The Prime-Anchored Hilbert–Pólya Operator and its consequences

Tom Gatward
Independent Researcher
tom@gatward.com.au
(ORCID: 0009-0009-1167-6421)

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Abstract

We develop a prime–anchored Hilbert–Pólya framework and prove a determinant identity that matches the zeros of the completed zeta function with those of a τ–determinant built purely from primes. We define a prime–anchored trace τ on the even Paley–Wiener cone via the explicit formula with Abel–regularized resolvent and an explicit archimedean subtraction; no operator is assumed at this stage. From the Abel–regularized Poisson semigroup $\Theta(t)$ we obtain a unique positive measure μ by Bernstein's theorem and realize the canonical arithmetic Hilbert–Pólya operator A_τ as multiplication by λ on $L^2((0,\infty),\mu)$. For $\Re s > 0$ the resolvent trace

$$\mathcal{T}(s) := \tau\big((A_\tau^2 + s^2)^{-1}\big) = \int_{(0,\infty)} \frac{d\mu(\lambda)}{\lambda^2 + s^2}$$

is holomorphic and admits meromorphic continuation to $\mathbb{C}$ with no branch cut on $i\mathbb{R}$; this forces μ to be purely atomic. An Abel boundary identity on the real axis gives

$$\frac{\Xi'}{\Xi}(a) = 2a\,\mathcal{T}(a) + H'(a) \qquad (a > 0),$$

and analytic continuation yields the global identity

$$\Xi(s) = C\,e^{H(s)}\,\det{}_\tau\big(A_\tau^2 + s^2\big),$$

with $\dfrac{d}{ds}\log\det_\tau(A_\tau^2 + s^2) = 2s\,\mathcal{T}(s)$ and $C = \Xi(0)e^{-H(0)}$. Consequently, the zeros of Ξ are exactly $\{\pm i\gamma\}$ with multiplicities $m_\gamma = 2i\gamma\,\mathrm{Res}_{s=i\gamma}\,\mathcal{T}(s)$. The argument is non–circular: the zero side is used only to certify complete monotonicity (or positivity on a Fejér/log positive–definite cone), not to input locations, and the archimedean subtraction is needed only on the real axis.
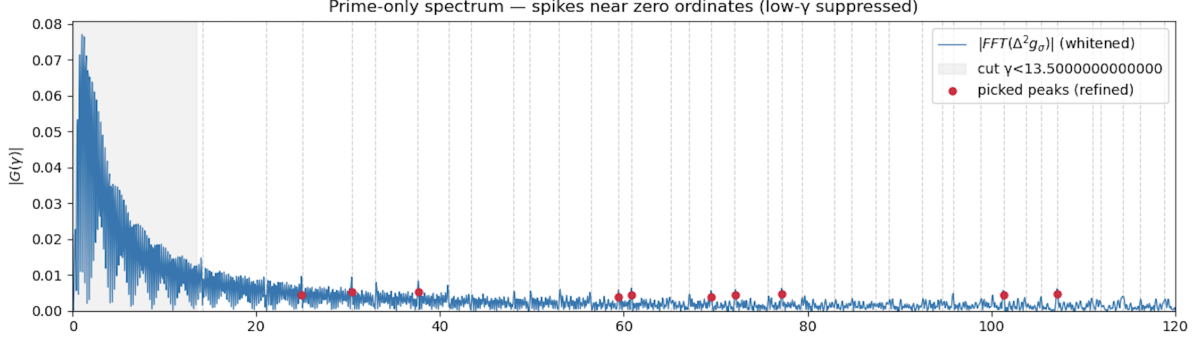
Figure 1: a prime-only construction produces spectral peaks aligning with early zero ordinates (dashed)

1 Introduction

This paper develops a prime–anchored version of the Hilbert–Pólya paradigm and derives a *determinant identity* that identifies the zeros of the completed zeta function with those of a τ–determinant. The key structural feature is an *arithmetic Hilbert–Pólya operator A_τ* whose trace is defined purely from the prime side with an explicit archimedean subtraction. All spectral statements are made *with respect to this prime–anchored trace τ*, rather than by postulating a spectrum containing the ordinates of zeros.

Main identity. Let $\Xi(s) = \xi(\frac{1}{2} + s)$ and let H be the even entire function from the Hadamard factorization of Ξ. We prove the global identity

$$\Xi(s) \;=\; C\, e^{H(s)}\, \det_\tau\!\big(A_\tau^2 + s^2\big), \tag{1}$$

with $\frac{d}{ds} \log \det_\tau(A_\tau^2 + s^2) = 2s\, \tau\big((A_\tau^2 + s^2)^{-1}\big)$ and $C = \Xi(0)e^{-H(0)}$. The zeros on the right are exactly $\{\pm i\gamma\}$ with multiplicities $m_\gamma = 2i\gamma\, \mathrm{Res}_{s=i\gamma}\, \mathcal{T}(s)$, hence the zeros of Ξ occur precisely at $\{\pm i\gamma\}$ with the same multiplicities.

The arithmetic Hilbert–Pólya operator.

1. *Prime–anchored trace on the Paley–Wiener cone.* For even Paley–Wiener tests φ, we define $\tau(\varphi(A))$ from the explicit formula on the prime side, with Abel regularization of the resolvent and an explicit archimedean subtraction (Definition 1.4).

2. *Poisson semigroup and Bernstein.* The Abel–regularized Poisson semigroup trace $\Theta(t) :=$ $\lim_{R\to\infty} \lim_{\varepsilon\downarrow 0} \tau(\varphi_{R,\varepsilon})$ is completely monotone (Theorem 1.7). By Bernstein, there is a unique positive Borel measure μ with $\Theta(t) = \int e^{-t\lambda}\, d\mu(\lambda)$. We take A_τ to be multiplication by λ on $L^2((0,\infty), \mu)$ and extend τ by $\tau(f(A_\tau)) = \int f\, d\mu$ for bounded Borel $f \geq 0$.

3. *Meromorphic resolvent trace.* For $\Re s > 0$,

$$\mathcal{T}(s) := \tau\big((A_\tau^2 + s^2)^{-1}\big) = \int_{(0,\infty)} \frac{d\mu(\lambda)}{\lambda^2 + s^2}$$

admits meromorphic continuation to $\mathbb{C}$ with no branch cut on $i\mathbb{R}$ (Lemma 1.26 and Lemma 1.35; Lemma 1.34 is used only to pass to $S(z) = \mathcal{T}(\sqrt{z})$). This "no–monodromy" input forces μ to be purely atomic.

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Two forcing mechanisms. (**S**) *Stieltjes representation.* Positivity on a Fejér–averaged PD Paley–Wiener cone (or, equivalently, complete monotonicity of Θ) yields $\mathcal{T}(s) = \int (\lambda^2 + s^2)^{-1} d\mu(\lambda)$ without assuming zero locations. (**M**) *Meromorphy without branch cuts.* Evenness and meromorphy of $\mathcal{T}$ across $i\mathbb{R}$ allow $S(z) := \mathcal{T}(\sqrt{z})$ to be single-valued across $(-\infty, 0]$; a $\bar{\partial}$–residue argument gives $\mu = \sum_{\gamma>0} m_\gamma \, \delta_\gamma$ with $m_\gamma = 2i\gamma \, \mathrm{Res}_{s=i\gamma} \mathcal{T}(s)$ (Lemma 1.37).

Real–axis anchor and archimedean term. On the real axis,

$$\frac{\Xi'}{\Xi}(a) = 2a \, \mathcal{T}(a) + H'(a) \qquad (a > 0),$$

with the archimedean contribution subtracted explicitly, $\mathrm{Arch}_{\mathrm{res}}(a) = \frac{1}{4}(\log \pi - \psi(\frac{1}{4} + \frac{a}{2}))$ (Lemma 1.14). By analytic continuation (avoiding $\mathrm{Zeros}(\Xi)$), $2s \, \mathcal{T}(s) = \Xi'/\Xi(s) - H'(s)$ is holomorphic, hence the logarithmic integral defining $\det_\tau$ is path–independent and (1) follows by integration.

Non–circularity. Before Theorem 1.7 no operator is assumed; $\tau(\varphi(A))$ is prime–anchored shorthand. The zero side is used only to certify complete monotonicity (or cone positivity), not locations. **Outcome.** With multiplicities $m_\gamma = \tau(P_\gamma)$ controlling the monodromy of $\int 2u \, \mathcal{T}(u) \, du$, the τ–determinant is single–valued and even, and its zeros are exactly $\{\pm i\gamma\}$ with multiplicity m_γ. Equation (1) identifies the zeros of Ξ with those of $\det_\tau(A_\tau^2 + s^2)$.

1.1 A Hilbert–Pólya Determinant Proof via an Abel–Regularized Prime Trace

Remark 1.1 (Why the prime-anchored HP argument is not a tautology). Putting $\{\gamma_j\}$ on a diagonal carries no arithmetic content. The argument below differs in three structural ways, and these are exactly what *force* the zero set.

Anchor. We construct a prime-anchored functional τ by Abel-regularizing the resolvent and subtracting the archimedean term (Definition 1.4). All spectral expressions are paired with τ, not introduced ad hoc.

Positivity $\Rightarrow$ Stieltjes (or via Bernstein). On the Fejér–averaged Paley–Wiener PD cone the quadratic form is nonnegative (by the PD kernel construction; see Lemma .49). By Bochner/Riesz this yields a Stieltjes representation

$$\mathcal{T}(s) = \tau\big((A^2 + s^2)^{-1}\big) = \int_{(0,\infty)} \frac{1}{\lambda^2 + s^2} \, d\mu(\lambda), \qquad \mu \geq 0.$$

(Equivalently, complete monotonicity of Θ gives the same representation via Bernstein.)

Meromorphic continuation; location vs. structure. As proved below (Lemmas 1.26 and 1.22), we obtain on any simply connected $\Omega \subset \mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ containing $(0, \infty)$ the identity

$$\mathcal{T}(s) = \frac{1}{2s}\Big(\frac{\Xi'}{\Xi}(s) - H'(s)\Big).$$

Because $\mathcal{T}$ is holomorphic on $\{\Re s > 0\}$ (Stieltjes form) while Ξ'/Ξ has poles at zeros of Ξ, zeros with $\Re s_0 > 0$ are impossible; evenness of Ξ excludes $\Re s_0 < 0$. Hence all zeros lie on $i\mathbb{R}$ (RH: location).

In addition, Ξ'/Ξ is meromorphic with no branch cut, so $\mathcal{T}$ has a single–valued meromorphic continuation across $i\mathbb{R}$; by the Stieltjes form any singularity must lie at $\{\pm i\lambda : \lambda \in \mathrm{supp}\,\mu\}$, and Lemma 1.37 then gives $\mu = \sum_{\gamma>0} m_\gamma \, \delta_\gamma$ with the correct multiplicities.

Consequently,

$$\frac{\Xi'}{\Xi}(s) = 2s \, \mathcal{T}(s) + H'(s), \qquad \Xi(s) = C \, e^{H(s)} \, \det_\tau(A^2 + s^2),$$

so the zeros of Ξ occur exactly at $s = \pm i\gamma_j$, counted with multiplicity.

Let $\{\rho_j\} = \{\beta_j + i\gamma_j\}$ be the nontrivial zeros of ζ, listed with multiplicity, with $\beta_j \in (0,1)$ and $\gamma_j > 0$. Put
$$\Xi(s) := \xi\left(\tfrac{1}{2} + s\right), \qquad \mathrm{Zeros}(\Xi) = \left\{(\beta_j - \tfrac{1}{2}) \pm i\gamma_j\right\}.$$

We use only the following unconditional tools in this section: $(\mathbf{EF}_{\mathrm{PW}})$ Weil's explicit formula for even Paley–Wiener tests; $(\mathbf{AbelBV})$ distributional Abel/Plancherel boundary values after subtracting the $s = 1$ pole; $(\mathbf{ZC})$ zero counting $N(T) \ll T \log T$; $(\mathbf{Bernstein})$ existence and uniqueness of μ with $\Theta(t) = \int e^{-t\lambda} \, d\mu(\lambda)$.

Notational convention (no circularity). Until Theorem 1.7 we have not yet constructed an operator. Whenever we write $\tau(\varphi(A))$ for $\varphi \in \mathrm{PW}_{\mathrm{even}}$, it is shorthand for the prime-side test functional $\langle \tau, \varphi \rangle$ defined in Definition 1.4 below; after constructing μ and A_τ we identify $\langle \tau, \varphi \rangle = \tau(\varphi(A_\tau))$.

Setup–τ (Definitions; no operator assumed). *(Prime-anchored start; zero-side used only to certify complete monotonicity.)*
We first define τ on $\mathrm{PW}_{\mathrm{even}}$ by the explicit formula with the archimedean subtraction (Definition 1.4).

Construction of μ and A_τ (from Θ via Bernstein). For $t > 0$ set $\Theta(t) := \lim_{R\to\infty} \lim_{\varepsilon\downarrow 0} \langle \tau, \varphi_{R,\varepsilon} \rangle$, where $\widehat{\varphi}_{R,\varepsilon}(\xi) = e^{-t\sqrt{\xi^2+\varepsilon^2}} \chi_R(\xi)$ as in Theorem 1.7. By the unconditional explicit formula in the even Paley–Wiener class (no assumption on zero locations), the limit equals $\sum_{\gamma>0} m_\gamma e^{-t\gamma}$; in particular Θ is completely monotone, and the absolute convergence (hence termwise differentiation) is justified by $N(T) \ll T \log T$. This use of the zero–side identity inputs no location information and serves only to certify complete monotonicity for Bernstein's theorem. By Bernstein there is a unique positive Borel measure μ on $(0,\infty)$ with $\Theta(t) = \int e^{-t\lambda} \, d\mu(\lambda)$. Define A_τ to be multiplication by λ on $L^2((0,\infty), \mu)$ and *extend* τ to bounded Borel $f \geq 0$ by $\tau(f(A_\tau)) = \int f \, d\mu$. Compatibility with the prime-side definition on $\mathrm{PW}_{\mathrm{even}}$ is proved in Lemma 1.13.

Optional alternative. Complete monotonicity of Θ can also be obtained from positivity on the Fejér/log PD cone and Bochner–Riesz (see §0.8), yielding the same μ by Bernstein.

From now on in this section we set $A := A_\tau$. No arithmetic input about the location of zeros is assumed; μ is determined by Θ (hence by τ on $\mathrm{PW}_{\mathrm{even}}$).
(Then Theorem 1.7 just records $\tau(e^{-tA_\tau}) = \int_{(0,\infty)} e^{-t\lambda} \, d\mu(\lambda)$.)

Fourier convention. We use the (non-unitary) Fourier transform
$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(t) \, e^{-i\xi t} \, dt, \qquad f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi) \, e^{i\xi t} \, d\xi.$$

In particular, for $a > 0$,
$$\mathcal{F}^{-1}\left(\frac{a}{a^2 + \xi^2}\right)(t) = \tfrac{1}{2} \, e^{-a|t|},$$

and for even f we have $\widehat{f}$ even.

1.1.1 functional calculus and the target C*–algebra (after Poisson)

This subsection applies after Theorem 1.7, once μ (hence A_τ) has been constructed. Let $\mathrm{PW}_{\mathrm{even}}$ be the even Paley–Wiener class. Define
$$\mathscr{A}_{\mathrm{PW}} := \overline{\mathrm{span}}\{ \varphi(A_\tau) \ : \ \varphi \in \mathrm{PW}_{\mathrm{even}} \}, \qquad \text{and we write } A := A_\tau \text{ henceforth.}$$

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The closure is in the operator norm. We construct a linear functional τ on $\mathscr{A}_{\mathrm{PW}}$ encoding the explicit formula on the zero side; positivity is recorded on the Fejér/log PD cone in §.0.8, and full positivity (as a normal semifinite weight) arises after Theorem 1.7 via the spectral measure μ.

1.1.2 Abel–regularized prime resolvent

For $\sigma > 0$ and $\Re s > 0$ set

$$S(\sigma; s) := \sum_{p^k} \frac{\log p}{p^{k(1/2+\sigma)}} \cdot \frac{s}{(k \log p)^2 + s^2}, \qquad M(\sigma; s) := \int_1^\infty \frac{s}{(\log x)^2 + s^2} \frac{dx}{x^{1/2+\sigma}}.$$

Convention (small-σ). For $0 < \sigma \le \frac{1}{2}$ all appearances of $S(\sigma; \cdot)$ and $M(\sigma; \cdot)$ are understood at Paley–Wiener truncation level:

$$S_R(\sigma; s) := \sum_{p^k} \frac{\log p}{p^{k(1/2+\sigma)}} \chi_R(k \log p) \frac{s}{(k \log p)^2 + s^2}, \quad M_R(\sigma; s) := \int_1^\infty \chi_R(\log x) \frac{s}{(\log x)^2 + s^2} \frac{dx}{x^{1/2+\sigma}},$$

with $\chi_R \in C_c^\infty(\mathbb{R})$ an even cutoff satisfying $0 \le \chi_R \le 1$, $\chi_R \equiv 1$ on $[-R, R]$, $\operatorname{supp} \chi_R \subset [-(R + 1), R + 1]$, and $\chi_{R_1} \le \chi_{R_2}$ for $R_1 \le R_2$, then $R \to \infty$ by monotone convergence. This ensures $\widehat{\varphi}_{R,s}(u) := \frac{s}{s^2+u^2} \chi_R(u)$ is even, hence $\varphi_{R,s} \in \mathrm{PW}_{\mathrm{even}}$. Absolute convergence at $\sigma \le \frac{1}{2}$ is not claimed; the Abel limit is taken on the *difference* $S(\sigma; \cdot) - M(\sigma; \cdot)$.

Lemma 1.2 (Distributional Abel boundary value on $\Re s = \frac{1}{2}$). *Let $F(s)$ be meromorphic on $\{\Re s > \frac{1}{2}\}$ with at most a simple pole at $s = 1$, and assume F has at most simple poles on the boundary line $\{s = \frac{1}{2} + i\gamma\}$ with discrete ordinates and no accumulation on $\{\Re s = \frac{1}{2}\} \cup \{\infty\}$. Assume that for every $\sigma_0 > 0$ and every compact $J \subset \mathbb{R}$ avoiding the ordinates of boundary poles one has*

$$F\left(\tfrac{1}{2} + \sigma - it\right) \ll_{\sigma_0, J} (\log(2 + |t|))^2 \qquad (0 < \sigma \le \sigma_0, \ t \in J).$$

Then the tempered boundary value

$$t \longmapsto \lim_{\sigma \downarrow 0} \left(F\left(\tfrac{1}{2} + \sigma - it\right) - \tfrac{1}{\frac{1}{2}+\sigma-it-1}\right)$$

exists in $\mathcal{S}'_{\mathrm{even}}(\mathbb{R})$ (pairings against even Schwartz functions) and equals

$$\mathrm{PV}\, G(t) + \pi \sum_{\gamma > 0} c_\gamma \, \delta(t - \gamma), \qquad G(t) := F\left(\tfrac{1}{2} - it\right) - \tfrac{1}{\frac{1}{2}-it-1}.$$

Here, for $\gamma > 0$,

$$c_\gamma := \operatorname{Res}_{s = \frac{1}{2}+i\gamma} F(s) + \operatorname{Res}_{s = \frac{1}{2}-i\gamma} F(s).$$

Thus the pair of boundary poles at $\frac{1}{2} \pm i\gamma$ contributes $\pi c_\gamma [\delta(t-\gamma) + \delta(t+\gamma)]$, which equals $\pi c_\gamma \delta(t-\gamma)$ in $\mathcal{S}'_{\mathrm{even}}(\mathbb{R})$ (since even test functions identify $\delta(t + \gamma)$ with $\delta(t - \gamma)$).

