1	2	−1
3	H	A	–	H	2	1	+1
4	A	H	A	–	1	2	−1

	1	2	3	4	5	h_i	a_i	Δ_i
1	–	H	A	H	A	2	2	0
2	A	–	H	A	H	2	2	0
3	H	A	–	H	A	2	2	0
4	A	H	A	–	H	2	2	0
5	H	A	H	A	–	2	2	0

Fig. 1. Balanced home–away assignments X^0 for $n = 4, 5$.

and Woeginger [11]. Moreover, contrary to their situations in our model minimizing breaks is not an important issue (breaks may be penalized in the objective function).

Usually, for fairness reasons, it is required that each team plays approximately half of its games at home and the other half away. If n is odd, this means that for each team the number of home games equals the number of away games, i.e. each team plays exactly $\frac{n-1}{2}$ games at home and $\frac{n-1}{2}$ games away. If n is even, for each team the numbers of home and away games may differ by at most one, i.e. each team must have $\lceil \frac{n-1}{2} \rceil$ games of one type and $\lfloor \frac{n-1}{2} \rfloor$ games of the other.

In the remainder of the paper we study in more detail how such balanced mode assignments can be found. Contrary to the general multi-mode RCPSP where it is already NP-hard to determine a feasible mode assignment (respecting all non-renewable resource constraints), we will show that it is easy to find balanced mode assignments. In Section 2 we deal with the question how an arbitrary unbalanced mode assignment can be changed such that it becomes balanced. In Section 3 we consider the situation where a balanced mode assignment is given and we want to change the mode of some game. If after the change the new assignment is unbalanced, we apply a repairing mechanism which repairs the assignment with a minimal number of additional mode changes. Then, in Section 4 we propose neighborhoods on the set of balanced mode assignments and show that these neighborhoods are connected, i.e. each solution can be reached from any other solution by a finite number of steps in the corresponding neighborhood. Finally, in Section 5 we study situations with preassignments where for certain games the modes are fixed in advance. We give necessary and sufficient conditions in order to decide whether a balanced HAA exists respecting the given modes.

2. Balanced home–away assignments

In this section we consider the problem of constructing balanced home–away assignments for a single round robin tournament or the first half series of a double round robin tournament. We are given n teams and have to assign to each of the $\frac{n(n-1)}{2}$ games $[i, j]$ with $i < j$ the mode H (indicating a home game for team i) or the mode A (indicating a home game for team j). Such an assignment is also called **home–away assignment** (HAA). It can be represented by an $n \times n$-matrix X with $X_{ij} = H$ if the game between teams i and j is played at team i and A otherwise. Note that this matrix is antisymmetric in the sense that $X_{ij} = H$ if and only if $X_{ji} = A$.

For each team $i = 1, \ldots, n$ we denote by h_i the number of its home games in X, by a_i the number of its away games and by $\Delta_i := h_i - a_i$ their difference. Due to the antisymmetry of X the number of home games equals the number of away games in a tournament and we have $\sum_{i=1}^{n} \Delta_i = \sum_{i=1}^{n} h_i - \sum_{i=1}^{n} a_i = 0$. Note that for even n the Δ_i-values are always odd, for odd n they are always even. A home–away assignment X is called **balanced** if for each team the numbers of home and away games differ by at most one, i.e. if $|\Delta_i| \leq 1$ for $i = 1, \ldots, n$ holds. If n is odd, this condition is equivalent to $\Delta_i = 0$ for $i = 1, \ldots, n$; if n is even, it is equivalent to $\Delta_i \in \{+1, -1\}$ for $i = 1, \ldots, n$.

It is easy to see that balanced home–away assignments exist. If we assign game $[i, j]$ with $i < j$ to mode H if $i + j$ is odd and to mode A otherwise, the corresponding HAA X^0 is obviously balanced (cf. Fig. 1 for the cases $n = 4, 5$).

Home–away assignments can also be represented by orientations of the complete graph K_n where the edge $i - j$ represents the game between teams i and j. If in a HAA the game $i - j$ is played at team i, we orient the edge into (j, i), if the game is played at team j, we orient the edge into (i, j). A balanced HAA corresponds to an orientation of K_n where for each vertex the number of incoming edges differs from the number of outgoing edges by at most one. For example, in Fig. 2 for $n = 5$ the balanced HAA

$$X^0 = \{(2, 1), (1, 3), (4, 1), (1, 5), (3, 2), (2, 4), (5, 2), (4, 3), (3, 5), (5, 4)\}$$

is depicted.

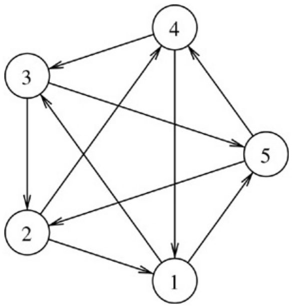

Fig. 2. Balanced home–away assignment X^0 for $n = 5$.

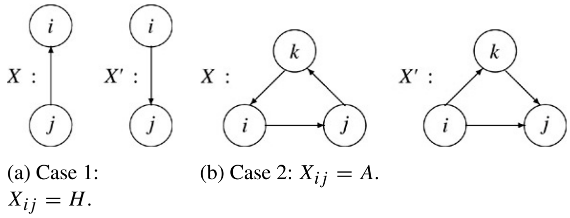

Fig. 3. Constructing X' from X.

In the following we show how an arbitrary unbalanced home–away assignment can be changed such that it becomes balanced. Recall that a HAA is balanced if and only if for $i = 1, \dots, n$ the condition $|\Delta_i| = 0$ holds in the case that n is odd and $|\Delta_i| = 1$ holds in the case that n is even. For an arbitrary HAA X let $\delta := \sum_{\{i:|\Delta_i|>0\}} |\Delta_i|$ if n is odd and $\delta := \sum_{\{i:|\Delta_i|>1\}} |\Delta_i|$ if n is even. The value $\delta \geq 0$ measures the 'imbalance' of X. Obviously, X is balanced if and only if $\delta = 0$ holds. On the other hand, the largest value δ for an unbalanced HAA is $2\sum_{i=1}^{(n-1)/2}(2i) = \frac{1}{2}(n^2 - 1)$ if n is odd and $2\sum_{i=1}^{n/2-1}(2i + 1) = \frac{1}{2}(n^2 - 4)$ if n is even.

