The Prime-Anchored Hilbert–Pólya Operator and its consequences

Tom Gatward
Independent Researcher
tom@gatward.com.au
(ORCID: 0009-0009-1167-6421)

Public release: 24th October 2025

Abstract

We develop a prime–anchored Hilbert–Pólya framework and prove a determinant identity that matches the zeros of the completed zeta function with those of a τ–determinant built purely from primes. We define a prime–anchored trace τ on the even Paley–Wiener cone via the explicit formula with Abel–regularized resolvent and an explicit archimedean subtraction; no operator is assumed at this stage. From the Abel–regularized Poisson semigroup $\Theta(t)$ we obtain a unique positive measure μ by Bernstein's theorem and realize the canonical arithmetic Hilbert–Pólya operator A_τ as multiplication by λ on $L^2((0,\infty),\mu)$. For $\Re s > 0$ the resolvent trace

$$\mathcal{T}(s) := \tau\big((A_\tau^2 + s^2)^{-1}\big) = \int_{(0,\infty)} \frac{d\mu(\lambda)}{\lambda^2 + s^2}$$

is holomorphic and admits meromorphic continuation to $\mathbb{C}$ with no branch cut on $i\mathbb{R}$; this forces μ to be purely atomic. An Abel boundary identity on the real axis gives

$$\frac{\Xi'}{\Xi}(a) = 2a\,\mathcal{T}(a) + H'(a) \qquad (a > 0),$$

and analytic continuation yields the global identity

$$\Xi(s) = C\, e^{H(s)}\, \det\nolimits_\tau\big(A_\tau^2 + s^2\big),$$

with $\dfrac{d}{ds} \log \det_\tau(A_\tau^2 + s^2) = 2s\,\mathcal{T}(s)$ and $C = \Xi(0)e^{-H(0)}$. Consequently, the zeros of Ξ are exactly $\{\pm i\gamma\}$ with multiplicities $m_\gamma = 2i\gamma\, \mathrm{Res}_{s=i\gamma}\, \mathcal{T}(s)$. The argument is non–circular: the zero side is used only to certify complete monotonicity (or positivity on a Fejér/log positive–definite cone), not to input locations, and the archimedean subtraction is needed only on the real axis.

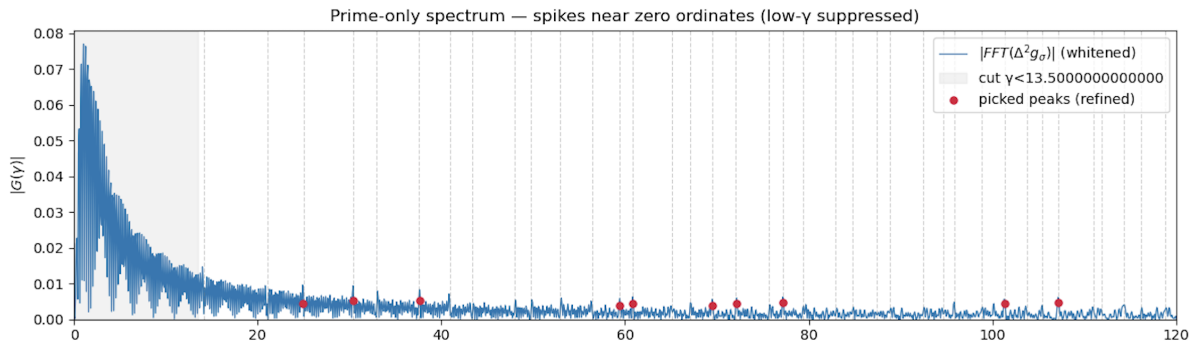

Figure 1: a prime-only construction produces spectral peaks aligning with early zero ordinates (dashed)

1 Introduction

This paper develops a prime–anchored version of the Hilbert–Pólya paradigm and derives a *determinant identity* that identifies the zeros of the completed zeta function with those of a τ–determinant. The key structural feature is an *arithmetic Hilbert–Pólya operator* A_τ whose trace is defined purely from the prime side with an explicit archimedean subtraction. All spectral statements are made *with respect to this prime–anchored trace* τ, rather than by postulating a spectrum containing the ordinates of zeros.

Main identity. Let $\Xi(s) = \xi(\frac{1}{2} + s)$ and let H be the even entire function from the Hadamard factorization of Ξ. We prove the global identity

$$\Xi(s) \;=\; C\, e^{H(s)}\, \det_\tau\big(A_\tau^2 + s^2\big), \tag{1}$$

with $\frac{d}{ds} \log \det_\tau(A_\tau^2 + s^2) \;=\; 2s\, \tau\big((A_\tau^2 + s^2)^{-1}\big)$ and $C \;=\; \Xi(0)e^{-H(0)}$. The zeros on the right are exactly $\{\pm i\gamma\}$ with multiplicities $m_\gamma \;=\; 2i\gamma\, \mathrm{Res}_{s=i\gamma}\, \mathcal{T}(s)$, hence the zeros of Ξ occur precisely at $\{\pm i\gamma\}$ with the same multiplicities.

The arithmetic Hilbert–Pólya operator.

1. *Prime–anchored trace on the Paley–Wiener cone.* For even Paley–Wiener tests φ, we define $\tau(\varphi(A))$ from the explicit formula on the prime side, with Abel regularization of the resolvent and an explicit archimedean subtraction (Definition 1.4).

2. *Poisson semigroup and Bernstein.* The Abel–regularized Poisson semigroup trace $\Theta(t) := \lim_{R\to\infty} \lim_{\varepsilon\downarrow 0} \tau(\varphi_{R,\varepsilon})$ is completely monotone (Theorem 1.10). By Bernstein, there is a unique positive Borel measure μ with $\Theta(t) = \int e^{-t\lambda}\, d\mu(\lambda)$. We take A_τ to be multiplication by λ on $L^2((0,\infty),\mu)$ and extend τ by $\tau(f(A_\tau)) = \int f\, d\mu$ for bounded Borel $f \geq 0$.

3. *Meromorphic resolvent trace.* For $\Re s > 0$,

$$\mathcal{T}(s) := \tau\big((A_\tau^2 + s^2)^{-1}\big) = \int_{(0,\infty)} \frac{d\mu(\lambda)}{\lambda^2 + s^2}$$

admits meromorphic continuation to $\mathbb{C}$ with no branch cut on $i\mathbb{R}$ (Lemma 1.20 and Lemma 1.25; Lemma 1.24 is used only to pass to $S(z) = \mathcal{T}(\sqrt{z})$). This "no–monodromy" input forces μ to be purely atomic.

Two forcing mechanisms. (**S**) *Stieltjes representation.* Positivity on a Fejér–averaged PD Paley–Wiener cone (or, equivalently, complete monotonicity of Θ) yields $\mathcal{T}(s) = \int(\lambda^2+s^2)^{-1}\,d\mu(\lambda)$ without assuming zero locations. (**M**) *Meromorphy without branch cuts.* Evenness and meromorphy of $\mathcal{T}$ across $i\mathbb{R}$ allow $S(z) := \mathcal{T}(\sqrt{z})$ to be single-valued across $(-\infty,0]$; a $\bar\partial$–residue argument gives $\mu = \sum_{\gamma>0} m_\gamma\,\delta_\gamma$ with $m_\gamma = 2i\gamma\ \mathrm{Res}_{s=i\gamma}\,\mathcal{T}(s)$ (Lemma 1.27).

Real–axis anchor and archimedean term. On the real axis,

$$\frac{\Xi'}{\Xi}(a) = 2a\,\mathcal{T}(a) + H'(a) \qquad (a>0),$$

with the archimedean contribution subtracted explicitly, $\mathrm{Arch}_{\mathrm{res}}(a) = \frac{1}{4}(\log\pi - \psi(\frac{1}{4}+\frac{a}{2}))$ (Lemma 1.14). By analytic continuation (avoiding $\mathrm{Zeros}(\Xi)$), $2s\,\mathcal{T}(s) = \Xi'/\Xi(s) - H'(s)$ is holomorphic, hence the logarithmic integral defining $\det_\tau$ is path–independent and (1) follows by integration.

Non–circularity. Before Theorem 1.10 no operator is assumed; $\tau(\varphi(A))$ is prime–anchored shorthand. The zero side is used only to certify complete monotonicity (or cone positivity), not locations. **Outcome.** With multiplicities $m_\gamma = \tau(P_\gamma)$ controlling the monodromy of $\int 2u\,\mathcal{T}(u)\,du$, the τ–determinant is single–valued and even, and its zeros are exactly $\{\pm i\gamma\}$ with multiplicity m_γ. Equation (1) identifies the zeros of Ξ with those of $\det_\tau(A_\tau^2+s^2)$.

1.1 A Hilbert–Pólya Determinant Proof via an Abel–Regularized Prime Trace

Remark 1.1 (Why the prime-anchored HP argument is not a tautology). Putting $\{\gamma_j\}$ on a diagonal carries no arithmetic content. The argument below differs in three structural ways, and these are exactly what *force* the zero set.

Anchor. We construct a prime-anchored functional τ by Abel-regularizing the resolvent and subtracting the archimedean term (Definition 1.4). All spectral expressions are paired with τ, not introduced ad hoc.

Positivity $\Rightarrow$ Stieltjes (or via Bernstein). On the Fejér–averaged Paley–Wiener PD cone the quadratic form is nonnegative (by the PD kernel construction; see Lemma 1.6). By Bochner/Riesz this yields a Stieltjes representation

$$\mathcal{T}(s) = \tau\big((A^2+s^2)^{-1}\big) = \int_{(0,\infty)} \frac{1}{\lambda^2+s^2}\,d\mu(\lambda), \qquad \mu\geq 0.$$

(Equivalently, complete monotonicity of Θ gives the same representation via Bernstein.)

Meromorphic continuation; location vs. structure. As proved below (Lemmas 1.20 and 1.16), we obtain on any simply connected $\Omega\subset\mathbb{C}\setminus\mathrm{Zeros}(\Xi)$ containing $(0,\infty)$ the identity

$$\mathcal{T}(s) = \frac{1}{2s}\Big(\frac{\Xi'}{\Xi}(s) - H'(s)\Big).$$

Because $\mathcal{T}$ is holomorphic on $\{\Re s>0\}$ (Stieltjes form) while Ξ'/Ξ has poles at zeros of Ξ, zeros with $\Re s_0>0$ are impossible; evenness of Ξ excludes $\Re s_0<0$. Hence all zeros lie on $i\mathbb{R}$ (RH: location).

In addition, Ξ'/Ξ is meromorphic with no branch cut, so $\mathcal{T}$ has a single–valued meromorphic continuation across $i\mathbb{R}$; by the Stieltjes form any singularity must lie at $\{\pm i\lambda : \lambda\in\operatorname{supp}\mu\}$, and Lemma 1.27 then gives $\mu=\sum_{\gamma>0} m_\gamma\,\delta_\gamma$ with the correct multiplicities.

Consequently,

$$\frac{\Xi'}{\Xi}(s) = 2s\,\mathcal{T}(s) + H'(s), \qquad \Xi(s) = C\,e^{H(s)}\,\det_\tau(A^2+s^2),$$

so the zeros of Ξ occur exactly at $s=\pm i\gamma_j$, counted with multiplicity.

Let $\{\rho_j\} = \{\beta_j + i\gamma_j\}$ be the nontrivial zeros of ζ, listed with multiplicity, with $\beta_j \in (0,1)$ and $\gamma_j > 0$. Put

$$\Xi(s) := \xi\big(\tfrac{1}{2} + s\big), \qquad \mathrm{Zeros}(\Xi) = \big\{(\beta_j - \tfrac{1}{2}) \pm i\gamma_j\big\}.$$

We use only the following unconditional tools in this section: **($\mathbf{EF}_{\mathrm{PW}}$)** Weil's explicit formula for even Paley–Wiener tests; **(AbelBV)** distributional Abel/Plancherel boundary values after subtracting the $s = 1$ pole; **(ZC)** zero counting $N(T) \ll T \log T$; **(Bernstein)** existence and uniqueness of μ with $\Theta(t) = \int e^{-t\lambda}\, d\mu(\lambda)$.

Notational convention (no circularity). Until Theorem 1.10 we have not yet constructed an operator. Whenever we write $\tau(\varphi(A))$ for $\varphi \in \mathrm{PW}_{\mathrm{even}}$, it is shorthand for the prime-side test functional $\langle \tau, \varphi \rangle$ defined in Definition 1.4 below; after constructing μ and A_τ we identify $\langle \tau, \varphi \rangle = \tau(\varphi(A_\tau))$.

Setup–τ (Definitions; no operator assumed). *(Prime-anchored start; zero-side used only to certify complete monotonicity.)*

We first define τ on $\mathrm{PW}_{\mathrm{even}}$ by the explicit formula with the archimedean subtraction (Definition 1.4).

Construction of μ and A_τ (from Θ via Bernstein). For $t > 0$ set $\Theta(t) := \lim_{R\to\infty} \lim_{\varepsilon\downarrow 0} \langle \tau, \varphi_{R,\varepsilon} \rangle$, where $\widehat{\varphi}_{R,\varepsilon}(\xi) = e^{-t\sqrt{\xi^2+\varepsilon^2}}\chi_R(\xi)$ as in Theorem 1.10. By the unconditional explicit formula in the even Paley–Wiener class (no assumption on zero locations), the limit equals $\sum_{\gamma>0} m_\gamma\, e^{-t\gamma}$; in particular Θ is completely monotone, and the absolute convergence (hence termwise differentiation) is justified by $N(T) \ll T \log T$. This use of the zero–side identity inputs no location information and serves only to certify complete monotonicity for Bernstein's theorem. By Bernstein there is a unique positive Borel measure μ on $(0, \infty)$ with $\Theta(t) = \int e^{-t\lambda}\, d\mu(\lambda)$. Define A_τ to be multiplication by λ on $L^2((0,\infty), \mu)$ and *extend* τ to bounded Borel $f \geq 0$ by $\tau(f(A_\tau)) = \int f\, d\mu$. Compatibility with the prime-side definition on $\mathrm{PW}_{\mathrm{even}}$ is proved in Lemma 1.13.

Optional alternative. Complete monotonicity of Θ can also be obtained from positivity on the Fejér/log PD cone and Bochner–Riesz (see §1.1.3), yielding the same μ by Bernstein.

From now on in this section we set $A := A_\tau$. No arithmetic input about the location of zeros is assumed; μ is determined by Θ (hence by τ on $\mathrm{PW}_{\mathrm{even}}$).

(Then Theorem 1.10 just records $\tau(e^{-tA_\tau}) = \int_{(0,\infty)} e^{-t\lambda}\, d\mu(\lambda)$.)

Fourier convention. We use $\widehat{f}(\xi) = \int_{\mathbb{R}} f(u)\, e^{-i\xi u}\, du$; for even f we have $\widehat{f}$ even.

1.1.1 Abelian functional calculus and the target C*–algebra (after Poisson)

This subsection applies after Theorem 1.10, once μ (hence A_τ) has been constructed. Let $\mathrm{PW}_{\mathrm{even}}$ be the even Paley–Wiener class. Define

$$\mathscr{A}_{\mathrm{PW}} \;:=\; \overline{\mathrm{span}}\{\ \varphi(A_\tau)\ :\ \varphi \in \mathrm{PW}_{\mathrm{even}}\ \}, \qquad \text{and we write } A := A_\tau \text{ henceforth.}$$

The closure is in the operator norm. We construct a linear functional τ on $\mathscr{A}_{\mathrm{PW}}$ encoding the explicit formula on the zero side; positivity is recorded on the Fejér/log PD cone in §1.1.3, and full positivity (as a normal semifinite weight) arises after Theorem 1.10 via the spectral measure μ.