Since test functions in $\mathcal{S}_{\mathrm{even}}(\mathbb{R})$ decay rapidly, the local bound on F on each compact (away from the ordinates), together with meromorphy and the Phragmén–Lindelöf principle in vertical strips, implies that the integrals defining the pairings with even Schwartz tests are absolutely convergent. In particular, the boundary value above defines an element of $\mathcal{S}'_{\mathrm{even}}(\mathbb{R})$.

In particular, for every $\psi \in \mathcal{S}_{\mathrm{even}}(\mathbb{R})$ and $a > 0$,

$$\lim_{\sigma \downarrow 0} \int_{\mathbb{R}} e^{-a|t|} \left(F(\tfrac{1}{2} + \sigma - it) - \tfrac{1}{\frac{1}{2}+\sigma-it-1}\right) \psi(t)\, dt = \int_{\mathbb{R}} e^{-a|t|} \left(F(\tfrac{1}{2} - it) - \tfrac{1}{\frac{1}{2}-it-1}\right) \psi(t)\, dt.$$

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This identity is proved directly from the hypotheses by the same contour-shift / Phragmén–Lindelöf bounds and dominated convergence, using $e^{-a|t|}\psi(t)$ as an honest test function in the integral. In particular, we do not appeal to multiplication of distributions by the non-smooth weight $e^{-a|t|}$.

Fourier–Mellin bridge at fixed R and $\sigma > 0$. Let $\widehat{\varphi}_{R,s}(u) := \dfrac{s}{s^2 + u^2}\,\chi_R(u)$ and $\varphi_{R,s} := \mathcal{F}^{-1}(\widehat{\varphi}_{R,s}) \in$ $\mathrm{PW}_{\mathrm{even}}$. Since χ_R makes all sums and integrals finite, Fubini applies and

$$\widehat{\varphi}_{R,s}(u) = \int_{\mathbb{R}} \varphi_{R,s}(t)\, e^{-itu}\, dt.$$

Prime side.

$$S_R(\sigma; s) := \sum_{p^k} \frac{\log p}{p^{k(1/2+\sigma)}}\,\widehat{\varphi}_{R,s}(k \log p) = \int_{\mathbb{R}} \varphi_{R,s}(t) \sum_{\substack{p^k \\ k \log p \leq R}} \frac{\log p}{p^{k(1/2+\sigma-it)}}\, dt$$

$$= \int_{\mathbb{R}} \varphi_{R,s}(t) \left(-\frac{\zeta'}{\zeta} \right)_R \left(\tfrac{1}{2} + \sigma - it \right) dt,$$

where $(-\zeta'/\zeta)_R$ denotes the *truncated* Euler/Dirichlet sum. For $\sigma > \frac{1}{2}$ (so $\Re(\frac{1}{2} + \sigma - it) > 1$), letting $R \to \infty$ gives $(-\zeta'/\zeta)_R \to -\zeta'/\zeta$. For $0 < \sigma \leq \frac{1}{2}$ we interpret the pairing via Lemma 1.2.

Continuous side. With $x = e^u$,

$$M_R(\sigma; s) = \int_0^\infty \chi_R(u)\, \frac{s}{s^2 + u^2}\, e^{(1/2-\sigma)u}\, du = \int_{\mathbb{R}} \varphi_{R,s}(t) \left[\int_0^\infty \chi_R(u)\, e^{-(\sigma-\frac{1}{2}+it)u}\, du \right] dt.$$

Write $\alpha := \sigma - \frac{1}{2} + it$. Then

$$\int_0^\infty \chi_R(u)\, e^{-\alpha u}\, du = \frac{1}{\alpha} - \int_0^\infty \left(1 - \chi_R(u) \right) e^{-\alpha u}\, du.$$

Hence

$$M_R(\sigma; s) = \int_{\mathbb{R}} \varphi_{R,s}(t) \left[\frac{1}{\alpha} - E_R^{(\infty)}(t) \right] dt, \qquad E_R^{(\infty)}(t) := \int_0^\infty \left(1 - \chi_R(u) \right) e^{-\alpha u}\, du.$$

For $\sigma > \frac{1}{2}$ we have

$$\sup_{t \in \mathbb{R}} \left| E_R^{(\infty)}(t) \right| \leq \int_R^\infty e^{-(\sigma-\frac{1}{2})u}\, du = \frac{e^{-(\sigma-\frac{1}{2})R}}{\sigma - \frac{1}{2}}.$$

By the Fourier inversion bound $\|f\|_\infty \leq \frac{1}{2\pi}\|\widehat{f}\|_1$ (for our convention with inverse factor $(2\pi)^{-1}$), we have

$$\|\varphi_{R,s}\|_\infty \leq \frac{1}{2\pi}\,\|\widehat{\varphi}_{R,s}\|_1 \leq \frac{1}{2\pi} \int_{\mathbb{R}} \frac{s}{s^2 + u^2}\, du = \frac{1}{2},$$

uniformly in R.

Hence, for every $a > 0$,

$$\int_{\mathbb{R}} e^{-a|t|}\,|\varphi_{R,s}(t)|\, dt \leq \|\varphi_{R,s}\|_\infty \int_{\mathbb{R}} e^{-a|t|}\, dt = \frac{1}{2} \cdot \frac{2}{a} = \frac{1}{a}.$$

Hence, for any fixed $a > 0$,

$$\left| \int_{\mathbb{R}} e^{-a|t|} \varphi_{R,s}(t)\, E_R^{(\infty)}(t)\, dt \right| \leq \sup_t \left| E_R^{(\infty)}(t) \right| \int_{\mathbb{R}} e^{-a|t|}\,|\varphi_{R,s}(t)|\, dt \ll_a \frac{e^{-(\sigma-\frac{1}{2})R}}{\sigma - \frac{1}{2}} \xrightarrow[R \to \infty]{} 0,$$

In particular, taking $a = s$ shows that the Abel–weighted error term tends to 0 as $R \to \infty$; by Lemma 1.2 the same decay holds after the Abel boundary passage $\sigma \downarrow 0$ in $\mathcal{S}'_{\text{even}}(\mathbb{R})$.

Conclusion. At fixed R,

$$S_R(\sigma; s) - M_R(\sigma; s) = \int_{\mathbb{R}} \varphi_{R,s}(t) \left[\left(-\tfrac{\zeta'}{\zeta}\right)_R \left(\tfrac{1}{2} + \sigma - it\right) - \frac{1}{\sigma - \tfrac{1}{2} + it} \right] dt \; + \; \underbrace{\int_{\mathbb{R}} \varphi_{R,s}(t) \, E_R^{(\infty)}(t) \, dt}_{=: E_R(\sigma; s)}.$$

Even–test conjugacy. Since $\varphi_{R,s}$ (and later φ_s) are even, we may replace $\frac{1}{\sigma - \tfrac{1}{2} + it}$ by its conjugate $\frac{1}{\sigma - \tfrac{1}{2} - it}$ inside all pairings without changing their value:

$$\int_{\mathbb{R}} \varphi_{R,s}(t) \, \frac{dt}{\sigma - \tfrac{1}{2} + it} = \int_{\mathbb{R}} \varphi_{R,s}(t) \, \frac{dt}{\sigma - \tfrac{1}{2} - it} \quad \text{(substitute } t \mapsto -t\text{).}$$

Thus the subtraction term here matches the one used in Lemma 1.2, and that lemma applies verbatim to the $\sigma \downarrow 0$ boundary passage.

Let $\varphi_s := \mathcal{F}^{-1}\left(u \mapsto \frac{s}{s^2 + u^2}\right)$, so $\varphi_s(t) = \tfrac{1}{2} e^{-s|t|}$. Since

$$\widehat{\varphi}_{R,s}(u) = \frac{s}{s^2 + u^2} \chi_R(u) \quad \text{and} \quad 0 \le \chi_R(u) \uparrow 1,$$

we have $\widehat{\varphi}_{R,s} \to \frac{s}{s^2 + u^2}$ in $L^1(\mathbb{R})$. By the inverse Fourier transform this implies $\varphi_{R,s}(t) \to \varphi_s(t)$ pointwise.

In fact, by our Fourier inversion estimate,

$$\sup_{t \in \mathbb{R}} |\varphi_{R,s}(t) - \varphi_s(t)| \; \le \; \frac{1}{2\pi} \left\| \widehat{\varphi}_{R,s} - \frac{s}{s^2 + u^2} \right\|_1 \; = \; \frac{1}{2\pi} \int_{\mathbb{R}} \frac{s}{s^2 + u^2} \left| 1 - \chi_R(u) \right| du \; \xrightarrow[R \to \infty]{} \; 0,$$

since $0 \le 1 - \chi_R \le 1$ and $\chi_R(u) \uparrow 1$ pointwise with $\frac{s}{s^2 + u^2} \in L^1(\mathbb{R})$.

For each fixed $a > 0$ the family $e^{-a|t|} \varphi_{R,s}(t)$ is dominated by an $L^1(\mathbb{R})$ function independent of R. Hence for every $\psi \in \mathcal{S}_{\text{even}}(\mathbb{R})$,

$$\int_{\mathbb{R}} e^{-a|t|} \varphi_{R,s}(t) \, \psi(t) \, dt \; \xrightarrow[R \to \infty]{} \; \int_{\mathbb{R}} e^{-a|t|} \varphi_s(t) \, \psi(t) \, dt,$$

i.e. $e^{-a|t|} \varphi_{R,s} \to e^{-a|t|} \varphi_s$ in $\mathcal{S}'_{\text{even}}(\mathbb{R})$.

Using the conclusion above at fixed $\sigma > \tfrac{1}{2}$ and letting $R \to \infty$ we obtain

$$S(\sigma; s) - M(\sigma; s) = \int_{\mathbb{R}} \varphi_s(t) \left[-\frac{\zeta'}{\zeta} \left(\tfrac{1}{2} + \sigma - it\right) - \frac{1}{\sigma - \tfrac{1}{2} - it} \right] dt,$$

where $S(\sigma; s)$ and $M(\sigma; s)$ denote the limits of $S_R(\sigma; s)$ and $M_R(\sigma; s)$ as $R \to \infty$, and we used the even–test conjugacy to rewrite the subtraction term with $\sigma - \tfrac{1}{2} - it$.

Order of limits. For each fixed R, the identity above expresses $S_R(\sigma; s) - M_R(\sigma; s)$ as a pairing against $(-\zeta'/\zeta)_R(\tfrac{1}{2} + \sigma - it)$ with the subtraction term normalized exactly as in Lemma 1.2, so that lemma applies at this fixed R and yields the Abel boundary value as $\sigma \downarrow 0$ in $\mathcal{S}'_{\text{even}}(\mathbb{R})$. To pass $R \to \infty$ under the Abel limit, note first that for any fixed $\sigma_0 > 0$ and away from the finitely many boundary poles, the hypothesis of Lemma 1.2 gives

$$\left(-\frac{\zeta'}{\zeta}\right)\left(\tfrac{1}{2} + \sigma - it\right) \ll_{\sigma_0} (\log(2 + |t|))^2 \qquad (0 < \sigma \le \sigma_0).$$

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Together with the subtraction term $\frac{1}{\sigma - \frac{1}{2} - it}$ and the uniform bound $|\varphi_{R,s}(t)| \leq \frac{1}{2}$, this shows that for $0 < \sigma \leq \sigma_0$ the Abel–weighted integrands are dominated by

$$e^{-s|t|}\big((\log(2 + |t|))^2 + 1\big) \ \in \ L^1(\mathbb{R}),$$

independently of R. Hence, by dominated convergence (using also $\widehat{\varphi}_{R,s} \to \frac{s}{s^2+u^2}$ in L^1 and the bounds on $E_R^{(\infty)}$ and on the Abel–weighted error term above on the continuous side), we may interchange the truncation limit $R \to \infty$ with the Abel limit $\sigma \downarrow 0$ in the definition of the Abel–regularized prime resolvent.

Lemma 1.3 (Abel boundary value; distributional form). *Fix $a > 0$ and let $\psi \in \mathcal{S}_{\text{even}}(\mathbb{R})$. Then*

$$\lim_{\sigma \downarrow 0} \int_0^\infty e^{-at}\left[\left(-\frac{\zeta'}{\zeta}\right)(s) - \frac{1}{s-1}\right]_{s=\frac{1}{2}+\sigma-it} \psi(t)\, dt = \int_0^\infty e^{-at}\left[\left(-\frac{\zeta'}{\zeta}\right)(s) - \frac{1}{s-1}\right]_{s=\frac{1}{2}-it} \psi(t)\, dt.$$

Consequently, for real $a > 0$,

$$\mathcal{R}(a) := \lim_{\sigma \downarrow 0} \big(S(\sigma; a) - M(\sigma; a)\big) = \Re \int_0^\infty e^{-at}\left[-\frac{\zeta'}{\zeta}\left(\tfrac{1}{2} - it\right) - \frac{1}{\frac{1}{2} - it - 1}\right] dt.$$

Explanation. *By the distributional identity*

$$\lim_{\sigma \downarrow 0} \frac{1}{\sigma - i(t - \gamma)} = \pi\, \delta(t - \gamma) \ + \ i\, \mathrm{PV} \frac{1}{t - \gamma},$$

the boundary value of $-\zeta'/\zeta$ along $\Re s = \frac{1}{2}$ decomposes as "$PV + \delta$".
   For $-\zeta'/\zeta$ the boundary poles occur precisely at zeros on the critical line. Let

$$d_\gamma \ := \ \mathrm{Res}_{s=\frac{1}{2}+i\gamma}\left(-\frac{\zeta'}{\zeta}\right) \ = \ -m_\gamma^{(1/2)} \in \mathbb{R}.$$

In the sense of Lemma 1.2 one then has the paired coefficient

$$c_\gamma \ = \ d_\gamma + \overline{d_\gamma} \ = \ 2d_\gamma \ = \ -2\, m_\gamma^{(1/2)}.$$

Hence the atomic contribution on $\mathbb{R}$ (tested against even functions) is

$$\pi \sum_{\gamma>0} c_\gamma\, \delta(t - \gamma) \ = \ -2\pi \sum_{\gamma>0} m_\gamma^{(1/2)}\, \delta(t - \gamma).$$

Passing to the half–line with the real part removes the factor 2, yielding the term $-\pi \sum_{\gamma>0} m_\gamma^{(1/2)} e^{-a\gamma}$ in the identity below.
   Moreover, for fixed $a > 0$ and $\psi \in \mathcal{S}_{\text{even}}$,

$$\int_0^\infty e^{-at}\big(1 + \log(2 + t)\big)^2 |\psi(t)|\, dt < \infty,$$

which ensures dominated convergence for the $\sigma \downarrow 0$ limit in the weighted pairings above.
Boundary–value reference. *By Hörmander's Fourier–Laplace boundary–value theorem [1, Thm. 7.4.2] (cf. also [1, Thm. 3.1.15]), after removing the pole at $s = 1$ the limit*

$$\lim_{\sigma \downarrow 0} \left(-\frac{\zeta'}{\zeta}\left(\tfrac{1}{2} + \sigma - it\right) - \frac{1}{\frac{1}{2}+\sigma-it-1}\right)$$

exists in $\mathcal{S}'(\mathbb{R})$ and equals the tempered boundary distribution

$$t \;\mapsto\; -\frac{\zeta'}{\zeta}\big(\tfrac{1}{2} - it\big) - \frac{1}{\frac{1}{2} - it - 1}.$$

At boundary simple poles the PV $+\,\delta$ *decomposition follows from [1, §3.2].*

Justification (distributional). *The function* $-\zeta'/\zeta$ *is meromorphic with a simple pole at 1 and, for each fixed $\sigma > 0$, satisfies $-\zeta'/\zeta(\frac{1}{2} + \sigma - it) \ll (\log(2 + |t|))^2$ (uniformly on compact t–sets avoiding ordinates; see Titchmarsh [3, Thm. 9.6(A); see also Thm. 9.2 and (9.6.1)–(9.6.3)]; cf. Iwaniec–Kowalski [2, §5.2]). Uniform L^1 domination in $\sigma \downarrow 0$ may fail near ordinates, so we appeal to the cited Fourier–Laplace boundary–value theorem in $\mathcal{S}'(\mathbb{R})$ after subtracting the pole at 1. The Abel weight e^{-at} ensures absolute convergence of the pairings with $\psi \in \mathcal{S}_{\mathrm{even}}$, yielding the claimed limit.*

Growth control used (away from ordinates). *For each fixed $\sigma_0 > 0$ and every compact $J \subset \mathbb{R}$ avoiding ordinates,*

$$-\frac{\zeta'}{\zeta}\Big(\tfrac{1}{2} + \sigma - it\Big) \;\ll_{\sigma_0, J}\; (\log(2 + |t|))^2 \qquad (0 < \sigma \le \sigma_0, \; t \in J).$$

(See Titchmarsh [3, Thm. 9.6(A); see also Thm. 9.2 and (9.6.1)–(9.6.3)]; cf. Iwaniec–Kowalski [2, §5.2].)

We do not rely on a global uniform bound as $\sigma \downarrow 0$; instead we use the Abel–Plancherel boundary theorem for tempered distributions after subtracting the pole at 1, and the Abel weight e^{-at} guarantees absolute integrability of the pairing.

Approximation to $\psi \equiv 1$. Let $\psi_n(t) := e^{-(t/n)^2}$. Then $\psi_n \in \mathcal{S}_{\mathrm{even}}$, $0 \le \psi_n \le 1$, and $\psi_n \uparrow 1$ pointwise as $n \to \infty$. Write the boundary distribution (after subtracting the $s = 1$ pole) as the sum of a principal–value part and a discrete atomic part supported at ordinates:

$$\Big[-\frac{\zeta'}{\zeta}\big(\tfrac{1}{2} - it\big) - \frac{1}{\frac{1}{2} - it - 1}\Big] \;=\; \mathrm{PV}\,G(t) \;+\; \pi \sum_{\gamma > 0} c_\gamma\, \delta(t - \gamma) \quad \text{in } \mathcal{S}'_{\mathrm{even}}(\mathbb{R}),$$

with $c_\gamma = \mathrm{Res}_{s = \frac{1}{2} + i\gamma}\big(-\zeta'/\zeta\big) = -m_\gamma^{(1/2)}$.

Then for $a > 0$,

$$\int_0^\infty e^{-at}\, \mathrm{PV}\,G(t)\, \psi_n(t)\, dt \;\xrightarrow[n \to \infty]{}\; \int_0^\infty e^{-at}\, \mathrm{PV}\,G(t)\, dt,$$

Since $\int_0^\infty e^{-at}(\log(2 + t))^2\, dt < \infty$, dominated convergence applies to the PV part on each compact avoiding ordinates, and a diagonal argument yields the limit as $\psi_n \uparrow 1$. while

$$\sum_{\gamma > 0} \pi\, c_\gamma\, e^{-a\gamma}\, \psi_n(\gamma) \;\longrightarrow\; \sum_{\gamma > 0} \pi\, c_\gamma\, e^{-a\gamma}$$

by dominated convergence, since $|\psi_n(\gamma)| \le 1$ and $\sum_{\gamma > 0} |c_\gamma|\, e^{-a\gamma} < \infty$ (here $|c_\gamma| = m_\gamma^{(1/2)} \le m_\gamma$, and $N(T) \ll T \log T$). Hence, combining the PV dominated convergence and the dominated convergence of the atomic part, the boundary identity tested against $e^{-at}\psi_n$ passes to the case $\psi \equiv 1$.

Archimedean correction (real axis only). For $a > 0$ define the real–axis scalar

$$\mathrm{Arch}_{\mathrm{res}}(a) := \int_0^\infty e^{-at}\,\mathrm{Arch}\big[\cos(t\cdot)\big]\,dt.$$

This is the archimedean contribution in the explicit formula tested against the cosine kernel with Abel weight; it is used only on the real axis. We do *not* view $\mathrm{Arch}_{\mathrm{res}}$ as a holomorphic function of s.