Theorem 1. *Each unbalanced HAA X with $\delta > 0$ can be transformed into a balanced HAA X' by changing the modes of at most $\lfloor \frac{\delta}{2} \rfloor$ games.*

Proof. At first we consider the case that n is odd. For each team i let $H_i := \{j \neq i \mid X_{ij} = H\}$ be the set of teams against which i plays at home in X and $A_i := \{j \neq i \mid X_{ij} = A\}$ be the set of teams against which i plays away. If X is not balanced, due to $\sum_{i=1}^{n} \Delta_i = 0$ we have at least one team i with $\Delta_i > 0$ (i.e. $\Delta_i \geq 2$ since all Δ-values are even) and at least one team j with $\Delta_j < 0$ (i.e. $\Delta_j \leq -2$). These two teams i, j satisfy $h_i > \frac{n-1}{2}$, $a_i < \frac{n-1}{2}$, and $a_j > \frac{n-1}{2}$, $h_j < \frac{n-1}{2}$ where $a_i = |A_i|$ and $h_i = |H_i|$.

- Case 1: $X_{ij} = H$, i.e. the game between teams i and j is played at team i. If we change the mode of the game $i - j$ into $X'_{ij} = A$, the values h_i, a_j are decreased by 1 and the values a_i, h_j are increased by 1 (cf. Fig. 3(a)). Thus, for the resulting HAA X' we get $\Delta'_i = h'_i - a'_i = \Delta_i - 2 \geq 0$ and $\Delta'_j = \Delta_j + 2 \leq 0$ implying $\delta' = \delta - 4 \geq 0$.
- Case 2: $X_{ij} = A$, i.e. the game between teams i and j is played at team j. In this situation changing the mode of the game $i - j$ does not reduce the imbalance δ. But, in the following we will show that another team k exists such that changing the modes of the two games $i - k$ and $j - k$ reduces δ.
  Due to $i, j \notin H_i \cup A_j$ we have $|H_i \cup A_j| \leq n - 2$. This implies $H_i \cap A_j \neq \emptyset$ since
  $$|H_i \cap A_j| = h_i + a_j - |H_i \cup A_j| > \frac{n-1}{2} + \frac{n-1}{2} - (n-2) = 1,$$
  i.e. a team $k \in H_i \cap A_j$ exists. If we change the modes of the two games $i - k$ and $j - k$ into $X'_{ik} = A$ and $X'_{jk} = H$ (cf. Fig. 3(b)), for the resulting HAA X' we get $\Delta'_i = \Delta_i - 2 \geq 0$, $\Delta'_j = \Delta_j + 2 \leq 0$ and $\Delta'_k = \Delta_k$ implying $\delta' = \delta - 4 \geq 0$.

	1	2	3	4	5	6	7	8	9	Δ_i
1	–	H	H	H	H	H	H	A	A	+4
2	A	–	H	A	H	A	H	A	H	0
3	A	A	–	H	H	H	A	A	H	0
4	A	H	A	–	A	A	H	H	H	0
5	A	A	A	H	–	H	A	H	H	0
6	A	H	A	H	A	–	H	H	A	0
7	A	A	H	A	H	A	–	H	H	0
8	H	H	H	A	A	A	A	–	A	−2
9	H	A	A	A	A	H	A	H	–	−2

(a) Unbalanced HAA for $n = 9$.

	1	2	3	4	5	6	7	8	9	Δ_i
1	–	$\underline{A}$	H	$\underline{A}$	H	H	H	A	A	0
2	$\underline{H}$	–	H	A	H	A	H	A	$\underline{A}$	0
3	A	A	–	H	H	H	A	A	H	0
4	$\underline{H}$	H	A	–	A	A	H	$\underline{A}$	H	0
5	A	A	A	H	–	H	A	H	H	0
6	A	H	A	H	A	–	H	H	A	0
7	A	A	H	A	H	A	–	H	H	0
8	H	H	H	$\underline{H}$	A	A	A	–	A	0
9	H	$\underline{H}$	A	A	A	H	A	H	–	0

(b) Balanced HAA for $n = 9$.

	1	2	3	4	5	6	7	8	Δ_i
1	–	A	A	A	A	A	H	H	−3
2	H	–	A	H	A	H	A	A	−1
3	H	H	–	A	H	A	A	A	−1
4	H	A	H	–	A	H	A	A	−1
5	H	H	A	H	–	A	A	A	−1
6	H	A	H	A	H	–	A	A	−1
7	A	H	H	H	H	H	–	A	+3
8	A	H	H	H	H	H	H	–	+5

(c) Unbalanced HAA for $n = 8$.

	1	2	3	4	5	6	7	8	Δ_i
	–	$\underline{H}$	A	A	A	A	H	H	−1
2	$\underline{A}$	–	A	H	A	H	$\underline{H}$	A	−1
3	H	H	–	A	H	A	A	$\underline{H}$	+1
4	H	A	H	–	A	H	A	$\underline{H}$	+1
5	H	H	A	H	–	A	A	A	−1
6	H	A	H	A	H	–	A	A	−1
7	A	$\underline{A}$	H	H	H	H	–	A	+1
8	A	H	$\underline{A}$	$\underline{A}$	H	H	H	–	+1

(d) Balanced HAA for $n = 8$.

Fig. 4. Transforming unbalanced into balanced home–away assignments.

In both cases the measure of imbalance is reduced by 4. By iterating these changes we obtain a balanced HAA X' with $\delta' = 0$ after $\frac{\delta}{4}$ iterations. Since in each iteration the modes of at most two games are changed, in total at most $\frac{\delta}{2}$ mode changes are performed.

