First Problem in LaTeX

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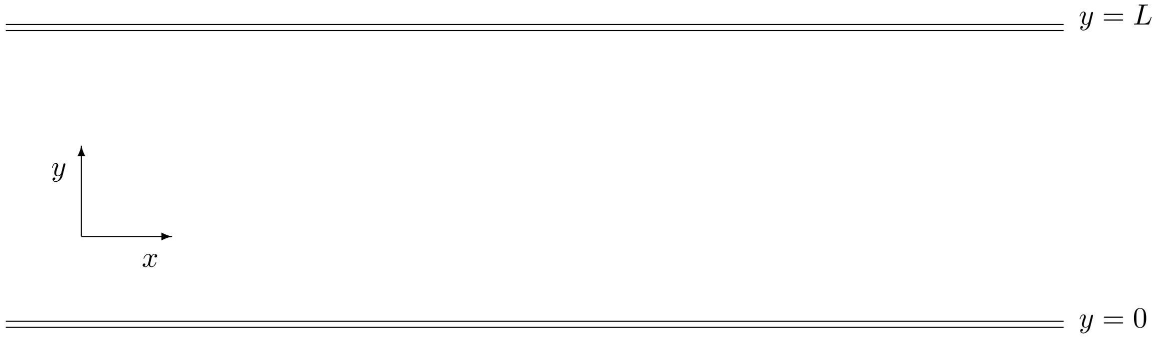
1 First Problem



$$\nabla \cdot \left(\frac{1}{\rho}\nabla P\right) + \frac{\omega^2}{B}P = 0$$
$$\text{So} \qquad \nabla^2 P + k^2 P = 0$$
$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + k^2 P = 0$$

$$\text{So if } P = X(x)Y(y), \text{ then } \nabla^2 P = X''Y + XY''$$
$$\text{So} \qquad \frac{X''}{X} + \frac{Y''}{Y} + k^2 = 0$$
$$i.e. \qquad \frac{X''}{X} + k^2 = -\frac{Y''}{Y} = \alpha^2$$

So we have two 2nd order linear ODEs:

$$\frac{X''}{X} + k^2 = \alpha^2$$
$$X'' + (k^2 - \alpha^2)X = 0 \tag{1}$$

$$\text{and} \qquad -\frac{Y''}{Y} = \alpha^2$$
$$Y'' + \alpha^2 Y = 0 \tag{2}$$

Solving (2) gives:

$$Y = a_1 cos(\alpha y) + a_2 sin(\alpha y)$$

And given Neumann boundary conditions:

$$\frac{\partial P}{\partial \overrightarrow{n}} = 0 \quad \forall \mathbf{x} \in \partial P \implies (\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}) \cdot (0,1) = 0$$
$$\implies \frac{\partial P}{\partial y} = 0 \quad \text{for } y = 0, L$$
$$\implies Y'(0) = Y'(L) = 0$$

$$Y'(y) = -\alpha a_1 \sin(\alpha y) + \alpha a_2 \cos(\alpha y)$$
$$\text{So} \qquad Y'(0) = \alpha a_2 = 0 \implies a_2 = 0$$
$$\text{and} \qquad Y'(L) = -\alpha a_1 \sin(\alpha y) = 0 \implies \alpha = n\pi/L \qquad \text{for } n \in \mathbb{N}$$

So as there are still unknown constants in X, set $Y_n = \cos(\frac{n\pi y}{L})$

Now with $\alpha_n = n\pi/L$, (1) becomes:

$$X'' + (k^2 - \left(\frac{n\pi}{L}\right)^2)X = 0 \qquad \forall n \in \mathbb{N}$$

$$\text{so} \qquad X_n = b_{n+}e^{i\beta_n x} + b_{n-}e^{-i\beta_n x} \qquad \left(\text{where } \beta_n = \sqrt{k^2 - \left(\frac{n\pi}{L}\right)^2}\right)$$

$$\text{so} \qquad P_n = X_n Y_n = \left(b_{n+}e^{i\beta_n x} + b_{n-}e^{-i\beta_n x}\right)\cos(\alpha_n y)$$

$$\text{set } k = 1 \implies P_n = \left(b_{n+}e^{i\sqrt{1-\left(\frac{n\pi}{L}\right)^2}x} + b_{n-}e^{-i\sqrt{1-\left(\frac{n\pi}{L}\right)^2}x}\right)\cos(\frac{n\pi y}{L})$$

$$\text{Now if we write} \qquad Y_n = \cos(\alpha_n y) = \frac{1}{2}\left(e^{i\alpha_n y} + e^{-i\alpha_n y}\right)$$

$$\begin{aligned}
\text{So } P_n &= X_n Y_n e^{-i\omega t} \\
&= \frac{1}{2}\left(b_{n+}e^{i\beta_n x} + b_{n-}e^{-i\beta_n x}\right)\left(e^{i\alpha_n y} + e^{-i\alpha_n y}\right)e^{-i\omega t} \\
&= \frac{1}{2}\Big(b_{n+}e^{i\beta_n x + i\alpha_n y - i\omega t} + b_{n+}e^{i\beta_n x - i\alpha_n y - i\omega t} \\
&\quad + b_{n-}e^{-i\beta_n x + i\alpha_n y - i\omega t} + b_{n-}e^{-i\beta_n x - i\alpha_n y - i\omega t}\Big)
\end{aligned}$$