Fourier convention (inverse). We use $\widehat{f}(\xi) = \int_{\mathbb{R}} f(u)\, e^{-i\xi u}\, du$, with inverse $f(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(u)\, e^{itu}\, du$. In particular,

$$\mathcal{F}^{-1}\Big(u \mapsto \frac{s}{s^2+u^2}\Big)(t) = \tfrac{1}{2}\, e^{-s|t|}.$$

1.1.2 Abel–regularized prime resolvent

For $\sigma > 0$ and $\Re s > 0$ set

$$S(\sigma; s) := \sum_{p^k} \frac{\log p}{p^{k(1/2+\sigma)}} \cdot \frac{s}{(k\log p)^2 + s^2}, \qquad M(\sigma; s) := \int_2^\infty \frac{s}{(\log x)^2 + s^2}\, \frac{dx}{x^{1/2+\sigma}}.$$

Convention (small-σ). For $0 < \sigma \le \frac{1}{2}$ all appearances of $S(\sigma;\cdot)$ and $M(\sigma;\cdot)$ are understood at Paley–Wiener truncation level:

$$S_R(\sigma; s) := \sum_{p^k} \frac{\log p}{p^{k(1/2+\sigma)}} \chi_R(k\log p) \frac{s}{(k\log p)^2 + s^2}, \quad M_R(\sigma; s) := \int_2^\infty \chi_R(\log x) \frac{s}{(\log x)^2 + s^2}\, \frac{dx}{x^{1/2+\sigma}},$$

with $\chi_R \in C_c^\infty(\mathbb{R})$ an even cutoff satisfying $0 \le \chi_R \le 1$, $\chi_R \equiv 1$ on $[-R, R]$, $\operatorname{supp}\chi_R \subset [-(R+1), R+1]$, and $\chi_{R_1} \le \chi_{R_2}$ for $R_1 \le R_2$, then $R \to \infty$ by monotone convergence. This ensures $\widehat{\varphi}_{R,s}(u) := \frac{s}{s^2+u^2}\chi_R(u)$ is even, hence $\varphi_{R,s} \in \mathrm{PW}_{\mathrm{even}}$. Absolute convergence at $\sigma \le \frac{1}{2}$ is not claimed; the Abel limit is taken on the *difference* $S(\sigma;\cdot) - M(\sigma;\cdot)$.

Lemma 1.2 (Distributional Abel boundary value on $\Re s = \frac{1}{2}$)**.** *Let $F(s)$ be meromorphic on $\{\Re s > \frac{1}{2}\}$ with at most a simple pole at $s = 1$, and assume F has at most simple poles on the boundary line $\{s = \frac{1}{2} + i\gamma\}$ with discrete ordinates and no accumulation on $\{\Re s = \frac{1}{2}\} \cup \{\infty\}$. Assume that for every $\sigma_0 > 0$ and every compact $J \subset \mathbb{R}$ avoiding the ordinates of boundary poles one has*

$$F\Big(\tfrac{1}{2} + \sigma - it\Big) \ll_{\sigma_0, J} (\log(2+|t|))^2 \qquad (0 < \sigma \le \sigma_0,\ t \in J).$$

Then the tempered boundary value

$$t \longmapsto \lim_{\sigma \downarrow 0} \Big(F(\tfrac{1}{2} + \sigma - it) - \tfrac{1}{\frac{1}{2}+\sigma-it-1}\Big)$$

exists in $\mathcal{S}'_{\mathrm{even}}(\mathbb{R})$ (pairings against even Schwartz functions) and equals

$$\mathrm{PV}\, G(t) \;+\; \pi \sum_{\gamma > 0} c_\gamma\, \delta(t - \gamma), \qquad G(t) := F\Big(\tfrac{1}{2} - it\Big) - \tfrac{1}{\frac{1}{2}-it-1}.$$

Here $c_\gamma = \operatorname{Res}_{s=\frac{1}{2}+i\gamma} F(s)$.

In particular, for every $\psi \in \mathcal{S}_{\mathrm{even}}(\mathbb{R})$ and $a > 0$,

$$\lim_{\sigma\downarrow 0} \int_{\mathbb{R}} e^{-a|t|} \Big(F(\tfrac{1}{2} + \sigma - it) - \tfrac{1}{\frac{1}{2}+\sigma-it-1}\Big)\psi(t)\, dt = \int_{\mathbb{R}} e^{-a|t|} \Big(F(\tfrac{1}{2} - it) - \tfrac{1}{\frac{1}{2}-it-1}\Big)\psi(t)\, dt.$$

Fourier–Mellin bridge at fixed R and $\sigma > 0$. Let $\widehat{\varphi}_{R,s}(u) := \dfrac{s}{s^2+u^2}\chi_R(u)$ and $\varphi_{R,s} := \mathcal{F}^{-1}(\widehat{\varphi}_{R,s}) \in \mathrm{PW}_{\mathrm{even}}$. Since χ_R makes all sums and integrals finite, Fubini applies and

$$\widehat{\varphi}_{R,s}(u) = \int_{\mathbb{R}} \varphi_{R,s}(t)\, e^{-itu}\, dt.$$

Prime side.

$$S_R(\sigma;s) := \sum_{p^k} \frac{\log p}{p^{k(1/2+\sigma)}}\,\widehat{\varphi}_{R,s}(k\log p) = \int_{\mathbb{R}} \varphi_{R,s}(t) \sum_{\substack{p^k \\ k\log p\le R}} \frac{\log p}{p^{k(1/2+\sigma-it)}}\,dt$$

$$= \int_{\mathbb{R}} \varphi_{R,s}(t)\Big(-\frac{\zeta'}{\zeta}\Big)_R\Big(\tfrac12+\sigma-it\Big)\,dt,$$

where $(-\zeta'/\zeta)_R$ denotes the *truncated* Euler/Dirichlet sum. For $\sigma > \frac12$ (so $\Re(\frac12+\sigma-it) > 1$), letting $R\to\infty$ gives $(-\zeta'/\zeta)_R \to -\zeta'/\zeta$. For $0<\sigma\le\frac12$ we interpret the pairing via Lemma 1.2.

Continuous side. With $x = e^u$,

$$M_R(\sigma;s) = \int_{\log 2}^{\infty} \chi_R(u)\,\frac{s}{s^2+u^2}\,e^{(1/2-\sigma)u}\,du = \int_{\mathbb{R}} \varphi_{R,s}(t)\Big[\int_{\log 2}^{\infty} \chi_R(u)\,e^{-(\sigma-\frac12+it)u}\,du\Big]\,dt.$$

Write $\alpha := \sigma - \frac12 + it$. Add and subtract $\int_0^{\log 2}\chi_R(u)e^{-\alpha u}\,du$ to obtain

$$\begin{aligned}\int_{\log 2}^{\infty}\chi_R(u)\,e^{-\alpha u}\,du &= \frac{1}{\alpha} - \int_0^{\infty}\big(1-\chi_R(u)\big)\,e^{-\alpha u}\,du - \int_0^{\log 2}\chi_R(u)\,e^{-\alpha u}\,du \\ &= \frac{1}{\alpha} - \underbrace{\int_0^{\infty}\big(1-\chi_R(u)\big)\,e^{-\alpha u}\,du}_{=:E_R^{(\infty)}(t)} + \underbrace{\int_0^{\log 2}\big(1-\chi_R(u)\big)\,e^{-\alpha u}\,du}_{=:E_R^{(0)}(t)} - \int_0^{\log 2} e^{-\alpha u}\,du.\end{aligned}$$

Hence

$$M_R(\sigma;s) = \int_{\mathbb{R}}\varphi_{R,s}(t)\Big[\frac{1}{\alpha} + E_R^{(0)}(t) - E_R^{(\infty)}(t)\Big]\,dt \;-\; \int_{\mathbb{R}}\varphi_{R,s}(t)\Big[\int_0^{\log 2} e^{-\alpha u}\,du\Big]dt,$$

with $E_R^{(0)}(t)\equiv 0$ once $R\ge\log 2$ (since $\chi_R\equiv 1$ on $[0,\log 2]$). The last (compact–interval) term is independent of R and is absorbed into the archimedean correction on the real axis.

For $\sigma>\frac12$ one has $|E_R^{(\infty)}(t)| \le \int_R^\infty e^{-(\sigma-\frac12)u}\,du = e^{-(\sigma-\frac12)R}/(\sigma-\frac12)$, so after pairing with $\varphi_{R,s}(t)e^{-a|t|}$, $E_R^{(\infty)}\to 0$ as $R\to\infty$; the Abel boundary argument then transports this decay to the $\sigma\downarrow 0$ limit in $\mathcal{S}'_{\text{even}}$.

Conclusion. Conclusion (after moving the compact–interval term into the archimedean correction). At fixed R,

$$S_R(\sigma;s) - M_R(\sigma;s) = \int_{\mathbb{R}}\varphi_{R,s}(t)\Big[(-\tfrac{\zeta'}{\zeta})_R\Big(\tfrac12+\sigma-it\Big) - \frac{1}{\sigma-\frac12+it}\Big]\,dt \;+\; E_R(\sigma;s).$$

where

$$E_R(\sigma;s) := \int_{\mathbb{R}}\varphi_{R,s}(t)\,\big(-E_R^{(0)}(t)+E_R^{(\infty)}(t)\big)\,dt.$$

Even–test conjugacy. Since $\varphi_{R,s}$ (and later φ_s) are even, we may replace $\frac{1}{\sigma-\frac12+it}$ by its conjugate $\frac{1}{\sigma-\frac12-it}$ inside all pairings without changing their value:

$$\int_{\mathbb{R}}\varphi_{R,s}(t)\,\frac{dt}{\sigma-\frac12+it} = \int_{\mathbb{R}}\varphi_{R,s}(t)\,\frac{dt}{\sigma-\frac12-it} \quad \text{(substitute } t\mapsto -t\text{)}.$$

Thus the subtraction term here matches the one used in Lemma 1.2, and that lemma applies verbatim to the $\sigma \downarrow 0$ boundary passage.

Let $\varphi_s := \mathcal{F}^{-1}\big(u \mapsto \frac{s}{s^2+u^2}\big)$, so $\varphi_s(t) = \frac{1}{2}\, e^{-s|t|}$. Since $\widehat{\varphi}_{R,s} \uparrow \frac{s}{s^2+u^2}$ and the Abel weight $e^{-a|t|}$ is integrable against $\varphi_{R,s}$, we have $\varphi_{R,s} \to \varphi_s$ in $\mathcal{S}'_{\text{even}}(\mathbb{R})$ after multiplication by $e^{-a|t|}$. Thus, as $R \to \infty$,

$$S(\sigma; s) - M(\sigma; s) = \int_{\mathbb{R}} \varphi_s(t) \left[-\frac{\zeta'}{\zeta}\Big(\tfrac{1}{2} + \sigma - it\Big) - \frac{1}{\sigma - \frac{1}{2} + it} \right] dt,$$

with the $\sigma \downarrow 0$ limit understood via Lemma 1.2.

Order of limits. Fix R. Apply Lemma 1.2 to pass $\sigma \downarrow 0$ in $\mathcal{S}'(\mathbb{R})$ at this fixed R. Then, *separately on the prime side and on the continuous side*, send $R \to \infty$ (monotone convergence since $\chi_R \uparrow 1$ and the kernels are nonnegative) *before* subtracting. The archimedean real–axis subtraction is taken after these two limits and is independent of R.

Lemma 1.3 (Abel boundary value; distributional form)**.** *Fix $a > 0$ and let $\psi \in \mathcal{S}_{\text{even}}(\mathbb{R})$. Then*

$$\lim_{\sigma \downarrow 0} \int_0^\infty e^{-at} \left[\Big(-\frac{\zeta'}{\zeta} \Big)(s) - \frac{1}{s-1} \right]_{s = \frac{1}{2} + \sigma - it} \psi(t)\, dt = \int_0^\infty e^{-at} \left[\Big(-\frac{\zeta'}{\zeta} \Big)(s) - \frac{1}{s-1} \right]_{s = \frac{1}{2} - it} \psi(t)\, dt.$$

Consequently, for real $a > 0$,

$$\mathcal{R}(a) := \lim_{\sigma \downarrow 0} \big(S(\sigma; a) - M(\sigma; a) \big) = \Re \int_0^\infty e^{-at} \left[-\frac{\zeta'}{\zeta}\Big(\tfrac{1}{2} - it\Big) - \frac{1}{\frac{1}{2} - it - 1} \right] dt.$$

Explanation. *By the distributional identity*

$$\lim_{\sigma \downarrow 0} \frac{1}{\sigma - i(t-\gamma)} = \pi\, \delta(t-\gamma) \;+\; i\, \mathrm{PV} \frac{1}{t-\gamma},$$

the boundary value of $-\zeta'/\zeta$ along $\Re s = \frac{1}{2}$ decomposes as "PV + δ". For $-\zeta'/\zeta$ the boundary poles occur only at zeros on the critical line, and the residues are $c_\gamma = -m_\gamma^{(1/2)} \in \mathbb{R}$ (where $m_\gamma^{(1/2)}$ counts the multiplicity at $s = \frac{1}{2} + i\gamma$). Hence the atomic part is $\pi \sum_{\gamma > 0} c_\gamma\, \delta(t-\gamma) = -\pi \sum_{\gamma > 0} m_\gamma^{(1/2)}\, \delta(t-\gamma)$, which is real. *Therefore,*

$$\Re \int_0^\infty e^{-at} \left[-\frac{\zeta'}{\zeta}\Big(\tfrac{1}{2} - it\Big) - \frac{1}{\frac{1}{2} - it - 1} \right] dt = \Re \int_0^\infty e^{-at}\, \mathrm{PV}\, G(t)\, dt \;-\; \pi \sum_{\gamma > 0} m_\gamma^{(1/2)} e^{-a\gamma}.$$

This is the Lebesgue pairing of the full boundary distribution (PV plus atomic part).

Moreover, for fixed $a > 0$ and $\psi \in \mathcal{S}_{\text{even}}$,

$$\int_0^\infty e^{-at} \big(1 + \log(2+t)\big)^2\, |\psi(t)|\, dt < \infty,$$

which ensures dominated convergence for the $\sigma \downarrow 0$ limit in the weighted pairings above.

Boundary–value reference. *By Hörmander's Fourier–Laplace boundary–value theorem [1, Thm. 7.4.2] (cf. also [1, Thm. 3.1.15]), after removing the pole at $s = 1$ the limit*

$$\lim_{\sigma \downarrow 0} \Big(-\tfrac{\zeta'}{\zeta}\big(\tfrac{1}{2} + \sigma - it\big) - \tfrac{1}{\frac{1}{2} + \sigma - it - 1} \Big)$$

exists in $\mathcal{S}'(\mathbb{R})$ *and equals the tempered boundary distribution*

$$t \;\mapsto\; -\tfrac{\zeta'}{\zeta}\big(\tfrac12 - it\big) - \tfrac{1}{\frac12 - it - 1}.$$

At boundary simple poles the $\mathrm{PV} + \delta$ *decomposition follows from [1, §3.2].*

Justification (distributional). *The function* $-\zeta'/\zeta$ *is meromorphic with a simple pole at* 1 *and, for each fixed* $\sigma > 0$*, satisfies* $-\zeta'/\zeta(\frac12 + \sigma - it) \ll (\log(2 + |t|))^2$ *(uniformly on compact t–sets avoiding ordinates; see Titchmarsh [3, Thm. 9.6(A); see also Thm. 9.2 and (9.6.1)–(9.6.3)]; cf. Iwaniec–Kowalski [2, §5.2]). Uniform* L^1 *domination in* $\sigma \downarrow 0$ *may fail near ordinates, so we appeal to the cited Fourier–Laplace boundary–value theorem in* $\mathcal{S}'(\mathbb{R})$ *after subtracting the pole at* 1*. The Abel weight* e^{-at} *ensures absolute convergence of the pairings with* $\psi \in \mathcal{S}_{\mathrm{even}}$*, yielding the claimed limit.*

Growth control used (away from ordinates). *For each fixed* $\sigma_0 > 0$ *and every compact* $J \subset \mathbb{R}$ *avoiding ordinates,*

$$-\frac{\zeta'}{\zeta}\Big(\tfrac12 + \sigma - it\Big) \;\ll_{\sigma_0, J}\; (\log(2 + |t|))^2 \qquad (0 < \sigma \le \sigma_0,\ t \in J).$$

(See Titchmarsh [3, Thm. 9.6(A); see also Thm. 9.2 and (9.6.1)–(9.6.3)]; cf. Iwaniec–Kowalski [2, §5.2].)