Now we consider the case that n is even. Let $\vartheta := |\{l \mid \Delta_l \geq 1\}| - |\{l \mid \Delta_l \leq -1\}|$ be the difference between the numbers of teams with positive and negative Δ-values. W.l.o.g. assume $\vartheta \leq 0$, i.e. $|\{l \mid \Delta_l \geq 1\}| \leq \frac{n}{2}$ and $|\{l \mid \Delta_l \leq -1\}| \geq \frac{n}{2}$ (the other case can be treated symmetrically). Let i be a team with $\Delta_i > 1$. Such a team exists due to $\sum_{\{l \mid \Delta_l > 0\}} \Delta_l = \sum_{\{l \mid \Delta_l < 0\}} |\Delta_l| > \frac{n}{2}$ (since X is unbalanced) and $|\{l \mid \Delta_l \geq 1\}| \leq \frac{n}{2}$.

Due to $h_i > \lceil \frac{n-1}{2} \rceil$ at least $\frac{n}{2}$ teams $k \neq i$ exist which play at team i. Among these teams k at least one team j must satisfy $\Delta_j \leq -1$ since $|\{l \neq i \mid \Delta_l \geq 1\}| < \frac{n}{2}$. If we change the mode of the game $i - j$ into $X'_{ij} = A$, for the resulting HAA X' we get $\Delta'_i = \Delta_i - 2 \geq 1$ and $\Delta'_j = \Delta_j + 2 \leq 1$ implying $\delta' \leq \delta - 2$.

By iterating these changes we obtain a balanced HAA X' with $\delta' = 0$ after at most $\lfloor \frac{\delta}{2} \rfloor$ iterations because the measure of imbalance is reduced by at least 2 in each iteration. Since in each iteration the mode of one game is changed, in total at most $\lfloor \frac{\delta}{2} \rfloor$ mode changes are performed. □

Note that in our transformation procedure the number of balanced teams never decreases since teams l which are already balanced (i.e. satisfy $|\Delta_l| \leq 1$) never become unbalanced in an iteration. In the case that n is odd, we have $\Delta'_l = \Delta_l = 0$ for all these teams l, in the case that n is even, the Δ_l-values may change between $+1$ and -1.

Example 1. For an odd number of teams consider the HAA X with $\delta = 4 + 2 + 2 = 8$ for $n = 9$ shown in Fig. 4(a). We have one team $i = 1$ with $\Delta_i > 0$ and two teams $j = 8, 9$ with $\Delta_j < 0$. Due to $X_{18} = X_{19} = A$ we may not change the modes of the games $1 - 8$ and $1 - 9$ directly (i.e. we have Case 2). Since $H_1 \cap A_8 = \{4, 5, 6, 7\}$ we may choose the team $k = 4$ and change the modes of the games $1 - 4$ and $4 - 8$. Afterwards, due to $H_1 \cap A_9 = \{2, 3, 5, 7\}$ we may choose $k = 2$ and change the modes of the games $1 - 2$ and $2 - 9$ leading to the balanced HAA shown in Fig. 4(b).

For an even number of teams consider the HAA X with $\delta = 3 + 3 + 5 = 11$ for $n = 8$ shown in Fig. 4(c). We have $\vartheta = 2 - 6 = -4 < 0$ and $i = 7$ is a team with $\Delta_i > 0$. In the set $H_7 = \{2, 3, 4, 5, 6\}$ all teams $j \in H_7$ satisfy

	1	2	3	4	h_i	a_i	Δ_i
1	–	$\underline{A}$	A	H	1	2	−1
2	$\underline{H}$	–	H	A	2	1	+1
3	H	A	–	H	2	1	+1
4	A	H	A	–	1	2	−1

(a) Changing the mode of $1-2$.

	1	2	3	4	h_i	a_i	Δ_i
1	–	H	$\underline{H}$	H	3	0	+3
2	A	–	H	A	1	2	−1
3	$\underline{A}$	A	–	H	1	2	−1
4	A	H	A	–	1	2	−1

(b) Changing the mode of $1-3$.

Fig. 5. Balanced and unbalanced home–away assignment after changing one mode.

$\Delta_j \le -1$, i.e. we may choose an arbitrary team $j \in H_7$ and change the mode of the game $i - j$ into $X'_{ij} = A$. Let us choose $j = 2$. Then we get $\vartheta = 3 - 5 = -2 < 0$ and $i = 8$ is a team with $\Delta_i > 0$. In the set $H_8 = \{2, 3, 4, 5, 6, 7\}$ the teams $j = 3, 4, 5, 6$ satisfy $\Delta_j \le -1$. Thus, in the next two iterations we may swap the modes of the games $3-8$ and $4-8$. Afterwards, we have $\vartheta = 5 - 3 = 2 > 0$ and $i = 1$ is a team with $\Delta_i < 0$. In the set $A_1 = \{2, 3, 4, 5, 6\}$ the teams $j = 2, 3, 4$ satisfy $\Delta_j \ge 1$. When we change the mode of the game $1-2$, we get the balanced HAA shown in Fig. 4(d).

Note that in the first iteration we could not start with a team i with $\Delta_i < 0$ since all teams $j \in A_1 = \{2, 3, 4, 5, 6\}$ also satisfy $\Delta_j < 0$. This observation shows that for even n the ϑ-value is important and has to be taken into account for choosing a team to start with. □

3. Changing balanced home–away assignments

In this section we consider the situation where a balanced HAA X is given and the mode of a single game is changed. Afterwards, the resulting HAA may be unbalanced. Then we want to obtain again a balanced HAA X' by performing a minimal number of additional changes. If we change the mode of one game in a balanced HAA for an odd number of teams, obviously the resulting HAA X' becomes unbalanced (since for the two teams $\lambda = i, j$ their numbers a_λ, h_λ change from $\frac{n-1}{2}$ to $\frac{n-1}{2} + 1$ and $\frac{n-1}{2} - 1$, respectively, i.e. for X' we get $|\Delta'_i| = |\Delta'_j| = 2 \ne 0$).

On the other hand, for an even number of teams, the resulting HAA X' may be balanced or not. For example, if we change in the HAA X^0 for $n = 4$ (cf. Fig. 1) the mode of game $1-2$ from H to A, we get the balanced HAA shown in Fig. 5(a). On the other hand, if we change the mode of game $1-3$ from A to H, we get the HAA shown in Fig. 5(b), which is not balanced for team 1.

In the following we will show how a HAA which becomes unbalanced after a mode change can be repaired with a minimal number of additional mode changes.