Now, $\alpha_n = \frac{n\pi}{L}$ and $\beta_n = \sqrt{1 - \left(\frac{n\pi}{L}\right)^2} = \sqrt{1 - \alpha_n^2}$. What we want is for terms of P_n to be of the form $ae^{i\cos(\psi)x + i\sin(\psi)y - i\omega t}$ as this is a simple plane wave traveling at an angle of ψ to the x axis

$$\begin{aligned}
\text{so if} \quad \alpha_n &= \sin(\psi_n) \qquad \implies \alpha_n < 1 \\
\text{then} \quad \beta_n &= \sqrt{1 - \alpha_n^2} \\
&= \sqrt{1 - sin(\psi_n)^2} \\
&= \cos(\psi_n) \qquad \text{for } -\pi/2 \geq \psi_n \geq \pi/2
\end{aligned}$$

$$\begin{aligned}
\implies P_n &= \frac{1}{2}\Big(b_{n+}e^{i\cos(\psi_n)x + i\sin(\psi_n)y - i\omega t} + b_{n+}e^{i\cos(\psi_n)x - i\sin(\psi_n)y - i\omega t} \\
&\quad + b_{n-}e^{-i\cos(\psi_n)x + i\sin(\psi_n)y - i\omega t} + b_{n-}e^{-i\cos(\psi_n)x - i\sin(\psi_n)y - i\omega t}\Big) \\
&= \frac{1}{2}\Big(b_{n+}e^{i\cos(\psi_n)x + i\sin(\psi_n)y - i\omega t} + b_{n+}e^{i\cos(-\psi_n)x + i\sin(-\psi_n)y - i\omega t} \\
&\quad + b_{n-}e^{-(i\cos(-\psi_n)x + i\sin(-\psi_n)y + i\omega t)} + b_{n-}e^{-(i\cos(\psi_n)x + i\sin(\psi_n)y + i\omega t)}\Big)
\end{aligned}$$

So if $b_{n+} = 1$ and $b_{n-} = 0$ and if $n < L/\pi$ then:
$P_n = \frac{1}{2}\Big(e^{i\cos(\psi_n)x + i\sin(\psi_n)y - i\omega t} + e^{i\cos(-\psi_n)x + i\sin(-\psi_n)y - i\omega t}\Big)$ which is the sum of two plane waves traveling at angles of ψ_n and $-\psi_n$ to the x axis, with $-\pi/2 \geq \psi_n \geq \pi/2$. This corresponds to a

traveling wave with nodal lines, where $P_n = 0$ and $t = 0$ at:

$$x = \frac{\pi(2l-1)}{2\cos(\psi_n)} \qquad \text{for } l \in \mathbb{Z}$$

$$\text{and} \qquad y = \frac{\pi(2k-1)}{2\sin(\psi_n)} \qquad \text{for } k = 1, \dots, n \text{ and } n > 0$$

For $n = 0$ there are no nodal lines in the x direction as the two plane waves are equal so the resultant wave is the same plane wave.

As $|e^{i\cos(\psi_n)x+i\sin(\psi_n)y-i\omega t}|$ and $|e^{i\cos(-\psi_n)x+i\sin(-\psi_n)y-i\omega t}|$ are bounded by 1, half their sum, i.e. $|P|$ is bounded by 1.

For $\alpha_n > 1$ we have $\beta_n = \sqrt{1-\alpha_n^2} = i\sqrt{\alpha_n^2 - 1} = i\gamma_n$ where γ_n is some real constant. So for $b_{n+} = 1$ and $b_{n-} = 0$:

$$P_n = \frac{1}{2}\left(e^{-\gamma_n x + i\alpha_n y - i\omega t} + e^{-\gamma_n x - i\alpha_n y - i\omega t}\right)$$

Which has no wave property in the x direction but now has exponential growth/decay.

2 Boundary Conditions

$$P_{n\pm}(x,y) = b_{n\pm}e^{\pm i\beta_n x}\cos(\frac{n\pi y}{L})$$

$$P(x,y) = \sum_{n\in\mathbb{N}} P_{n+}(x,y) + P_{n-}(x,y)$$

If we add two more boundary conditions we can solve for the final two constants, so set $P(a,y) = f(y)$ and $P(b,y) = g(y)$. so at $x = a$:

$$P(a,x) = \sum_{n\in\mathbb{N}} P_{n+}(a,y) + P_{n-}(a,y) = f(y)$$

$$\implies \sum_{n\in\mathbb{N}} \left(b_{n+}e^{i\beta_n a} + b_{n-}e^{-i\beta_n a}\right)\cos(\frac{n\pi y}{L}) = f(x)$$

$$\text{So} \sum_{n\in\mathbb{N}} A_n \cos(\frac{n\pi y}{L}) = f(x)$$

which is the fourier series for $f(x)$

$$\text{So} \qquad A_0 = \frac{1}{L}\int_0^L f(y)dy \qquad \text{and} \qquad A_{m\neq 0} = \frac{2}{L}\int_0^L f(y)\cos(\frac{m\pi y}{L})dy$$

And similarly for B_m at $x = b$

Now:

$$b_{n+}e^{i\beta_n a} + b_{n-}e^{-i\beta_n a} = A_n \qquad \text{and} \qquad b_{n+}e^{i\beta_n b} + b_{n-}e^{-i\beta_n b} = B_n$$

which is just a system of two equations with two unknowns