Definition 1.4 (Prime-side scalar and prime weight)**.** For real $a > 0$ define the scalar

$$\mathcal{T}_{\mathrm{pr}}(a) := \mathcal{R}(a) - \mathrm{Arch}_{\mathrm{res}}(a).$$

For $\varphi \in \mathrm{PW}_{\mathrm{even}}$ set

$$\tau\big(\varphi(A)\big) := \lim_{\sigma \downarrow 0}\left(\sum_{p^k} \frac{\log p}{p^{k(1/2+\sigma)}}\,\widehat{\varphi}(k \log p) - \int_1^\infty \widehat{\varphi}(\log x)\,\frac{dx}{x^{1/2+\sigma}}\right) - \mathrm{Arch}[\varphi].$$

By Weil's explicit formula for even Paley–Wiener tests (unconditional),

$$\tau\big(\varphi(A)\big) = \sum_{\substack{\rho \\ \Im\rho>0}} \widehat{\varphi}(\Im\rho) \qquad (\varphi \in \mathrm{PW}_{\mathrm{even}}). \tag{2}$$

We use τ on the algebraic span $\mathrm{span}\{\varphi(A) : \varphi \in \mathrm{PW}_{\mathrm{even}}\}$. A normal semifinite positive extension to the von Neumann algebra generated by $\{f(A)\}$ will be obtained after Theorem 1.7 via the spectral measure μ.

Scope. The scalar $\mathcal{T}_{\mathrm{pr}}(a)$ is used only for the real-axis Poisson/Abel pairing with $\varphi_a(t) = \frac{1}{2}e^{-a|t|}$; it does not modify the general Paley–Wiener definition of τ.

1.1.3 Prime weight on $\mathrm{PW}_{\mathrm{even}}$: well-definedness and EF identity

Goal. Verify that the prime-anchored functional τ from Definition 1.4 is well defined on $\mathrm{PW}_{\mathrm{even}}$ and matches the zero-side Paley–Wiener pairing via Weil's explicit formula (cf. Lemma 1.3).

Proposition 1.5 (Prime weight on $\mathrm{PW}_{\mathrm{even}}$)**.** *The functional τ of Definition 1.4 is well defined on* $\mathrm{PW}_{\mathrm{even}}$ *and satisfies*
$$\tau\big(\varphi(A)\big) = \sum_{\substack{\rho \\ \Im\rho>0}} \widehat{\varphi}(\Im\rho) \qquad (\varphi \in \mathrm{PW}_{\mathrm{even}}).$$

Proof. Let $\varphi \in \mathrm{PW}_{\mathrm{even}}$, so $\widehat{\varphi} \in C_c^\infty(\mathbb{R})$ is even with $\mathrm{supp}\,\widehat{\varphi} \subset [-R, R]$ for some $R > 0$. For $\sigma > 0$ set

$$\widehat{\varphi}_\sigma(u) := e^{-\sigma|u|}\,\widehat{\varphi}(u), \qquad \varphi_\sigma := \mathcal{F}^{-1}(\widehat{\varphi}_\sigma).$$

Then $\varphi_\sigma \in \mathrm{PW}_{\mathrm{even}}$, $\widehat{\varphi}_\sigma$ is even, smooth, compactly supported in $[-R, R]$, and $\widehat{\varphi}_\sigma \to \widehat{\varphi}$ pointwise as $\sigma \downarrow 0$ with $|\widehat{\varphi}_\sigma| \le |\widehat{\varphi}|$.

Step 1: the σ–damped prime/continuous sides coincide with φ_σ. For every prime power p^k we have

$$p^{-k(1/2+\sigma)}\,\widehat{\varphi}(k \log p) = p^{-k/2}\,e^{-\sigma k \log p}\,\widehat{\varphi}(k \log p) = p^{-k/2}\,\widehat{\varphi}_\sigma(k \log p),$$

and, with the change of variables $x = e^u$,

$$\int_1^\infty \widehat{\varphi}(\log x)\,\frac{dx}{x^{1/2+\sigma}} = \int_0^\infty \widehat{\varphi}(u)\,e^{(1/2-\sigma)u}\,du = \int_0^\infty \widehat{\varphi}_\sigma(u)\,e^{u/2}\,du = \int_1^\infty \widehat{\varphi}_\sigma(\log x)\,\frac{dx}{x^{1/2}}.$$

Hence, for each fixed $\sigma > 0$, the *prime* and *continuous* pieces satisfy

$$\sum_{p^k} \frac{\log p}{p^{k(1/2+\sigma)}}\,\widehat{\varphi}(k\log p) \;-\; \int_1^\infty \widehat{\varphi}(\log x)\,\frac{dx}{x^{1/2+\sigma}} \;=\; \sum_{p^k} \frac{\log p}{p^{k/2}}\,\widehat{\varphi}_\sigma(k\log p) \;-\; \int_1^\infty \widehat{\varphi}_\sigma(\log x)\,\frac{dx}{x^{1/2}}.$$

The archimedean correction in the explicit formula is linear and continuous in the test function, so at level σ it is $\mathrm{Arch}[\varphi_\sigma]$ (not $\mathrm{Arch}[\varphi]$). Since $\widehat{\varphi}_\sigma$ has compact support, both the prime sum and the integral are *finite* sums/integrals and thus absolutely convergent; no rearrangement issues arise.

Step 2: explicit formula at fixed $\sigma > 0$. Weil's explicit formula (in the even Paley–Wiener class and with the normalizations used to define $\mathrm{Arch}[\cdot]$) gives, for each $\sigma > 0$,

$$\sum_{\substack{\rho \\ \Im\rho > 0}} \widehat{\varphi}_\sigma(\Im\rho) \;=\; \sum_{p^k} \frac{\log p}{p^{k/2}}\,\widehat{\varphi}_\sigma(k\log p) \;-\; \int_1^\infty \widehat{\varphi}_\sigma(\log x)\,\frac{dx}{x^{1/2}} \;-\; \mathrm{Arch}[\varphi_\sigma]. \tag{3}$$

(See, e.g., Weil; or Iwaniec–Kowalski, *Analytic Number Theory*, Thm. 5.12/Prop. 5.15, for this normalization with even tests and compactly supported Fourier transform. Evenness halves the zero-side sum to $\Im\rho > 0$.)

Combining the previous display with (3), for every $\sigma > 0$ we have

$$\boxed{\;\sum_{\substack{\rho \\ \Im\rho > 0}} \widehat{\varphi}_\sigma(\Im\rho) \;=\; \sum_{p^k} \frac{\log p}{p^{k(1/2+\sigma)}}\,\widehat{\varphi}(k\log p) \;-\; \int_1^\infty \widehat{\varphi}(\log x)\,\frac{dx}{x^{1/2+\sigma}} \;-\; \mathrm{Arch}[\varphi_\sigma].\;}$$

Step 3: letting $\sigma \downarrow 0$. Because $\mathrm{supp}\,\widehat{\varphi} \subset [-R, R]$, only zeros with $0 < \Im\rho \leq R$ contribute to $\sum_{\Im\rho > 0} \widehat{\varphi}_\sigma(\Im\rho)$, and there are finitely many of them. Hence $\widehat{\varphi}_\sigma(\Im\rho) \to \widehat{\varphi}(\Im\rho)$ termwise, and

$$\lim_{\sigma \downarrow 0} \sum_{\substack{\rho \\ \Im\rho > 0}} \widehat{\varphi}_\sigma(\Im\rho) \;=\; \sum_{\substack{\rho \\ \Im\rho > 0}} \widehat{\varphi}(\Im\rho).$$

Since $\widehat{\varphi}_\sigma \to \widehat{\varphi}$ pointwise with $|\widehat{\varphi}_\sigma| \leq |\widehat{\varphi}|$ and $\mathrm{supp}\,\widehat{\varphi} \subset [-R, R]$, the archimedean functional is continuous on $\mathrm{PW}_{\mathrm{even}}$, hence

$$\mathrm{Arch}[\varphi_\sigma] \;\xrightarrow[\sigma \downarrow 0]{}\; \mathrm{Arch}[\varphi].$$

On the prime/continuous side, Step 1 showed that for each $\sigma > 0$ the two expressions are finite; moreover, $\widehat{\varphi}_\sigma \to \widehat{\varphi}$ pointwise with $|\widehat{\varphi}_\sigma| \leq |\widehat{\varphi}|$, so the (finite) sums/integrals converge to the corresponding ones with $\sigma = 0$ and φ in place of φ_σ. Therefore, taking $\sigma \downarrow 0$ in the boxed identity yields

$$\lim_{\sigma \downarrow 0} \left(\sum_{p^k} \frac{\log p}{p^{k(1/2+\sigma)}}\,\widehat{\varphi}(k\log p) - \int_1^\infty \widehat{\varphi}(\log x)\,\frac{dx}{x^{1/2+\sigma}} \right) - \mathrm{Arch}[\varphi] \;=\; \sum_{\substack{\rho \\ \Im\rho > 0}} \widehat{\varphi}(\Im\rho).$$

By Definition 1.4 of τ on $\mathrm{PW}_{\mathrm{even}}$, the left-hand side is precisely $\tau\big(\varphi(A)\big)$, which proves

$$\tau\big(\varphi(A)\big) \;=\; \sum_{\substack{\rho \\ \Im\rho > 0}} \widehat{\varphi}(\Im\rho).$$

$\square$

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1.1.4 Technical bounds and integral interchanges

Lemma 1.6 (Operator and scalar bounds). *All implied constants below may be taken uniform in* $\sigma \in (0,1]$ *where such a parameter appears later.*

For $a > 0$,

$$\|(A^2 + a^2)^{-1}\| \le a^{-2}, \qquad \text{and for fixed } t > 0 \text{ and all } u \ge 0, \quad \left\|\frac{\cos(uA)}{t^2 + u^2}\right\| \le \frac{1}{t^2 + u^2}.$$

Moreover, there exists $C > 0$ *such that, uniformly for* $a \ge 1$,

$$\left|\tau\big((A^2 + a^2)^{-1}\big)\right| \le \frac{C(1 + \log a)}{a}.$$

Proof. Recall $A = A_\tau$ acts by multiplication by λ on $L^2((0,\infty), \mu)$ from Theorem 1.7, so the following operator-norm bounds are immediate by spectral calculus.

For the scalar bound we appeal to the prime–side representation proved below in Lemma 1.13: for real $a > 0$,
$$a\,\tau\big((A^2 + a^2)^{-1}\big) = \mathcal{T}_{\mathrm{pr}}(a) = \mathcal{R}(a) - \mathrm{Arch}_{\mathrm{res}}(a),$$
Thus it suffices to bound $\mathcal{T}_{\mathrm{pr}}(a)/a$; we do not use any properties of A at this point.

$M(\sigma; a)$ converges absolutely for $\sigma \ge \frac{1}{2}$. For $0 < \sigma < \frac{1}{2}$ we interpret both $M(\sigma; a)$ and $S(\sigma; a)$ via the same σ-damped Paley–Wiener truncation (finite for each cutoff) and pass to the limit using the explicit formula / Stieltjes integration by parts.

PW–truncation convention. All estimates below are performed at the Paley–Wiener truncation level (finite sums/integrals) with $\widehat{\psi}_R(\xi) = \dfrac{a}{a^2 + \xi^2}\chi_R(\xi)$ as in Lemma 1.13; the $R \to \infty$ limit is taken by monotone convergence. No unconditional absolute convergence at $\sigma \le \frac{1}{2}$ is claimed a priori.

Moreover, for $\sigma > \frac{1}{2}$,

$$|M(\sigma; a)| = \int_1^\infty \frac{a}{(\log x)^2 + a^2}\,\frac{dx}{x^{1/2+\sigma}} = \int_0^\infty \frac{a}{u^2 + a^2}\,e^{(1/2-\sigma)u}\,du \ll 1 \qquad (\sigma > \tfrac{1}{2}),$$

uniformly in $a \ge 1$. For $\sigma = \frac{1}{2}$,

$$|M(\tfrac{1}{2}; a)| = \int_1^\infty \frac{a}{(\log x)^2 + a^2}\,\frac{dx}{x} = \int_0^\infty \frac{a}{u^2 + a^2}\,du = \frac{\pi}{2} \ll 1.$$

For $0 < \sigma < \frac{1}{2}$ we work at the σ-damped Paley–Wiener truncation level and pass to the limit using the explicit formula / Stieltjes integration by parts, which yields an $O(1)$ bound uniformly in $a \ge 1$. Using partial summation with the trivial bound $\psi(x) = \sum_{n \le x} \Lambda(n) \le x \log x$,

$$S(\sigma; a) \ll \int_{\log 2}^\infty \frac{2a}{u^2 + a^2}\,e^{-(\frac{1}{2}+\sigma)u}\,(1 + u)\,du \ll 1 + \log a,$$

uniformly for $a \ge 1$, and, by Lemma 1.14,

$$\mathrm{Arch}_{\mathrm{res}}(a) = \frac{1}{4}\left(\log \pi - \psi\Big(\frac{1}{4} + \frac{a}{2}\Big)\right) = -\frac{1}{4}\log a + O(1) \qquad (a \to \infty).$$

Thus $|\mathrm{Arch}_{\mathrm{res}}(a)| \ll 1 + \log a$ uniformly for $a \ge 1$.

Temperedness of the archimedean term. The distribution $\mathrm{Arch}[\cos(t\cdot)]$ is a finite linear combination of derivatives of $\log \Gamma$ evaluated on even tests (hence tempered). The Abel weight e^{-at} ensures absolute convergence; together with Lemma 1.14 this yields the uniform bound $|\mathrm{Arch}_{\mathrm{res}}(a)| \ll 1 + \log a$. $\qquad \square$

1.1.5 Poisson semigroup identity

Theorem 1.7 (Poisson semigroup identity). *(Here A_τ is the canonical multiplication operator constructed below.)*

For every $t > 0$,

$$\tau\big(e^{-tA_\tau}\big) \;=\; \int_{(0,\infty)} e^{-t\lambda}\, d\mu(\lambda).$$

In particular, after Lemma 1.37 (atomicity), $\tau(e^{-tA_\tau}) = \sum_{\gamma>0} m_\gamma e^{-t\gamma}$.

Proof. Fix $t > 0$. Let $\chi_R \in C_c^\infty(\mathbb{R})$ be even with $0 \le \chi_R \le 1$, $\chi_R \equiv 1$ on $[-R, R]$, and $\chi_{R_1} \le \chi_{R_2}$ for $R_1 \le R_2$. For $\varepsilon \in (0,1]$ set

$$\widehat{\varphi}_{R,\varepsilon}(\xi) := e^{-t\sqrt{\xi^2+\varepsilon^2}} \chi_R(\xi), \qquad \varphi_{R,\varepsilon} := \mathcal{F}^{-1}(\widehat{\varphi}_{R,\varepsilon}) \in \mathrm{PW}_{\mathrm{even}}.$$

Then by (2),

$$\tau(\varphi_{R,\varepsilon}) = \sum_{\gamma>0} m_\gamma\, \widehat{\varphi}_{R,\varepsilon}(\gamma),$$

Monotonicity for MCT. For fixed $t > 0$,

$$\widehat{\varphi}_{R,\varepsilon}(\xi) = e^{-t\sqrt{\xi^2+\varepsilon^2}} \chi_R(\xi) \ge 0,$$

and it is monotone in both parameters: if $0 < \varepsilon_1 < \varepsilon_2$ then $e^{-t\sqrt{\xi^2+\varepsilon_1^2}} \ge e^{-t\sqrt{\xi^2+\varepsilon_2^2}}$ so $\widehat{\varphi}_{R,\varepsilon_1}(\xi) \ge \widehat{\varphi}_{R,\varepsilon_2}(\xi)$; and if $R_1 < R_2$ then $\chi_{R_1} \le \chi_{R_2}$ so $\widehat{\varphi}_{R_1,\varepsilon}(\xi) \le \widehat{\varphi}_{R_2,\varepsilon}(\xi)$. Hence, for each $\gamma > 0$, the terms $\widehat{\varphi}_{R,\varepsilon}(\gamma)$ increase as $\varepsilon \downarrow 0$ and as $R \uparrow \infty$. Therefore, by the monotone convergence theorem,

$$\sum_{\gamma>0} m_\gamma\, \widehat{\varphi}_{R,\varepsilon}(\gamma) \;\xrightarrow[\varepsilon\downarrow 0]{\mathrm{MCT}}\; \sum_{\gamma>0} m_\gamma\, e^{-t\gamma} \chi_R(\gamma) \;\xrightarrow[R\to\infty]{\mathrm{MCT}}\; \sum_{\gamma>0} m_\gamma\, e^{-t\gamma} \;=:\; \Theta(t).$$

Thus $\lim_{R\to\infty} \lim_{\varepsilon\downarrow 0} \tau(\varphi_{R,\varepsilon}) = \Theta(t)$. For each $n \ge 0$ and $t > 0$ the series $\sum_{\gamma>0} m_\gamma \gamma^n e^{-t\gamma}$ converges absolutely: since $N(T) \ll T \log T$, we have

$$\sum_{\gamma>0} m_\gamma\, \gamma^n e^{-t\gamma} \;\ll\; \int_0^\infty (1 + u\log(2+u))\, u^n e^{-tu}\, du < \infty.$$

Thus differentiation under the sum is justified by dominated convergence, giving

$$(-1)^n \Theta^{(n)}(t) = \sum_{\gamma>0} m_\gamma\, \gamma^n e^{-t\gamma} \;\ge\; 0.$$

Hence Θ is completely monotone, and by Bernstein's theorem there exists a unique positive Borel measure μ on $(0,\infty)$ with $\Theta(t) = \int_{(0,\infty)} e^{-t\lambda}\, d\mu(\lambda)$.

Define A_τ as multiplication by λ on $L^2((0,\infty), \mu)$ and extend τ by $\tau(f(A_\tau)) := \int f\, d\mu$ for bounded Borel $f \ge 0$. Taking $f(\lambda) = e^{-t\lambda}$ gives

$$\tau(e^{-tA_\tau}) = \int_{(0,\infty)} e^{-t\lambda}\, d\mu(\lambda).$$

$\square$

In particular, for the canonical operator A_τ,

$$\tau(e^{-tA_\tau}) = \Theta(t) = \int_{(0,\infty)} e^{-t\lambda} \, d\mu(\lambda) \qquad (t > 0),$$

and after Lemma 1.37 this equals $\sum_{\gamma > 0} m_\gamma e^{-t\gamma}$.

Corollary 1.8 (Identification of the spectral measure). *After Theorem 1.7, there is a unique positive Borel measure μ on $(0, \infty)$ with*

$$\tau(e^{-tA}) = \int_{(0,\infty)} e^{-t\lambda} \, d\mu(\lambda) \qquad (t > 0).$$

After Lemma 1.37 we identify $\mu = \sum_{\gamma>0} m_\gamma \, \delta_\gamma$, and for every bounded Borel $f \geq 0$ we then have $\tau(f(A)) = \int f \, d\mu = \sum_{\gamma>0} m_\gamma f(\gamma)$.

Lemma 1.9 (Daniell extension for τ). *If $\tau(e^{-tA}) = \int_{(0,\infty)} e^{-t\lambda} \, d\mu(\lambda)$ for all $t > 0$ and τ is normal and positive on the functional calculus, then for every bounded Borel $f \geq 0$,*

$$\tau(f(A)) = \int_{(0,\infty)} f(\lambda) \, d\mu(\lambda).$$

Proof. Let $\mathcal{M}$ be the class of bounded Borel $f \geq 0$ on $(0, \infty)$ such that $\tau(f(A)) = \int f \, d\mu$. By normality of τ on the functional calculus and the monotone convergence theorem, $\mathcal{M}$ is a *monotone class*: if $0 \leq f_n \uparrow f$ pointwise with each $f_n \in \mathcal{M}$, then $\tau(f_n(A)) \uparrow \tau(f(A))$ and $\int f_n \, d\mu \uparrow \int f \, d\mu$, hence $f \in \mathcal{M}$.