Theorem 2. *Consider a balanced HAA X in which the mode of one game $i - j$ is changed. Then a balanced HAA X' can be obtained from X by changing at most two additional modes.*

Proof. At first we consider the case that n is odd. Since X is balanced, we have $\Delta_i = \Delta_j = 0$, i.e. $h_i = a_i = h_j = a_j = \frac{n-1}{2}$. W.l.o.g. we assume $X_{ij} = H$ (the other case can be treated symmetrically). As mentioned above, after changing the mode of the game $i - j$ into $X'_{ij} = A$, teams i and j satisfy $|\Delta'_i| = |\Delta'_j| = 2 \ne 0$, i.e. both teams are no longer balanced. In the following we will show that another team k exists such that by changing the modes of the two games $i - k$ and $j - k$, team k remains balanced and teams i, j become balanced.

Due to $i, j \notin A_i \cup H_j$ we have $|A_i \cup H_j| \le n - 2$. This implies $A_i \cap H_j \ne \emptyset$ since

$$|A_i \cap H_j| = a_i + h_j - |A_i \cup H_j| \ge \frac{n-1}{2} + \frac{n-1}{2} - (n-2) = 1,$$

i.e. a team $k \in A_i \cap H_j$ exists. Thus, if we additionally change the modes of the two games $i - k$ and $j - k$ into $X'_{ik} = H$ and $X'_{jk} = A$, we get $\Delta'_i = \Delta'_j = \Delta'_k = 0$ for the resulting HAA X', i.e. X' is balanced.

Now we consider the case that n is even. Again w.l.o.g. assume $\mathrm{X}_{ij} = H$.

- Case 1: $\Delta_i = +1$, $\Delta_j = -1$. In this case we may simply change the mode of the game $i - j$ into $X'_{ij} = A$ and obtain $\Delta'_i = -1$, $\Delta'_j = +1$, i.e. X' is again balanced.

	1	2	3	4	Δ_i
1	–	H	A	H	+1
2	A	–	$\underline{A}$	A	−3
3	H	$\underline{H}$	–	H	+3
4	A	H	A	–	−1

(a) Changing the mode of $2-3$.

	1	2	3	4	Δ_i
1	–	$\underline{A}$	$\underline{H}$	H	+1
2	$\underline{H}$	–	A	A	−1
3	$\underline{A}$	H	–	H	+1
4	A	H	A	–	−1

(b) Changing $1-2$, $1-3$.

Fig. 6. Repairing a home–away assignment with two additional mode changes.

- Case 2: $\Delta_i = -1\,, \Delta_j = +1$. In this case by changing the mode of game $i - j$ into $X'_{ij} = A$ we get $\Delta'_i = -3\,, \Delta'_j = +3$, i.e. both teams i and j get unbalanced. But, as in the case where n is odd, we find a team $k \in A_i \cap H_j$ since
$$|A_i \cap H_j| = a_i + h_j - |A_i \cup H_j| = \left\lceil \frac{n-1}{2} \right\rceil + \left\lceil \frac{n-1}{2} \right\rceil - (n-2) = 2.$$
By additionally changing the modes of the two games $i - k$ and $j - k$, we obtain $\Delta'_i = \Delta_i\,, \Delta'_j = \Delta_j, \Delta'_k = \Delta_k$, i.e. X' is balanced.
- Case 3: $\Delta_i = \Delta_j = -1$. In this case by changing the mode of game $i-j$ into $X'_{ij} = A$ we get $\Delta'_i = -3\,, \Delta'_j = +1$, i.e. only team i gets unbalanced. In this case we can find a team $k \in A_i$ with $\Delta_k = +1$ such that changing the mode of game $i - k$ into $X'_{ik} = H$ leads to a balanced HAA X' with $\Delta'_i = -1\,, \Delta'_j = +1$, $\Delta'_k = -1$. Such a team k exists since due to $a_i = \lceil \frac{n-1}{2} \rceil = \frac{n}{2}$ we have $\frac{n}{2}$ teams $k \neq i, j$ with $X_{ik} = A$. Among these teams k at least one team must satisfy $\Delta_k = +1$ since $\Delta_i = \Delta_j = -1$ and $|\{l \neq i, j \mid \Delta_l = -1\}| = \frac{n}{2} - 2 < \frac{n}{2}$.
- Case 4: $\Delta_i = \Delta_j = +1$. This case can be treated symmetrically to Case 3.

It is easy to see that if both teams i, j become unbalanced after changing the mode of the game $i - j$, at least two additional modes have to be changed in order to get a balanced HAA. If only one team becomes unbalanced, at least one additional mode change is necessary. Thus, in all cases above the number of additional mode changes is minimal. □

Example 2. We consider the case that n is even. An example for Case 1 was already considered in Fig. 5(a), an example for Case 4 in Fig. 5(b). The HAA in (b) can be repaired by one additional mode change (e.g. by setting $X'_{12} = A$ which leads to $\Delta'_1 = \Delta'_2 = +1$, $\Delta'_3 = \Delta'_4 = -1$). An example for Case 2 can be obtained if in the HAA X^0 for $n = 4$ the mode of the game $2 - 3$ is changed from H to A. Then we get the unbalanced HAA shown in Fig. 6(a) which is unbalanced for teams 2 and 3. In this situation, changing the modes of the two games $1 - 2$ and $1 - 3$ leads to a balanced HAA (cf. Fig. 6(b)).

4. Connected neighborhoods

In this section we discuss neighborhoods on the set of balanced HAAs. Furthermore, we will show that these neighborhood structures are connected, i.e. it is possible to reach any balanced HAA from any other by a finite number of steps in the corresponding neighborhood. We will again distinguish the two cases of an even and an odd number n of teams.