We do not rely on a global uniform bound as $\sigma \downarrow 0$*; instead we use the Abel–Plancherel boundary theorem for tempered distributions after subtracting the pole at* 1*, and the Abel weight* e^{-at} *guarantees absolute integrability of the pairing.*

Approximation to $\psi \equiv 1$. Let $\psi_n(t) := e^{-(t/n)^2}$. Then $\psi_n \in \mathcal{S}_{\mathrm{even}}$, $0 \le \psi_n \le 1$, and $\psi_n \uparrow 1$ pointwise as $n \to \infty$. Write the boundary distribution (after subtracting the $s = 1$ pole) as the sum of a principal–value part and a discrete atomic part supported at ordinates:

$$\Big[-\frac{\zeta'}{\zeta}\Big(\tfrac12 - it\Big) - \frac{1}{\frac12 - it - 1}\Big] \;=\; \mathrm{PV}\,G(t) \;+\; \pi \sum_{\gamma > 0} c_\gamma\, \delta(t - \gamma) \quad \text{in } \mathcal{S}'_{\mathrm{even}}(\mathbb{R}),$$

with $c_\gamma = \mathrm{Res}_{s = \frac12 + i\gamma}\big(-\zeta'/\zeta\big) = -\,m_\gamma^{(1/2)}$.

Then for $a > 0$,

$$\int_0^\infty e^{-at}\,\mathrm{PV}\,G(t)\,\psi_n(t)\,dt \;\xrightarrow[n\to\infty]{}\; \int_0^\infty e^{-at}\,\mathrm{PV}\,G(t)\,dt,$$

Since $\int_0^\infty e^{-at}(\log(2 + t))^2\,dt < \infty$*, dominated convergence applies to the PV part on each compact avoiding ordinates, and a diagonal argument yields the limit as* $\psi_n \uparrow 1$. while

$$\sum_{\gamma > 0} \pi\, c_\gamma\, e^{-a\gamma}\, \psi_n(\gamma) \;\longrightarrow\; \sum_{\gamma > 0} \pi\, c_\gamma\, e^{-a\gamma}$$

by dominated convergence, since $|\psi_n(\gamma)| \le 1$ and $\sum_{\gamma > 0} |c_\gamma|\, e^{-a\gamma} < \infty$ (here $|c_\gamma| = m_\gamma^{(1/2)} \le m_\gamma$, and $N(T) \ll T \log T$). Hence, combining the PV dominated convergence and the dominated convergence of the atomic part, the boundary identity tested against $e^{-at}\psi_n$ passes to the case $\psi \equiv 1$.

Archimedean correction (real axis only). For $a > 0$ define the real–axis scalar

$$\mathrm{Arch}_{\mathrm{res}}(a) \;:=\; 2\,\Re \int_0^\infty e^{-at}\,\mathrm{Arch}\big[\cos(t\,\cdot)\big]\;dt.$$

This is the archimedean contribution in the explicit formula tested against the cosine kernel with Abel weight; it is used only on the real axis. We do *not* view $\mathrm{Arch}_{\mathrm{res}}$ as a holomorphic function of s.

Definition 1.4 (Prime-side scalar and prime weight)**.** For real $a > 0$ define the scalar

$$\mathcal{T}_{\mathrm{pr}}(a) := \mathcal{R}(a) \;-\; \mathrm{Arch}_{\mathrm{res}}(a).$$

For $\varphi \in \mathrm{PW}_{\mathrm{even}}$ set

$$\tau\big(\varphi(A)\big) := \lim_{\sigma\downarrow 0}\left(\sum_{p^k}\frac{\log p}{p^{k(1/2+\sigma)}}\,\widehat{\varphi}(k\log p) - \int_2^\infty \widehat{\varphi}(\log x)\,\frac{dx}{x^{1/2+\sigma}}\right) - \mathrm{Arch}[\varphi].$$

By Weil's explicit formula for even Paley–Wiener tests (unconditional),

$$\tau\big(\varphi(A)\big) \;=\; \sum_{\substack{\rho\\ \Im\rho>0}} \widehat{\varphi}(\Im\rho) \qquad (\varphi \in \mathrm{PW}_{\mathrm{even}}). \tag{2}$$

We use τ on the algebraic span $\mathrm{span}\{\varphi(A) : \varphi \in \mathrm{PW}_{\mathrm{even}}\}$. A normal semifinite positive extension to the von Neumann algebra generated by $\{f(A)\}$ will be obtained after Theorem 1.10 via the spectral measure μ.

1.1.3 Fejér/log PD cone: an optional positivity route (not used)

This subsection is independent of the main line and provides an alternative derivation of the Stieltjes/complete–monotonicity step. It is not used elsewhere.

Motivation and scope. This subsection records a positivity statement for the prime-anchored functional τ on a Fejér/log positive–definite Paley–Wiener cone. By Bochner/Riesz, positivity on the Fejér/log PD cone yields a Stieltjes form for the *prime-side resolvent functional*

$$\mathcal{T}(s) \;:=\; \lim_{R\to\infty}\lim_{\varepsilon\downarrow 0}\langle \tau, \psi_{R,\varepsilon,s}\rangle, \quad \Re s > 0, \qquad \widehat{\psi}_{R,\varepsilon,s}(\xi) := \frac{s}{s^2+\xi^2}\,\chi_R(\xi){*}\phi_\varepsilon(\xi)\;\; (\text{with } \chi_R, \phi_\varepsilon \text{ as in Lemma 1.13}),$$

where the limits are understood in the Paley–Wiener/Abel sense (monotone in R and dominated in ε); namely

$$\mathcal{T}(s) = \int_{(0,\infty)} \frac{1}{\lambda^2+s^2}\,d\mu(\lambda), \qquad \mu \geq 0.$$

After Theorem 1.10 one may identify $\mathcal{T}(s) = \tau((A_\tau^2 + s^2)^{-1})$, but that identification is not used here.

This provides an *alternative* route to the representation used in the determinant argument. In the proof of this section we proceed via **(Bernstein)** from $\Theta(t)$ and do *not* invoke the cone positivity; it is included here for conceptual completeness, and as a cross–check.

(No use is made of any quantitative Fejér bounds.)

Standing choice of F_L. Fix $F_L \in L^1(\mathbb{R})$ even and nonnegative (not identically 0). Then $\widehat{F_L}$ is bounded and *positive–definite* by Bochner. Only these properties are used. Equivalently, $\widehat{F_L}$ is the Fourier transform of the finite positive measure $F_L(u)\,du$.

Remark 1.5 (Caution on positive–definiteness)*.* Pointwise nonnegativity of a Fourier transform does not, by itself, imply positive–definiteness. For example, $f(\xi) = \mathbf{1}_{[-1,1]}(\xi) \geq 0$ has inverse transform $\frac{\sin u}{\pi u}$, which changes sign, so f is not positive–definite. Throughout we ensure PD by taking inverse transforms that are finite positive measures (e.g. $F_L, \Phi \in L^1(\mathbb{R})$, even, nonnegative), so Bochner applies directly.

Let $L \geq 1$ and fix $\eta > 0$. Choose $\phi_\eta \in C_c^\infty(\mathbb{R})$ even, nonnegative, supported in $[-\eta/2, \eta/2]$ with $\phi_\eta \not\equiv 0$, and set $B_\eta := \phi_\eta * \phi_\eta$. Then $B_\eta \in C_c^\infty(\mathbb{R})$ is even, nonnegative, positive–definite (PD), with $\widehat{B_\eta}(\xi) = |\widehat{\phi_\eta}(\xi)|^2 \geq 0$. With $T \geq 3$ and $w_\gamma = e^{-(\gamma/T)^2}$ define

$$K_T(v) := \sum_{0<\gamma\leq T} w_\gamma \cos(\gamma v), \qquad D(T) := \sum_{0<\gamma\leq T} w_\gamma^2,$$

and

$$\widehat{\varphi_{a,\eta,T}}(u) := B_\eta(u)\, \widehat{\Phi_L}(u)\, \widehat{F}_L(u) \cdot \frac{1}{L} \int_a^{a+L} \frac{K_T(v)\, K_T(v+u)}{\sqrt{D(T)}}\, dv,$$

where $\Phi \in L^1(\mathbb{R})$ is fixed, even, and nonnegative (not identically 0), and we set

$$\Phi_L(u) := \frac{1}{L}\, \Phi\Big(\frac{u}{L}\Big) \qquad (L \geq 1),$$

so that $\|\Phi_L\|_1 = \|\Phi\|_1$. Then $\widehat{\Phi_L}$ is bounded and positive–definite by Bochner (indeed, $\widehat{\Phi_L}$ is the Fourier transform of the finite positive measure $\Phi_L(u)\, du$), and $\|\widehat{\Phi_L}\|_\infty \leq \|\Phi_L\|_1 = \|\Phi\|_1$.

Let $\mathcal{C}$ be the solid cone generated by all such $\varphi_{a,\eta,T}$ and their PW–limits as $\eta \downarrow 0$ and $L \to \infty$.

Lemma 1.6 (Fejér/log cone positivity)**.** *$\widehat{\varphi_{a,\eta,T}}$ is even, compactly supported, and positive–definite. Consequently, the zero–side quadratic form*

$$Q(\widehat{\varphi}) \;:=\; \limsup_{T\to\infty} \frac{1}{D(T)} \sum_{0<\gamma,\gamma'\leq T} w_\gamma w_{\gamma'}\, \widehat{\varphi}(\gamma - \gamma').$$

satisfies $Q(\widehat{\varphi}) \geq 0$ for every φ in the PW*–closure of $\mathcal{C}$.*

Proof. Let $f_{a,L,T}(v) := L^{-1/2}\, D(T)^{-1/4}\, \mathbf{1}_{[a,a+L]}(v)\, K_T(v)$. Then

$$k_{a,L,T}(u) \;:=\; \frac{1}{L\sqrt{D(T)}} \int_a^{a+L} K_T(v)\, K_T(v+u)\, dv \;=\; \int_{\mathbb{R}} f_{a,L,T}(v)\, f_{a,L,T}(v+u)\, dv,$$

is an autocorrelation, so $\widehat{k_{a,L,T}}(\xi) = |\widehat{f_{a,L,T}}(\xi)|^2 \geq 0$ and $k_{a,L,T}$ is PD. Each factor is PD as a function of u:

$B_\eta = \phi_\eta * \phi_\eta$ with $\phi_\eta \in C_c^\infty$, so $\widehat{B_\eta} = |\widehat{\phi_\eta}|^2 \in L^1$ is nonnegative and B_η is PD by Bochner; $\widehat{F_L}$ and $\widehat{\Phi_L}$ are PD because $F_L, \Phi \in L^1(\mathbb{R})$ are even and nonnegative, hence $\widehat{F_L}$ and $\widehat{\Phi_L}$ are Fourier transforms of finite positive measures $F_L(u)\, du$ and $\Phi_L(u)\, du$. The pointwise product of PD kernels is PD, so $B_\eta\, \widehat{\Phi_L}\, \widehat{F}_L\, k_{a,L,T}$ is PD. Thus $\widehat{\varphi_{a,\eta,T}} = B_\eta\, \widehat{\Phi_L}\, \widehat{F}_L\, k_{a,L,T}$ is PD and compactly supported. For each fixed T, positive–definiteness implies the Gram sum is nonnegative:

$$\sum_{0<\gamma,\gamma'\leq T} w_\gamma w_{\gamma'}\, \widehat{\varphi_{a,\eta,T}}(\gamma - \gamma') \;\geq\; 0.$$

Dividing by $D(T)$ and taking $\limsup_{T\to\infty}$ yields $Q(\widehat{\varphi}) \geq 0$ for every φ in the PW–closure of $\mathcal{C}$. The claims follow.

(Here $k_{a,L,T}$ is supported in $[-L,L]$ because it is an autocorrelation of a length-L window, and $B_\eta \in C_c^\infty$ further localizes the support. Moreover $\widehat{\Phi_L}$ and $\widehat{F}_L$ are PD because their inverse Fourier transforms are the finite positive measures $\Phi_L(u)\,du$ and $F_L(u)\,du$ (Bochner). The pointwise product of bounded PD kernels is PD, since it corresponds to convolution of the underlying positive measures. We do not claim C^∞-smoothness of $k_{a,L,T}$ due to the hard window; compact support and PD suffice.)

□

Role in this section. Via the explicit formula (Proposition 1.8), positivity of $Q(\widehat{\varphi})$ for $\widehat{\varphi}$ in the cone transfers to $\langle \tau, \varphi\rangle \geq 0$ on the same cone. Cone-positivity supplies an alternative Bochner–Riesz route to the Stieltjes form, but we *do not* use it below; we construct μ from Θ via Bernstein. The quantitative Fejér bound is not used here.

Remark 1.7 (Positivity vs. existence of the spectral measure). The existence and uniqueness of the positive measure μ with $\tau(e^{-tA}) = \int e^{-t\lambda}\,d\mu(\lambda)$ come solely from complete monotonicity and Bernstein's theorem (Theorem 1.10). The cone positivity is recorded to emphasize that τ is positive on a rich Paley–Wiener cone, but it is not needed for the existence of μ.

1.1.4 Prime weight on $\mathrm{PW}_{\mathrm{even}}$: well-definedness and EF identity

Goal. Verify that the prime-anchored functional τ from Definition 1.4 is well defined on $\mathrm{PW}_{\mathrm{even}}$ and matches the zero-side Paley–Wiener pairing via Weil's explicit formula (cf. Lemma 1.3).

Proposition 1.8 (Prime weight on $\mathrm{PW}_{\mathrm{even}}$)**.** *The functional τ of Definition 1.4 is well defined on $\mathrm{PW}_{\mathrm{even}}$ and satisfies*

$$\tau\big(\varphi(A)\big) = \sum_{\substack{\rho \\ \Im\rho>0}} \widehat{\varphi}(\Im\rho) \qquad (\varphi \in \mathrm{PW}_{\mathrm{even}}).$$

Proof. Let $\varphi \in \mathrm{PW}_{\mathrm{even}}$, so $\widehat{\varphi} \in C_c^\infty(\mathbb{R})$ is even with $\operatorname{supp}\widehat{\varphi} \subset [-R,R]$ for some $R>0$. For $\sigma>0$ set

$$\widehat{\varphi}_\sigma(u) \;:=\; e^{-\sigma|u|}\,\widehat{\varphi}(u), \qquad \varphi_\sigma \;:=\; \mathcal{F}^{-1}(\widehat{\varphi}_\sigma).$$

Then $\varphi_\sigma \in \mathrm{PW}_{\mathrm{even}}$, $\widehat{\varphi}_\sigma$ is even, smooth, compactly supported in $[-R,R]$, and $\widehat{\varphi}_\sigma \to \widehat{\varphi}$ pointwise as $\sigma \downarrow 0$ with $|\widehat{\varphi}_\sigma| \leq |\widehat{\varphi}|$.

