Universität Bremen

Center for Industrial Mathematics (ZeTeM)

**Faculty 03**

Mathematics / Computer science

# Outlook: DeepONets

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Bremen, 21.07.2023

# What did we do so far?

Learned PDEs for single parameter

a) For fixed $D \in \mathbb{R}$ solve parachute example:

$$\begin{cases} \partial_t^2 u(t) &= D\big(\partial_t u(t)\big)^2 - g \quad \text{for} \quad t \in [0,3] \\ u(0) &= H \\ \partial_t u(0) &= 0 \end{cases}$$

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b) For fixed $D \in \mathbb{R}$ solve heat equation:

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!! Bonus exercises: PINNs can solve for multiple $D \in [D_{min}, D_{max}]$

$$u_\theta(D, x, t) \approx u_D(x, t)$$

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## Learned PDE for single parameter function

- Room-with-heater example

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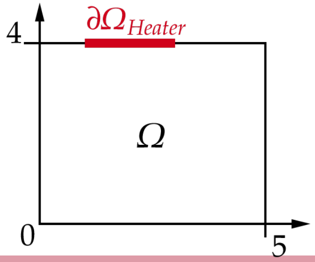

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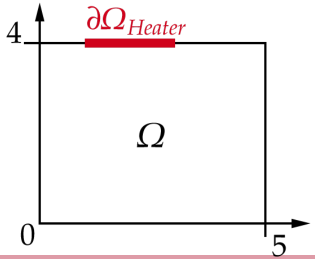
- ... start new training with PINNs :(



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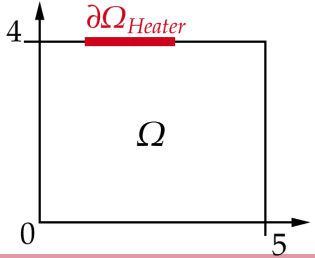
- **Goal:** Solve PDE for many $\tilde{h}(t)$ with ONE network



# Simpler Example for Illustration

- **Goal:** For **multiple** functions $f : [0,1] \to \mathbb{R}$ use **single** NN to solve

$$\begin{cases} u^{(1)}(x) & = f(x) \quad \text{for} \quad x \in [0,1] \\ u(0) & = 0 \end{cases}$$

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$\rightarrow$ Want NN that learns the mapping

$$\mathcal{A} : f \mapsto u_f \qquad \text{"}f \text{ is mapped to corr. solution } u_f\text{"}$$

**Problem:** Inputs of NNs can only be vectors in $\mathbb{R}^m$, but $\mathcal{A}$ receives a function!

# Operator Learning

- Want to learn *solution operator* $\mathcal{A} : f \mapsto u_f$ for

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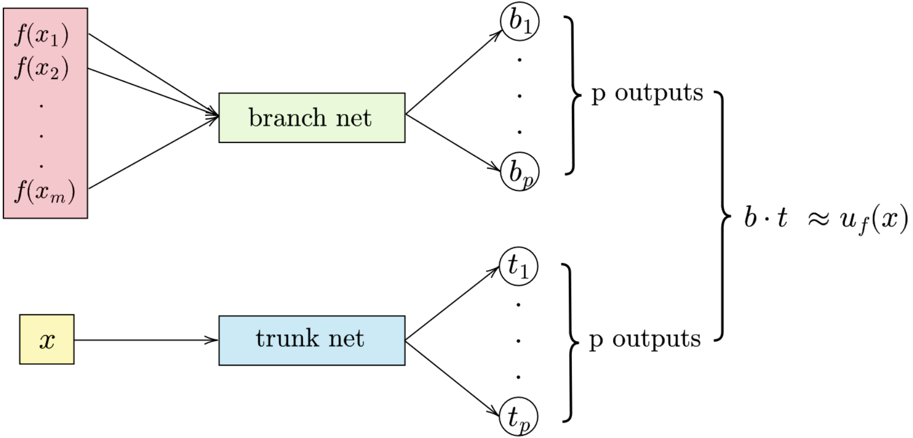
**Idea:**

- Fix points $x_1, \dots, x_m$ in domain of $f$
- Consider NNs of the form

$$u_\theta : \mathbb{R}^m \times [0, 1] \longrightarrow \mathbb{R}$$
$$\big(f(x_1), f(x_2), \dots, f(x_m), x\big) \longmapsto u(x)$$

# DeepONets - Divide and Conquer

(Lu, Jin, Karniadakis; 2019)



# DeepONets - Construction Steps

## Construction of DeepONet

$$u_\theta : \mathbb{R}^m \times [0,1] \longrightarrow \mathbb{R}$$
$$\Big(f(x_1), f(x_2), ..., f(x_m), x\Big) \longmapsto b \cdot t \approx u(x)$$

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- Build architecture of branch/trunk net, e.g. fully-connected

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- Data points look like $f(x_1), \ldots, f(x_m)$ and $u_f(\tilde{x}_1), \ldots, u_f(\tilde{x}_k)$
- Loss function for single tuple $(f, u_f)$:

$$\frac{1}{k}\sum_{j=1}^{k}\Big\|u_\theta\big(f(x_1), \ldots, f(x_m), \tilde{x}_j\big) - u_f(\tilde{x}_j)\Big\|^2$$

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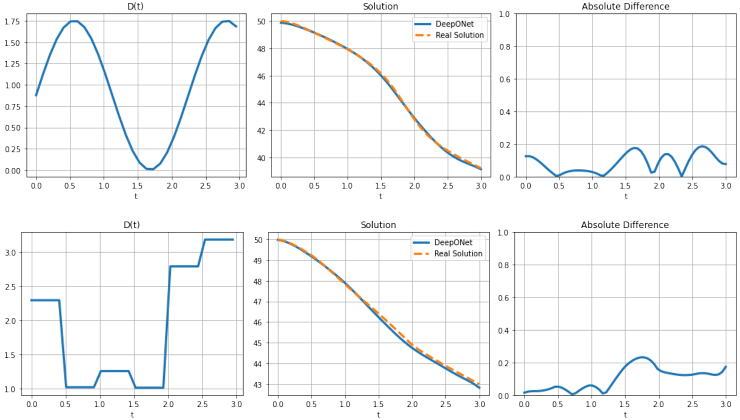
- Time dependent $D$ "=" parachute is opening and closing while falling
- Analytic solution is unknown $\rightarrow$ employ classical schemes to create training data
- Trained with TorchPhysics

# DeepONet - Parachute Example

## Evaluation on test set



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- Mesh-independent
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- Needs lot of data for training:
  - Expensive and time consuming to obtain
  - Physics-informed DeepONets
- May need problem specific adaptions
- Often trial and error for finding good parameters

# The End

Thank you for participating! :)

Thanks to Tom for his support!