By hypothesis, $e^{-t\lambda} \in \mathcal{M}$ for every $t > 0$. Let $\mathcal{A}$ be the algebra generated by $\{e^{-t\lambda} : t > 0\}$ with *nonnegative* coefficients. Then $\mathcal{A} \subset \mathcal{M}$ since $\tau\left(\sum_i c_i e^{-t_i A}\right) = \sum_i c_i \tau(e^{-t_i A}) = \sum_i c_i \int e^{-t_i \lambda} \, d\mu = \int \sum_i c_i e^{-t_i \lambda} \, d\mu$, and $\mathcal{A}$ is closed under pointwise products because $e^{-t\lambda} e^{-s\lambda} = e^{-(t+s)\lambda}$.

The family $\{e^{-t\lambda} : t > 0\}$ separates points on $(0, \infty)$, so the σ-algebra generated by $\mathcal{A}$ is the full Borel σ-algebra. By the functional monotone class theorem, the smallest monotone class containing $\mathcal{A}$ is exactly the set of all bounded Borel functions. Hence every bounded Borel $f \geq 0$ lies in $\mathcal{M}$, i.e. $\tau(f(A)) = \int f \, d\mu$. $\qquad\square$

Remark 1.10 (Use of Lemma 1.9). In our application, there is no circularity. We first construct the measure μ (from the prime-side semigroup and Bernstein), then define the canonical model A_τ and set

$$\tau(f(A_\tau)) := \int_{(0,\infty)} f(\lambda) \, d\mu(\lambda)$$

for bounded Borel $f \geq 0$. With this definition, τ is automatically normal and positive on the Borel functional calculus of A_τ, and the identity $\tau(e^{-tA_\tau}) = \int e^{-t\lambda} \, d\mu(\lambda)$ holds by construction.

Lemma 1.9 is therefore used only in the formal direction stated: it records that any normal positive τ agreeing with μ on the Poisson/Laplace semigroup is uniquely determined on all bounded Borel functions. We do *not* invoke the lemma to deduce normality or to construct μ.

Canonical resolvent trace. With μ and A_τ as in Corollary 1.8, define for $\Re s > 0$

$$\mathcal{T}(s) := \tau\left((A_\tau^2 + s^2)^{-1}\right) = \int_{(0,\infty)} \frac{1}{\lambda^2 + s^2} \, d\mu(\lambda).$$

For real $a > 0$, the compatibility lemma (proved in Lemma 1.13 below) yields

$$a\,\mathcal{T}(a) = \mathcal{T}_{\mathrm{pr}}(a) \qquad \text{so} \qquad \mathcal{T}(a) = \mathcal{T}_{\mathrm{pr}}(a)/a.$$

From now on we write $A := A_\tau$.

Arch continuity for the PW approximants. For even Paley–Wiener tests we have $\mathrm{Arch}[\varphi] = \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{\varphi}(\xi)\,G(\xi)\,d\xi$ with $G(\xi) = \frac{1}{2}\log\pi - \frac{1}{2}\Re\psi(\frac{1}{4} + \frac{i\xi}{2})$ and $|G(\xi)| \ll 1 + \log(2+|\xi|)$. For $\widehat{\varphi}_{R,\varepsilon} = \left(\frac{a}{a^2+\xi^2}\chi_R\right) * \phi_\varepsilon$, dominated convergence applies since $\left|\widehat{\varphi}_{R,\varepsilon}(\xi)\,G(\xi)\right| \le \frac{a}{a^2+\xi^2}\left(1 + \log(2+|\xi|)\right)$ and $\frac{a}{a^2+\xi^2}\left(1 + \log(2+|\xi|)\right) \in L^1(\mathbb{R})$; thus

$$\lim_{\varepsilon\downarrow 0}\lim_{R\to\infty}\mathrm{Arch}[\varphi_{R,\varepsilon}] = \frac{1}{2\pi}\int_{\mathbb{R}}\frac{a}{a^2+\xi^2}G(\xi)\,d\xi =: \mathrm{Arch}_{\mathrm{res}}(a),$$

Remark 1.11 (Consistency with Definition 1.4). The formula above agrees with the earlier definition

$$\mathrm{Arch}_{\mathrm{res}}(a) = \int_0^\infty e^{-at}\,\mathrm{Arch}[\cos(t\,\cdot)]\,dt.$$

Indeed, using $\int_0^\infty e^{-at}\cos(t\xi)\,dt = \frac{a}{a^2+\xi^2}$ and $\mathrm{Arch}[\varphi] = \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{\varphi}(\xi)\,G(\xi)\,d\xi$ with $G(\xi) = \frac{1}{2}\log\pi - \frac{1}{2}\Re\psi(\frac{1}{4} + \frac{i\xi}{2})$, we swap t– and ξ–integrals by dominated convergence (since $\frac{a}{a^2+\xi^2}(1+\log(2+|\xi|)) \in L^1(\mathbb{R})$) to get

$$\mathrm{Arch}_{\mathrm{res}}(a) = \frac{1}{2\pi}\int_{\mathbb{R}}\frac{a}{a^2+\xi^2}\,G(\xi)\,d\xi,$$

which Lemma 1.14 evaluates as $\frac{1}{4}\left(\log\pi - \psi(\frac{1}{4} + \frac{a}{2})\right)$.

Weighted prime/continuous resolvents. For bounded Borel $g : \mathbb{R}_{\ge 0} \to \mathbb{R}$ with compact support and for $\Re s > 0$, $\sigma > 0$, set

$$S_g(\sigma; s) := \sum_{p^k}\frac{\log p}{p^{k(1/2+\sigma)}}\,g(k\log p)\,\frac{s}{(k\log p)^2 + s^2}, \qquad M_g(\sigma; s) := \int_1^\infty g(\log x)\,\frac{s}{(\log x)^2 + s^2}\,\frac{dx}{x^{1/2+\sigma}}.$$

We write $S(\sigma; s) := S_{\mathbf{1}}(\sigma; s)$ and $M(\sigma; s) := M_{\mathbf{1}}(\sigma; s)$.

Remark 1.12 (Spectrum of A vs. spectrum of A^2). By construction the canonical model acts as multiplication by λ on $L^2((0,\infty), d\mu(\lambda))$, so $A \ge 0$ and

$$\sigma(A) = \mathrm{supp}(\mu) \subset [0,\infty).$$

Hence

$$A^2 \text{ corresponds to multiplication by } \lambda^2, \qquad \sigma(A^2) = \{\lambda^2 : \lambda \in \sigma(A)\},$$

and conversely A is the unique positive square root of A^2, so

$$\sigma(A) = \{\sqrt{x} : x \in \sigma(A^2)\}.$$

Thus the spectral data of A are completely encoded by the resolvent $(A^2 + s^2)^{-1}$ used below.

Lemma 1.13 (Compatibility: prime-side and measure-side resolvents agree). *For every $a > 0$,*

$$a\,\tau\big((A^2 + a^2)^{-1}\big) = \mathcal{T}_{\mathrm{pr}}(a) \qquad i.e. \qquad \mathcal{T}(a) = \mathcal{T}_{\mathrm{pr}}(a)/a.$$

Proof. Fix $a > 0$. Choose $\chi_R \in C_c^\infty(\mathbb{R})$ even with $0 \le \chi_R \le 1$, $\chi_R \equiv 1$ on $[-R, R]$, $\chi_R \uparrow 1$, and let $\phi_\varepsilon \in C_c^\infty(\mathbb{R})$ be an even mollifier with $\int\phi_\varepsilon = 1$, $\mathrm{supp}\,\phi_\varepsilon \subset [-\varepsilon, \varepsilon]$. Define

$$\widehat{\psi}_R(\xi) := \frac{a}{a^2+\xi^2}\,\chi_R(\xi), \qquad \widehat{\varphi}_{R,\varepsilon} := \widehat{\psi}_R * \phi_\varepsilon, \qquad \varphi_{R,\varepsilon} := \mathcal{F}^{-1}(\widehat{\varphi}_{R,\varepsilon}) \in \mathrm{PW}_{\mathrm{even}}.$$

Note that $\widehat{\varphi}_{R,\varepsilon} = \widehat{\psi}_R * \phi_\varepsilon \geq 0$, $\|\widehat{\varphi}_{R,\varepsilon}\|_\infty \leq \|\widehat{\psi}_R\|_\infty$ (since ϕ_ε has unit mass and is nonnegative), and $\operatorname{supp} \widehat{\varphi}_{R,\varepsilon} \subset \operatorname{supp} \widehat{\psi}_R + [-\varepsilon, \varepsilon] \subset [-R-1, R+1]$ for $\varepsilon \leq 1$. Moreover $\widehat{\varphi}_{R,\varepsilon} \to \widehat{\psi}_R$ pointwise (and in L^1_{loc}) as $\varepsilon \downarrow 0$.

   Measure side. By the definition of τ on the canonical model (see Corollary 1.8), for every bounded Borel $f \geq 0$ we have

$$\tau(f(A)) = \int_{(0,\infty)} f(\lambda) \, d\mu(\lambda).$$

In particular,

$$\tau(\varphi_{R,\varepsilon}(A)) = \int_{(0,\infty)} \widehat{\varphi}_{R,\varepsilon}(\lambda) \, d\mu(\lambda).$$

   Fix R and $0 < \varepsilon \leq 1$. Since $\widehat{\varphi}_{R,\varepsilon} = \widehat{\psi}_R * \phi_\varepsilon \geq 0$, $\|\widehat{\varphi}_{R,\varepsilon}\|_\infty \leq \|\widehat{\psi}_R\|_\infty$, and $\operatorname{supp} \widehat{\varphi}_{R,\varepsilon} \subset [0, R+1]$ on $(0, \infty)$, we may apply dominated convergence (dominated by $\|\widehat{\psi}_R\|_\infty \mathbf{1}_{[0,R+1]}(\lambda)$) to let $\varepsilon \downarrow 0$. To justify integrability of the dominator, use Bernstein's representation $\Theta(t) = \int_{(0,\infty)} e^{-t\lambda} \, d\mu(\lambda) < \infty$ for every $t > 0$. Then for fixed $t > 0$ and $R \geq 0$,

$$\mu([0, R+1]) \ \leq \ e^{t(R+1)} \int_{(0,\infty)} e^{-t\lambda} \, d\mu(\lambda) = e^{t(R+1)} \, \Theta(t) \ < \ \infty.$$

Hence $\|\widehat{\psi}_R\|_\infty \mathbf{1}_{[0,R+1]}(\lambda)$ is an integrable dominator and dominated convergence applies as $\varepsilon \downarrow 0$, giving

$$\int_{(0,\infty)} \widehat{\varphi}_{R,\varepsilon}(\lambda) \, d\mu(\lambda) \xrightarrow[\varepsilon \downarrow 0]{} \int_{(0,\infty)} \widehat{\psi}_R(\lambda) \, d\mu(\lambda).$$

   Now let $R \to \infty$. Because $\widehat{\psi}_R(\lambda) \uparrow \frac{a}{a^2+\lambda^2}$ pointwise and ≥ 0, monotone convergence yields

$$\int_{(0,\infty)} \widehat{\psi}_R(\lambda) \, d\mu(\lambda) \xrightarrow[R \to \infty]{} \int_{(0,\infty)} \frac{a}{a^2 + \lambda^2} \, d\mu(\lambda) = a \, \tau\big((A^2 + a^2)^{-1}\big).$$

   Prime side. By Definition 1.4 and (2),

$$\tau(\varphi_{R,\varepsilon}(A)) = \lim_{\sigma \downarrow 0} \big(S_{\widehat{\varphi}_{R,\varepsilon}}(\sigma; a) - M_{\widehat{\varphi}_{R,\varepsilon}}(\sigma; a)\big) - \operatorname{Arch}[\varphi_{R,\varepsilon}].$$

Let $\varepsilon \downarrow 0$. For fixed R, $\widehat{\varphi}_{R,\varepsilon}$ has compact support, so the prime sum and the $\log x$–integral are finite. Since $\widehat{\varphi}_{R,\varepsilon} \to \widehat{\psi}_R$ pointwise and the index sets are finite, the limit $\varepsilon \downarrow 0$ passes inside the sum and the integral. Now let $R \to \infty$. For the prime sum and the $\log x$–integral (both nonnegative), since $\widehat{\psi}_R \uparrow a/(a^2 + \xi^2)$, the monotone convergence theorem gives, for each fixed $\sigma > 0$,

$$\lim_{R \to \infty} \lim_{\varepsilon \downarrow 0} \Big(S_{\widehat{\varphi}_{R,\varepsilon}}(\sigma; a) - M_{\widehat{\varphi}_{R,\varepsilon}}(\sigma; a)\Big) = S(\sigma; a) - M(\sigma; a).$$

Consequently,

$$\lim_{\sigma \downarrow 0} \lim_{R \to \infty} \lim_{\varepsilon \downarrow 0} \Big(S_{\widehat{\varphi}_{R,\varepsilon}}(\sigma; a) - M_{\widehat{\varphi}_{R,\varepsilon}}(\sigma; a)\Big) = \lim_{\sigma \downarrow 0} \big(S(\sigma; a) - M(\sigma; a)\big).$$

(For the archimedean term we use dominated convergence, as noted below, to obtain $\operatorname{Arch}[\varphi_{R,\varepsilon}] \to \operatorname{Arch}_{\mathrm{res}}(a)$.)

   For the archimedean term, recall that for even Paley–Wiener tests

$$\operatorname{Arch}[\varphi] \ = \ \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\varphi}(\xi) \, G(\xi) \, d\xi, \qquad G(\xi) = \tfrac{1}{2} \log \pi - \tfrac{1}{2} \Re \, \psi\big(\tfrac{1}{4} + \tfrac{i\xi}{2}\big).$$

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Since $\widehat{\varphi}_{R,\varepsilon} = \widehat{\psi}_R * \phi_\varepsilon$ with $\phi_\varepsilon \geq 0$ of unit mass, we have

$$0 \leq \widehat{\varphi}_{R,\varepsilon}(\xi) \leq \widehat{\psi}_R(\xi) \leq \frac{a}{a^2 + \xi^2} \qquad (\xi \in \mathbb{R}).$$

Moreover $|G(\xi)| \ll 1 + \log(2 + |\xi|)$ and

$$\frac{a}{a^2 + \xi^2}\left(1 + \log(2 + |\xi|)\right) \in L^1(\mathbb{R}).$$

Hence by dominated convergence,

$$\text{Arch}[\varphi_{R,\varepsilon}] \xrightarrow[\varepsilon\downarrow 0]{} \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\psi}_R(\xi)\, G(\xi)\, d\xi, \qquad \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\psi}_R(\xi)\, G(\xi)\, d\xi \xrightarrow[R\to\infty]{} \frac{1}{2\pi}\int_{\mathbb{R}} \frac{a}{a^2 + \xi^2}\, G(\xi)\, d\xi.$$

By Lemma 1.14, the last integral equals $\text{Arch}_{\text{res}}(a)$.

Combining these,

$$\lim_{R\to\infty} \lim_{\varepsilon\downarrow 0} \tau(\varphi_{R,\varepsilon}(A)) = \lim_{\sigma\downarrow 0}\left(S(\sigma;a) - M(\sigma;a)\right) - \text{Arch}_{\text{res}}(a) =: T_{\text{pr}}(a).$$

Comparing the two limits gives

$$a\,\tau\big((A^2 + a^2)^{-1}\big) = T_{\text{pr}}(a), \qquad \text{i.e.} \qquad \mathcal{T}(a) = \frac{T_{\text{pr}}(a)}{a}.$$

This proves the claim.

$\square$

Explicit archimedean subtraction and the Hadamard term. Folding identity (zeta + gamma into Ξ). Recall

$$\Lambda(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s), \qquad \xi(s) := \tfrac{1}{2}s(s-1)\Lambda(s), \qquad \Xi(s) := \xi\left(\tfrac{1}{2} + s\right),$$

and write $\psi = \Gamma'/\Gamma$.

Taking a logarithmic derivative and shifting $s \mapsto \frac{1}{2} + s$ yields the exact identity

$$\frac{\Xi'}{\Xi}(s) = \frac{\zeta'}{\zeta}\left(\tfrac{1}{2} + s\right) + \frac{1}{s + \frac{1}{2}} + \frac{1}{s - \frac{1}{2}} - \frac{1}{2}\log\pi + \frac{1}{2}\psi\left(\frac{\frac{1}{2} + s}{2}\right). \tag{4}$$

(Here the rational terms come from $s(s-1)$ after the shift, the $-\frac{1}{2}\log\pi$ from $\pi^{-s/2}$, and the ψ term from $\Gamma(s/2)$.) This is the formula we use to fold the archimedean and elementary factors into Ξ'/Ξ on the real axis in what follows.

Write the Hadamard–log–derivative decomposition as

$$\frac{\Xi'}{\Xi}(s) = 2s\sum_\rho \frac{1}{s^2 - \rho^2} + H'_{\text{Had}}(s). \tag{5}$$

where H_{Had} is even entire.

Equivalently, since $\frac{1}{s-\rho} + \frac{1}{s+\rho} = \frac{2s}{s^2-\rho^2}$,

$$\frac{\Xi'}{\Xi}(s) = H'(s) + \sum_{\rho \neq 0} m_\rho\left(\frac{1}{s - \rho} + \frac{1}{s + \rho}\right). \tag{6}$$

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For $a > 0$, recall $\text{Arch}_{\text{res}}(a)$ from the real-axis definition above. Using the representation of $\text{Arch}[\varphi]$ for even Paley–Wiener tests and Lemma 1.14, we have the equivalent formula

$$\text{Arch}_{\text{res}}(a) = \int_0^\infty e^{-at}\left(\tfrac{1}{2}\log\pi - \tfrac{1}{2}\Re\,\psi(\tfrac{1}{4} + \tfrac{it}{2})\right)dt,$$

where $\psi = \Gamma'/\Gamma$.

We henceforth fix $H = H_{\text{Had}}$, the even entire Hadamard part of Ξ, so that $H'_{\text{Had}} \equiv 0$. The archimedean contribution is accounted for explicitly by the fold identity (4); on the real axis its Abel/Poisson transform is encoded by $\text{Arch}_{\text{res}}(a)$ (Lemma 1.14), and Lemmas 1.17–1.18 verify that these descriptions agree.

Lemma 1.14 (Archimedean real-axis computation). *For $a > 0$, with $G(\xi) = \tfrac{1}{2}\log\pi - \tfrac{1}{2}\Re\,\psi(\tfrac{1}{4} + \tfrac{i\xi}{2})$,*

$$\text{Arch}_{\text{res}}(a) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{a}{a^2+\xi^2}\,G(\xi)\,d\xi \overset{(*)}{=} \frac{1}{4}\left(\log\pi - \psi\left(\frac{1}{4} + \frac{a}{2}\right)\right) = -\frac{1}{4}\log a + O(1) \qquad (a \to \infty).$$

Proof. (1) $\displaystyle\int_0^\infty e^{-at}\cos(t\xi)\,dt = \frac{a}{a^2+\xi^2}$ *turns* $\text{Arch}_{\text{res}}(a)$ *into the displayed ξ–integral.*

(2) *Insert* $\Re\,\psi\left(\tfrac{1}{4} + \tfrac{i\xi}{2}\right) = \displaystyle\int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-t/4}\cos\left(\frac{\xi t}{2}\right)}{1 - e^{-t}}\right)dt$, *and swap the t– and ξ–integrals by dominated convergence since* $G(\xi) = \tfrac{1}{2}\log\pi - \tfrac{1}{2}\Re\,\psi(\tfrac{1}{4} + \tfrac{i\xi}{2}) = O(\log(2 + |\xi|))$ *and* $\frac{a}{a^2+\xi^2} \in L^1(\mathbb{R})$, *hence* $\frac{a}{a^2+\xi^2}G(\xi) \in L^1(\mathbb{R})$. *Then use* $\displaystyle\int_{\mathbb{R}} \frac{a}{a^2+\xi^2}\cos\left(\frac{\xi t}{2}\right)d\xi = \pi e^{-at/2}$.