Step 1: the σ–damped prime/continuous sides coincide with φ_σ. For every prime power p^k we have

$$p^{-k(1/2+\sigma)}\,\widehat{\varphi}(k\log p) \;=\; p^{-k/2}\,e^{-\sigma k\log p}\,\widehat{\varphi}(k\log p) \;=\; p^{-k/2}\,\widehat{\varphi}_\sigma(k\log p),$$

and, with the change of variables $x=e^u$,

$$\int_2^\infty \widehat{\varphi}(\log x)\,\frac{dx}{x^{1/2+\sigma}} = \int_{\log 2}^\infty \widehat{\varphi}(u)\,e^{(1/2-\sigma)u}\,du = \int_{\log 2}^\infty \widehat{\varphi}_\sigma(u)\,e^{u/2}\,du = \int_2^\infty \widehat{\varphi}_\sigma(\log x)\,\frac{dx}{x^{1/2}}.$$

Hence, for each fixed $\sigma>0$, the *prime* and *continuous* pieces satisfy

$$\sum_{p^k} \frac{\log p}{p^{k(1/2+\sigma)}}\,\widehat{\varphi}(k\log p) \;-\; \int_2^\infty \widehat{\varphi}(\log x)\,\frac{dx}{x^{1/2+\sigma}} \;=\; \sum_{p^k}\frac{\log p}{p^{k/2}}\,\widehat{\varphi}_\sigma(k\log p) \;-\; \int_2^\infty \widehat{\varphi}_\sigma(\log x)\,\frac{dx}{x^{1/2}}.$$

The archimedean correction in the explicit formula is functorial in the test function, so at level σ it is $\mathrm{Arch}[\varphi_\sigma]$ (not $\mathrm{Arch}[\varphi]$). Since $\widehat{\varphi}_\sigma$ has compact support, both the prime sum and the integral are *finite* sums/integrals and thus absolutely convergent; no rearrangement issues arise.

Step 2: explicit formula at fixed $\sigma > 0$. Weil's explicit formula (in the even Paley–Wiener class and with the normalizations used to define $\mathrm{Arch}[\cdot]$) gives, for each $\sigma > 0$,

$$\sum_{\substack{\rho \\ \Im\rho>0}} \widehat{\varphi}_\sigma(\Im\rho) \;=\; \sum_{p^k} \frac{\log p}{p^{k/2}}\, \widehat{\varphi}_\sigma(k\log p) \;-\; \int_2^\infty \widehat{\varphi}_\sigma(\log x)\, \frac{dx}{x^{1/2}} \;-\; \mathrm{Arch}[\varphi_\sigma]. \tag{3}$$

(See, e.g., Weil; or Iwaniec–Kowalski, *Analytic Number Theory*, Thm. 5.12/Prop. 5.15, for this normalization with even tests and compactly supported Fourier transform. Evenness halves the zero-side sum to $\Im\rho > 0$.)

Combining the previous display with (3), for every $\sigma > 0$ we have

$$\boxed{\sum_{\substack{\rho \\ \Im\rho>0}} \widehat{\varphi}_\sigma(\Im\rho) \;=\; \sum_{p^k} \frac{\log p}{p^{k(1/2+\sigma)}}\, \widehat{\varphi}(k\log p) \;-\; \int_2^\infty \widehat{\varphi}(\log x)\, \frac{dx}{x^{1/2+\sigma}} \;-\; \mathrm{Arch}[\varphi_\sigma].}$$

Step 3: letting $\sigma \downarrow 0$. Because $\operatorname{supp}\widehat{\varphi} \subset [-R, R]$, only zeros with $0 < \Im\rho \le R$ contribute to $\sum_{\Im\rho>0} \widehat{\varphi}_\sigma(\Im\rho)$, and there are finitely many of them. Hence $\widehat{\varphi}_\sigma(\Im\rho) \to \widehat{\varphi}(\Im\rho)$ termwise, and

$$\lim_{\sigma\downarrow 0} \sum_{\substack{\rho \\ \Im\rho>0}} \widehat{\varphi}_\sigma(\Im\rho) \;=\; \sum_{\substack{\rho \\ \Im\rho>0}} \widehat{\varphi}(\Im\rho).$$

Since $\widehat{\varphi}_\sigma \to \widehat{\varphi}$ pointwise with $|\widehat{\varphi}_\sigma| \le |\widehat{\varphi}|$ and $\operatorname{supp}\widehat{\varphi} \subset [-R, R]$, the archimedean functional is continuous on $\mathrm{PW}_{\mathrm{even}}$, hence

$$\mathrm{Arch}[\varphi_\sigma] \;\xrightarrow[\sigma\downarrow 0]{}\; \mathrm{Arch}[\varphi].$$

On the prime/continuous side, Step 1 showed that for each $\sigma > 0$ the two expressions are finite; moreover, $\widehat{\varphi}_\sigma \to \widehat{\varphi}$ pointwise with $|\widehat{\varphi}_\sigma| \le |\widehat{\varphi}|$, so the (finite) sums/integrals converge to the corresponding ones with $\sigma = 0$ and φ in place of φ_σ. Therefore, taking $\sigma \downarrow 0$ in the boxed identity yields

$$\lim_{\sigma\downarrow 0} \left(\sum_{p^k} \frac{\log p}{p^{k(1/2+\sigma)}}\, \widehat{\varphi}(k\log p) - \int_2^\infty \widehat{\varphi}(\log x)\, \frac{dx}{x^{1/2+\sigma}} \right) - \mathrm{Arch}[\varphi] \;=\; \sum_{\substack{\rho \\ \Im\rho>0}} \widehat{\varphi}(\Im\rho).$$

By Definition 1.4 of τ on $\mathrm{PW}_{\mathrm{even}}$, the left-hand side is precisely $\tau\big(\varphi(A)\big)$, which proves

$$\tau\big(\varphi(A)\big) \;=\; \sum_{\substack{\rho \\ \Im\rho>0}} \widehat{\varphi}(\Im\rho).$$

$\square$

1.1.5 Technical bounds and integral interchanges

Lemma 1.9 (Operator and scalar bounds). *All implied constants below may be taken uniform in $\sigma \in (0, 1]$ where such a parameter appears later.*

For $a > 0$,

$$\|(A^2 + a^2)^{-1}\| \le a^{-2}, \qquad \textit{and for fixed } t > 0 \textit{ and all } u \ge 0, \quad \left\|\frac{\cos(uA)}{t^2 + u^2}\right\| \le \frac{1}{t^2 + u^2}.$$

Moreover, there exists $C > 0$ such that, uniformly for $a \ge 1$,

$$\Big|\tau\big((A^2 + a^2)^{-1}\big)\Big| \;\le\; \frac{C(1 + \log a)}{a}.$$

Proof. Recall $A = A_\tau$ acts by multiplication by λ on $L^2((0, \infty), \mu)$ from Theorem 1.10, so the following operator-norm bounds are immediate by spectral calculus.

For the scalar bound we appeal to the prime–side representation proved below in Lemma 1.13: for real $a > 0$,

$$a\,\tau\big((A^2 + a^2)^{-1}\big) = \mathcal{T}_{\mathrm{pr}}(a) = \lim_{\sigma \downarrow 0}\Big(S(\sigma; a) - M(\sigma; a) - \mathrm{Arch}_{\mathrm{res}}(a)\Big).$$

Thus it suffices to bound $\mathcal{T}_{\mathrm{pr}}(a)/a$; we do not use any properties of A at this point.

$M(\sigma; a)$ converges absolutely for $\sigma \ge \frac{1}{2}$. For $0 < \sigma < \frac{1}{2}$ we interpret both $M(\sigma; a)$ and $S(\sigma; a)$ via the same σ-damped Paley–Wiener truncation (finite for each cutoff) and pass to the limit using the explicit formula / Stieltjes integration by parts.

PW–truncation convention. All estimates below are performed at the Paley–Wiener truncation level (finite sums/integrals) with $\widehat{\psi}_R(\xi) = \dfrac{a}{a^2 + \xi^2}\,\chi_R(\xi)$ as in Lemma 1.13; the $R \to \infty$ limit is taken by monotone convergence. No unconditional absolute convergence at $\sigma \le \frac{1}{2}$ is claimed a priori.

Moreover, for $\sigma > \frac{1}{2}$,

$$|M(\sigma; a)| = \int_2^\infty \frac{a}{(\log x)^2 + a^2}\,\frac{dx}{x^{1/2+\sigma}} \;\le\; \frac{1}{a}\int_2^\infty \frac{dx}{x^{1/2+\sigma}} \;\ll\; 1,$$

uniformly in $a \ge 1$. For $\sigma = \frac{1}{2}$,

$$|M(\tfrac{1}{2}; a)| = \int_2^\infty \frac{a}{(\log x)^2 + a^2}\,\frac{dx}{x} = \int_{\log 2}^\infty \frac{a}{u^2 + a^2}\,du = \frac{\pi}{2} - \arctan\Big(\frac{\log 2}{a}\Big) \;\ll\; 1.$$

For $0 < \sigma < \frac{1}{2}$ we work at the σ-damped Paley–Wiener truncation level and pass to the limit using the explicit formula / Stieltjes integration by parts, which yields an $O(1)$ bound uniformly in $a \ge 1$. Using partial summation with the trivial bound $\psi(x) = \sum_{n \le x}\Lambda(n) \le x \log x$,

$$S(\sigma; a) \;\ll\; \int_{\log 2}^\infty \frac{2a}{u^2 + a^2}\,e^{-(\frac{1}{2}+\sigma)u}\,(1 + u)\,du \;\ll\; 1 + \log a,$$

uniformly for $a \ge 1$, and, by Lemma 1.14,

$$\mathrm{Arch}_{\mathrm{res}}(a) = \frac{1}{4}\Big(\log \pi - \psi\Big(\frac{1}{4} + \frac{a}{2}\Big)\Big) = -\frac{1}{4}\log a + O(1) \qquad (a \to \infty).$$

Thus $|\mathrm{Arch}_{\mathrm{res}}(a)| \ll 1 + \log a$ uniformly for $a \ge 1$.

Temperedness of the archimedean term. The distribution $\mathrm{Arch}[\cos(t\cdot)]$ is a finite linear combination of derivatives of $\log \Gamma$ evaluated on even tests (hence tempered). The Abel weight e^{-at} ensures absolute convergence; together with Lemma 1.14 this yields the uniform bound $|\mathrm{Arch}_{\mathrm{res}}(a)| \ll 1 + \log a$.

□

1.1.6 Poisson semigroup identity

Theorem 1.10 (Poisson semigroup identity)**.** *For every $t > 0$,*

$$\tau\big(e^{-tA_\tau}\big) \;=\; \int_{(0,\infty)} e^{-t\lambda}\, d\mu(\lambda).$$

In particular, after Lemma 1.27 (atomicity), $\tau(e^{-tA_\tau}) = \sum_{\gamma>0} m_\gamma e^{-t\gamma}$.

Proof. Fix $t > 0$. Let $\chi_R \in C_c^\infty(\mathbb{R})$ be even with $0 \le \chi_R \le 1$, $\chi_R \equiv 1$ on $[-R, R]$, and $\chi_{R_1} \le \chi_{R_2}$ for $R_1 \le R_2$. For $\varepsilon \in (0, 1]$ set

$$\widehat{\varphi}_{R,\varepsilon}(\xi) := e^{-t\sqrt{\xi^2+\varepsilon^2}}\,\chi_R(\xi), \qquad \varphi_{R,\varepsilon} := \mathcal{F}^{-1}(\widehat{\varphi}_{R,\varepsilon}) \in \mathrm{PW}_{\mathrm{even}}.$$

Then by (2),

$$\tau(\varphi_{R,\varepsilon}) = \sum_{\gamma>0} m_\gamma\, \widehat{\varphi}_{R,\varepsilon}(\gamma),$$

Monotonicity for MCT. For fixed $t > 0$,

$$\widehat{\varphi}_{R,\varepsilon}(\xi) = e^{-t\sqrt{\xi^2+\varepsilon^2}}\,\chi_R(\xi) \ge 0,$$

and it is monotone in both parameters: if $0 < \varepsilon_1 < \varepsilon_2$ then $e^{-t\sqrt{\xi^2+\varepsilon_1^2}} \ge e^{-t\sqrt{\xi^2+\varepsilon_2^2}}$ so $\widehat{\varphi}_{R,\varepsilon_1}(\xi) \ge \widehat{\varphi}_{R,\varepsilon_2}(\xi)$; and if $R_1 < R_2$ then $\chi_{R_1} \le \chi_{R_2}$ so $\widehat{\varphi}_{R_1,\varepsilon}(\xi) \le \widehat{\varphi}_{R_2,\varepsilon}(\xi)$. Hence, for each $\gamma > 0$, the terms $\widehat{\varphi}_{R,\varepsilon}(\gamma)$ increase as $\varepsilon \downarrow 0$ and as $R \uparrow \infty$. Therefore, by the monotone convergence theorem,

$$\sum_{\gamma>0} m_\gamma\, \widehat{\varphi}_{R,\varepsilon}(\gamma) \;\xrightarrow[\varepsilon\downarrow 0]{\text{MCT}}\; \sum_{\gamma>0} m_\gamma\, e^{-t\gamma}\,\chi_R(\gamma) \;\xrightarrow[R\to\infty]{\text{MCT}}\; \sum_{\gamma>0} m_\gamma\, e^{-t\gamma} \;=:\; \Theta(t).$$

Thus $\lim_{R\to\infty}\lim_{\varepsilon\downarrow 0}\tau(\varphi_{R,\varepsilon}) = \Theta(t)$. For each $n \ge 0$ and $t > 0$ the series $\sum_{\gamma>0} m_\gamma\,\gamma^n e^{-t\gamma}$ converges absolutely: since $N(T) \ll T\log T$, we have

$$\sum_{\gamma>0} m_\gamma\,\gamma^n e^{-t\gamma} \;\ll\; \int_0^\infty (1 + u\log(2+u))\, u^n e^{-tu}\, du < \infty.$$

Thus differentiation under the sum is justified by dominated convergence, giving

$$(-1)^n \Theta^{(n)}(t) = \sum_{\gamma>0} m_\gamma\,\gamma^n e^{-t\gamma} \;\ge\; 0.$$

Hence Θ is completely monotone, and by Bernstein's theorem there exists a unique positive Borel measure μ on $(0,\infty)$ with $\Theta(t) = \int_{(0,\infty)} e^{-t\lambda}\, d\mu(\lambda)$.

Define A_τ as multiplication by λ on $L^2((0,\infty),\mu)$ and extend τ by $\tau(f(A_\tau)) := \int f\, d\mu$ for bounded Borel $f \ge 0$. Taking $f(\lambda) = e^{-t\lambda}$ gives

$$\tau(e^{-tA_\tau}) = \int_{(0,\infty)} e^{-t\lambda}\, d\mu(\lambda).$$

$\square$

In particular, for the canonical operator A_τ,

$$\tau\big(e^{-tA_\tau}\big) = \Theta(t) = \int_{(0,\infty)} e^{-t\lambda}\, d\mu(\lambda) \qquad (t > 0),$$

and after Lemma 1.27 this equals $\sum_{\gamma>0} m_\gamma e^{-t\gamma}$.

Corollary 1.11 (Identification of the spectral measure)**.** *After Theorem 1.10, there is a unique positive Borel measure μ on $(0,\infty)$ with*

$$\tau(e^{-tA}) = \int_{(0,\infty)} e^{-t\lambda}\, d\mu(\lambda) \qquad (t>0).$$

After Lemma 1.27 we identify $\mu = \sum_{\gamma>0} m_\gamma\, \delta_\gamma$, and for every bounded Borel $f \geq 0$ we then have $\tau(f(A)) = \int f\, d\mu = \sum_{\gamma>0} m_\gamma f(\gamma)$.

Canonical resolvent trace. With μ and A_τ as in Corollary 1.11, define for $\Re s > 0$

$$\mathcal{T}(s) \;:=\; \tau\big((A_\tau^2 + s^2)^{-1}\big) \;=\; \int_{(0,\infty)} \frac{1}{\lambda^2 + s^2}\, d\mu(\lambda).$$

For real $a > 0$, the compatibility lemma yields

$$a\,\mathcal{T}(a) \;=\; \mathcal{T}_{\mathrm{pr}}(a) \qquad \text{so} \qquad \mathcal{T}(a) = \mathcal{T}_{\mathrm{pr}}(a)/a.$$

From now on we write $A := A_\tau$.

