First try at LaTeX

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1 First Problem

text text text

$$\nabla \cdot \left(\frac{1}{\rho} \nabla P \right) + \frac{\omega^2}{B} P = 0$$

as ρ is a constant it can be taken out of the gradient and multiplied through the equation.

$$\nabla^2 P + k^2 p = 0$$

$$i.e. \qquad \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + k^2 p = 0$$

If we assume a solution of the form $P(x, y) = X(x)Y(y)$, substituting in to the equation gives:

$$X''Y + XY'' + k^2 XY = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} + k^2 = 0$$

$$\frac{X''}{X} + k^2 = -\frac{Y''}{Y} = \alpha^2$$

$$\text{so} \qquad Y'' + \alpha^2 Y = 0$$

so Y takes the form $Y = Ae^{icy} \implies Y' = Aice^{icy} \implies Y'' = -Ac^2 e^{icy}$

$$\text{so} \qquad -Ac^2 e^{icy} + \alpha^2 A e^{icy} = 0$$

$$c^2 = \alpha^2$$

$$c = \pm\alpha$$

2 giving it another go . . .

text text text

$$\nabla \cdot \left(\frac{1}{\rho} \nabla P \right) + \frac{\omega^2}{B} P \quad = \quad 0$$

as ρ is a constant it can be taken out of the gradient and multiplied through the equation.

$$\begin{aligned} \nabla^2 P + k^2 p &= 0 \\ \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + k^2 p &= 0 \end{aligned}$$

If we assume a solution of the form $P(x,y) = X(x)Y(y)$, substituting in to the equation gives:

$$\begin{aligned}
X''Y + XY'' + k^2XY &= 0 \\
\frac{X''}{X} + \frac{Y''}{Y} + k^2 &= 0 \\
\frac{X''}{X} + k^2 &= -\frac{Y''}{Y} = \alpha^2 \\
\text{so} \qquad Y'' + \alpha^2 Y &= 0
\end{aligned}$$

so Y takes the form $Y = Ae^{icy} \implies Y' = Aice^{icy} \implies Y'' = -Ac^2e^{icy}$

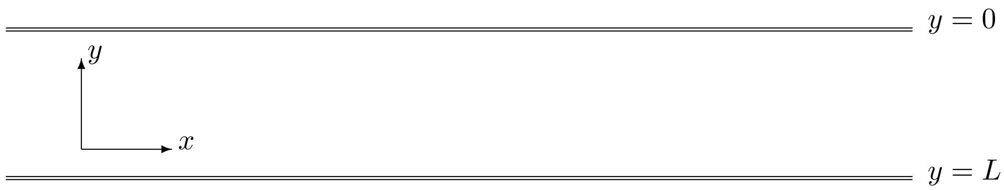
$$\begin{aligned}
\text{so} \qquad -Ac^2e^{icy} + \alpha^2 Ae^{icy} &= 0 \\
c^2 &= \alpha^2 \\
c &= \pm\alpha \\
\therefore \qquad Y &= A_1e^{i\alpha y} + A_2e^{-i\alpha y}
\end{aligned}$$

With Neumann boundary conditions: $P_x(x,0) = P_x(x,L) = 0 \quad i.e. \quad X'(x)Y(0) = X'(x)Y(L) = 0$
As X'(x) is not necessarily zero for all x this implies $Y(0) = Y(L) = 0$

$$\begin{aligned}
\text{so} \qquad Y(0) = A_1e^{i\alpha 0} + A_2e^{-i\alpha 0} &= Y(L) = A_1e^{i\alpha L} + A_2e^{-i\alpha L} = 0 \\
A_1 + A_2 &= 0 \\
\implies \qquad A_1 &= -A_2 \quad (set A_1 = A) \\
\text{then} \qquad Ae^{i\alpha L} - Ae^{-i\alpha L} &= 0 \\
Ae^{i\alpha L} &= Ae^{-i\alpha L} \\
\alpha &= -\alpha \\
\alpha &= 0
\end{aligned}$$

Therefore $Y = A_1 + A_2 = C \quad \forall y \in [0,L]$
Which implies $P(x,y) = P(x)$

3 restarting...



$$\nabla \cdot \left(\frac{1}{\rho}\nabla P\right) + \frac{\omega^2}{B} P = 0$$

$$\text{So} \qquad \nabla^2 P + k^2 P = 0$$

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + k^2 P = 0$$

So if $P = X(x)Y(y)$, then $\nabla^2 P = X''Y + XY''$

$$\text{So} \qquad \frac{X''}{X} + \frac{Y''}{Y} + k^2 = 0$$

$$i.e. \qquad \frac{X''}{X} + k^2 = -\frac{Y''}{Y} = \alpha^2$$

So we have two 2nd order linear ODEs:

$$\frac{X''}{X} + k^2 = \alpha^2$$

$$X'' + (k^2 - \alpha^2)X = 0 \tag{1}$$

$$\text{and} \qquad -\frac{Y''}{Y} = \alpha^2$$

$$Y'' + \alpha^2 Y = 0 \tag{2}$$

Solving (2) gives:

$$Y = a_1 cos(\alpha y) + a_2 sin(\alpha y)$$

And given Neumann boundary conditions:

$$\frac{\partial P}{\partial \overrightarrow{n}} = 0 \quad \forall \mathbf{x} \in \partial P \implies \left(\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}\right) \cdot (0,1) = 0$$

$$\implies \frac{\partial P}{\partial y} = 0 \quad \text{for } y = 0, L$$

$$\implies Y'(0) = Y'(L) = 0$$

$$Y'(y) = -\alpha a_1 \sin(\alpha y) + \alpha a_2 \cos(\alpha y)$$

$$\text{So} \qquad Y'(0) = \alpha a_2 = 0 \implies a_2 = 0$$

$$\text{and} \qquad Y'(L) = -\alpha a_1 \sin(\alpha y) = 0 \implies \alpha = n\pi/L \qquad \text{for } n \in \mathbb{N}$$