(3) *Recognize the t–integral via* $\psi(z) = \displaystyle\int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1 - e^{-t}}\right)dt$ *at* $z = \tfrac{1}{4} + \tfrac{a}{2}$, *yielding* $\text{Arch}_{\text{res}}(a) = \tfrac{1}{4}\left(\log\pi - \psi(\tfrac{1}{4} + \tfrac{a}{2})\right)$.

Remark 1.15 (Asymptotics). As $a \to \infty$,

$$\text{Arch}_{\text{res}}(a) = \frac{1}{4}\left(\log\pi - \psi\left(\frac{1}{4} + \frac{a}{2}\right)\right) = -\frac{1}{4}\log a + O(1).$$

In particular $|\text{Arch}_{\text{res}}(a)| \ll 1 + \log a$ uniformly for $a \geq 1$.

Lemma 1.16 (Growth for Poisson reproduction). *Fix $\varepsilon \in (1/2, 1)$ and put $\delta := \varepsilon - \tfrac{1}{2} > 0$. For $x \in [\varepsilon, 1]$ and $s = x + iy$, define*

$$F_{\text{reg}}(s) := \frac{\Xi'}{\Xi}(s) - H'_{\text{Had}}(s) = \frac{\Xi'}{\Xi}(s).$$

Then

$$F_{\text{reg}}(s) = O_\varepsilon\big(\log(2 + |y|)\big) \qquad (|y| \to \infty),$$

uniformly in x.

Proof. From the exact fold,

$$\frac{\Xi'}{\Xi}(s) = \frac{\zeta'}{\zeta}\left(\tfrac{1}{2} + s\right) + \frac{1}{s + \tfrac{1}{2}} + \frac{1}{s - \tfrac{1}{2}} - \tfrac{1}{2}\log\pi + \tfrac{1}{2}\,\psi\left(\frac{\tfrac{1}{2} + s}{2}\right).$$

Here $\sigma := \Re(\tfrac{1}{2} + s) = \tfrac{1}{2} + x \geq 1 + \delta$. By Titchmarsh [3, Thm. 9.6(A)] (see also Iwaniec–Kowalski [2, §5.2]),

$$\frac{\zeta'}{\zeta}(\sigma + it) = O_\delta\big(\log(2 + |t|)\big) \qquad (\sigma \geq 1 + \delta).$$

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Moreover, for fixed u in a compact set and $|v| \to \infty$,

$$\psi(u + iv) = \log|u + iv| + O\left(\frac{1}{|v|}\right) = \log(2 + |v|) + O(1),$$

and

$$\frac{1}{s + \frac{1}{2}} + \frac{1}{s - \frac{1}{2}} = O\left(\frac{1}{|y|}\right).$$

Hence $\Xi'/\Xi(x + iy) = O_\varepsilon(\log(2 + |y|))$.

For $|y| \le 1$ all terms are $O_\varepsilon(1)$, so the bound $O_\varepsilon(\log(2 + |y|))$ holds uniformly in $x \in [\varepsilon, 1]$.

For Ξ, the canonical Hadamard product (order 1, genus 1) with paired zeros $\{\pm\rho\}$ reads

$$\Xi(s) = e^A \prod_\rho \left(1 - \frac{s^2}{\rho^2}\right),$$

so the entire Hadamard part is constant and hence $H'_{\text{Had}} \equiv 0$ (see e.g. [3, §§2.13–2.16]).

Therefore $F_{\text{reg}}(x + iy) = \Xi'(x + iy)/\Xi(x + iy) = O_\varepsilon(\log(2 + |y|))$ uniformly in $x \in [\varepsilon, 1]$. $\qquad\square$

Right-half-plane Poisson (recall). If U is harmonic on $\{\Re s > 0\}$ and $\Re U(it) = O(\log(2 + |t|))$ as $|t| \to \infty$, then for every $a > 0$,

$$\Re U(a) \;=\; \frac{1}{\pi} \int_\mathbb{R} \frac{a}{a^2 + t^2} \, \Re U(it) \, dt,$$

and the integral converges absolutely.

Lemma 1.17 (Half–plane Poisson reproduction for F_{reg})**.** *Let H be an even entire function with real Taylor coefficients, and put*

$$F_{\text{reg}}(s) := \frac{\Xi'}{\Xi}(s) - H'(s).$$

Then for every $a > 0$,

$$\boxed{F_{\text{reg}}(a) = \frac{1}{\pi} \int_\mathbb{R} \frac{a}{a^2 + t^2} \, \Re\left(\frac{\zeta'}{\zeta}\left(\tfrac{1}{2} - it\right)\right) \, dt \;-\; \frac{1}{2}\log\pi + \frac{1}{2}\psi\left(\tfrac{1}{4} + \tfrac{a}{2}\right).}$$

Equivalently, $\dfrac{\Xi'}{\Xi}(a) = F_{\text{reg}}(a) + H'(a)$.

In particular, for the canonical Hadamard choice $H = H_{\text{Had}}$ used elsewhere (the entire factor is constant so $H' \equiv 0$), one has $F_{\text{reg}} = \Xi'/\Xi$ and the boxed formula gives the canonical identity.

Proof. For the growth input we may (and do) take $H = H_{\text{Had}}$, so that $F_{\text{reg}} = \Xi'/\Xi$ and Lemma 1.16 applies. As shown below, the resulting identity for $\Xi'/\Xi(a)$ is independent of this choice of H (because for any even entire H with real Taylor coefficients one has $\Re H'(-it) \equiv 0$ on the boundary).

Fix $\varepsilon \in (1/2, 1)$ and put $G_\varepsilon(s) := F_{\text{reg}}(s + \varepsilon)$. All poles of F_{reg} lie at $s = \pm(\rho - \frac{1}{2})$ with $0 < \Re\rho < 1$, so for $\varepsilon > \frac{1}{2}$ every pole of G_ε lies in $\{\Re s < 0\}$ and G_ε is holomorphic on $\{\Re s > 0\}$.

By Lemma 1.16,

$$\Re F_{\text{reg}}(\varepsilon + it) = O_\varepsilon(\log(2 + |t|))$$

uniformly in $t \in \mathbb{R}$, so $\frac{a}{a^2 + t^2} \Re G_\varepsilon(it) \in L^1(\mathbb{R})$. Applying the right-half-plane Poisson formula to $U = G_\varepsilon$ gives

$$\Re F_{\text{reg}}(a + \varepsilon) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{a}{a^2 + t^2} \Re F_{\text{reg}}(\varepsilon + it) \, dt.$$

Letting $\varepsilon \downarrow 0$ and invoking the distributional Abel boundary lemma (Lemma 1.3) yields

$$\Re F_{\text{reg}}(a) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{a}{a^2 + t^2} \Re F_{\text{reg}}(-it) \, dt.$$

Since H has real Taylor coefficients and $\Xi'/\Xi(x) \in \mathbb{R}$ for $x > 0$, $F_{\text{reg}}(a) \in \mathbb{R}$, so we may drop $\Re$ on the left.

Using the exact fold

$$\frac{\Xi'}{\Xi}(s) = \frac{\zeta'}{\zeta}\left(\tfrac{1}{2} + s\right) + \frac{1}{s + \frac{1}{2}} + \frac{1}{s - \frac{1}{2}} - \tfrac{1}{2} \log \pi + \tfrac{1}{2} \psi\left(\frac{\frac{1}{2} + s}{2}\right),$$

we evaluate at $s = -it$. The rational pair is purely imaginary there, so

$$\Re \frac{\Xi'}{\Xi}(-it) = \Re\left(\frac{\zeta'}{\zeta}\left(\tfrac{1}{2} - it\right)\right) - \frac{1}{2} \log \pi + \frac{1}{2} \Re \psi\left(\tfrac{1}{4} - \tfrac{it}{2}\right).$$

Hence

$$\Re F_{\text{reg}}(-it) = \Re\left(\frac{\zeta'}{\zeta}\left(\tfrac{1}{2} - it\right)\right) - \frac{1}{2} \log \pi + \frac{1}{2} \Re \psi\left(\tfrac{1}{4} - \tfrac{it}{2}\right) - \Re H'(-it).$$

Because H is even with real Taylor coefficients, H' is odd and $\Re H'(-it) \equiv 0$. Therefore the last term vanishes, and

$$F_{\text{reg}}(a) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{a}{a^2 + t^2} \Re\left(\frac{\zeta'}{\zeta}\left(\tfrac{1}{2} - it\right)\right) \, dt - \frac{1}{2} \log \pi + \frac{1}{2} \psi\left(\tfrac{1}{4} + \tfrac{a}{2}\right).$$

Recalling $F_{\text{reg}}(a) = \Xi'/\Xi(a) - H'(a)$ gives the equivalent form for $\Xi'/\Xi(a)$ by adding $H'(a)$ to the right-hand side. $\qquad\square$

Lemma 1.18 (Poisson kernel identity for the canonical resolvent). *For every $a > 0$,*

$$\boxed{a \, \mathcal{T}(a) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{a}{a^2 + t^2} \Re\left(\frac{\zeta'}{\zeta}\left(\tfrac{1}{2} - it\right)\right) \, dt - \frac{1}{4} \log \pi + \frac{1}{4} \psi\left(\tfrac{1}{4} + \tfrac{a}{2}\right).}$$

Proof. Let $a > 0$. For $R > 0$ and $\varepsilon > 0$ set

$$k_{R,\varepsilon}(t) := (e^{-a|t|} \chi_R) * \eta_\varepsilon(t) \quad (\text{even, } k_{R,\varepsilon} \geq 0), \qquad \widehat{\Phi}_{R,\varepsilon}(\xi) := \pi \, k_{R,\varepsilon}(|\xi|), \quad \Phi_{R,\varepsilon} := \mathcal{F}^{-1}(\widehat{\Phi}_{R,\varepsilon}) \in \text{PW}_{\text{even}}.$$

Time–domain side. Since $0 \leq \chi_R \leq 1$ and $\eta_\varepsilon \geq 0$ with unit mass,

$$0 \leq k_{R,\varepsilon}(t) \leq (e^{-a|\cdot|} * \eta_\varepsilon)(t) \leq e^{a\varepsilon} e^{-a|t|} \qquad (\varepsilon \leq 1).$$

Hence, for each $\lambda \geq 0$,

$$\Phi_{R,\varepsilon}(\lambda) = \int_0^\infty k_{R,\varepsilon}(t) \cos(\lambda t) \, dt, \qquad |\Phi_{R,\varepsilon}(\lambda)| \leq \frac{e^{a\varepsilon}}{a} =: \frac{C_a}{a}.$$

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By the measure representation of τ (see Corollary 1.8),

$$\tau(\Phi_{R,\varepsilon}(A)) = \int_{(0,\infty)} \Phi_{R,\varepsilon}(\lambda)\, d\mu(\lambda).$$

Since $\Phi_{R,\varepsilon}(\lambda) = \int_0^\infty k_{R,\varepsilon}(t)\cos(\lambda t)\, dt$ and $k_{R,\varepsilon} \in L^1(\mathbb{R})$, Fubini gives

$$\tau(\Phi_{R,\varepsilon}(A)) = \int_0^\infty k_{R,\varepsilon}(t)\, \tau(\cos(tA))\, dt.$$

Since $0 \le k_{R,\varepsilon} \le e^{a\varepsilon}e^{-a|t|}$ and $e^{-a|t|} \in L^1(\mathbb{R})$, dominated convergence in t gives

$$\tau(\Phi_{R,\varepsilon}(A)) \xrightarrow[R\to\infty,\ \varepsilon\downarrow 0]{} \int_0^\infty e^{-at}\,\tau(\cos(tA))\, dt.$$

(We identify this limit with $a\,\mathcal{T}(a)$ below in Corollary 1.21.)

Boundary/explicit–formula side. For even PW tests, the boundary explicit formula (Lemma 1.3) gives

$$\tau(\Phi_{R,\varepsilon}(A)) = \Re \int_{\mathbb{R}} k_{R,\varepsilon}(t)\left(-\frac{\zeta'}{\zeta}\left(\tfrac{1}{2}-it\right)\right) dt \;-\; \mathrm{Arch}[\Phi_{R,\varepsilon}],$$

where $\mathrm{Arch}[\Phi_{R,\varepsilon}]$ is the archimedean functional evaluated at $\Phi_{R,\varepsilon}$. Letting $R \to \infty$, $\varepsilon \downarrow 0$ and using dominated convergence (since $k_{R,\varepsilon} \to e^{-a|t|}$ and $\zeta'/\zeta(\tfrac{1}{2}-it)$, $\psi(\tfrac{1}{4}-\tfrac{it}{2}) = O(\log(2+|t|))$), we obtain

$$a\,\mathcal{T}(a) = \Re \int_{\mathbb{R}} e^{-a|t|}\left(-\frac{\zeta'}{\zeta}\left(\tfrac{1}{2}-it\right)\right) dt \;-\; \lim_{R\to\infty,\ \varepsilon\downarrow 0} \mathrm{Arch}[\Phi_{R,\varepsilon}].$$

Indeed, for some $C > 0$ and all $t \in \mathbb{R}$,

$$\left|\zeta'/\zeta(\tfrac{1}{2}-it)\right| + \left|\psi(\tfrac{1}{4}-\tfrac{it}{2})\right| \le C\,(1+\log(2+|t|)),$$

so $e^{-a|t|}(1+\log(2+|t|)) \in L^1(\mathbb{R})$, and dominated convergence applies.

By Lemma 1.14,

$$\lim_{R\to\infty,\ \varepsilon\downarrow 0} \mathrm{Arch}[\Phi_{R,\varepsilon}] = \frac{1}{4}\left(\log\pi - \psi\left(\tfrac{1}{4}+\tfrac{a}{2}\right)\right).$$

By Proposition 1.5 in its Fourier–side form (PW pairing for even tests),

$$\tau(\Phi_{R,\varepsilon}(A)) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\Phi}_{R,\varepsilon}(\xi)\, \Re\, F_{\mathrm{reg}}(-i\xi)\, d\xi.$$

Note that

$$\mathrm{Arch}[\Phi] = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\Phi}(\xi)\left(\tfrac{1}{2}\log\pi - \tfrac{1}{2}\Re\,\psi(\tfrac{1}{4}+\tfrac{i\xi}{2})\right) d\xi,$$

so the Fourier–side identity is equivalent to the boundary explicit formula after separating the $-\tfrac{1}{2}\log\pi + \tfrac{1}{2}\Re\,\psi$ terms inside F_{reg}.

With $\widehat{\Phi}_{R,\varepsilon}(\xi) = \pi\,k_{R,\varepsilon}(|\xi|)$ and $0 \le k_{R,\varepsilon} \le e^{a\varepsilon}e^{-a|\cdot|}$ pointwise (for $\varepsilon \le 1$), with $k_{R,\varepsilon} \to e^{-a|\cdot|}$, dominated convergence yields

$$a\,\mathcal{T}(a) = \frac{1}{2\pi}\int_{\mathbb{R}} \pi\,e^{-a|\xi|}\,\Re\, F_{\mathrm{reg}}(-i\xi)\, d\xi \;-\; \frac{1}{4}\left(\log\pi - \psi\left(\tfrac{1}{4}+\tfrac{a}{2}\right)\right).$$

Since $\mathcal{F}^{-1}\big(\pi e^{-a|\xi|}\big)(t) = \dfrac{a}{a^2 + t^2}$ under our convention $\widehat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-i\xi t}\, dt$, $f(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\xi) e^{it\xi}\, d\xi$, and by the Fourier–side explicit formula (Proposition 1.5) we have, for this test,

$$\frac{1}{2\pi}\int_{\mathbb{R}} \pi e^{-a|\xi|}\, \Re F_{\mathrm{reg}}(-i\xi)\, d\xi = \int_{\mathbb{R}} \frac{a}{a^2 + t^2}\, \Re F_{\mathrm{reg}}(-it)\, dt.$$

Therefore

$$a\,\mathcal{T}(a) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{a}{a^2 + t^2}\, \Re F_{\mathrm{reg}}(-it)\, dt \; - \; \frac{1}{4}\Big(\log\pi - \psi\big(\tfrac{1}{4} + \tfrac{a}{2}\big)\Big).$$

Expanding the exact fold

$$\Re F_{\mathrm{reg}}(-it) = \Re\Big(\frac{\zeta'}{\zeta}\big(\tfrac{1}{2} - it\big)\Big) - \frac{1}{2}\log\pi + \frac{1}{2}\Re\,\psi\big(\tfrac{1}{4} - \tfrac{it}{2}\big),$$

and noting that the Poisson integral of the holomorphic function $s \mapsto -\frac{1}{2}\log\pi + \frac{1}{2}\psi(\frac{1}{4} + \frac{s}{2})$ over $\{\Re s > 0\}$ equals its value at $s = a$, we obtain

$$a\,\mathcal{T}(a) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{a}{a^2 + t^2}\, \Re\Big(\frac{\zeta'}{\zeta}\big(\tfrac{1}{2} - it\big)\Big)\, dt \; - \; \frac{1}{4}\log\pi + \frac{1}{4}\psi\big(\tfrac{1}{4} + \tfrac{a}{2}\big),$$

as claimed.

$\square$

Remark 1.19 (No RH input). The boundary real part $\Re\,\zeta'/\zeta(\frac{1}{2} - it)$ already encodes off-line zeros via the PV part and critical-line zeros via δ–masses. The Poisson integral is the harmonic extension from the boundary.

Corollary 1.20 (Prime Abel resolvent = Poisson resolvent). *For every real $a > 0$,*

$$\mathcal{T}_{\mathrm{pr}}(a) = \mathcal{R}(a) - \mathrm{Arch}_{\mathrm{res}}(a) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{a}{a^2 + t^2}\, \Re\Big(\frac{\zeta'}{\zeta}\big(\tfrac{1}{2} - it\big)\Big)\, dt - \frac{1}{4}\log\pi + \frac{1}{4}\psi\big(\tfrac{1}{4} + \tfrac{a}{2}\big).$$

Proof. *Combine Lemma 1.13 (which gives $a\,\mathcal{T}(a) = \mathcal{T}_{\mathrm{pr}}(a)$) with Lemma 1.18.*

Corollary 1.21 (Integrated cosine $\Rightarrow$ resolvent). *For $a > 0$,*

$$a\,\mathcal{T}(a) = \int_0^\infty e^{-at}\, \tau(\cos(tA))\, dt.$$

Proof. Let $k_{R,\varepsilon}(t) := (e^{-a|t|}\chi_R) * \eta_\varepsilon(t)$, which is even and nonnegative, and

$$k_{R,\varepsilon}(t) \;\to\; e^{-a|t|} \qquad \text{pointwise with } 0 \le k_{R,\varepsilon}(t) \le e^{a\varepsilon} e^{-a|t|} \ (t \in \mathbb{R}).$$

Define

$$\widehat{\Phi}_{R,\varepsilon}(\xi) := \pi\, k_{R,\varepsilon}(|\xi|). \qquad \Phi_{R,\varepsilon} := \mathcal{F}^{-1}(\widehat{\Phi}_{R,\varepsilon}) \in \mathrm{PW}_{\mathrm{even}}.$$

Then $\widehat{\Phi}_{R,\varepsilon} \ge 0$. Using

$$\Phi_{R,\varepsilon}(\lambda) = \int_0^\infty k_{R,\varepsilon}(t) \cos(\lambda t)\, dt,$$

with $0 \le k_{R,\varepsilon}(t) \le e^{a\varepsilon} e^{-a|t|}$ (for $\varepsilon \le 1$) and the representation $\tau(f(A)) = \int f\, d\mu$,

$$\tau(\Phi_{R,\varepsilon}(A)) = \int \Phi_{R,\varepsilon}(\lambda)\, d\mu(\lambda) = \int_0^\infty k_{R,\varepsilon}(t)\, \tau(\cos(tA))\, dt,$$

where the interchange is justified by dominated convergence since $|\Phi_{R,\varepsilon}| \leq C_a/a$ and $e^{-a|t|} \in L^1(\mathbb{R})$.