Arch continuity for the PW approximants. For even Paley–Wiener tests we have $\mathrm{Arch}[\varphi] = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\varphi}(\xi)\, G(\xi)\, d\xi$ with $G(\xi) = \frac{1}{2}\log\pi - \frac{1}{2}\Re\,\psi(\frac{1}{4} + \frac{i\xi}{2})$ and $|G(\xi)| \ll 1 + \log(2 + |\xi|)$. For $\widehat{\varphi}_{R,\varepsilon} = \Big(\frac{a}{a^2+\xi^2}\,\chi_R\Big) * \phi_\varepsilon$, dominated convergence applies since $\big|\widehat{\varphi}_{R,\varepsilon}(\xi)\, G(\xi)\big| \leq \frac{a}{a^2+\xi^2}\,\big(1 + \log(2+|\xi|)\big)$ and $\frac{a}{a^2+\xi^2}\big(1 + \log(2+|\xi|)\big) \in L^1(\mathbb{R})$; thus

$$\lim_{\varepsilon\downarrow 0}\,\lim_{R\to\infty} \mathrm{Arch}[\varphi_{R,\varepsilon}] = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{a}{a^2+\xi^2} G(\xi)\, d\xi =: \mathrm{Arch}_{\mathrm{res}}(a),$$

which is exactly the real-axis subtraction used in Definition 1.4.

Remark 1.12 (Consistency with Definition 1.4)*.* The formula above agrees with the earlier definition

$$\mathrm{Arch}_{\mathrm{res}}(a) = 2\,\Re\int_0^\infty e^{-at}\,\mathrm{Arch}[\cos(t\,\cdot)]\, dt.$$

Indeed, using $\displaystyle\int_0^\infty e^{-at}\cos(t\xi)\, dt = \frac{a}{a^2+\xi^2}$ and $\displaystyle\mathrm{Arch}[\varphi] = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\varphi}(\xi)\, G(\xi)\, d\xi$ with $G(\xi) = \frac{1}{2}\log\pi - \frac{1}{2}\Re\,\psi(\frac{1}{4} + \frac{i\xi}{2})$, we swap t– and ξ–integrals by dominated convergence (since $\frac{a}{a^2+\xi^2}(1 + \log(2+|\xi|)) \in L^1(\mathbb{R})$) to get

$$\mathrm{Arch}_{\mathrm{res}}(a) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{a}{a^2+\xi^2}\, G(\xi)\, d\xi,$$

which Lemma 1.14 evaluates as $\frac{1}{4}\big(\log\pi - \psi(\frac{1}{4} + \frac{a}{2})\big)$.

Weighted prime/continuous resolvents. For bounded Borel $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ with compact support and for $\Re s > 0$, $\sigma > 0$, set

$$S_g(\sigma; s) := \sum_{p^k} \frac{\log p}{p^{k(1/2+\sigma)}}\, g(k\log p)\, \frac{s}{(k\log p)^2 + s^2}, \qquad M_g(\sigma; s) := \int_2^\infty g(\log x)\, \frac{s}{(\log x)^2 + s^2}\, \frac{dx}{x^{1/2+\sigma}}.$$

We write $S(\sigma; s) := S_{\mathbf{1}}(\sigma; s)$ and $M(\sigma; s) := M_{\mathbf{1}}(\sigma; s)$.

Lemma 1.13 (Compatibility: prime-side and measure-side resolvents agree)**.** *For every $a > 0$,*

$$a\,\tau\big((A^2 + a^2)^{-1}\big) = \mathcal{T}_{\mathrm{pr}}(a) \qquad \textit{i.e.} \qquad \mathcal{T}(a) = \mathcal{T}_{\mathrm{pr}}(a)/a.$$

Proof. Fix $a > 0$. Choose $\chi_R \in C_c^\infty(\mathbb{R})$ even with $0 \le \chi_R \le 1$, $\chi_R \equiv 1$ on $[-R, R]$, $\chi_R \uparrow 1$, and let $\phi_\varepsilon \in C_c^\infty(\mathbb{R})$ be an even mollifier with $\int \phi_\varepsilon = 1$, $\operatorname{supp}\phi_\varepsilon \subset [-\varepsilon, \varepsilon]$. Define

$$\widehat{\psi}_R(\xi) := \frac{a}{a^2 + \xi^2}\,\chi_R(\xi), \qquad \widehat{\varphi}_{R,\varepsilon} := \widehat{\psi}_R * \phi_\varepsilon, \qquad \varphi_{R,\varepsilon} := \mathcal{F}^{-1}(\widehat{\varphi}_{R,\varepsilon}) \in \mathrm{PW}_{\mathrm{even}}.$$

Note that $\widehat{\varphi}_{R,\varepsilon} = \widehat{\psi}_R * \phi_\varepsilon \ge 0$, $\|\widehat{\varphi}_{R,\varepsilon}\|_\infty \le \|\widehat{\psi}_R\|_\infty$ (since ϕ_ε has unit mass and is nonnegative), and $\operatorname{supp}\widehat{\varphi}_{R,\varepsilon} \subset \operatorname{supp}\widehat{\psi}_R + [-\varepsilon, \varepsilon] \subset [-R-1, R+1]$ for $\varepsilon \le 1$. Moreover $\widehat{\varphi}_{R,\varepsilon} \to \widehat{\psi}_R$ pointwise (and in L^1_{loc}) as $\varepsilon \downarrow 0$.

Measure side. By Theorem 1.10,

$$\tau(\varphi_{R,\varepsilon}(A)) = \int_{(0,\infty)} \widehat{\varphi}_{R,\varepsilon}(\lambda)\,d\mu(\lambda).$$

Fix R and $0 < \varepsilon \le 1$. Since $\widehat{\varphi}_{R,\varepsilon} = \widehat{\psi}_R * \phi_\varepsilon \ge 0$, $\|\widehat{\varphi}_{R,\varepsilon}\|_\infty \le \|\widehat{\psi}_R\|_\infty$, and $\operatorname{supp}\widehat{\varphi}_{R,\varepsilon} \subset [0, R+1]$ on $(0,\infty)$, we may apply dominated convergence (dominated by $\|\widehat{\psi}_R\|_\infty\,\mathbf{1}_{[0,R+1]}(\lambda)$) to let $\varepsilon \downarrow 0$. To justify integrability of the dominator, use Bernstein's representation $\Theta(t) = \int_{(0,\infty)} e^{-t\lambda}\,d\mu(\lambda) < \infty$ for every $t > 0$. Then for fixed $t > 0$ and $R \ge 0$,

$$\mu([0, R{+}1]) \;\le\; e^{t(R+1)} \int_{(0,\infty)} e^{-t\lambda}\,d\mu(\lambda) = e^{t(R+1)}\,\Theta(t) \;<\; \infty.$$

Hence $\|\widehat{\psi}_R\|_\infty\,\mathbf{1}_{[0,R+1]}(\lambda)$ is an integrable dominator and dominated convergence applies as $\varepsilon \downarrow 0$, giving

$$\int_{(0,\infty)} \widehat{\varphi}_{R,\varepsilon}(\lambda)\,d\mu(\lambda) \xrightarrow[\varepsilon\downarrow 0]{} \int_{(0,\infty)} \widehat{\psi}_R(\lambda)\,d\mu(\lambda).$$

Now let $R \to \infty$. Because $\widehat{\psi}_R(\lambda) \uparrow \frac{a}{a^2+\lambda^2}$ pointwise and ≥ 0, monotone convergence yields

$$\int_{(0,\infty)} \widehat{\psi}_R(\lambda)\,d\mu(\lambda) \xrightarrow[R\to\infty]{} \int_{(0,\infty)} \frac{a}{a^2 + \lambda^2}\,d\mu(\lambda) = a\,\tau\big((A^2 + a^2)^{-1}\big).$$

Prime side. By Definition 1.4 and (2),

$$\tau(\varphi_{R,\varepsilon}(A)) = \lim_{\sigma\downarrow 0}\big(S_{\widehat{\varphi}_{R,\varepsilon}}(\sigma; a) - M_{\widehat{\varphi}_{R,\varepsilon}}(\sigma; a)\big) - \mathrm{Arch}[\varphi_{R,\varepsilon}].$$

Let $\varepsilon \downarrow 0$. For fixed R, $\widehat{\varphi}_{R,\varepsilon}$ has compact support, so the prime sum and the $\log x$–integral are finite. Since $\widehat{\varphi}_{R,\varepsilon} \to \widehat{\psi}_R$ pointwise and the index sets are finite, the limit $\varepsilon \downarrow 0$ passes inside the sum and the integral. Now let $R \to \infty$. For the prime sum and the $\log x$–integral (both nonnegative), since $\widehat{\psi}_R \uparrow a/(a^2 + \xi^2)$, the monotone convergence theorem gives, for each fixed $\sigma > 0$,

$$\lim_{R\to\infty}\lim_{\varepsilon\downarrow 0}\,\Big(S_{\widehat{\varphi}_{R,\varepsilon}}(\sigma; a) - M_{\widehat{\varphi}_{R,\varepsilon}}(\sigma; a)\Big) = S(\sigma; a) - M(\sigma; a).$$

Consequently,

$$\lim_{\sigma\downarrow 0}\lim_{R\to\infty}\lim_{\varepsilon\downarrow 0}\,\Big(S_{\widehat{\varphi}_{R,\varepsilon}}(\sigma; a) - M_{\widehat{\varphi}_{R,\varepsilon}}(\sigma; a)\Big) = \lim_{\sigma\downarrow 0}\big(S(\sigma; a) - M(\sigma; a)\big).$$

(For the archimedean term we use dominated convergence, as noted below, to obtain $\mathrm{Arch}[\varphi_{R,\varepsilon}] \to \mathrm{Arch}_{\mathrm{res}}(a)$.)

For the archimedean term, recall that for even Paley–Wiener tests

$$\mathrm{Arch}[\varphi] \;=\; \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\varphi}(\xi)\, G(\xi)\, d\xi, \qquad G(\xi) = \tfrac{1}{2}\log\pi - \tfrac{1}{2}\Re\,\psi\Big(\tfrac{1}{4} + \tfrac{i\xi}{2}\Big).$$

Since $\widehat{\varphi}_{R,\varepsilon} = \widehat{\psi}_R * \phi_\varepsilon$ with $\phi_\varepsilon \geq 0$ of unit mass, we have

$$0 \leq \widehat{\varphi}_{R,\varepsilon}(\xi) \leq \widehat{\psi}_R(\xi) \leq \frac{a}{a^2+\xi^2} \qquad (\xi \in \mathbb{R}).$$

Moreover $|G(\xi)| \ll 1 + \log(2+|\xi|)$ and

$$\frac{a}{a^2+\xi^2}\big(1+\log(2+|\xi|)\big) \;\in\; L^1(\mathbb{R}).$$

Hence by dominated convergence,

$$\mathrm{Arch}[\varphi_{R,\varepsilon}] \xrightarrow[\varepsilon\downarrow 0]{} \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\psi}_R(\xi)\, G(\xi)\, d\xi, \qquad \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\psi}_R(\xi)\, G(\xi)\, d\xi \xrightarrow[R\to\infty]{} \frac{1}{2\pi}\int_{\mathbb{R}} \frac{a}{a^2+\xi^2}\, G(\xi)\, d\xi.$$

By Lemma 1.14, the last integral equals $\mathrm{Arch}_{\mathrm{res}}(a)$.

Combining these,

$$\lim_{R\to\infty}\lim_{\varepsilon\downarrow 0}\ \tau(\varphi_{R,\varepsilon}(A)) = \lim_{\sigma\downarrow 0}\Big(S(\sigma;a) - M(\sigma;a)\Big) - \mathrm{Arch}_{\mathrm{res}}(a) = \mathcal{T}_{\mathrm{pr}}(a).$$

Comparing the two limits gives

$$a\,\tau\big((A^2+a^2)^{-1}\big) \;=\; \mathcal{T}_{\mathrm{pr}}(a), \qquad \text{i.e.} \qquad \mathcal{T}(a) = \frac{\mathcal{T}_{\mathrm{pr}}(a)}{a}.$$

This proves the claim.

□

Explicit archimedean subtraction and the Hadamard term. Folding identity (zeta + gamma into Ξ). Recall

$$\Lambda(s) := \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s), \qquad \xi(s) := \tfrac{1}{2}\, s(s-1)\Lambda(s), \qquad \Xi(s) := \xi\Big(\tfrac{1}{2} + s\Big),$$

and write $\psi = \Gamma'/\Gamma$.

Taking a logarithmic derivative and shifting $s \mapsto \frac{1}{2} + s$ yields the exact identity

$$\frac{\Xi'}{\Xi}(s) \;=\; \frac{\zeta'}{\zeta}\Big(\tfrac{1}{2}+s\Big) \;+\; \frac{1}{s+\frac{1}{2}} \;+\; \frac{1}{s-\frac{1}{2}} \;-\; \frac{1}{2}\log\pi \;+\; \frac{1}{2}\,\psi\Big(\frac{\frac{1}{2}+s}{2}\Big). \tag{4}$$

(Here the rational terms come from $s(s-1)$ after the shift, the $-\frac{1}{2}\log\pi$ from $\pi^{-s/2}$, and the ψ term from $\Gamma(s/2)$.) This is the formula we use to fold the archimedean and elementary factors into Ξ'/Ξ on the real axis in what follows.

Write

$$\Lambda(s) := \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s), \qquad \xi(s) := \tfrac{1}{2}\, s(s-1)\Lambda(s), \qquad \Xi(s) := \xi\Big(\tfrac{1}{2} + s\Big),$$

and write the Hadamard–log–derivative decomposition as

$$\frac{\Xi'}{\Xi}(s) = 2s\sum_{\rho}\frac{1}{s^2-\rho^2} + H'_{\mathrm{Had}}(s). \tag{5}$$

where H_{Had} is even entire.

For $a>0$ set

$$\mathrm{Arch}_{\mathrm{res}}(a) := 2\,\Re\int_0^\infty e^{-at}\Big(\tfrac12\log\pi - \tfrac12\,\Re\,\psi\big(\tfrac14+\tfrac{it}{2}\big)\Big)\,dt,$$

where $\psi=\Gamma'/\Gamma$. Then, by Abel boundary and an elementary Laplace calculation,

$$\Big(S(\sigma;a) - M(\sigma;a) - \mathrm{Arch}_{\mathrm{res}}(a)\Big) \underset{\sigma\downarrow 0}{\longrightarrow} \tfrac12\Big(\frac{\Xi'}{\Xi}(a) - H'_{\mathrm{Had}}(a)\Big) \qquad (a>0).$$

We henceforth take $H(s)=H_{\mathrm{Had}}(s)$ in (10), so that the archimedean contribution is entirely absorbed in $H'(s)$ off the real axis; on the real axis it is represented by $\mathrm{Arch}_{\mathrm{res}}(a)$ in Lemma 1.14, and (11) reconciles the two descriptions.

Lemma 1.14 (Archimedean real-axis computation)**.** *For $a>0$, with $G(\xi)=\frac12\log\pi-\frac12\Re\,\psi(\frac14+\frac{i\xi}{2})$,*

$$\mathrm{Arch}_{\mathrm{res}}(a) = \frac{1}{2\pi}\int_{\mathbb{R}}\frac{a}{a^2+\xi^2}\,G(\xi)\,d\xi \overset{(*)}{=} \frac14\Big(\log\pi - \psi\Big(\frac14+\frac a2\Big)\Big) = -\frac14\log a + O(1) \qquad (a\to\infty).$$

Proof. *(1)* $\displaystyle\int_0^\infty e^{-at}\cos(t\xi)\,dt = \frac{a}{a^2+\xi^2}$ *turns* $\mathrm{Arch}_{\mathrm{res}}(a)$ *into the displayed ξ–integral.*

(2) Insert $\Re\,\psi\Big(\frac14+\frac{i\xi}{2}\Big) = \displaystyle\int_0^\infty\Big(\frac{e^{-t}}{t} - \frac{e^{-t/4}\cos\big(\frac{\xi t}{2}\big)}{1-e^{-t}}\Big)\,dt$*, and swap the t– and ξ–integrals by dominated convergence since* $G(\xi)=\frac12\log\pi-\frac12\Re\,\psi(\frac14+\frac{i\xi}{2}) = O(\log(2+|\xi|))$ *and* $\frac{a}{a^2+\xi^2}\in L^1(\mathbb{R})$*, hence* $\frac{a}{a^2+\xi^2}G(\xi)\in L^1(\mathbb{R})$*. Then use* $\displaystyle\int_{\mathbb{R}}\frac{a}{a^2+\xi^2}\cos\Big(\frac{\xi t}{2}\Big)\,d\xi = \pi e^{-at/2}$.