Let X be a balanced HAA for an even number n of teams, represented by the underlying orientation $G = (V, E_G)$ of the complete graph K_n. Then the neighborhood $\mathcal{N}_1(X)$ is defined as the set of all balanced HAAs X' which can be obtained from X by changing the mode of a single game (i.e. by changing the orientation of one edge in the corresponding graph). If we consider the corresponding neighborhood graph (i.e. each vertex represents a balanced HAA and two vertices are linked by an edge if the corresponding HAAs can be transformed into each other by changing the mode of a single game), we can show that this graph is connected:

Theorem 3. *The neighborhood structure $\mathcal{N}_1$ is connected, i.e. it is possible to transform any balanced HAA into any other by a finite number of steps in $\mathcal{N}_1$.*

Proof. We consider two arbitrary balanced HAAs X and X', represented by the corresponding orientations $G = (V, E_G)$ and $G' = (V, E_{G'})$ of the complete graph K_n. Let $E_{G\setminus G'} := E_G \setminus E_{G'}$ be the set of all directed edges in G which do not belong to G' (i.e. which have opposite directions in G and G'). Furthermore, let $A_i^{G\setminus G'} := \{j \mid (i, j) \in E_{G\setminus G'}\}$ be the set of teams j for which i plays away in G and i plays at home in G'. Symmetrically, let $H_i^{G\setminus G'} := \{j \mid (j, i) \in E_{G\setminus G'}\}$.

We have to show that we can transform X into X' by a finite number of steps in the neighborhood $\mathcal{N}_1$. We will prove this by induction on $e := |E_{G\setminus G'}|$. If $e = 0$, then all edges in G also belong to G' and vice versa. Thus, we have $X = X'$, i.e. the claim is true in this case. So let us now suppose $e > 0$ and consider the following two cases:

- Case 1: There exists an edge $(i, j) \in E_{G\setminus G'}$ with $\Delta_i^G = -1$ and $\Delta_j^G = +1$ (where Δ_i^G is defined as the Δ_i-value with respect to G). By changing the orientation of the edge (i, j) into (j, i) we obtain a new balanced HAA X with $E_X = (E_G \setminus \{(i, j)\}) \cup \{(j, i)\}$, $\Delta_i^X = +1$, $\Delta_j^X = -1$ and $|E_{X\setminus G'}| = |E_{G\setminus G'}| - 1$. Therefore, by induction the claim is true.
- Case 2: There exists no edge $(i, j) \in E_{G\setminus G'}$ with $\Delta_i^G = -1$ and $\Delta_j^G = +1$. Since $e > 0$, at least one edge $(i, j) \in E_G$ exists which does not belong to $E_{G'}$. W.l.o.g. let $\Delta_i^G = +1$ (the case $\Delta_i^G = -1$ can be treated symmetrically). According to the identity

$$\Delta_i^{G'} = \Delta_i^G + 2(|A_i^{G\setminus G'}| - |H_i^{G\setminus G'}|)$$

and because $\Delta_i^{G'} \in \{+1, -1\}$ and $|A_i^{G\setminus G'}| \geq 1$, we may conclude that a $k \in H_i^{G\setminus G'}$ exists, i.e. $(k, i) \in E_{G\setminus G'}$. Because we are in Case 2, it follows that $\Delta_k^G = +1$.

By changing the roles of i and k, we are in the same situation as before and we may conclude that an edge $(k_1, k) \in E_{G\setminus G'}$ exists. Because we are in Case 2, again it follows that $\Delta_{k_1}^G = +1$. Since $\Delta_{k_1}^G = +1$ and $|\{l \mid \Delta_l^G = -1\}| = \frac{n}{2}$, also an edge $(k_2, k_1) \in E_G$ exists with $\Delta_{k_2}^G = -1$ and $(k_2, k_1) \in E_{G'}$ (since we are in Case 2, it can be excluded that $(k_2, k_1) \notin E_{G'}$).

Consider the following three HAAs which are successively obtained from X by a step in $\mathcal{N}_1$, i.e. by changing the mode of a single game: $E_{X_1} := (E_G \setminus \{(k_2, k_1)\}) \cup \{(k_1, k_2)\}$, $E_{X_2} := (E_{X_1} \setminus \{(k_1, k)\}) \cup \{(k, k_1)\}$ and $E_{X_3} := (E_{X_2} \setminus \{(k, i)\}) \cup \{(i, k)\}$.

Due to the Δ-values X_1, X_2, X_3 are balanced and

$$E_{X_3} := (E_G \setminus \{(k_2, k_1), (k_1, k), (k, i)\}) \cup \{(k_1, k_2), (k, k_1), (i, k)\}.$$

It follows that $|E_{X_3\setminus G'}| = |E_{G\setminus G'}| - 2 + 1$. Therefore, by induction the claim is true. □

It is easy to see that the neighborhood $\mathcal{N}_1$ is not suitable in the case of an odd number n (since changing the mode of a single game in a balanced HAA always results in an unbalanced HAA, cf. Section 3). Based on the techniques in the proof of Theorem 2 we will define a neighborhood where the modes of three games are changed. More specifically, for an odd number n the neighborhood $\mathcal{N}_2(X)$ of a balanced HAA X is defined as the set of all HAAs X' which can be obtained from X by changing the modes of games belonging to a directed cycle of length three (i.e. by substituting a directed cycle $(i, j), (j, k), (k, i)$ by the cycle $(j, i), (i, k), (k, j)$ in the corresponding orientation $G = (V, E_G)$). Note that such a move always transforms a balanced HAA into another balanced HAA because changing the orientation of all edges on a directed cycle does not change the Δ-values.

Before we can prove that this neighborhood is connected, we prove the following result:

Lemma 1. *Let C be a directed cycle of arbitrary length in an orientation $G = (V, E_G)$ of the complete graph K_n. Then it is possible to reverse the orientation of all edges on C by successively reversing the orientation of edges in directed cycles with length three.*

Proof. Let C be a directed cycle of arbitrary length l in G. We will prove the claim by induction on l. If C is a cycle of length three, the claim is obviously true.

So let $l > 3$. Further on let $(i, j), (j, k) \in C \subset E_G$ be two edges on the cycle C. Because G is an orientation of the complete graph K_n, we have either $(k, i) \in E_G$ or $(i, k) \in E_G$.