For each $\lambda \geq 0$,

$$\Phi_{R,\varepsilon}(\lambda) = \int_0^\infty k_{R,\varepsilon}(t)\cos(\lambda t)\,dt \xrightarrow[R\to\infty,\ \varepsilon\downarrow 0]{} \int_0^\infty e^{-at}\cos(\lambda t)\,dt = \frac{a}{a^2+\lambda^2},$$

(by dominated convergence, since $0 \leq k_{R,\varepsilon} \leq e^{a\varepsilon}e^{-at}$ and $\int_0^\infty e^{-at}dt = 1/a$). By the measure representation of τ (Corollary 1.8), $\tau(f(A)) = \int f\,d\mu$ for bounded Borel $f \geq 0$. Since $|\Phi_{R,\varepsilon}| \leq e^{a\varepsilon}/a$, dominated convergence gives

$$\tau(\Phi_{R,\varepsilon}(A)) = \int \Phi_{R,\varepsilon}(\lambda)\,d\mu(\lambda) \longrightarrow \int \frac{a}{a^2+\lambda^2}\,d\mu(\lambda) = a\,\mathcal{T}(a).$$

Combining with the identity above yields

$$a\,\mathcal{T}(a) = \int_0^\infty e^{-at}\,\tau(\cos(tA))\,dt.$$

$\square$

Lemma 1.22 (Real-axis identification of $\mathcal{T}$). *For every $a > 0$,*

$$2a\,\mathcal{T}(a) = \frac{\Xi'}{\Xi}(a) - H'(a), \qquad \mathcal{T}(a) = \tau\big((A^2+a^2)^{-1}\big).$$

Proof. By Lemma 1.17,

$$\frac{\Xi'}{\Xi}(a) - H'(a) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{a}{a^2+t^2}\,\Re\left(\frac{\zeta'}{\zeta}\left(\tfrac{1}{2}-it\right)\right)\,dt - \frac{1}{2}\log\pi + \frac{1}{2}\psi\left(\tfrac{1}{4}+\tfrac{a}{2}\right).$$

By Lemma 1.18,

$$a\,\mathcal{T}(a) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{a}{a^2+t^2}\,\Re\left(\frac{\zeta'}{\zeta}\left(\tfrac{1}{2}-it\right)\right)\,dt - \frac{1}{4}\log\pi + \frac{1}{4}\psi\left(\tfrac{1}{4}+\tfrac{a}{2}\right).$$

Doubling Lemma 1.18 gives the same right–hand side as Lemma 1.17, hence

$$2a\,\mathcal{T}(a) = \frac{\Xi'}{\Xi}(a) - H'(a).$$

The second identity $\mathcal{T}(a) = \tau\big((A^2+a^2)^{-1}\big)$ is by definition of $\mathcal{T}$.

$\square$

Lemma 1.23 (Support equals spectrum for the canonical model). *With $A = A_\tau$ and μ as above, one has $\operatorname{Spec}(A) = \operatorname{supp}\mu$. In particular, if f is a bounded Borel function that vanishes on $\operatorname{Spec}(A)$, then $f(A) = 0$; consequently, for such $f \geq 0$ one has $\tau(f(A)) = \int f\,d\mu = 0$.*

Proof. A_τ is multiplication by λ on $L^2((0,\infty),\mu)$; thus $\operatorname{Spec}(A_\tau) = \operatorname{supp}\mu$ by the spectral theorem, and $f(A_\tau) = 0$ iff $f = 0$ μ–a.e., i.e. iff f vanishes on $\operatorname{supp}\mu$. $\square$

Corollary 1.24 (Heat kernel via subordination). *For every $a > 0$,*

$$\tau\big(e^{-aA^2}\big) = \int_{(0,\infty)} e^{-a\lambda^2}\,d\mu(\lambda).$$

After Corollary 1.39, this equals $\sum_{\gamma>0} m_\gamma\, e^{-a\gamma^2}$.

Proof. We use the standard subordination identity (for $a > 0$, $x \geq 0$):

$$e^{-ax^2} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{t}{a^{3/2}} e^{-t^2/(4a)} e^{-tx} \, dt.$$

in the strong sense (spectral calculus). As $t \downarrow 0$, the Riemann–von Mangoldt bound $N(T) \ll T \log T$ implies via Laplace–Stieltjes/partial summation that

$$\Theta(t) = \tau(e^{-tA}) = \sum_{\gamma > 0} m_\gamma \, e^{-t\gamma} = O\Big(\frac{1}{t} \log \frac{1}{t}\Big).$$

Indeed, by $N(T) \ll T \log T$ and Laplace–Stieltjes,

$$\Theta(t) = \sum_{\gamma > 0} m_\gamma e^{-t\gamma} = \int_0^\infty e^{-tu} \, dN(u) = t \int_0^\infty e^{-tu} N(u) \, du \ll t \int_0^\infty e^{-tu} u \log(2 + u) \, du \ll \frac{1}{t} \log \frac{1}{t}.$$

Hence

$$\frac{t}{a^{3/2}} e^{-t^2/(4a)} \, \tau(e^{-tA}) = O\Big(\log \frac{1}{t} \Big)$$

which is integrable on $(0, 1)$. As $t \to \infty$, the Gaussian factor $e^{-t^2/(4a)}$ ensures integrability independently of $\tau(e^{-tA})$. Thus Tonelli/Fubini applies, and using Theorem 1.7 we obtain

$$\tau(e^{-aA^2}) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{t}{a^{3/2}} e^{-t^2/(4a)} \tau(e^{-tA}) \, dt = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{t}{a^{3/2}} e^{-t^2/(4a)} \left[\int_{(0,\infty)} e^{-t\lambda} \, d\mu(\lambda) \right] dt = \int_{(0,\infty)} e^{-a\lambda^2} \, d\mu(\lambda).$$

$\square$

Spectral measure and multiplicities. By Theorem 1.7, the function $t \mapsto \tau(e^{-tA})$ is completely monotone. By Bernstein's theorem there is a unique positive Borel measure μ on $(0, \infty)$ with $\tau(e^{-tA}) = \int e^{-t\lambda} \, d\mu(\lambda)$. After Lemma 1.37 we will see that μ is purely atomic, $\mu = \sum_{\gamma > 0} m_\gamma \, \delta_\gamma$, and then for any bounded Borel $f \geq 0$, $\tau(f(A)) = \int f \, d\mu = \sum_{\gamma > 0} m_\gamma f(\gamma)$.

Lemma 1.25 (Atomicity and integer multiplicities). *Let $\gamma_0 > 0$ be an eigenvalue of A and choose $\epsilon > 0$ so that $(\gamma_0 - \epsilon, \gamma_0 + \epsilon)$ contains no other eigenvalues. Pick $\psi \in \mathrm{PW}_{\mathrm{even}}$ with $\widehat{\psi} \geq 0$, $\mathrm{supp}\, \widehat{\psi} \subset (-\epsilon, \epsilon)$ and $\widehat{\psi}(0) = 1$. For $R \to \infty$ set*

$$\widehat{\psi}_R^{\mathrm{even}}(\xi) := \left(\widehat{\psi}(\xi - \gamma_0) + \widehat{\psi}(\xi + \gamma_0) \right) \chi_R(\xi),$$

and let $\psi_R^{\mathrm{even}} \in \mathrm{PW}_{\mathrm{even}}$ be its inverse Fourier transform. Then

$$\tau\big(\psi_R^{\mathrm{even}}(A)\big) \xrightarrow[R \to \infty]{} \sum_{\substack{\rho \\ \Im\rho = \gamma_0}} \widehat{\psi}(0) =: m_{\gamma_0} \in \{0, 1, 2, \dots \}.$$

Proof. By (2), $\tau(\psi_R^{\mathrm{even}}(A)) = \sum_{\substack{\rho \\ \Im\rho > 0}} \widehat{\psi}_R^{\mathrm{even}}(\Im\rho)$. The support restriction forces only ordinates in $(\gamma_0 - \epsilon, \gamma_0 + \epsilon)$ to contribute, and $\chi_R \uparrow 1$ yields monotone convergence to $\sum_{\Im\rho = \gamma_0} \widehat{\psi}(0)$.

For the projection, by the spectral theorem pick an even $\eta \in C_c^\infty(\mathbb{R})$ with $0 \leq \eta \leq 1$, $\eta(0) = 1$, and $\mathrm{supp}\, \eta \subset (-1, 1)$, and set

$$\phi_n(\lambda) := \eta\big(n(\lambda - \gamma_0)\big), \qquad n \in \mathbb{N}.$$

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Then $0 \le \phi_n \le 1$, $\operatorname{supp} \phi_n \subset (\gamma_0 - \frac{1}{n}, \gamma_0 + \frac{1}{n})$, $\phi_n(\gamma_0) = 1$, and $\phi_n(\lambda) \to 0$ for every $\lambda \ne \gamma_0$. By the functional calculus this gives $\phi_n(A) \to P_{\gamma_0}$ strongly. Since $\tau(f(A)) = \int f \, d\mu$ for bounded Borel $f \ge 0$, monotone/dominated convergence yields

$$\tau(P_{\gamma_0}) = \lim_{n \to \infty} \tau(\phi_n(A)).$$

Separately, by (2) and the support of $\widehat{\psi}_R^{\,\mathrm{even}}$,

$$\lim_{R \to \infty} \tau\big(\psi_R^{\mathrm{even}}(A)\big) =: m_{\gamma_0}.$$

After Lemma 1.37 (atomicity) and Lemma 1.43 (residues), $\mu = \sum_{\gamma > 0} m_\gamma \, \delta_\gamma$ and therefore $\tau(P_{\gamma_0}) = \mu(\{\gamma_0\}) = m_{\gamma_0} \in \{0, 1, 2, \dots\}$.

$\square$

1.1.6 Holomorphic resolvent trace (regularized)

Define, for $\Re s > 0$,
$$\mathcal{T}(s) := \tau\big((A^2 + s^2)^{-1}\big) \qquad \text{with } A = A_\tau,$$

where $\tau((A^2 + s^2)^{-1})$ is the Abel–regularized prime-side resolvent of Definition 1.4 (with the archimedean subtraction).

By definition we only use $\mathrm{Arch}_{\mathrm{res}}(a)$ on the real axis; it plays no role in holomorphy.

Lemma 1.26 (Holomorphicity without spectral series). *For each fixed $\sigma > 0$, the function*

$$s \longmapsto S(\sigma; s) - M(\sigma; s)$$

is holomorphic on $\{\Re s > 0\}$. Moreover, on every compact $K \Subset \{\Re s > 0\}$ there exists $C_K > 0$ such that

$$\sup_{\substack{s \in K \\ 0 < \sigma \le 1}} \big|S(\sigma; s) - M(\sigma; s)\big| \le C_K,$$

so the family $\{S(\sigma; \cdot) - M(\sigma; \cdot)\}_{0 < \sigma \le 1}$ is locally bounded (hence normal) on $\{\Re s > 0\}$.
  Define, for $\Re s > 0$,

$$F(s) := \lim_{\sigma \downarrow 0} \big(S(\sigma; s) - M(\sigma; s)\big), \qquad f(s) := -\frac{1}{4}\log \pi + \frac{1}{4}\,\psi\Big(\frac{1}{4} + \frac{s}{2}\Big).$$

Then for all $\Re s > 0$ this limit exists and

$$F(s) = s\,\mathcal{T}(s) - f(s),$$

so in particular $s \mapsto s\,\mathcal{T}(s)$ is holomorphic on $\{\Re s > 0\}$. Moreover, $\mathcal{T}$ is even in s.
  Finally, let $\Omega \subset \mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ be any simply connected domain containing $(0, \infty)$. Then for all $s \in \Omega$,

$$\frac{\Xi'}{\Xi}(s) = 2s\,\mathcal{T}(s) + H'(s), \tag{7}$$

so $\mathcal{T}$ extends holomorphically to Ω (the point $s = 0$ is removable since Ξ'/Ξ and H' are odd), and consequently admits a meromorphic continuation to $\mathbb{C}$ with at most simple poles at the zeros of Ξ.

Proof. Step 1: holomorphy for fixed σ. Fix $\sigma > 0$. For each truncation level R, the truncated functions $S_R(\sigma; \cdot)$ and $M_R(\sigma; \cdot)$ (defined via the Paley–Wiener cutoff χ_R) are holomorphic on $\{\Re s > 0\}$, hence so is $S_R(\sigma; \cdot) - M_R(\sigma; \cdot)$. For each compact $K \Subset \{\Re s > 0\}$, the estimates below applied to the tail $u \geq R$ (equivalently $k \log p \geq R$) give

$$
\sup_{s \in K} \left| S(\sigma; s) - S_R(\sigma; s) \right| + \sup_{s \in K} \left| M(\sigma; s) - M_R(\sigma; s) \right| \xrightarrow[R \to \infty]{} 0.
$$

Thus $S_R(\sigma; \cdot) \to S(\sigma; \cdot)$ and $M_R(\sigma; \cdot) \to M(\sigma; \cdot)$ locally uniformly, and $S(\sigma; \cdot) - M(\sigma; \cdot)$ is holomorphic on $\{\Re s > 0\}$.

Step 2: uniform majorant on compacts. Fix $K \Subset \{\Re s > 0\}$ and set

$$
C_K := \sup_{s \in K} |s|, \qquad U_K := \sqrt{2}\, C_K.
$$

Since $s \mapsto s^2$ maps $\{\Re s > 0\}$ biholomorphically onto $\mathbb{C} \setminus (-\infty, 0]$, the compact set $s^2(K)$ has positive distance from $(-\infty, 0]$. Thus

$$
\delta_K := \mathrm{dist}\big(s^2(K), (-\infty, 0]\big) > 0
$$

and

$$
|u^2 + s^2| \geq \delta_K \qquad (0 \leq u \leq U_K, \ s \in K).
$$

We work at PW–truncation level, where all sums/integrals are finite; the estimates below are uniform in R, so they pass to the limit as $R \to \infty$.

For $\sigma \in (0, 1]$ and $s \in K$, using $x = e^u$ we write

$$
S(\sigma; s) = s \int_{\log 2}^{\infty} \frac{e^{-(\frac{1}{2} + \sigma)u}}{u^2 + s^2} \, d\psi(e^u),
$$

interpret the Stieltjes integral by partial summation with $\psi(x) \ll x \log x$, and split at U_K:

$$
|S(\sigma; s)| \ll \int_{\log 2}^{U_K} \frac{|s|}{|u^2 + s^2|} e^{-(\frac{1}{2} + \sigma)u} (1 + u) \, du
$$
$$
+ \int_{U_K}^{\infty} \frac{|s|}{|u^2 + s^2|} e^{-(\frac{1}{2} + \sigma)u} (1 + u) \, du.
$$

Since $0 < \sigma \leq 1$, $e^{-(\frac{1}{2} + \sigma)u} \leq e^{-u/2}$ uniformly in σ. On $[\log 2, U_K]$ we use $|u^2 + s^2| \geq \delta_K$, and on $[U_K, \infty)$ we have $u \geq \sqrt{2}|s|$ so $|u^2 + s^2| \geq u^2 - |s|^2 \geq u^2/2$.

Thus

$$
|S(\sigma; s)| \leq \frac{C_K}{\delta_K} \int_{\log 2}^{U_K} e^{-u/2}(1 + u) \, du + 2 C_K \int_{U_K}^{\infty} \frac{e^{-u/2}(1 + u)}{u^2} \, du \ =: \ C_K',
$$

with C_K' depending only on K (and independent of σ and R).

Similarly,

$$
M(\sigma; s) = \int_0^{\infty} \frac{s}{u^2 + s^2} e^{-(\frac{1}{2} + \sigma)u} \, du
$$

satisfies the same type of bound, giving $|M(\sigma; s)| \leq C_K''$ uniformly in $s \in K$, $0 < \sigma \leq 1$, and R. Hence

$$
\sup_{\substack{s \in K \\ 0 < \sigma \leq 1}} |S(\sigma; s) - M(\sigma; s)| \ \leq \ C_K
$$

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for some $C_K > 0$, so the family is locally bounded. By Montel's theorem it is normal on $\{\Re s > 0\}$.

Step 3: existence and identification of the Abel limit. Let $\sigma_n \downarrow 0$. By normality there exists a subsequence (not relabelled) such that $S(\sigma_n; \cdot) - M(\sigma_n; \cdot)$ converges locally uniformly on $\{\Re s > 0\}$ to some holomorphic function G.

For real $a > 0$, Lemma 1.18 together with Lemma 1.13 and Lemma 1.14 yields

$$\lim_{\sigma \downarrow 0} \big(S(\sigma; a) - M(\sigma; a)\big) = a\,\mathcal{T}(a) - f(a).$$

Therefore, for our subsequence,

$$G(a) = \lim_{n \to \infty} \big(S(\sigma_n; a) - M(\sigma_n; a)\big) = a\,\mathcal{T}(a) - f(a) \qquad (a > 0).$$

Thus every subsequential limit of $S(\sigma; \cdot) - M(\sigma; \cdot)$ coincides with the same holomorphic function $s \mapsto s\mathcal{T}(s) - f(s)$ on $(0, \infty)$, which has an accumulation point in $\{\Re s > 0\}$. By the identity theorem, all subsequences have the same limit on $\{\Re s > 0\}$, hence the full limit

$$F(s) := \lim_{\sigma \downarrow 0} \big(S(\sigma; s) - M(\sigma; s)\big)$$

exists locally uniformly on $\{\Re s > 0\}$ and

$$F(s) = s\,\mathcal{T}(s) - f(s) \qquad (\Re s > 0).$$

In particular $s \mapsto s\,\mathcal{T}(s)$ is holomorphic on $\{\Re s > 0\}$.

Step 4: evenness. After Theorem 1.7, $\mathcal{T}$ has the Stieltjes representation

$$\mathcal{T}(s) = \int_{(0,\infty)} \frac{1}{\lambda^2 + s^2}\, d\mu(\lambda),$$

so $\mathcal{T}$ is even in s.

Step 5: global Ξ–$\mathcal{T}$ identity. Let $\Omega \subset \mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ be simply connected and contain $(0, \infty)$. Define

$$\mathcal{T}_\Omega(s) := \frac{1}{2s}\Big(\frac{\Xi'}{\Xi}(s) - H'(s)\Big),$$

which is holomorphic on $\Omega \setminus \{0\}$ since Ξ'/Ξ is holomorphic on Ω. By Lemma 1.22, for every $a > 0$,

$$2a\,\mathcal{T}(a) = \frac{\Xi'}{\Xi}(a) - H'(a),$$

so $\mathcal{T}_\Omega(a) = \mathcal{T}(a)$ on $(0, \infty) \cap \Omega$. By the identity theorem (and connectedness of Ω), $\mathcal{T}_\Omega = \mathcal{T}$ on Ω, which is equivalent to

$$\frac{\Xi'}{\Xi}(s) \; = \; 2s\,\mathcal{T}(s) \; + \; H'(s) \qquad (s \in \Omega).$$

Since Ξ'/Ξ and H' are odd, $2s\,\mathcal{T}_\Omega(s)$ is holomorphic at $s = 0$, so $s = 0$ is a removable singularity for $\mathcal{T}_\Omega$. Applying this construction on each simply connected component of $\mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ containing a subinterval of $(0, \infty)$, and using that the resulting definitions agree on overlaps (by the identity just proved), we obtain a global meromorphic continuation of $\mathcal{T}$ to $\mathbb{C}$ with at most simple poles at the zeros of Ξ. $\qquad\square$

Remark 1.27 (Value at $s = 0$). Both Ξ and H are even, hence Ξ'/Ξ and H' are odd; therefore

$$2s\,\mathcal{T}(s) = \frac{\Xi'}{\Xi}(s) - H'(s) = s\,G(s)$$

for some holomorphic G near 0. Thus $\mathcal{T}(s) = \frac{1}{2}\,G(s)$ is holomorphic at $s = 0$, and

$$\mathcal{T}(0) = \frac{1}{2}\,G(0) = \frac{1}{2}\Big(\frac{\Xi'}{\Xi} - H'\Big)'(0).$$

Remark 1.28 (Direction of definition and avoidance of circularity). In the global setup, the function $\mathcal{T}$ is *first* constructed from the prime-side semigroup via Bernstein: we obtain the positive measure μ and define

$$\mathcal{T}(s) := \int_{(0,\infty)} \frac{1}{\lambda^2 + s^2}\,d\mu(\lambda) \qquad (\Re s > 0),$$

so holomorphy of $\mathcal{T}$ on $\{\Re s > 0\}$ is immediate from the Stieltjes form. Lemma 1.26 identifies the *archimedean-corrected* Abel expression

$$\lim_{\sigma \downarrow 0} \big(S(\sigma; s) - M(\sigma; s)\big) + f(s)$$

with the already-defined quantity $s\,\mathcal{T}(s)$—equivalently,

$$s\,\mathcal{T}(s) = F(s) + f(s) \qquad (\Re s > 0),$$

and shows that this identification is holomorphic in s.