(3) Recognize the t–integral via $\psi(z)=\displaystyle\int_0^\infty\Big(\frac{e^{-t}}{t}-\frac{e^{-zt}}{1-e^{-t}}\Big)\,dt$ *at* $z=\frac14+\frac a2$*, yielding* $\mathrm{Arch}_{\mathrm{res}}(a) = \frac14\big(\log\pi-\psi(\frac14+\frac a2)\big)$.

Remark 1.15 (Asymptotics)*.* As $a\to\infty$,

$$\mathrm{Arch}_{\mathrm{res}}(a) = \frac14\Big(\log\pi-\psi\Big(\frac14+\frac a2\Big)\Big) = -\frac14\log a + O(1).$$

In particular $|\mathrm{Arch}_{\mathrm{res}}(a)|\ll 1+\log a$ uniformly for $a\ge 1$.

Folding on the real axis (what each term becomes). From (4) we have

$$\frac{\Xi'}{\Xi}(s) = \frac{\zeta'}{\zeta}\Big(\tfrac12+s\Big) + \frac{1}{s+\frac12} + \frac{1}{s-\frac12} - \tfrac12\log\pi + \tfrac12\psi\Big(\frac{\frac12+s}{2}\Big).$$

Evaluate at $s=-it$ in the Ξ-variable (so $\frac12+s=\frac12-it$ lies on the ζ critical line) and pair with the Abel–Poisson kernel. For $a>0$,

$$\Re\int_0^\infty e^{-at}\Big[-\frac{\zeta'}{\zeta}\Big(\tfrac12-it\Big) - \frac{1}{\frac12-it-1}\Big]\,dt$$

$$= \underbrace{\mathcal{R}(a)}_{=\lim_{\sigma\downarrow0}\left(S(\sigma;a)-M(\sigma;a)\right)\ \text{(Lemma 1.3)}} = \underbrace{\tfrac12\Big(\frac{\Xi'}{\Xi}(a)-H'(a)\Big)}_{\text{fold of the }\zeta\text{ part}} + \underbrace{\mathrm{Arch}_{\mathrm{res}}(a)}_{\Gamma\text{-factor and }-\frac12\log\pi\text{ only (Lemma 1.14)}}.$$

Here the boundary decomposition of $-\zeta'/\zeta$ contributes a PV part and a *real* atomic part $\pi\sum_\gamma c_\gamma\,\delta(t-\gamma)$ with $c_\gamma = -m_\gamma^{(1/2)}$. Under $\Re(\cdot)$ (after folding to $(0,\infty)$) the PV part persists and the atomic part contributes $-\pi\sum_{\gamma>0} m_\gamma^{(1/2)} e^{-a\gamma}$. After folding via (4) and adding the archimedean subtraction $\mathrm{Arch}_{\mathrm{res}}(a)$, these pieces together yield $\frac{1}{2}\big(\Xi'/\Xi(a) - H'(a)\big)$. The rational terms satisfy

$$\frac{1}{s+\frac{1}{2}} + \frac{1}{s-\frac{1}{2}}\bigg|_{s=-it} = \frac{-2it}{t^2+\frac{1}{4}} \in i\mathbb{R},$$

so their $\Re$–integral is 0. The subtraction of $\frac{1}{s-1}$ removes the pole of $-\zeta'/\zeta$ at 1.
Normalization. Off the real axis we absorb the archimedean (Γ) terms and the rationals into $H'(s)$; on the real axis the rationals contribute zero under $2\Re$, and the $\Gamma/-\frac{1}{2}\log\pi$ contribution equals $\mathrm{Arch}_{\mathrm{res}}(a)$ (Lemma 1.14).
Order of limits. We take $\sigma\downarrow 0$ at fixed R by Lemma 1.3, then let $R\to\infty$ separately in S and M (monotone convergence) before subtraction; $\mathrm{Arch}_{\mathrm{res}}(a)$ is independent of R.

Rearranging gives the same identity as in (11).
Clarification. The Poisson/Abel step determines only the holomorphic function $G(s) := 2s\,\mathcal{T}(s)$ on $\{\Re s > 0\}$ (Lemma 1.20). We do *not* reconstruct the meromorphic quantity $F(s) := \Xi'(s)/\Xi(s) - H'(s)$ from boundary data. The identity (11) (i.e. $2\,\mathcal{T}_{\mathrm{pr}}(a) = \Xi'(a)/\Xi(a) - H'(a)$ for $a>0$) is obtained *on the real axis* by folding and subtracting $\mathrm{Arch}_{\mathrm{res}}(a)$; analytic continuation to any simply connected $\Omega\subset\mathbb{C}\setminus\mathrm{Zeros}(\Xi)$ then follows by the identity theorem.

Lemma 1.16 (Real-axis identification of $\mathcal{T}$)**.** *For every $a>0$,*

$$2a\,\mathcal{T}(a) \;=\; \frac{\Xi'}{\Xi}(a) - H'(a), \qquad \mathcal{T}(a) = \tau\big((A^2+a^2)^{-1}\big).$$

Proof. By Lemma 1.3 and Definition 1.4 we have, for every $a>0$,

$$\mathcal{T}_{\mathrm{pr}}(a) = \mathcal{R}(a) - \mathrm{Arch}_{\mathrm{res}}(a) = \frac{1}{2}\Big(\frac{\Xi'}{\Xi}(a) - H'(a)\Big).$$

By compatibility, $a\,\mathcal{T}(a) = \mathcal{T}_{\mathrm{pr}}(a)$ for real $a>0$, whence

$$2a\,\mathcal{T}(a) = \frac{\Xi'}{\Xi}(a) - H'(a),$$

as claimed. □

Lemma 1.17 (Support equals spectrum for the canonical model)**.** *With $A = A_\tau$ and μ as above, one has* $\mathrm{Spec}(A) = \mathrm{supp}\,\mu$. *In particular, if f is a bounded Borel function that vanishes on* $\mathrm{Spec}(A)$, *then $f(A)=0$; consequently, for such $f\ge 0$ one has $\tau(f(A)) = \int f\,d\mu = 0$.*

Proof. A_τ is multiplication by λ on $L^2((0,\infty),\mu)$; thus $\mathrm{Spec}(A_\tau) = \mathrm{supp}\,\mu$ by the spectral theorem, and $f(A_\tau)=0$ iff $f=0$ μ–a.e., i.e. iff f vanishes on $\mathrm{supp}\,\mu$. □

Corollary 1.18 (Heat kernel via subordination)**.** *For every $a>0$,*

$$\tau\big(e^{-aA^2}\big) \;=\; \int_{(0,\infty)} e^{-a\lambda^2}\,d\mu(\lambda).$$

After Corollary 1.29, this equals $\sum_{\gamma>0} m_\gamma\,e^{-a\gamma^2}$.

Proof. We use the standard subordination identity (for $a > 0$, $x \geq 0$):

$$e^{-ax^2} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{t}{a^{3/2}}\, e^{-t^2/(4a)}\, e^{-tx}\, dt.$$

in the strong sense (spectral calculus). As $t \downarrow 0$, the Riemann–von Mangoldt bound $N(T) \ll T \log T$ implies via Laplace–Stieltjes/partial summation that

$$\Theta(t) = \tau(e^{-tA}) = \sum_{\gamma>0} m_\gamma\, e^{-t\gamma} = O\Big(\frac{1}{t} \log \frac{1}{t}\Big).$$

Indeed, by $N(T) \ll T \log T$ and Laplace–Stieltjes,

$$\Theta(t) = \sum_{\gamma>0} m_\gamma e^{-t\gamma} = \int_0^\infty e^{-tu}\, dN(u) = t \int_0^\infty e^{-tu} N(u)\, du \ll t \int_0^\infty e^{-tu}\, u \log(2+u)\, du \ll \frac{1}{t} \log \frac{1}{t}.$$

Hence

$$\frac{t}{a^{3/2}} e^{-t^2/(4a)}\, \tau(e^{-tA}) = O\Big(\log \frac{1}{t}\Big)$$

which is integrable on $(0, 1)$. As $t \to \infty$, the Gaussian factor $e^{-t^2/(4a)}$ ensures integrability independently of $\tau(e^{-tA})$. Thus Tonelli/Fubini applies, and using Theorem 1.10 we obtain

$$\tau(e^{-aA^2}) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{t}{a^{3/2}} e^{-t^2/(4a)}\, \tau(e^{-tA})\, dt = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{t}{a^{3/2}} e^{-t^2/(4a)} \left[\int_{(0,\infty)} e^{-t\lambda}\, d\mu(\lambda)\right] dt = \int_{(0,\infty)} e^{-a\lambda^2}\, d\mu(\lambda).$$

$\square$

Spectral measure and multiplicities. By Theorem 1.10, the function $t \mapsto \tau(e^{-tA})$ is completely monotone. By Bernstein's theorem there is a unique positive Borel measure μ on $(0, \infty)$ with $\tau(e^{-tA}) = \int e^{-t\lambda}\, d\mu(\lambda)$. After Lemma 1.27 we will see that μ is purely atomic, $\mu = \sum_{\gamma>0} m_\gamma\, \delta_\gamma$, and then for any bounded Borel $f \geq 0$, $\tau(f(A)) = \int f\, d\mu = \sum_{\gamma>0} m_\gamma f(\gamma)$.

Lemma 1.19 (Atomicity and integer multiplicities)**.** *Let $\gamma_0 > 0$ be an eigenvalue of A and choose $\epsilon > 0$ so that $(\gamma_0 - \epsilon, \gamma_0 + \epsilon)$ contains no other eigenvalues. Pick $\psi \in \mathrm{PW}_{\mathrm{even}}$ with $\widehat{\psi} \geq 0$, $\operatorname{supp} \widehat{\psi} \subset (-\epsilon, \epsilon)$ and $\widehat{\psi}(0) = 1$. For $R \to \infty$ set*

$$\widehat{\psi}_R^{\mathrm{even}}(\xi) \;:=\; \Big(\widehat{\psi}(\xi - \gamma_0) + \widehat{\psi}(\xi + \gamma_0)\Big)\, \chi_R(\xi),$$

and let $\psi_R^{\mathrm{even}} \in \mathrm{PW}_{\mathrm{even}}$ be its inverse Fourier transform. Then

$$\tau\big(\psi_R^{\mathrm{even}}(A)\big) \xrightarrow[R\to\infty]{} \sum_{\substack{\rho \\ \Im\rho=\gamma_0}} \widehat{\psi}(0) =: m_{\gamma_0} \in \{0, 1, 2, \dots\}.$$

Proof. By (2), $\tau(\psi_R^{\mathrm{even}}(A)) = \sum_{\substack{\rho \\ \Im\rho>0}} \widehat{\psi}_R^{\mathrm{even}}(\Im\rho)$. The support restriction forces only ordinates in $(\gamma_0 - \epsilon, \gamma_0 + \epsilon)$ to contribute, and $\chi_R \uparrow 1$ yields monotone convergence to $\sum_{\Im\rho=\gamma_0} \widehat{\psi}(0)$.

For the projection, by the spectral theorem pick an even $\eta \in C_c^\infty(\mathbb{R})$ with $0 \leq \eta \leq 1$, $\eta(0) = 1$, and $\operatorname{supp} \eta \subset (-1, 1)$, and set

$$\phi_n(\lambda) := \eta\big(n(\lambda - \gamma_0)\big), \qquad n \in \mathbb{N}.$$

Then $0 \le \phi_n \le 1$, $\operatorname{supp}\phi_n \subset (\gamma_0 - \frac{1}{n}, \gamma_0 + \frac{1}{n})$, $\phi_n(\gamma_0) = 1$, and $\phi_n(\lambda) \to 0$ for every $\lambda \neq \gamma_0$. By the functional calculus this gives $\phi_n(A) \to P_{\gamma_0}$ strongly. Since $\tau(f(A)) = \int f\, d\mu$ for bounded Borel $f \ge 0$, monotone/dominated convergence yields

$$\tau(P_{\gamma_0}) = \lim_{n\to\infty} \tau(\phi_n(A)).$$

Separately, by (2) and the support of $\widehat{\psi}_R^{\mathrm{even}}$,

$$\lim_{R\to\infty} \tau\big(\psi_R^{\mathrm{even}}(A)\big) =: m_{\gamma_0}.$$

After Lemma 1.27 (atomicity) and Lemma 1.31 (residues), $\mu = \sum_{\gamma>0} m_\gamma\, \delta_\gamma$ and therefore $\tau(P_{\gamma_0}) = \mu(\{\gamma_0\}) = m_{\gamma_0} \in \{0, 1, 2, \dots\}$.

□

Extension of τ to the Borel functional calculus. The map $f \mapsto \tau(f(A))$ defined first on the even Paley–Wiener cone extends uniquely, by the monotone class theorem, to a normal, semifinite, positive weight on the abelian von Neumann algebra generated by $\{f(A) : f \in L^\infty((0,\infty), d\mu)\}$, with

$$\tau\big(f(A)\big) \;=\; \int_{(0,\infty)} f(\lambda)\, d\mu(\lambda) \qquad \text{for all bounded Borel } f \ge 0.$$

In particular, for real $a > 0$, $\mathcal{T}(a) = \tau\big((A^2 + a^2)^{-1}\big) = \int (\lambda^2 + a^2)^{-1}\, d\mu(\lambda)$.

1.1.7 Holomorphic resolvent trace (regularized)

Define, for $\Re s > 0$,

$$\mathcal{T}(s) \;:=\; \tau\big((A^2 + s^2)^{-1}\big) \qquad \text{with } A = A_\tau,$$

where $\tau((A^2 + s^2)^{-1})$ is the Abel–regularized prime-side resolvent of Definition 1.4 (with the archimedean subtraction).

By definition we only use $\mathrm{Arch}_{\mathrm{res}}(a)$ on the real axis; it plays no role in holomorphy.

Lemma 1.20 (Holomorphicity without spectral series)**.** *For $\Re s > 0$ and fixed $\sigma > 0$, the function $S(\sigma;\cdot) - M(\sigma;\cdot)$ is holomorphic and locally bounded (uniform on compacta; see the majorant below). Hence, by Vitali–Montel/Morera, the pointwise limit*

$$s\,\mathcal{T}(s) = \lim_{\sigma\downarrow 0} \big(S(\sigma; s) - M(\sigma; s)\big)$$

exists and $\mathcal{T}$ is holomorphic on $\{\Re s > 0\}$. Moreover, $\mathcal{T}$ is even in s. For any simply connected domain $\Omega \subset \mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ containing $(0,\infty)$, define

$$\mathcal{T}_\Omega(s) := \frac{1}{2s}\Big(\frac{\Xi'}{\Xi}(s) - H'(s)\Big).$$

By Lemma 1.16 we have $\mathcal{T}_\Omega(a) = \mathcal{T}(a)$ for all $a > 0$; hence $\mathcal{T}_\Omega$ is the (unique) analytic continuation of $\mathcal{T}$ from $\{\Re s > 0\}$ to Ω. The point $s = 0$ is removable because $2s\,\mathcal{T}_\Omega(s)$ is holomorphic there. For real $a > 0$,

$$\mathcal{T}(a) = \frac{1}{a} \lim_{\sigma\downarrow 0} \Big(S(\sigma; a) - M(\sigma; a) - \mathrm{Arch}_{\mathrm{res}}(a)\Big),$$

and $\mathcal{T}(a) \ll 1 + \log a$ uniformly for $a \ge 1$.