- Case 1: $(k, i) \in E_G$

  The set of edges $D := \{(i, j), (j, k), (k, i)\} \subset E_G$ constitutes a directed cycle of length three which can be substituted by the cycle $D' = \{(j, i), (k, j), (i, k)\}$ in one step. This leads to the new orientation G' with

$E_{G'} = (E_G \setminus D) \cup D'$. The graph G' contains the directed cycle $F := (C \setminus D) \cup \{(i,k)\}$ of length $l-1$. By induction the orientation of all edges in F can be reversed by successively reversing the orientation of edges in directed cycles with length three. The resulting graph equals the graph which is achieved by reversing all edges in the cycle C. Thus, the claim is true in this case.

- Case 2: $(i,k) \in E_G$
  The set $D := (C \setminus \{(i,j),(j,k)\}) \cup \{(i,k)\}$ constitutes a directed cycle of length $l-1$. By induction, this cycle can be reversed as described above. The resulting orientation G' with $E_{G'} = (E_G \setminus D) \cup \{(p,q) \mid (q,p) \in D\}$ contains the cycle $F := \{(i,j),(j,k),(k,i)\}$. Obviously, one step substitutes this cycle and transforms the graph G' into the desired graph G'' with $E_{G''} := (E_G \setminus C) \cup \{(p,q) \mid (q,p) \in C\}$. Thus, the claim is also true in this case. □

Theorem 4. *The neighborhood structure $\mathcal{N}_2$ is connected.*

Proof. As in the case where n is even we will prove the claim by induction on $e := |E_{G\setminus G'}|$. If $e = 0$, we are done. So let $e > 0$. Then by definition, an edge $(i,j) \in E_{G\setminus G'}$ exists. Because $\Delta_k^G = 0$ for all k and

$$\Delta_k^{G'} = \Delta_k^G + 2(|A_k^{G\setminus G'}| - |H_k^{G\setminus G'}|)$$

for all k, it follows that

$$|A_k^{G\setminus G'}| = |H_k^{G\setminus G'}|$$

for all k. Therefore, a directed cycle $C \subset E_{G\setminus G'}$ of length $l > 0$ containing the edge (i,j) exists. Due to Lemma 1 performing a certain number of steps in the neighborhood $\mathcal{N}_2$ leads to a new graph G'' such that $E_{G''} = (E_G \setminus C) \cup \{(p,q) \mid (q,p) \in C\}$. For G'' we get

$$|E_{G''\setminus G'}| = |E_{G\setminus G'}| - l.$$

Since by induction we can transform G'' into G' by a finite number of steps in $\mathcal{N}_2$, the claim is true. □

Note that the neighborhood $\mathcal{N}_1$ is only defined by mode changes of a single game which lead to balanced home–away assignments (if a mode change leads to an unbalanced HAA, this change is not applied). According to the results of Section 3, we may enlarge the neighborhood as follows: $\mathcal{N}_1'(X)$ contains all home–away assignments X' which can be obtained from X by changing the mode of a single game and applying the repairing mechanism of Theorem 2 (if necessary) afterwards. Since $\mathcal{N}_1$ is contained in $\mathcal{N}_1'$, the neighborhood $\mathcal{N}_1'$ is also connected.

5. Balanced home–away assignments with preassignments

In this section we study situations with preassignments where for certain games the modes are fixed in advance. We are interested in deciding whether a balanced HAA respecting the given modes exists. We assume that we are given a set F of fixed games (i,j) which have to be a home game for team i and ask for a balanced HAA X satisfying $X_{ij} = H$ for all $(i,j) \in F$. Due to the antisymmetry $X_{ij} = H \Leftrightarrow X_{ji} = A$ a necessary condition for the existence of such a HAA is that $(j,i) \notin F$ when $(i,j) \in F$ (a game cannot be a home and an away game simultaneously). Thus, w.l.o.g. we may assume that the set F fulfills this condition.

If we ignore the antisymmetry condition, related problems have been studied in the context of binary matrices. In this situation a $m \times n$-matrix $A = (a_{ij})$ with $a_{ij} \in \{0,1\}$ has to be determined where the row sums $R_i = \sum_{j=1}^n a_{ij}$ $(i = 1,\ldots,m)$ and the column sums $S_j = \sum_{i=1}^n a_{ij}$ $(j = 1,\ldots,n)$ are given. In 1957 Gale [7] and Ryser [12] independently obtained necessary and sufficient conditions for the existence of a $m \times n$-matrix A with given row and column sums (see also [2] for an excellent overview about the whole topic of binary matrices with given row and column sums).

A more complicated situation arises if additionally some entries a_{ij} of the matrix are fixed in advance and we search for a matrix with given row and column sums which is compatible with the fixed entries. W.l.o.g. we may assume that all fixed entries are zero. Otherwise, a fixed entry $a_{ij} = 1$ may be converted to the entry $a_{ij} = 0$ by setting $R_i := R_i - 1$ and $S_j := S_j - 1$. While also explicit necessary and sufficient conditions for the existence of a corresponding binary matrix exist (cf. Mirsky [9]), in general these conditions cannot be checked in polynomial time. But, although no efficient algorithm checking only explicit formulas is known, this problem can efficiently be solved by network flow techniques, providing also a solution (if it exists). Associated with such an instance is a capacitated network $G = (V, A)$ with

- vertices $V = \{s_1, \ldots, s_m\} \cup \{t_1, \ldots, t_n\} \cup \{s, t\}$, and
- arcs $A = \{(s, s_i) \mid i = 1, \ldots, m\} \cup \{(t_j, t) \mid j = 1, \ldots, n\} \cup \{(s_i, t_j) \mid i = 1, \ldots, m; j = 1, \ldots, n\}$.

The arcs (s, s_i) get the required row sums R_i as capacities, and the arcs (t_j, t) get the required column sums S_j as capacities. Furthermore, the arc (s_i, t_j) corresponds to the entry a_{ij} of the matrix. Thus, for all fixed entries a_{ij} the arc (s_i, t_j) gets the capacity 0, for all other (i, j) the capacity of the arc (s_i, t_j) is 1.

It is easy to see that a binary matrix satisfying the required constraints exists if and only if in G the maximal flow value is $\sum_{i=1}^{m} R_i = \sum_{j=1}^{n} S_j$. Furthermore, due to the integrality of an optimal flow, from such a flow x the entries a_{ij} may be derived by setting $a_{ij} := x_{s_i,t_j} \in \{0, 1\}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Efficient algorithms for solving maximum flow problems can be found in Ahuja et al. [1].