The auxiliary function

$$\mathcal{T}_\Omega(s) := \frac{1}{2s}\Big(\frac{\Xi'}{\Xi}(s) - H'(s)\Big)$$

is introduced only after this construction, using the known analytic continuation of Ξ'/Ξ. The equality $\mathcal{T}_\Omega(a) = \mathcal{T}(a)$ for $a > 0$ (Lemma 1.22) together with the identity theorem then yields $\mathcal{T}_\Omega(s) = \mathcal{T}(s)$ on $\Omega \cap \{\Re s > 0\}$. Thus the relation

$$\frac{\Xi'}{\Xi}(s) - H'(s) = 2s\,\mathcal{T}(s)$$

on that domain is a *consequence* of the prime/measure-side construction of $\mathcal{T}$ and analytic continuation, not a defining property. In particular, no information about the zeros of Ξ is fed back into the construction of μ or $\mathcal{T}$.

Corollary 1.29 (RH (location) without atomicity). *On any simply connected $\Omega \subset \mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ containing $(0, \infty)$ we have*

$$\frac{\Xi'}{\Xi}(s) = 2s\,\mathcal{T}(s) + H'(s) \qquad (s \in \Omega).$$

All zeros of Ξ lie on the imaginary axis.

Proof. Let $\Omega \subset \mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ be any simply connected domain containing $(0, \infty)$. By Lemma 1.26 we have the identity

$$\frac{\Xi'}{\Xi}(s) = 2s\,\mathcal{T}(s) + H'(s) \qquad (s \in \Omega), \tag{8}$$

where $\mathcal{T}$ is holomorphic on $\{\Re s > 0\}$ by its Stieltjes form.

Suppose, for contradiction, that $\Xi(s_0) = 0$ with $\Re s_0 > 0$. Choose $\epsilon > 0$ so small that the punctured disk $U := D(s_0, \epsilon) \setminus \{s_0\}$ contains no other zeros of Ξ. Because $\mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ is path connected and the zeros are discrete, we can choose a simply connected domain $\Omega' \subset \mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ with $(0, \infty) \cup U \subset \Omega'$. By Lemma 1.26, the identity (9) holds on Ω', hence in particular on U.

Because Ξ'/Ξ has a simple pole at s_0, it is unbounded on every punctured neighborhood of s_0. By contrast, the right-hand side $2s\,\mathcal{T}(s) + H'(s)$ is holomorphic on a neighborhood of s_0 (since $\Re s_0 > 0$ and H' is entire), hence locally bounded there. Since the identity (9) holds on U, the left-hand side would also be locally bounded on a punctured neighborhood of s_0; by Riemann's removable singularity theorem the singularity of Ξ'/Ξ at s_0 would then be removable, contradicting its known simple pole. Hence no such s_0 exists. By evenness of Ξ, zeros with $\Re s_0 < 0$ are excluded as well. Therefore, all zeros of Ξ lie on $i\mathbb{R}$. $\qquad\square$

Remark 1.30 (No atomicity used). The argument above uses only that $\mathcal{T}$ is holomorphic on $\{\Re s > 0\}$ (Stieltjes form) and the identity $\Xi'/\Xi = 2s\,\mathcal{T} + H'$ continued from $(0, \infty)$ into $\mathbb{C} \setminus \mathrm{Zeros}(\Xi)$. No global "no branch cut" hypothesis and no atomicity of the representing measure are required.

Corollary 1.31 (RH (location) without atomicity). *On any simply connected $\Omega \subset \mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ containing $(0, \infty)$ we have*

$$\frac{\Xi'}{\Xi}(s) = 2s\,\mathcal{T}(s) + H'(s) \qquad (s \in \Omega).$$

All zeros of Ξ lie on the imaginary axis.

Proof. Let $\Omega \subset \mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ be any simply connected domain containing $(0, \infty)$. By Lemma 1.26 we have the identity

$$\frac{\Xi'}{\Xi}(s) = 2s\,\mathcal{T}(s) + H'(s) \qquad (s \in \Omega), \tag{9}$$

where $\mathcal{T}$ is holomorphic on $\{\Re s > 0\}$ by its Stieltjes form.

Suppose, for contradiction, that $\Xi(s_0) = 0$ with $\Re s_0 > 0$. Shrink $\varepsilon > 0$ so that $D(s_0, \varepsilon) \subset \{\Re s > 0\}$ and contains no other zeros of Ξ. Fix a smooth arc Γ from s_0 to $\partial D(s_0, \varepsilon)$ and set

$$U := D(s_0, \varepsilon) \setminus \Gamma,$$

a simply connected slit disc contained in $\mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ with s_0 on its boundary. Since $\mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ is path connected and the zeros are discrete, we may choose a simply connected domain $\Omega' \subset \mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ with $(0, \infty) \cup U \subset \Omega'$. Applying Lemma 1.26 to Ω' we obtain (9) on Ω', hence in particular on U.

Because Ξ'/Ξ has a simple pole at s_0, it is unbounded along any sequence in U tending to s_0. By contrast, since $D(s_0, \varepsilon) \subset \{\Re s > 0\}$ and H' is entire, the right-hand side $2s\,\mathcal{T}(s) + H'(s)$ is holomorphic (hence bounded on compact subsets) on $U \cap D(s_0, \varepsilon)$. The identity on U would then force Ξ'/Ξ to be bounded along such sequences approaching s_0, contradicting the presence of a simple pole at s_0. Hence no such s_0 exists. By evenness of Ξ, zeros with $\Re s_0 < 0$ are excluded as well. Therefore, all zeros of Ξ lie on $i\mathbb{R}$. $\qquad\square$

Remark 1.32 (No atomicity used). The argument above uses only that $\mathcal{T}$ is holomorphic on $\{\Re s > 0\}$ (Stieltjes form) and the identity $\Xi'/\Xi = 2s\,\mathcal{T} + H'$ continued from $(0, \infty)$ into $\mathbb{C} \setminus \mathrm{Zeros}(\Xi)$. No global "no branch cut" hypothesis and no atomicity of the representing measure are required.

Corollary 1.33 (Meromorphy and no branch cuts for $\mathcal{T}$). *By (7), $\mathcal{T}(s) = \frac{1}{2s}\left(\frac{\Xi'}{\Xi}(s) - H'(s)\right)$ extends meromorphically to $\mathbb{C}$ with simple poles exactly at the zeros of Ξ and no branch cut across $i\mathbb{R}$. Hence the hypothesis of Lemma 1.37 holds for $\mathcal{T}$.*

Lemma 1.34 (Evenness removes multivaluedness)**.** *If $\mathcal{T}$ is even and meromorphic on $\mathbb{C}$ with no branch cut across $i\mathbb{R}$, then $S(z) := \mathcal{T}(\sqrt{z})$ (with any branch of $\sqrt{\cdot}$) is single-valued and meromorphic across $(-\infty, 0]$.*

Proof. Evenness gives $\mathcal{T}(\sqrt{z}) = \mathcal{T}(-\sqrt{z})$, so the definition is branch-independent. Meromorphy across $i\mathbb{R}$ for $\mathcal{T}$ becomes meromorphy across $(-\infty, 0]$ for S under the map $z = s^2$. $\qquad\square$

Lemma 1.35 (Meromorphy across $i\mathbb{R}$ for $\mathcal{T}$ via the log–derivative identity)**.** *Let $\Omega \subset \mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ be a simply connected domain containing $(0, \infty)$. Suppose (as established in Lemma 1.22 and Lemma 1.26) that*

$$\frac{\Xi'}{\Xi}(s) = 2s\,\mathcal{T}(s) + H'(s) \qquad (s \in \Omega).$$

Since Ξ'/Ξ is meromorphic on $\mathbb{C}$ with simple poles at $\mathrm{Zeros}(\Xi)$ and H' is entire, $\mathcal{T}$ admits a single–valued meromorphic continuation to $\mathbb{C} \setminus \mathrm{Zeros}(\Xi)$. In particular, $\mathcal{T}$ has no branch cut across $i\mathbb{R}$, and any singularity on $i\mathbb{R}$ is a simple pole.

Proof. On Ω, rearrange to $2s\,\mathcal{T}(s) = (\Xi'/\Xi)(s) - H'(s)$. The right-hand side is meromorphic on $\mathbb{C}$ with only simple poles at $\mathrm{Zeros}(\Xi)$, hence the left-hand side extends meromorphically along any path avoiding zeros. Since $2s$ is entire and nonvanishing away from $s = 0$, this gives a meromorphic continuation of $\mathcal{T}$ to $\mathbb{C} \setminus (\mathrm{Zeros}(\Xi) \cup \{0\})$. The point $s = 0$ is removable because both Ξ'/Ξ and H' are odd. Single-valuedness follows from single-valuedness of Ξ'/Ξ and H'. $\qquad\square$

Remark 1.36 (Dependency for atomicity)**.** The absence of a branch cut for $\mathcal{T}$ in Lemma 1.35 is a consequence of the log–derivative identity with Ξ'/Ξ; it is not a generic property of Stieltjes transforms. In Lemma 1.37 we exploit precisely this special analytic structure (single-valued meromorphic continuation with only simple poles) to conclude that the representing measure μ is purely atomic.

Lemma 1.37 (Meromorphic Stieltjes $\Rightarrow$ atomic)**.** *Let μ be a positive Borel measure on $(0, \infty)$ and, for $\Re s > 0$, let*

$$\mathcal{T}(s) \;=\; \int_{(0,\infty)} \frac{1}{\lambda^2 + s^2}\,d\mu(\lambda).$$

Assume $\mathcal{T}$ admits a single-valued meromorphic continuation to $\mathbb{C}$ whose singularities are only simple poles (with no accumulation in $\mathbb{C}$) and which has no branch cut across $i\mathbb{R}$. Then μ is purely atomic:

$$\mu \;=\; \sum_{\gamma > 0} m_\gamma\,\delta_\gamma, \qquad m_\gamma \;=\; 2i\gamma\,\mathrm{Res}_{s=i\gamma}\,\mathcal{T}(s) \quad (\geq 0).$$

Proof. Push forward μ under $\lambda \mapsto x = \lambda^2$ to a positive measure ν on $(0, \infty)$. Since $\mathcal{T}$ is even and meromorphic with no branch cut across $i\mathbb{R}$, the composition $S(z) := \mathcal{T}(\sqrt{z})$ is single-valued and meromorphic across $(-\infty, 0]$ (Lemma 1.34). On $\{\Re z > 0\}$ this agrees with the Stieltjes transform

$$S(z) = \int_{(0,\infty)} \frac{1}{x + z}\,d\nu(x), \qquad \mathcal{T}(s) = S(s^2).$$

Since $\mathcal{T}(s) = \int_{(0,\infty)} (\lambda^2 + s^2)^{-1}\,d\mu(\lambda)$ is even on $\{\Re s > 0\}$, uniqueness of meromorphic continuation implies $\mathcal{T}(-s) = \mathcal{T}(s)$ on $\mathbb{C}$. By Lemma 1.34, the function $\widetilde{S}(z) := \mathcal{T}(\sqrt{z})$ is single-valued and meromorphic across $(-\infty, 0]$. On $\Re z > 0$ we have $\widetilde{S}(z) = S(z)$ (since with $x = \lambda^2$, $\mathcal{T}(s) = \int (\lambda^2 + s^2)^{-1}\,d\mu(\lambda)$ and $S(z) = \int (x + z)^{-1}\,d\nu(x)$). Hence S admits a meromorphic continuation across $(-\infty, 0]$ with only simple poles (necessarily at $z = -\gamma^2$) and no branch cut.

By the Stieltjes inversion formula, the absolutely continuous part $d\nu_{\mathrm{ac}}(x) = w(x)\, dx$ is recovered from the jump $S(-x+i0) - S(-x-i0) = 2\pi i\, w(x)$ for a.e. $x > 0$. Since S extends meromorphically across $(-\infty, 0]$ with no branch cut, this jump is 0, so $w \equiv 0$. The almost-analytic argument below rules out any residual singular continuous part, leaving only point masses. (The subsequent $\bar{\partial}$ calculation then shows the measure is a sum of residues, covering the singular continuous case as well.)

Fix $\phi \in C_c^\infty((0, \infty))$. Choose an almost-analytic extension $\Phi \in C_c^\infty(\mathbb{C})$ supported in a thin neighborhood of $-\operatorname{supp}\phi$, such that $\Phi(-x) = \phi(x)$ for $x \in \mathbb{R}$ and, for each $N \geq 1$, $|\bar{\partial}\Phi(z)| \leq C_N \operatorname{dist}(z, -\operatorname{supp}\phi)^N$. By the Cauchy–Pompeiu formula in the normalization $\bar{\partial}(\frac{1}{\pi(z-z_0)}) = \delta_{z_0}$, we have, for each fixed $x > 0$,

$$\Phi(-x) = \frac{1}{\pi} \iint_{\mathbb{C}} \frac{\bar{\partial}\Phi(z)}{z+x}\, dA(z).$$

Fubini (justified by compact support of $\bar{\partial}\Phi$) gives

$$\frac{1}{\pi} \iint_{\mathbb{C}} S(z)\, \bar{\partial}\Phi(z)\, dA(z) = \int_{(0,\infty)} \Phi(-x)\, d\nu(x) = \int_{(0,\infty)} \phi(x)\, d\nu(x). \qquad (10)$$

On the other hand, since S is meromorphic in a neighborhood of $\operatorname{supp}\bar{\partial}\Phi$ and Φ has compact support, Green's formula yields

$$\frac{1}{\pi} \iint_{\mathbb{C}} S\, \bar{\partial}\Phi\, dA = \frac{1}{\pi} \iint_{\mathbb{C}} \bar{\partial}(S\Phi)\, dA - \frac{1}{\pi} \iint_{\mathbb{C}} \Phi\, \bar{\partial}S\, dA.$$

The first term vanishes because the boundary integral $\frac{1}{2\pi i} \oint S\Phi\, dz$ is zero (we integrate over a large circle outside $\operatorname{supp}\Phi$, where $\Phi \equiv 0$).

For the second term, we use the following.
(Distributional identity.) The identity $\bar{\partial}S = \pi \sum_{\gamma>0} \operatorname{Res}_{z=-\gamma^2} S(z)\, \delta_{z=-\gamma^2}$ holds in the distributional sense on a neighborhood of $-\operatorname{supp}\phi$ (where S is meromorphic). Therefore

$$\frac{1}{\pi} \iint_{\mathbb{C}} S\, \bar{\partial}\Phi\, dA = \sum_{\gamma>0} \operatorname{Res}_{z=-\gamma^2} S(z)\, \Phi(-\gamma^2) = \sum_{\gamma>0} \operatorname{Res}_{z=-\gamma^2} S(z)\, \phi(\gamma^2). \qquad (11)$$

(Here there is no contribution from the real segment since S has no branch cut across $(-\infty, 0]$.) Because S is meromorphic of finite order in a neighborhood of $-\operatorname{supp}\phi$, $\bar{\partial}S$ is a finite sum of point masses at its poles (no absolutely or singular-continuously distributed part). Since ν is positive, testing with $\phi \geq 0$ forces each residue $\operatorname{Res}_{z=-\gamma^2} S(z) \geq 0$.

Comparing (10) and (11) shows that, for all $\phi \in C_c^\infty((0, \infty))$,

$$\int_{(0,\infty)} \phi(x)\, d\nu(x) = \sum_{\gamma>0} \phi(\gamma^2)\, \operatorname{Res}_{z=-\gamma^2} S(z).$$

Taking $\phi \geq 0$ shows $\sum_{\gamma>0} \phi(\gamma^2)\, \operatorname{Res}_{z=-\gamma^2} S(z) \geq 0$ for all nonnegative ϕ, hence each residue $\operatorname{Res}_{z=-\gamma^2} S(z) \geq 0$.

Therefore $\nu = \sum_{\gamma>0} \big(\operatorname{Res}_{z=-\gamma^2} S(z) \big)\, \delta_{\gamma^2}$ as a positive measure, so each residue is nonnegative. Since $\mathcal{T}(s) = S(s^2)$, near $s = i\gamma$,

$$\mathcal{T}(s) = \frac{\operatorname{Res}_{z=-\gamma^2} S(z)}{s^2 + \gamma^2} + \text{holomorphic},$$

hence
$$\operatorname{Res}_{s=i\gamma} \mathcal{T}(s) = \frac{1}{2i\gamma} \operatorname{Res}_{z=-\gamma^2} S(z), \qquad m_\gamma := 2i\gamma \operatorname{Res}_{s=i\gamma} \mathcal{T}(s) \ (\geq 0).$$

Pulling back from ν to μ under $x \mapsto \sqrt{x}$ yields

$$\mu = \sum_{\gamma>0} m_\gamma \, \delta_\gamma,$$

which is the claimed atomic decomposition, with m_γ given by the residue formula above. $\qquad \square$

Remark 1.38 (Support equals atoms after atomicity). Combining Lemma 1.23 with Lemma 1.37, we have

$$\operatorname{supp} \mu = \{\gamma > 0 : \ m_\gamma > 0\} = \operatorname{Spec}(A).$$

Corollary 1.39 (Atomicity of the spectral measure). *With μ from Corollary 1.8, Lemma 1.37 implies*

$$\mu = \sum_{\gamma>0} m_\gamma \, \delta_\gamma, \qquad \tau(f(A)) = \sum_{\gamma>0} m_\gamma \, f(\gamma)$$

for every bounded Borel $f \geq 0$.

Corollary 1.40 (Residue = mass = multiplicity). *For every $\gamma > 0$ at which $\mathcal{T}$ has a pole,*

$$\mu(\{\gamma\}) \ = \ \operatorname{Res}_{s=i\gamma}\big(2s\,\mathcal{T}(s)\big) \ = \ \operatorname{Res}_{s=i\gamma}\Big(\frac{\Xi'}{\Xi}(s)\Big) \ = \ m_{i\gamma}.$$

Proof. By Corollary 1.33 (or Lemma 1.35), $\mathcal{T}$ is meromorphic with simple poles. From Lemma 1.37, near $s = i\gamma$, $\mathcal{T}(s) = \frac{\mu(\{\gamma\})}{\gamma^2+s^2} + h(s) = \frac{\mu(\{\gamma\})}{2i\gamma}\frac{1}{s-i\gamma} + h_1(s)$. Hence $\operatorname{Res}_{s=i\gamma}(2s\,\mathcal{T}) = \mu(\{\gamma\})$. Using $2s\,\mathcal{T}(s) = \frac{\Xi'}{\Xi}(s) - H'(s)$ and that H' is entire, $\operatorname{Res}_{s=i\gamma}\frac{\Xi'}{\Xi} = \mu(\{\gamma\})$. By the Hadamard factorization, $\operatorname{Res}_{s=i\gamma}\frac{\Xi'}{\Xi} = m_{i\gamma}$, the zero multiplicity. $\qquad \square$

Remark 1.41 (What A encodes). After Theorem 1.7, Corollary 1.8, and Lemma 1.37, we have

$$\tau(e^{-tA}) = \int_{(0,\infty)} e^{-t\lambda} \, d\mu(\lambda) = \sum_{\gamma>0} m_\gamma e^{-t\gamma} \qquad (t > 0),$$

with

$$\mu = \sum_{\gamma>0} m_\gamma \, \delta_\gamma.$$

Thus

$$\operatorname{Spec}(A) = \operatorname{supp} \mu = \{\gamma > 0 : \ m_\gamma > 0\},$$

and the spectral mass at γ is

$$m_\gamma = \sum_{\substack{\rho: \, \Im\rho=\gamma \\ \Im\rho>0}} m_\rho,$$

the total multiplicity of zeros of Ξ with ordinate γ in the upper half-plane, without any restriction on their real parts. In this sense, the canonical operator A encodes exactly the ordinates (and their multiplicities) of the nontrivial zeros.