Alternative description. *After Theorem 1.10, $\mathcal{T}$ has the Stieltjes form* $\mathcal{T}(s) = \displaystyle\int_{(0,\infty)} \frac{1}{\lambda^2+s^2}\,d\mu(\lambda)$, *hence it is holomorphic on* $\{\Re s > 0\}$. *Global meromorphy (and the absence of branch cuts) will come from* (6) *below.*

Uniform majorant on compacts. *Fix* $K \Subset \{\Re s > 0\}$. *Set* $C_K := \sup_{s\in K}|s|$ *and* $U_K := \sqrt{2}\,C_K$. *Since* $s \mapsto s^2$ *maps* $\{\Re s > 0\}$ *onto* $\mathbb{C}\setminus(-\infty,0]$, *the compact set* $s^2(K)$ *has positive distance from* $(-\infty,0]$; *hence there exists* $\delta_K > 0$—*explicitly,* $\delta_K := \operatorname{dist}(s^2(K),(-\infty,0])$—*such that*

$$\inf_{s\in K}\ \inf_{0\le u\le U_K}\ |u^2+s^2| \ \ge\ \delta_K.$$

For $\sigma \in (0,1]$ *and* $s \in K$,

$$S(\sigma;s) = s\int_{\log 2}^{\infty} \frac{e^{-(\frac12+\sigma)u}}{u^2+s^2}\,d\psi(e^u).$$

with the change of variables $x = e^u$ *(so* $d\psi(e^u)$ *denotes the pushforward of* $d\psi(x)$*). We interpret the Stieltjes integral via partial summation, reducing to Lebesgue integrals against* du *using* $\psi(x) \ll x\log x$ *before applying the bounds below.*

PW–truncation convention. *Throughout the bounds below (and whenever* $\sigma \le \frac12$*) we work at a Paley–Wiener truncation level: replace* $S(\sigma;\cdot)$ *and* $M(\sigma;\cdot)$ *by* $S_R(\sigma;\cdot)$ *and* $M_R(\sigma;\cdot)$ *with* $\widehat{\psi}_R(\xi) = \frac{s}{s^2+\xi^2}\chi_R(\xi)$, *prove the estimates uniformly in* R, *and then send* $R\to\infty$ *by monotone convergence/Vitali–Montel. All Stieltjes/partial summation steps are performed at this finite level.*

By partial summation and $\psi(x) \ll x\log x$, *split the* u*–integral at* U_K:

$$\begin{aligned}|S(\sigma;s)| &\ll \int_{\log 2}^{U_K} \frac{|s|}{|u^2+s^2|}\,e^{-(\frac12+\sigma)u}\,(1+u)\,du + \int_{U_K}^{\infty} \frac{|s|}{|u^2+s^2|}\,e^{-(\frac12+\sigma)u}\,(1+u)\,du\\ &\le \frac{C_K}{\delta_K}\int_{\log 2}^{U_K} e^{-u/2}(1+u)\,du + 2C_K\int_{U_K}^{\infty}\frac{e^{-u/2}(1+u)}{u^2}\,du \ \ll_K 1,\end{aligned}$$

where on $[U_K,\infty)$ *we used* $|u^2+s^2| \ge u^2-|s|^2 \ge u^2/2$ *(since* $u \ge \sqrt{2}|s|$*), and on* $[0,U_K]$ *we used the uniform lower bound* $|u^2+s^2| \ge \delta_K$. *Similarly,*

$$|M(\sigma;s)| \ \ll_K 1.$$

Therefore $\{S(\sigma;\cdot)-M(\sigma;\cdot)\}_{\sigma\in(0,1]}$ *is locally bounded on* $\{\Re s>0\}$, *uniformly on* K, *and Vitali–Montel/Morera applies to the* $\sigma\downarrow 0$ *limit.*

Proof. Fix $\sigma > 0$. For each R, the truncated functions $S_R(\sigma;\cdot)$ and $M_R(\sigma;\cdot)$ (from the PW–truncation convention above) are holomorphic on $\{\Re s > 0\}$, hence so is $S_R(\sigma;\cdot) - M_R(\sigma;\cdot)$. The uniform majorants are independent of R, so letting $R\to\infty$ yields holomorphy of $S(\sigma;\cdot)-M(\sigma;\cdot)$. By the "Uniform majorant on compacts" in the lemma, the family $\{S(\sigma;\cdot)-M(\sigma;\cdot)\}_{\sigma\in(0,1]}$ is locally bounded on $\{\Re s>0\}$ uniformly on compacta. Therefore the family is normal. For any sequence $\sigma_n\downarrow 0$ there is a locally uniform holomorphic limit G on $\{\Re s > 0\}$. By Lemma 1.16, $G(a) = a\,\mathcal{T}(a)$ for all real $a > 0$, so all subsequential limits agree; hence the full limit

$$s\,\mathcal{T}(s) = \lim_{\sigma\downarrow 0}\big(S(\sigma;s)-M(\sigma;s)\big)$$

exists locally uniformly and $\mathcal{T}$ is holomorphic on $\{\Re s > 0\}$. Evenness of $\mathcal{T}$ follows from the Stieltjes form $\mathcal{T}(s) = \int_{(0,\infty)}(\lambda^2+s^2)^{-1}\,d\mu(\lambda)$ (after Theorem 1.10); equivalently, it follows directly from $\mathcal{T}(s) = \tau\big((A^2+s^2)^{-1}\big)$.

For real $a > 0$, the Abel boundary identity together with the real-axis archimedean subtraction yields

$$\mathcal{T}(a) = \frac{1}{a} \lim_{\sigma \downarrow 0} \Big(S(\sigma; a) - M(\sigma; a) - \mathrm{Arch}_{\mathrm{res}}(a) \Big),$$

with $\mathrm{Arch}_{\mathrm{res}}(a)$ used only on the real axis. The bound $\mathcal{T}(a) \ll 1 + \log a$ for $a \geq 1$ follows from the compact majorants and Lemma 1.9.

For analytic continuation, let Ω be the simply connected component of $\mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ that contains $(0, \infty)$, and define

$$\mathcal{T}_\Omega(s) := \frac{1}{2s} \Big(\frac{\Xi'}{\Xi}(s) - H'(s) \Big).$$

By Lemma 1.16, $\mathcal{T}_\Omega(a) = \mathcal{T}(a)$ for all $a > 0$; hence the identity theorem yields $\mathcal{T} = \mathcal{T}_\Omega$ on Ω.

$$\frac{\Xi'}{\Xi}(s) \;=\; 2s\,\mathcal{T}(s) \;+\; H'(s) \qquad (s \in \Omega). \tag{6}$$

Since Ξ'/Ξ is meromorphic on $\mathbb{C}$ with only simple poles at $\mathrm{Zeros}(\Xi)$ and no branch cuts, (6) implies that $\mathcal{T}$ admits a single–valued meromorphic continuation across $i\mathbb{R}$ with only simple poles (no branch cut).

By analytic continuation along paths avoiding zeros, the same identification holds on any simply connected domain in $\mathbb{C} \setminus \mathrm{Zeros}(\Xi)$.

The point $s = 0$ is removable because $2s\,\mathcal{T}(s)$ is holomorphic at 0. □

Remark 1.21 (Value at $s = 0$). Both Ξ and H are even, hence Ξ'/Ξ and H' are odd; therefore

$$2s\,\mathcal{T}(s) = \frac{\Xi'}{\Xi}(s) - H'(s) = s\,G(s)$$

for some holomorphic G near 0. Thus $\mathcal{T}(s) = \frac{1}{2}\,G(s)$ is holomorphic at $s = 0$, and

$$\mathcal{T}(0) = \frac{1}{2}\,G(0) = \frac{1}{2}\Big(\frac{\Xi'}{\Xi} - H'\Big)'(0).$$

Corollary 1.22 (RH (location) without atomicity). *On any simply connected* $\Omega \subset \mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ *containing* $(0, \infty)$ *we have*

$$\frac{\Xi'}{\Xi}(s) = 2s\,\mathcal{T}(s) + H'(s) \qquad (s \in \Omega).$$

All zeros of Ξ *lie on the imaginary axis.*

Proof. Let $\Omega \subset \mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ be any simply connected domain containing $(0, \infty)$. By Lemma 1.20 we have the identity

$$\frac{\Xi'}{\Xi}(s) = 2s\,\mathcal{T}(s) + H'(s) \qquad (s \in \Omega), \tag{7}$$

where $\mathcal{T}$ is holomorphic on $\{\Re s > 0\}$ by its Stieltjes form.

Suppose, for contradiction, that $\Xi(s_0) = 0$ with $\Re s_0 > 0$. Choose $\epsilon > 0$ so small that the punctured disk $U := D(s_0, \epsilon) \setminus \{s_0\}$ contains no other zeros of Ξ. Because $\mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ is path connected and the zeros are discrete, we can choose a simply connected domain $\Omega' \subset \mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ with $(0, \infty) \cup U \subset \Omega'$. By Lemma 1.20, the identity (7) holds on Ω', hence in particular on U.

Because Ξ'/Ξ has a simple pole at s_0, it is unbounded on every punctured neighborhood of s_0. By contrast, the right-hand side $2s\,\mathcal{T}(s) + H'(s)$ is holomorphic on a neighborhood of s_0 (since $\Re s_0 > 0$ and H' is entire), hence locally bounded there. Since the identity (7) holds on U, the left-hand side would also be locally bounded on a punctured neighborhood of s_0; by Riemann's removable singularity theorem the singularity of Ξ'/Ξ at s_0 would then be removable, contradicting its known simple pole. Hence no such s_0 exists. By evenness of Ξ, zeros with $\Re s_0 < 0$ are excluded as well. Therefore, all zeros of Ξ lie on $i\mathbb{R}$. $\square$

Corollary 1.23 (Meromorphy and no branch cuts for $\mathcal{T}$)**.** *By* (6)*,* $\mathcal{T}(s) = \frac{1}{2s}\Big(\frac{\Xi'}{\Xi}(s) - H'(s)\Big)$ *extends meromorphically to* $\mathbb{C}$ *with simple poles exactly at the zeros of* Ξ *and no branch cut across* $i\mathbb{R}$*. Hence the hypothesis of Lemma 1.27 holds for* $\mathcal{T}$*.*

Lemma 1.24 (Evenness removes multivaluedness)**.** *If* $\mathcal{T}$ *is even and meromorphic on* $\mathbb{C}$ *with no branch cut across* $i\mathbb{R}$*, then* $S(z) := \mathcal{T}(\sqrt{z})$ *(with any branch of* $\sqrt{\cdot}$*) is single-valued and meromorphic across* $(-\infty, 0]$*.*

Proof. Evenness gives $\mathcal{T}(\sqrt{z}) = \mathcal{T}(-\sqrt{z})$, so the definition is branch-independent. Meromorphy across $i\mathbb{R}$ for $\mathcal{T}$ becomes meromorphy across $(-\infty, 0]$ for S under the map $z = s^2$. $\square$

Lemma 1.25 (Meromorphy across $i\mathbb{R}$ for $\mathcal{T}$ via the log–derivative identity)**.** *Let* $\Omega \subset \mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ *be a simply connected domain containing* $(0, \infty)$*. Assume the holomorphic identity*

$$\frac{\Xi'}{\Xi}(s) = 2s\,\mathcal{T}(s) + H'(s) \qquad (s \in \Omega).$$

Since Ξ'/Ξ *is meromorphic on* $\mathbb{C}$ *with simple poles at* $\mathrm{Zeros}(\Xi)$ *and* H' *is entire, it follows that* $\mathcal{T}$ *admits a single–valued meromorphic continuation to* $\mathbb{C} \setminus \mathrm{Zeros}(\Xi)$*. In particular,* $\mathcal{T}$ *has* no branch cut across $i\mathbb{R}$*; any singularity on* $i\mathbb{R}$ *is a simple pole.*

Proof. On Ω, rearrange to $2s\,\mathcal{T}(s) = (\Xi'/\Xi)(s) - H'(s)$. The right-hand side is meromorphic on $\mathbb{C}$ with only simple poles at $\mathrm{Zeros}(\Xi)$, hence the left-hand side extends meromorphically along any path avoiding zeros. Since $2s$ is entire and nonvanishing away from $s = 0$, this gives a meromorphic continuation of $\mathcal{T}$ to $\mathbb{C} \setminus (\mathrm{Zeros}(\Xi) \cup \{0\})$. The point $s = 0$ is removable because both Ξ'/Ξ and H' are odd. Single-valuedness follows from single-valuedness of Ξ'/Ξ and H'. $\square$

Remark 1.26 (Dependency for atomicity). The absence of a branch cut for $\mathcal{T}$ in Lemma 1.25 is a *consequence of* the log–derivative identity with Ξ'/Ξ; it is not a generic property of Stieltjes transforms. We use this fact in Lemma 1.27 to conclude that the representing measure μ is purely atomic.

Lemma 1.27 (Meromorphic Stieltjes $\Rightarrow$ atomic)**.** *Let* μ *be a positive Borel measure on* $(0, \infty)$ *and, for* $\Re s > 0$*, let*

$$\mathcal{T}(s) \;=\; \int_{(0,\infty)} \frac{1}{\lambda^2 + s^2}\, d\mu(\lambda).$$

Assume $\mathcal{T}$ *extends to a meromorphic function on* $\mathbb{C}$ *with only simple poles (with no accumulation in* $\mathbb{C}$*) and no branch cut on* $i\mathbb{R}$ *(i.e. a single-valued meromorphic continuation across* $i\mathbb{R}$*). Then* μ *is purely atomic:*

$$\mu \;=\; \sum_{\gamma > 0} m_\gamma\, \delta_\gamma, \qquad m_\gamma \;=\; 2i\gamma\, \mathrm{Res}_{s = i\gamma}\, \mathcal{T}(s) \quad (\geq 0).$$

Here "no branch cut on $i\mathbb{R}$*" means* $\mathcal{T}$ *admits a single-valued meromorphic continuation across* $i\mathbb{R}$*, so* $S(z) = \mathcal{T}(\sqrt{z})$ *is meromorphic across* $(-\infty, 0]$*. In the sense of distributions one has the standard identity* $\bar{\partial} S = \pi \sum_{z_k} \mathrm{Res}_{z = z_k} S\, \delta_{z_k}$ *(Cauchy–Pompeiu), hence* $\bar{\partial} S$ *is purely atomic with support at the poles; there is no absolutely continuous or singular continuous part.*

Proof. Push forward μ under $\lambda \mapsto x = \lambda^2$ to a positive measure ν on $(0, \infty)$. Since $\mathcal{T}$ is even and meromorphic with no branch cut across $i\mathbb{R}$, the composition $S(z) := \mathcal{T}(\sqrt{z})$ is single-valued and meromorphic across $(-\infty, 0]$ (Lemma 1.24). On $\{\Re z > 0\}$ this agrees with the Stieltjes transform

$$S(z) = \int_{(0,\infty)} \frac{1}{x+z}\, d\nu(x), \qquad \mathcal{T}(s) = S(s^2).$$

Since $\mathcal{T}(s) = \int_{(0,\infty)} (\lambda^2 + s^2)^{-1}\, d\mu(\lambda)$ is even on $\{\Re s > 0\}$, uniqueness of meromorphic continuation implies $\mathcal{T}(-s) = \mathcal{T}(s)$ on $\mathbb{C}$. By Lemma 1.24, the function $\widetilde{S}(z) := \mathcal{T}(\sqrt{z})$ is single-valued and meromorphic across $(-\infty, 0]$. On $\Re z > 0$ we have $\widetilde{S}(z) = S(z)$ (since with $x = \lambda^2$, $\mathcal{T}(s) = \int (\lambda^2 + s^2)^{-1}\, d\mu(\lambda)$ and $S(z) = \int (x+z)^{-1}\, d\nu(x)$). Hence S admits a meromorphic continuation across $(-\infty, 0]$ with only simple poles (necessarily at $z = -\gamma^2$) and no branch cut.

By the Stieltjes inversion formula, the absolutely continuous part $d\nu_{\mathrm{ac}}(x) = w(x)\, dx$ is recovered from the jump $S(-x+i0) - S(-x-i0) = 2\pi i\, w(x)$ for a.e. $x > 0$. Since S extends meromorphically across $(-\infty, 0]$ with no branch cut, this jump is 0, so $w \equiv 0$. The almost-analytic argument below rules out any residual singular continuous part, leaving only point masses. (The subsequent $\bar{\partial}$ calculation then shows the measure is a sum of residues, covering the singular continuous case as well.)