So as there are still unknown constants in X, set $Y_n = \cos(\dfrac{n\pi y}{L})$

Now with $\alpha_n = n\pi/L$, (1) becomes:

$$X'' + (k^2 - \left(\frac{n\pi}{L}\right)^2)X = 0 \qquad \forall n \in \mathbb{N}$$

$$\text{so} \qquad X_n = b_{n+}e^{i\beta_n x} + b_{n-}e^{-i\beta_n x} \qquad \left(\text{where } \beta_n = \sqrt{k^2 - \left(\frac{n\pi}{L}\right)^2}\right)$$

$$\text{so} \qquad P_n = X_nY_n = \left(b_{n+}e^{i\beta_n x} + b_{n-}e^{-i\beta_n x}\right)\cos(\alpha_n y)$$

$$\text{set } k = 1 \implies P_n = \left(b_{n+}e^{i\sqrt{1-(\frac{n\pi}{L})^2}x} + b_{n-}e^{-i\sqrt{1-(\frac{n\pi}{L})^2}x}\right)\cos(\frac{n\pi y}{L})$$

The Y_n component of P_n will always be $\cos(n\pi y/L)$ and then if $L > n$ then β_n is real so the X_n component takes the form $c_1\cos(\beta_n x) + c_2\sin(\beta_n x)$.

If $L < n$ then β_n is imaginary so the X_n component takes the form $b_{n+}e^{\gamma x} + b_{n-}e^{-\gamma x}$.

$$\begin{aligned}
Y_n = \cos(\alpha_n y) &= \frac{1}{2}\left(e^{i\alpha_n y} + e^{-i\alpha_n y}\right) \\
\text{So } P_n &= X_nY_ne^{-i\omega t} \\
&= \frac{1}{2}\left(b_{n+}e^{i\beta_n x} + b_{n-}e^{-i\beta_n x}\right)\left(e^{i\alpha_n y} + e^{-i\alpha_n y}\right)e^{-i\omega t} \\
&= \frac{1}{2}\Big(b_{n+}e^{i\beta_n x + i\alpha_n y - i\omega t} + b_{n+}e^{i\beta_n x - i\alpha_n y - i\omega t} \\
&\quad + b_{n-}e^{-i\beta_n x + i\alpha_n y - i\omega t} + b_{n-}e^{-i\beta_n x - i\alpha_n y - i\omega t}\Big)
\end{aligned}$$

Now, $\alpha_n = \frac{n\pi}{L}$ and $\beta_n = \sqrt{1 - \left(\frac{n\pi}{L}\right)^2} = \sqrt{1 - \alpha_n^2}$. What we want is for terms of P_n to be of the form $ae^{i\cos(\psi)x + i\sin(\psi)y - i\omega t}$ as this is a simple plane wave traveling at an angle of ψ to the x axis

$$\begin{aligned}
\text{so if } \alpha_n &= \sin(\psi_n) \\
\alpha_n^2 &= \sin^2(\psi_n) \\
\alpha_n^2 + \cos^2(\psi_n) &= 1 \\
\cos(\psi_n) &= \sqrt{1 - \alpha_n^2} \\
&= \beta_n
\end{aligned}$$

$$\begin{aligned}
\implies P_n &= \frac{1}{2}\Big(b_{n+}e^{i\cos(\psi_n)x + i\sin(\psi_n)y - i\omega t} + b_{n+}e^{i\cos(\psi_n)x - i\sin(\psi_n)y - i\omega t} \\
&\quad + b_{n-}e^{-i\cos(\psi_n)x + i\sin(\psi_n)y - i\omega t} + b_{n-}e^{-i\cos(\psi_n)x - i\sin(\psi_n)y - i\omega t}\Big) \\
&= \frac{1}{2}\Big(b_{n+}e^{i\cos(\psi_n)x + i\sin(\psi_n)y - i\omega t} + b_{n+}e^{i\cos(-\psi_n)x + i\sin(-\psi_n)y - i\omega t} \\
&\quad + b_{n-}e^{-(i\cos(-\psi_n)x + i\sin(-\psi_n)y + i\omega t)} + b_{n-}e^{-(i\cos(\psi_n)x + i\sin(\psi_n)y + i\omega t)}\Big)
\end{aligned}$$

So if $b_{n+} = 1$ and $b_{n-} = 0$ then:
$P = \frac{1}{2}\left(e^{i\cos(\psi_n)x + i\sin(\psi_n)y - i\omega t} + e^{i\cos(-\psi_n)x + i\sin(-\psi_n)y - i\omega t}\right)$ which is the sum of two plane waves traveling at angles of ψ_n and $-\psi_n$ to the x axis. This corresponds to a traveling wave with nodal lines, where $P = 0$ and $t = 0$ at:

$$\begin{aligned} y &= \frac{\pi(2k-1)}{2\alpha} \qquad \text{for } k = 1, \ldots, n \\ \text{and } x &= \frac{\pi(2l-1)}{2\beta} \qquad \text{for } l \in \mathbb{Z} \end{aligned}$$

As $|e^{i\cos(\psi_n)x + i\sin(\psi_n)y - i\omega t}|$ and $|e^{i\cos(-\psi_n)x + i\sin(-\psi_n)y - i\omega t}|$ are bounded by 1, half their sum, i.e. $|P|$ is bounded by 1.