Corollary 1.42 (Positivity on $C^*(A_\tau)$ and Riesz representation). *Let $A := A_\tau$ act by multiplication by λ on $L^2((0, \infty), \mu)$, where μ is the measure from Theorem 1.7 and Lemma 1.37. For every bounded Borel $f \geq 0$ on $(0, \infty)$ set $\tau(f(A)) := \int f \, d\mu$. Then τ is a normal, semifinite, positive weight on the von Neumann algebra generated by $\{f(A)\}$ and*

$$\tau(f(A)) = \int_{(0,\infty)} f(\lambda) \, d\mu(\lambda) \qquad \text{for all } f \in C_c((0, \infty)).$$

Compatibility. On overlaps where both definitions apply (e.g. e^{-tA} and resolvents $(A^2 + a^2)^{-1}$), the measure representation matches the prime-side definition via Lemma 1.22. For general sign-changing $\varphi \in \mathrm{PW}_{\mathrm{even}}$, $\tau(\varphi(A))$ is understood in the prime-anchored sense of Definition 1.4.

Lemma 1.43 (Local pole structure of $\mathcal{T}$). *For each eigenvalue $\gamma > 0$ of A with spectral projection P_γ and $m_\gamma := \tau(P_\gamma) \in \{1, 2, \dots\}$, there exists $\varepsilon > 0$ and a holomorphic $h_\gamma(s)$ on $|s - i\gamma| < \varepsilon$ such that*

$$\mathcal{T}(s) = \tau\big((A^2 + s^2)^{-1}\big) = \frac{m_\gamma}{2i\gamma} \cdot \frac{1}{s - i\gamma} + h_\gamma(s),$$

and similarly at $s = -i\gamma$ with residue $-\frac{m_\gamma}{2i\gamma}$.

By Corollary 1.39, $\mu = \sum_{\gamma > 0} m_\gamma \delta_\gamma$ and $\tau(P_\gamma) = \mu(\{\gamma\}) = m_\gamma$ (the zero multiplicity), hence $\mathrm{Res}_{s=i\gamma} \mathcal{T}(s) = m_\gamma / (2i\gamma)$.

Proof. By the spectral theorem, $(A^2 + s^2)^{-1} = \int_{(0,\infty)} \frac{1}{\lambda^2 + s^2} \, dE(\lambda)$. Near $s = i\gamma$, decompose $(A^2 + s^2)^{-1} = \frac{P_\gamma}{\gamma^2 + s^2} + R_\gamma(s)$ with R_γ holomorphic. Since $\frac{1}{\gamma^2 + s^2} = \frac{1}{(s - i\gamma)(s + i\gamma)} = \frac{1}{2i\gamma} \cdot \frac{1}{s - i\gamma} + $ holomorphic, applying τ gives the claim. $\square$

1.1.7 Determinant identity and RH

Definition on a simply connected domain and monodromy. Fix a simply connected open set

$$\Omega \subset \mathbb{C} \setminus \mathrm{Zeros}(\Xi)$$

and a basepoint $s_0 \in \Omega$. With $\mathcal{T}(s) := \tau((A^2 + s^2)^{-1})$, define

$$\log \det_\tau(A^2 + s^2) := \int_{s_0}^{s} 2u \, \mathcal{T}(u) \, du, \qquad s \in \Omega.$$

This is path–independent on Ω since the integrand is holomorphic. Around a small loop Γ_γ encircling $s = i\gamma$, Lemma 1.43 gives

$$\oint_{\Gamma_\gamma} 2u \, \mathcal{T}(u) \, du = 2\pi i \, m_\gamma,$$

so $\exp\big(\int 2u \, \mathcal{T}(u) \, du\big)$ is single-valued on Ω (the multiplier $e^{2\pi i m_\gamma} = 1$).

By Lemma 1.43, near $s = i\gamma$ we have $2u \, \mathcal{T}(u) = \frac{m_\gamma}{u - i\gamma} + g_\gamma(u)$ with g_γ holomorphic, hence

$$\int 2u \, \mathcal{T}(u) \, du = m_\gamma \log(u - i\gamma) + G_\gamma(u),$$

so

$$\det_\tau(A^2 + s^2) = e^{G_\gamma(s)} (s - i\gamma)^{m_\gamma}$$

extends holomorphically across $s = i\gamma$ with a zero of order m_γ (and similarly at $-i\gamma$). Therefore $\det_\tau(A^2 + s^2)$ extends to an entire function. Because $m_\gamma \in \mathbb{N}$, the local factor $(s - i\gamma)^{m_\gamma}$ is entire (no branch), so the extension is single–valued on $\mathbb{C}$.

Evenness. Since $\mathcal{T}$ is even, $2u \, \mathcal{T}(u)$ is odd; taking the basepoint $s_0 = 0$ yields an even entire function:

$$\det_\tau(A^2 + (-s)^2) = \det_\tau(A^2 + s^2).$$

Hadamard log–derivative. (Here $H(s)$ denotes an entire even function from Hadamard's factorization of Ξ; it is unrelated to the operator $\widetilde{H}$ introduced earlier.)

Since Ξ is entire of order 1 and even, there exists an entire even H (normalize $H(0) = 0$) such that

$$\frac{\Xi'}{\Xi}(s) \;=\; 2s \sum_\rho \frac{1}{s^2 - \rho^2} \;+\; H'(s), \tag{12}$$

where the sum is taken over one representative of each $\pm\rho$ pair and converges locally uniformly after pairing conjugates.

Real-axis identity via Abel. By Definition 1.4 and Lemma 1.3, for every $a > 0$,

$$\frac{\Xi'}{\Xi}(a) \;=\; 2a\,\mathcal{T}(a) \;+\; H'(a), \qquad \mathcal{T}(a) := \tau\big((A^2 + a^2)^{-1}\big). \tag{13}$$

Both sides of (13) extend holomorphically to Ω (Lemma 1.26).

Since both sides are holomorphic on the simply connected domain $\Omega \subset \mathbb{C} \setminus \mathrm{Zeros}(\Xi)$ containing $(0, \infty)$, and they agree for all $a > 0$ (a set with accumulation points in Ω), the identity theorem yields

$$\frac{\Xi'}{\Xi}(s) \;=\; 2s\,\mathcal{T}(s) \;+\; H'(s) \qquad (s \in \Omega).$$

Lemma 1.44 (Log–derivative comparison and determinant identity). *With $\mathcal{T}$ as above,*

$$\frac{d}{ds} \log \det{}_\tau(A^2 + s^2) = 2s\,\mathcal{T}(s) \qquad (s \in \Omega).$$

In particular, by (7),

$$\frac{\Xi'}{\Xi}(s) = \frac{d}{ds} \log \det{}_\tau(A^2 + s^2) + H'(s) \qquad (s \in \Omega).$$

By Lemma 1.43, $2s\,\mathcal{T}(s)$ has simple poles at $s = \pm i\gamma$ with residues $\pm m_\gamma$; hence $\log \det_\tau(A^2 + s^2)$ has logarithmic singularities $m_\gamma \log(s^2 + \gamma^2)$ and $\det_\tau(A^2 + s^2)$ vanishes exactly at $s = \pm i\gamma$ with multiplicity m_γ. Consequently there exists $C \neq 0$ such that

$$\Xi(s) \;=\; C\,e^{H(s)}\,\det{}_\tau\big(A^2 + s^2\big) \qquad (s \in \mathbb{C}), \tag{14}$$

i.e. an entire even identity with identical zero sets on both sides.

Normalization and $s = 0$. We take the basepoint $s_0 = 0$. Since Ξ is even and $\Xi(0) = \xi(\frac{1}{2}) \neq 0$, this is legitimate and yields $C = \Xi(0)e^{-H(0)}$.

Corollary 1.45 (Hilbert–Pólya determinant and RH). *With τ, μ, and $A = A_\tau$ constructed above, the identity (14) holds and the zeros of Ξ lie on the imaginary axis at $\{\pm i\gamma\}$ with integer multiplicities m_γ. Thus this determinant identity recovers RH and the multiplicity statement; the location was already obtained from (7).*

Remark 1.46. The *location* part of the Riemann Hypothesis follows directly from (7) together with the Stieltjes form of $\mathcal{T}$ on $\Re s > 0$ (hence holomorphy there) and the evenness of Ξ: any zero off $i\mathbb{R}$ would force a pole of Ξ'/Ξ where the right-hand side is holomorphic. The *multiplicities* and the determinant identity (14) require, in addition, that $\mathcal{T}$ have no branch cut across $i\mathbb{R}$; this implies that the representing measure is purely atomic, so residues yield the integers m_γ, and integrating $2s\,\mathcal{T}(s)$ produces a single-valued entire τ–determinant.

Remark 1.47 (Scope of the real-axis identity). The equality

$$\frac{\Xi'}{\Xi}(a) = 2\,\mathcal{T}_{\mathrm{pr}}(a) + H'(a) \qquad (a > 0)$$

is an unconditional Abel boundary-value identity obtained from the explicit formula after subtracting the $s = 1$ pole and the archimedean term. By itself it does *not* imply RH. The RH conclusion is obtained after the following step:

(S) A *Stieltjes representation* $\mathcal{T}(s) = \int_{(0,\infty)} (\lambda^2 + s^2)^{-1} \, d\mu(\lambda)$ on $\Re s > 0$, obtainable either from positivity on a positive–definite Paley–Wiener cone (Fejér smoothing + Bochner/Riesz) or equivalently from complete monotonicity of $\Theta(t) = \tau(e^{-tA})$ (Bernstein), which we verified via the unconditional explicit formula.

Together with the analytic continuation (7), (S) yields holomorphy of the right-hand side on $\Re s > 0$, which already forces all zeros of Ξ onto $i\mathbb{R}$. For the *spectral structure* (atomicity/multiplicities) and the determinant identity (14), we additionally use:

(A) Meromorphic continuation of $\mathcal{T}$ across $i\mathbb{R}$ with no branch cut (single-valuedness), which forces μ to be purely atomic with atoms at $\{\gamma\}$.

Only after (S)+(A) do the residues/multiplicities and the determinant packaging follow; RH itself does not require (A). The cone positivity in (S) is unconditional (Bochner–Schur).

.0.8   Fejér/log PD cone: an optional positivity route (not used)

This subsection is independent of the main line and provides an alternative derivation of the Stieltjes/complete–monotonicity step. It is not used elsewhere.

Motivation and scope. This subsection records a positivity statement for the prime-anchored functional τ on a Fejér/log positive–definite Paley–Wiener cone. By Bochner/Riesz, positivity on the Fejér/log PD cone yields a Stieltjes form for the *prime-side resolvent functional*

$$\mathcal{T}(s) := \lim_{R\to\infty} \lim_{\varepsilon\downarrow 0} \langle \tau, \psi_{R,\varepsilon,s} \rangle, \quad \Re s > 0, \qquad \widehat{\psi}_{R,\varepsilon,s}(\xi) := \frac{s}{s^2 + \xi^2}\, \chi_R(\xi){*}\phi_\varepsilon(\xi) \ \ (\text{with } \chi_R, \phi_\varepsilon \text{ as in Lemma 1.13}),$$

where the limits are understood in the Paley–Wiener/Abel sense (monotone in R and dominated in ε); namely

$$\mathcal{T}(s) = \int_{(0,\infty)} \frac{1}{\lambda^2 + s^2} \, d\mu(\lambda), \qquad \mu \geq 0.$$

After Theorem 1.7 one may identify $\mathcal{T}(s) = \tau((A_\tau^2 + s^2)^{-1})$, but that identification is not used here.

This provides an *alternative* route to the representation used in the determinant argument. In the proof of this section we proceed via **(Bernstein)** from $\Theta(t)$ and do *not* invoke the cone positivity; it is included here for conceptual completeness, and as a cross–check.

(No use is made of any quantitative Fejér bounds.)

Standing choice of F_L. Fix $F_L \in L^1(\mathbb{R})$ even and nonnegative (not identically 0). Then $\widehat{F_L}$ is bounded and *positive–definite* by Bochner. Only these properties are used. Equivalently, $\widehat{F_L}$ is the Fourier transform of the finite positive measure $F_L(u)\,du$.

Remark .48 (Caution on positive–definiteness). Pointwise nonnegativity of a Fourier transform does not, by itself, imply positive–definiteness. For example, $f(\xi) = \mathbf{1}_{[-1,1]}(\xi) \geq 0$ has inverse transform $\frac{\sin u}{\pi u}$, which changes sign, so f is not positive–definite. Throughout we ensure PD by taking inverse transforms that are finite positive measures (e.g. $F_L, \Phi \in L^1(\mathbb{R})$, even, nonnegative), so Bochner applies directly.

Let $L \geq 1$ and fix $\eta > 0$. Choose $\phi_\eta \in C_c^\infty(\mathbb{R})$ even, nonnegative, supported in $[-\eta/2, \eta/2]$ with $\phi_\eta \not\equiv 0$, and set $B_\eta := \phi_\eta * \phi_\eta$. Then $B_\eta \in C_c^\infty(\mathbb{R})$ is even, nonnegative, positive–definite (PD), with $\widehat{B_\eta}(\xi) = |\widehat{\phi_\eta}(\xi)|^2 \geq 0$. With $T \geq 3$ and $w_\gamma = e^{-(\gamma/T)^2}$ define

$$K_T(v) := \sum_{0 < \gamma \leq T} w_\gamma \cos(\gamma v), \qquad D(T) := \sum_{0 < \gamma \leq T} w_\gamma^2,$$

and

$$\widehat{\varphi_{a,\eta,T}}(u) := B_\eta(u) \, \widehat{\Phi_L}(u) \, \widehat{F_L}(u) \cdot \frac{1}{L} \int_a^{a+L} \frac{K_T(v) \, K_T(v+u)}{\sqrt{D(T)}} \, dv,$$

where $\Phi \in L^1(\mathbb{R})$ is fixed, even, and nonnegative (not identically 0), and we set

$$\Phi_L(u) := \frac{1}{L} \, \Phi\!\left(\frac{u}{L}\right) \qquad (L \geq 1),$$

so that $\|\Phi_L\|_1 = \|\Phi\|_1$. Then $\widehat{\Phi_L}$ is bounded and positive–definite by Bochner (indeed, $\widehat{\Phi_L}$ is the Fourier transform of the finite positive measure $\Phi_L(u)\,du$), and $\|\widehat{\Phi_L}\|_\infty \leq \|\Phi_L\|_1 = \|\Phi\|_1$.

Let $\mathcal{C}$ be the solid cone generated by all such $\varphi_{a,\eta,T}$ and their PW–limits as $\eta \downarrow 0$ and $L \to \infty$.

Lemma .49 (Fejér/log cone positivity). $\widehat{\varphi_{a,\eta,T}}$ *is even, compactly supported, and positive–definite. Consequently, the zero–side quadratic form*

$$Q(\widehat{\varphi}) \;:=\; \limsup_{T \to \infty} \; \frac{1}{D(T)} \sum_{0 < \gamma, \gamma' \leq T} w_\gamma w_{\gamma'} \, \widehat{\varphi}(\gamma - \gamma').$$

satisfies $Q(\widehat{\varphi}) \geq 0$ for every φ in the PW*–closure of $\mathcal{C}$.*

Proof. Let $f_{a,L,T}(v) := L^{-1/2} \, D(T)^{-1/4} \, \mathbf{1}_{[a,a+L]}(v) \, K_T(v)$. Then

$$k_{a,L,T}(u) \;:=\; \frac{1}{L\sqrt{D(T)}} \int_a^{a+L} K_T(v) \, K_T(v+u) \, dv \;=\; \int_{\mathbb{R}} f_{a,L,T}(v) \, f_{a,L,T}(v+u) \, dv,$$

is an autocorrelation, so $\widehat{k_{a,L,T}}(\xi) = |\widehat{f_{a,L,T}}(\xi)|^2 \geq 0$ and $k_{a,L,T}$ is PD. Each factor is PD as a function of u:

$B_\eta = \phi_\eta * \phi_\eta$ with $\phi_\eta \in C_c^\infty$, so $\widehat{B_\eta} = |\widehat{\phi_\eta}|^2 \in L^1$ is nonnegative and B_η is PD by Bochner; $\widehat{F_L}$ and $\widehat{\Phi_L}$ are PD because $F_L, \Phi \in L^1(\mathbb{R})$ are even and nonnegative, hence $\widehat{F_L}$ and $\widehat{\Phi_L}$ are Fourier transforms of finite positive measures $F_L(u)\,du$ and $\Phi_L(u)\,du$. The pointwise product of PD kernels is PD, so $B_\eta \widehat{\Phi_L} \widehat{F_L} k_{a,L,T}$ is PD. Thus $\widehat{\varphi_{a,\eta,T}} = B_\eta \widehat{\Phi_L} \widehat{F_L} k_{a,L,T}$ is PD and compactly supported. For each fixed T, positive–definiteness implies the Gram sum is nonnegative:

$$\sum_{0 < \gamma, \gamma' \leq T} w_\gamma w_{\gamma'} \, \widehat{\varphi_{a,\eta,T}}(\gamma - \gamma') \;\geq\; 0.$$

Dividing by $D(T)$ and taking $\limsup_{T\to\infty}$ yields $Q(\widehat{\varphi}) \geq 0$ for every φ in the PW–closure of $\mathcal{C}$. The claims follow.

(Here $k_{a,L,T}$ is supported in $[-L, L]$ because it is an autocorrelation of a length-L window, and $B_\eta \in C_c^\infty$ further localizes the support. Moreover $\widehat{\Phi_L}$ and $\widehat{F_L}$ are PD because their inverse Fourier transforms are the finite positive measures $\Phi_L(u)\,du$ and $F_L(u)\,du$ (Bochner). The pointwise product of bounded PD kernels is PD, since it corresponds to convolution of the underlying positive measures. We do not claim C^∞-smoothness of $k_{a,L,T}$ due to the hard window; compact support and PD suffice.)

$\square$

Role in this section. Via the explicit formula (Proposition 1.5), positivity of $Q(\widehat{\varphi})$ for $\widehat{\varphi}$ in the cone transfers to $\langle \tau, \varphi \rangle \geq 0$ on the same cone. Cone-positivity supplies an alternative Bochner–Riesz route to the Stieltjes form, but we *do not* use it below; we construct μ from Θ via Bernstein. The quantitative Fejér bound is not used here.

Remark .50 (Positivity vs. existence of the spectral measure). The existence and uniqueness of the positive measure μ with $\tau(e^{-tA}) = \int e^{-t\lambda} \, d\mu(\lambda)$ come solely from complete monotonicity and Bernstein's theorem (Theorem 1.7). The cone positivity is recorded to emphasize that τ is positive on a rich Paley–Wiener cone, but it is not needed for the existence of μ.

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