In the following we will show that in our situation (i.e. due to the balancing constraints) requiring antisymmetry does not make the problem harder. It is sufficient to consider the case that n is odd. If n is even, we may add a dummy team $n + 1$ to the problem which is not involved in any fixed game in F. It is easy to see that each balanced HAA for these $n+1$ teams can be converted into a balanced HAA for the original n teams (by eliminating team $n+1$) and vice versa. Thus, a balanced HAA respecting F for $n + 1$ teams exists if and only if a balanced HAA respecting F for n teams exists.

For odd n the balancing conditions for a balanced HAA X read $\sum_{i=1}^{n} h_i = \sum_{i=1}^{n} a_i = \frac{n-1}{2}$. If we indicate a home game by '1' and an away game by '0', this condition is equivalent to fixed row and column sums $R_i = S_i := \frac{n-1}{2}$ for $i = 1, \ldots, n$. Furthermore, the fixed games may be modeled by requiring $a_{ij} = 1$ and $a_{ji} = 0$ for all $(i, j) \in F$. Additionally, we set $a_{ii} := 0$ for $i = 1, \ldots, n$ and define $N := \{1, \ldots, n\}$.

In the following we will show that for balanced matrices the existence problems with and without antisymmetry are equivalent:

Theorem 5. *For odd n let $F \subset N \times N$ be a set such that $(i, j) \in F$ implies $(j, i) \notin F$. A binary matrix A with row and column sums $R_i = S_i = \frac{n-1}{2}$; $a_{ii} = 0$ for $i = 1, \ldots, n$ and $a_{ij} = 1$, $a_{ji} = 0$ for all $(i, j) \in F$ exists if and only if a balanced HAA X with $\mathrm{X}_{ij} = H$ for all $(i, j) \in F$ exists.*

Proof. The implication '$\Leftarrow$' just follows from the fact that each balanced HAA-matrix by definition is a special binary matrix with entries 'H' and 'A' instead of '0' and '1'.

We will now prove the implication '$\Rightarrow$'. Let $F \subset N \times N$ be a set such that $(i, j) \in F$ implies $(j, i) \notin F$. Furthermore, let A be a binary matrix with $a_{ij} = 1$ and $a_{ji} = 0$ for all $(i, j) \in F$; row and column sums $R_i = S_i = \frac{n-1}{2}$ and $a_{ii} = 0$ for $i = 1, \ldots, n$. If this matrix A fulfills the antisymmetry condition $a_{ij} \neq a_{ji}$ for all $i \neq j$, we are done. Otherwise, we will convert A into an equivalent matrix which is antisymmetric and hence induces a balanced HAA X with $\mathrm{X}_{ij} = H$ for all $(i, j) \in F$.

Let

$$\gamma(A) := |\{(i, j) \in N \times N, i \neq j \mid a_{ij} = a_{ji}\}|$$

be the number of entries in A which have to be repaired. If $\gamma(A) = 0$, then the regarded matrix A fulfills all criteria for being a balanced HAA-matrix, because then we have $a_{ij} \neq a_{ji}$ for all $i \neq j$. So let $\gamma(A) > 0$, i.e.

$$\Gamma(A) := \{(i, j) \in N \times N, i \neq j \mid a_{ij} = a_{ji}\} \neq \emptyset.$$

In the following we will identify a sequence of entries in $\Gamma(A)$ which can be flipped without changing the balancing constraints and the fixed games and which reduces the value $\gamma(A)$. In this process of converting the matrix A we will use the following property:

For each pair $(i, j) \in \Gamma(A)$ there exist pairs $(i, k), (l, j) \in \Gamma(A)$ with $k, l \in N \setminus \{i, j\}$ and $a_{ik} = a_{ki} = a_{lj} = a_{jl} = 1 - a_{ij}$.

In order to prove this property, assume to the contrary that no index $k \in N \setminus \{i, j\}$ with $a_{ik} = a_{ki} = 1 - a_{ij}$ exists. W.l.o.g. assume $a_{ij} = a_{ji} = 0$ (the case $a_{ij} = a_{ji} = 1$ can be treated symmetrically). Then for all $\lambda \in N \setminus \{i, j\}$ we have $a_{i\lambda} = 0$ or $a_{\lambda i} = 0$. Especially, $a_{i\lambda} = 1$ for some index $\lambda \in N \setminus \{i\}$ implies $a_{\lambda i} = 0$. Thus, the set $Y := \{\lambda \in N \setminus \{i\} \mid a_{i\lambda} = 1\}$ is a subset of the set $Z := \{\lambda \in N \setminus \{i\} \mid a_{\lambda i} = 0\}$. Due to $j \in Z$ and $j \notin Y$ we must have $|Z| > |Y|$. But this contradicts the assumption that A is balanced, since then $|Y| = |Z| = \frac{n-1}{2}$ holds.

By exchanging the roles of rows and sums we can also prove that an index $l \in N \setminus \{i, j\}$ with $a_{lj} = a_{jl} = 1 - a_{ij}$ exists.

Now we choose an arbitrary pair $(k_0, l_0) \in \Gamma(A)$. Due to the preceding property there exists a pair $(k_0, l_1) \in \Gamma(A)$ with $l_1 \neq l_0$ and $a_{k_0,l_0} \neq a_{k_0,l_1}$. For this pair also a pair $(k_1, l_1) \in \Gamma(A)$ exists with $k_1 \neq k_0$ and $a_{k_0,l_1} \neq a_{k_1,l_1}$. Once again it follows that we have a pair $(k_1, l_2) \in \Gamma(A)$ with $l_2 \neq l_1$ and $a_{k_1,l_2} \neq a_{k_1,l_1}$ and so on. Proceeding this way we get a sequence

$$\sigma : (k_0, l_0), (k_0, l_1), (k_1, l_1), (k_1, l_2), (k_2, l_2), (k_2, l_3), \ldots$$

in which alternately either the first entry of a pair is identical with the first entry of the following pair or the second entry equals the second entry of the following pair. Due to the fact that we regard a finite set of teams, there exist indices $e \neq f$ where for the first time we either have

(1) $k_e = k_f$ for an $e \in \{0, \ldots, f - 1\}$ or
(2) $l_e = l_f$ for an $e \in \{0, \ldots, f - 1\}$.