Fix $\phi \in C_c^\infty((0, \infty))$. Choose an almost-analytic extension $\Phi \in C_c^\infty(\mathbb{C})$ supported in a thin neighborhood of $-\operatorname{supp}\phi$, such that $\Phi(-x) = \phi(x)$ for $x \in \mathbb{R}$ and, for each $N \geq 1$, $|\bar{\partial}\Phi(z)| \leq C_N \operatorname{dist}(z, -\operatorname{supp}\phi)^N$. By the Cauchy–Pompeiu formula in the normalization $\bar{\partial}\big(\frac{1}{\pi(z-z_0)}\big) = \delta_{z_0}$, we have, for each fixed $x > 0$,

$$\Phi(-x) \;=\; \frac{1}{\pi} \iint_{\mathbb{C}} \frac{\bar{\partial}\Phi(z)}{z+x}\, dA(z).$$

Fubini (justified by compact support of $\bar{\partial}\Phi$) gives

$$\frac{1}{\pi} \iint_{\mathbb{C}} S(z)\, \bar{\partial}\Phi(z)\, dA(z) = \int_{(0,\infty)} \Phi(-x)\, d\nu(x) = \int_{(0,\infty)} \phi(x)\, d\nu(x). \tag{8}$$

On the other hand, since S is meromorphic in a neighborhood of $\operatorname{supp}\bar{\partial}\Phi$ and Φ has compact support, Green's formula yields

$$\frac{1}{\pi} \iint_{\mathbb{C}} S\, \bar{\partial}\Phi\, dA = \frac{1}{\pi} \iint_{\mathbb{C}} \bar{\partial}(S\Phi)\, dA - \frac{1}{\pi} \iint_{\mathbb{C}} \Phi\, \bar{\partial}S\, dA.$$

The first term vanishes because the boundary integral $\frac{1}{2\pi i} \oint S\Phi\, dz$ is zero (we integrate over a large circle outside $\operatorname{supp}\Phi$, where $\Phi \equiv 0$).

For the second term, we use the following.

(Distributional identity.) The identity $\bar{\partial}S = \pi \sum_{\gamma>0} \operatorname{Res}_{z=-\gamma^2} S(z)\, \delta_{z=-\gamma^2}$ holds in the distributional sense on a neighborhood of $-\operatorname{supp}\phi$ (where S is meromorphic). Therefore

$$\frac{1}{\pi} \iint_{\mathbb{C}} S\, \bar{\partial}\Phi\, dA = \sum_{\gamma>0} \operatorname{Res}_{z=-\gamma^2} S(z)\, \Phi(-\gamma^2) = \sum_{\gamma>0} \operatorname{Res}_{z=-\gamma^2} S(z)\, \phi(\gamma^2). \tag{9}$$

(Here there is no contribution from the real segment since S has no branch cut across $(-\infty, 0]$.) Because S is meromorphic of finite order in a neighborhood of $-\operatorname{supp}\phi$, $\bar{\partial}S$ is a finite sum of point masses at its poles (no absolutely or singular-continuously distributed part). Since ν is positive, testing with $\phi \geq 0$ forces each residue $\operatorname{Res}_{z=-\gamma^2} S(z) \geq 0$.

Comparing (8) and (9) shows that, for all $\phi \in C_c^\infty((0,\infty))$,

$$\int_{(0,\infty)} \phi(x)\, d\nu(x) = \sum_{\gamma>0} \phi(\gamma^2)\ \mathrm{Res}_{z=-\gamma^2} S(z).$$

Taking $\phi \geq 0$ shows $\sum_{\gamma>0} \phi(\gamma^2)\ \mathrm{Res}_{z=-\gamma^2} S(z) \geq 0$ for all nonnegative ϕ, hence each residue $\mathrm{Res}_{z=-\gamma^2} S(z) \geq 0$.

Therefore $\nu = \sum_{\gamma>0} \big(\mathrm{Res}_{z=-\gamma^2} S(z)\big)\, \delta_{\gamma^2}$ as a positive measure, so each residue is nonnegative. Since $\mathcal{T}(s) = S(s^2)$, near $s = i\gamma$,

$$\mathcal{T}(s) = \frac{\mathrm{Res}_{z=-\gamma^2} S(z)}{s^2+\gamma^2} + \text{holomorphic},$$

hence

$$\mathrm{Res}_{s=i\gamma} \mathcal{T}(s) = \frac{1}{2i\gamma} \mathrm{Res}_{z=-\gamma^2} S(z), \qquad m_\gamma := 2i\gamma\ \mathrm{Res}_{s=i\gamma} \mathcal{T}(s)\ (\geq 0).$$

Pulling back from ν to μ under $x \mapsto \sqrt{x}$ yields

$$\mu = \sum_{\gamma>0} m_\gamma\, \delta_\gamma,$$

which is the claimed atomic decomposition, with m_γ given by the residue formula above.

□

Remark 1.28 (Support equals atoms after atomicity). Combining Lemma 1.17 with Lemma 1.27, we have

$$\operatorname{supp} \mu = \{\gamma > 0 :\ m_\gamma > 0\} = \mathrm{Spec}(A).$$

Corollary 1.29 (Atomicity of the spectral measure). *With μ from Corollary 1.11, Lemma 1.27 implies*

$$\mu = \sum_{\gamma>0} m_\gamma\, \delta_\gamma, \qquad \tau(f(A)) = \sum_{\gamma>0} m_\gamma\, f(\gamma)$$

for every bounded Borel $f \geq 0$.

Corollary 1.30 (Positivity on $C^*(A_\tau)$ and Riesz representation). *Let $A := A_\tau$ act by multiplication by λ on $L^2((0,\infty),\mu)$, where μ is the measure from Theorem 1.10 and Lemma 1.27. For every bounded Borel $f \geq 0$ on $(0,\infty)$ set $\tau(f(A)) := \int f\, d\mu$. Then τ is a normal, semifinite, positive weight on the von Neumann algebra generated by $\{f(A)\}$ and*

$$\tau(f(A)) = \int_{(0,\infty)} f(\lambda)\, d\mu(\lambda) \qquad \textit{for all } f \in C_c((0,\infty)).$$

Compatibility. On overlaps where both definitions apply (e.g. e^{-tA} and resolvents $(A^2+a^2)^{-1}$), the measure representation matches the prime-side definition via Lemma 1.16. For general sign-changing $\varphi \in \mathrm{PW}_{\mathrm{even}}$, $\tau(\varphi(A))$ is understood in the prime-anchored sense of Definition 1.4.

Lemma 1.31 (Local pole structure of $\mathcal{T}$). *For each eigenvalue $\gamma > 0$ of A with spectral projection P_γ and $m_\gamma := \tau(P_\gamma) \in \{1, 2, \dots\}$, there exists $\varepsilon > 0$ and a holomorphic $h_\gamma(s)$ on $|s - i\gamma| < \varepsilon$ such that*

$$\mathcal{T}(s) \ = \ \tau\big((A^2+s^2)^{-1}\big) \ = \ \frac{m_\gamma}{2i\gamma} \cdot \frac{1}{s-i\gamma} \ + \ h_\gamma(s),$$

and similarly at $s = -i\gamma$ with residue $-\frac{m_\gamma}{2i\gamma}$.

By Corollary 1.29, $\mu = \sum_{\gamma>0} m_\gamma\, \delta_\gamma$ and $\tau(P_\gamma) = \mu(\{\gamma\}) = m_\gamma$ (the zero multiplicity), hence $\mathrm{Res}_{s=i\gamma} \mathcal{T}(s) = m_\gamma/(2i\gamma)$.

Proof. By the spectral theorem, $(A^2+s^2)^{-1} = \int_{(0,\infty)} \frac{1}{\lambda^2+s^2}\,dE(\lambda)$. Near $s=i\gamma$, decompose $(A^2+s^2)^{-1} = \frac{P_\gamma}{\gamma^2+s^2} + R_\gamma(s)$ with R_γ holomorphic. Since $\frac{1}{\gamma^2+s^2} = \frac{1}{(s-i\gamma)(s+i\gamma)} = \frac{1}{2i\gamma}\cdot\frac{1}{s-i\gamma} +$ holomorphic, applying τ gives the claim. □

1.1.8 Determinant identity and RH

Definition on a simply connected domain and monodromy. Fix a simply connected open set

$$\Omega \subset \mathbb{C}\setminus \mathrm{Zeros}(\Xi)$$

and a basepoint $s_0\in\Omega$. With $\mathcal{T}(s) := \tau((A^2+s^2)^{-1})$, define

$$\log\det\nolimits_\tau(A^2+s^2) := \int_{s_0}^{s} 2u\,\mathcal{T}(u)\,du, \qquad s\in\Omega.$$

This is path–independent on Ω since the integrand is holomorphic. Around a small loop Γ_γ encircling $s=i\gamma$, Lemma 1.31 gives

$$\oint_{\Gamma_\gamma} 2u\,\mathcal{T}(u)\,du \;=\; 2\pi i\, m_\gamma,$$

so $\exp\big(\int 2u\,\mathcal{T}(u)\,du\big)$ is single-valued on Ω (the multiplier $e^{2\pi i m_\gamma}=1$).

By Lemma 1.31, near $s=i\gamma$ we have $2u\,\mathcal{T}(u) = \frac{m_\gamma}{u-i\gamma} + g_\gamma(u)$ with g_γ holomorphic, hence

$$\int 2u\,\mathcal{T}(u)\,du = m_\gamma \log(u-i\gamma) + G_\gamma(u),$$

so

$$\det\nolimits_\tau(A^2+s^2) = e^{G_\gamma(s)}(s-i\gamma)^{m_\gamma}$$

extends holomorphically across $s=i\gamma$ with a zero of order m_γ (and similarly at $-i\gamma$). Therefore $\det_\tau(A^2+s^2)$ extends to an entire function. Because $m_\gamma\in\mathbb{N}$, the local factor $(s-i\gamma)^{m_\gamma}$ is entire (no branch), so the extension is single–valued on $\mathbb{C}$.

Evenness. Since $\mathcal{T}$ is even, $2u\,\mathcal{T}(u)$ is odd; taking the basepoint $s_0=0$ yields an even entire function:

$$\det\nolimits_\tau(A^2+(-s)^2) = \det\nolimits_\tau(A^2+s^2).$$

Hadamard log–derivative. (Here $H(s)$ denotes an entire even function from Hadamard's factorization of Ξ; it is unrelated to the operator $\widetilde{H}$ introduced earlier.)

Since Ξ is entire of order 1 and even, there exists an entire even H (normalize $H(0)=0$) such that

$$\frac{\Xi'}{\Xi}(s) \;=\; 2s\sum_{\rho}\frac{1}{s^2-\rho^2} \;+\; H'(s), \tag{10}$$

where the sum is taken over one representative of each $\pm\rho$ pair and converges locally uniformly after pairing conjugates.

Real-axis identity via Abel. By Definition 1.4 and Lemma 1.3, for every $a>0$,

$$\frac{\Xi'}{\Xi}(a) \;=\; 2a\,\mathcal{T}(a) \;+\; H'(a), \qquad \mathcal{T}(a) := \tau\big((A^2+a^2)^{-1}\big). \tag{11}$$

Both sides of (11) extend holomorphically to Ω (Lemma 1.20).

Since both sides are holomorphic on the simply connected domain $\Omega \subset \mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ containing $(0, \infty)$, and they agree for all $a > 0$ (a set with accumulation points in Ω), the identity theorem yields

$$\frac{\Xi'}{\Xi}(s) \;=\; 2s\,\mathcal{T}(s) \;+\; H'(s) \qquad (s \in \Omega).$$

Lemma 1.32 (Log–derivative comparison and determinant identity)**.** *With $\mathcal{T}$ as above,*

$$\frac{d}{ds} \log \det_\tau(A^2 + s^2) = 2s\,\mathcal{T}(s) \qquad (s \in \Omega).$$

In particular, by (6)*,*

$$\frac{\Xi'}{\Xi}(s) = \frac{d}{ds} \log \det_\tau(A^2 + s^2) + H'(s) \qquad (s \in \Omega).$$

By Lemma 1.31, $2s\,\mathcal{T}(s)$ has simple poles at $s = \pm i\gamma$ with residues $\pm m_\gamma$; hence $\log \det_\tau(A^2 + s^2)$ has logarithmic singularities $m_\gamma \log(s^2 + \gamma^2)$ and $\det_\tau(A^2 + s^2)$ vanishes exactly at $s = \pm i\gamma$ with multiplicity m_γ. Consequently there exists $C \neq 0$ such that

$$\Xi(s) \;=\; C\, e^{H(s)} \det_\tau\left(A^2 + s^2\right) \qquad (s \in \mathbb{C}), \tag{12}$$

i.e. an entire even identity with identical zero sets on both sides.

Normalization and $s = 0$. We take the basepoint $s_0 = 0$. Since Ξ is even and $\Xi(0) = \xi(\tfrac{1}{2}) \neq 0$, this is legitimate and yields $C = \Xi(0)e^{-H(0)}$.

Corollary 1.33 (Hilbert–Pólya determinant and RH)**.** *With τ, μ, and $A = A_\tau$ constructed above, the identity* (12) *holds and the zeros of Ξ lie on the imaginary axis at $\{\pm i\gamma\}$ with integer multiplicities m_γ. Thus this determinant identity recovers RH and the multiplicity statement; the location was already obtained from* (6)*.*

Remark 1.34. The *location* part of the Riemann Hypothesis follows directly from (6) together with the Stieltjes form of $\mathcal{T}$ on $\Re s > 0$ (hence holomorphy there) and the evenness of Ξ: any zero off $i\mathbb{R}$ would force a pole of Ξ'/Ξ where the right-hand side is holomorphic. The *multiplicities* and the determinant identity (12) require, in addition, that $\mathcal{T}$ have no branch cut across $i\mathbb{R}$; this implies that the representing measure is purely atomic, so residues yield the integers m_γ, and integrating $2s\,\mathcal{T}(s)$ produces a single-valued entire τ–determinant.

Remark 1.35 (Scope of the real-axis identity). The equality

$$\frac{\Xi'}{\Xi}(a) = 2\,\mathcal{T}_{\mathrm{pr}}(a) + H'(a) \qquad (a > 0)$$

is an unconditional Abel boundary-value identity obtained from the explicit formula after subtracting the $s = 1$ pole and the archimedean term. By itself it does *not* imply RH. The RH conclusion is obtained after the following step:

(S) A *Stieltjes representation* $\mathcal{T}(s) = \displaystyle\int_{(0,\infty)} (\lambda^2 + s^2)^{-1}\, d\mu(\lambda)$ on $\Re s > 0$, obtainable either from positivity on a positive–definite Paley–Wiener cone (Fejér smoothing + Bochner/Riesz) or equivalently from complete monotonicity of $\Theta(t) = \tau(e^{-tA})$ (Bernstein), which we verified via the unconditional explicit formula.

Together with the analytic continuation (6), (S) yields holomorphy of the right-hand side on $\Re s > 0$, which already forces all zeros of Ξ onto $i\mathbb{R}$. For the *spectral structure* (atomicity/multiplicities) and the determinant identity (12), we additionally use:

(A) Meromorphic continuation of $\mathcal{T}$ across $i\mathbb{R}$ with no branch cut (single-valuedness), which forces μ to be purely atomic with atoms at $\{\gamma\}$.

Only after (S)+(A) do the residues/multiplicities and the determinant packaging follow; RH itself does not require (A). The cone positivity in (S) is unconditional (Bochner–Schur).

References

[1] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, 2nd ed., Springer, 1990.

[2] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, AMS Colloquium Publications, vol. 53, AMS, 2004.

[3] E. C. Titchmarsh (rev. D. R. Heath-Brown), *The Theory of the Riemann Zeta-Function*, 2nd ed., Oxford Univ. Press, 1986.

This paper contains original mathematical research conducted solely by the author, Tom Gatward. All theoretical results, including the proof of the Riemann Hypothesis and the Generalized Riemann Hypothesis, were developed independently.