W.l.o.g. suppose that we have case (1), case (2) can be treated similarly. Then we regard the subsequence

$$\sigma' : (k_e, l_{e+1}), (k_{e+1}, l_{e+1}), \ldots, (k_{f-1}, l_f), (k_f, l_f)$$

where $k_e = k_f$. Furthermore, we have $l_g \neq l_h$ for all $g \neq h \in \{e + 1, \ldots, f\}$ and $k_g \neq k_h$ for all $g \neq h \in \{e + 1, \ldots, f\}$. We now claim that in σ' a pair $(i, j) \in \sigma'$ exists such that $(j, i) \notin \sigma'$ holds.

Otherwise, suppose that for all $(i, j) \in \sigma'$ also $(j, i) \in \sigma'$ holds. Then we have $(l_{e+1}, k_e), (l_{e+1}, k_{e+1}) \in \sigma'$ and therefore there exists an index $d < f$ such that $(k_d, l_d) = (l_{e+1}, k_e)$ and $(k_d, l_{d+1}) = (l_{e+1}, k_{e+1})$. By construction of the sequence σ we have $a_{k_g,l_g} \neq a_{k_h,l_{h+1}}$ for all indices g, h. This implies $a_{k_e,l_{e+1}} \neq a_{k_d,l_d} = a_{l_{e+1},k_e}$, which contradicts the fact that $(k_e, l_{e+1}), (l_{e+1}, k_e) \in \Gamma(A)$ (i.e. $a_{k_e,l_{e+1}} = a_{l_{e+1},k_e}$).

By flipping all entries in A corresponding to the subsequence σ' (i.e. by replacing '1' by '0' and vice versa), we get a matrix A' for which all row and column sums are equal to those of A. We have $\gamma(A') < \gamma(A)$, because as shown above at least one pair $(i, j) \in \sigma' \subset \Gamma(A)$ exists for which (j, i) is no element of σ'. Therefore, after flipping the corresponding entry of A the pair (i, j) is no element of $\Gamma(A')$. Furthermore, A' satisfies $a'_{ij} = 1$ and $a'_{ji} = 0$ for all $(i, j) \in F$, because this property was satisfied for A and the changed entries are all elements of the set $\Gamma(A)$ (which is disjoint to F).

By iterating this procedure we end up with a matrix X for which $r(X) = 0$ holds, i.e. which corresponds to a balanced HAA respecting the fixed games in the set F. □

Example 3. As an example consider the following 5×5-matrix

$$A = \begin{pmatrix} 0 & \underline{1} & 1 & 0 & \underline{0} \\ \underline{1} & 0 & 0 & 1 & \underline{0} \\ 0 & 1 & 0 & \underline{0} & \underline{1} \\ 1 & 0 & \underline{0} & 0 & \underline{1} \\ \underline{0} & \underline{0} & \underline{1} & \underline{1} & 0 \end{pmatrix},$$

where the $\gamma(A) = 12$ elements from the set $\Gamma(A)$ are underlined. Starting with the pair $(1, 2)$ we get the sequence

$$\sigma : (1, 2), (1, 5), (4, 5), (4, 3), (5, 3), (5, 1), (2, 1), (2, 5)$$

with $l_4 = l_1 = 5$. Thus, we have case (2) and get

$$\sigma' : (4, 5), (4, 3), (5, 3), (5, 1), (2, 1), (2, 5).$$

After flipping the 6 elements corresponding to this subsequence, we obtain the matrix

$$A' = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ \boxed{0} & 0 & 0 & 1 & \boxed{1} \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & \boxed{1} & 0 & \boxed{0} \\ \boxed{1} & 0 & \boxed{0} & 1 & 0 \end{pmatrix},$$

which fulfills the antisymmetry condition, i.e. has $\gamma(A') = 0$. □

Since the proof of Theorem 5 is a constructive one, it can directly be used for an algorithm in practice. Given a set F of fixed games, we solve the corresponding network flow problem ignoring the antisymmetry conditions. If no feasible solution exists, then due to Theorem 5 no balanced HAA respecting the given modes exists. If, on the other hand, a feasible flow exists, also a balanced HAA exists which can be obtained from the flow by the repairing mechanism of the proof of Theorem 5.

6. Concluding remarks

We have studied the subproblem of constructing and changing balanced home–away assignments which arises in the first stage of a two-stage approach to construct timetables for sports leagues. The results from Section 2 show that it is easy to find balanced HAAs. Furthermore, the procedure in the proof of Theorem 1 may be used to construct different balanced HAAs as starting solutions for a local search approach (any randomly generated unbalanced HAA can be repaired by this procedure). In Section 3 we have considered the situation that a balanced HAA is given and the mode of some game should be changed. Such a situation may be of interest in connection with a two-stage local search procedure where in the first stage the modes of some games are changed. We have shown how an unbalanced HAA after one mode change can be repaired with a minimal number of additional mode changes such that it becomes balanced. In Section 4 neighborhoods have been defined on the set of balanced HAAs. Since they are connected, it is possible to reach any balanced HAA from any other one with a local search procedure which is based on these neighborhoods. Finally, in Section 5 situations with preassignments have been studied. It has been shown that finding a balanced HAA respecting certain fixed modes (if it exists) can be done in polynomial time.

We already started to use the theoretical results from this paper in order to develop efficient local search algorithms for sports league scheduling problems in practice. In connection with the above mentioned scheduling problem in a non-professional table tennis league (cf. Knust [8]) we implemented a two-stage procedure (see Fiekers [6]). While in the first stage a search on different balanced HAAs is performed (using the proposed neighborhoods), in the second stage RCPSP schedules are constructed. More specifically, we use an adapted genetic algorithm for the RCPSP based on list scheduling techniques and an improvement phase where games are moved to other time periods in order to obtain an even distribution of the games over the whole time horizon.

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