CIS5200: Machine Learning	Spring 2025
Homework 4	
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Problem 1: AdaBoost

1.1 By definition, we have:

$$\text{error}(H_T) := \frac{1}{m}\sum_{i=1}^{m} \mathbb{1}[\text{sgn}(H_T(x_i)) \neq y_i] \tag{1}$$

To prove

$$\text{error}(H_T) \leq \frac{1}{m}\sum_{i=1}^{m} \exp(-y_i H_T(x_i)) \tag{2}$$

is equivalent to prove that

$$\sum_{i=1}^{m} \mathbb{1}[\text{sgn}(H_T(x_i)) \neq y_i] \leq \sum_{i=1}^{m} \exp(-y_i H_T(x_i)) \tag{3}$$

and noticing that for each data point x_i in the dataset X, its error in the T-th iteration is given by the term:

$$\mathbb{1}[\text{sgn}(H_T(x_i)) \neq y_i] \tag{4}$$

When the prediction of the predictor in the T-th iteration is correct, this term is equal to 0, for the product of the prediction and correcponding label is given by:

$$\begin{aligned} y_i H_T(x_i) &\leq 0 \\ -y_i H_T(x_i) &\geq 0 \end{aligned} \tag{5}$$

and thus, the exponential term is :

$$\begin{gathered} \exp(-y_i H_T(x_i)) \geq 1 \\ \mathbb{1}[\text{sgn}(H_T(x_i)) \neq y_i] = 1 \leq \exp(-y_i H_T(x_i)) \end{gathered} \tag{6}$$

When the prediction of the predictor in the T-th iteration is incorrect, this term is equal to 1, for the product of the prediction and correcponding label is given by:

$$\begin{aligned} y_i H_T(x_i) &\geq 0 \\ -y_i H_T(x_i) &\leq 0 \end{aligned} \tag{7}$$

and thus, the exponential term is :

$$\begin{gathered} 0 < \exp(-y_i H_T(x_i)) \leq 1 \\ \mathbb{1}[\text{sgn}(H_T(x_i)) \neq y_i] = 0 < \exp(-y_i H_T(x_i)) \leq 1 \end{gathered} \tag{8}$$

Thus, we have shown that for each data point x_i in the dataset X, the following holds:

$$\mathbb{1}[\text{sgn}(H_T(x_i)) \neq y_i] \leq \exp(-y_i H_T(x_i)) \tag{9}$$

and thus, we can sum over all data points x_i in the dataset X to obtain:

$$\sum_{i=1}^{m} \mathbb{1}[\text{sgn}(H_T(x_i)) \neq y_i] \leq \sum_{i=1}^{m} \exp(-y_i H_T(x_i)) \tag{10}$$

and thus, we have shown that:

$$\text{error}(H_T) \leq \frac{1}{m} \sum_{i=1}^{m} \exp(-y_i H_T(x_i)) \tag{11}$$

Proved.

1.2 Noticing that the deifintion of the predictor in the T-th iteration is defined as:

$$H_T(x) = \sum_{t=1}^{T} \alpha_t h_t(x) \tag{12}$$

, then, by defintion, the weight $w_{t+1,i}$ is given by:

$$w_{t+1,i} = \frac{w_{t,i}}{Z_t} \exp(-\alpha_t y_i h_t(x_i)) \tag{13}$$

, and thus using this recursive definition from $w_{1,i}$ to $w_{t+1,i}$, we have:

$$\begin{aligned} w_{t+1,i} &= \frac{w_{t,i}}{Z_t} \exp(-\alpha_t y_i h_t(x_i)) \\ &= \frac{w_{1,i}}{\prod_{t=1}^{T} Z_t} \prod_{t=1}^{T} \exp(-\alpha_t y_i h_t(x_i)) \\ &= \frac{1}{m \prod_{t=1}^{T} Z_t} \prod_{t=1}^{T} \exp(-\alpha_t y_i h_t(x_i)) \\ &= \frac{1}{m \prod_{t=1}^{T} Z_t} \exp(-y_i H_T(x_i)) \end{aligned} \tag{14}$$

which indicates that:

$$m w_{t+1,i} \prod_{t=1}^{T} Z_t = \exp(-y_i H_T(x_i)) \tag{15}$$

Then, we may sum all these terms over all data points x_i in the dataset X to obtain:

$$\begin{aligned}
\sum_{i=1}^{m} m w_{t+1,i} \prod_{t=1}^{T} Z_t &= \sum_{i=1}^{m} \exp(-y_i H_T(x_i)) \\
m \sum_{i=1}^{m} w_{t+1,i} \prod_{t=1}^{T} Z_t &= \sum_{i=1}^{m} \exp(-y_i H_T(x_i)) \\
m \prod_{t=1}^{T} Z_t &= \sum_{i=1}^{m} \exp(-y_i H_T(x_i)) \\
\prod_{t=1}^{T} Z_t &= \frac{1}{m} \sum_{i=1}^{m} \exp(-y_i H_T(x_i))
\end{aligned} \tag{16}$$

, which is exactly the same as the equation given in the problem, which means that we proved that:

$$\frac{1}{m} \sum_{i=1}^{m} \exp(-y_i H_T(x_i)) = \prod_{t=1}^{T} Z_t \tag{17}$$

1.3 From the definition of Z_t, we have:

$$Z_t = \sum_{j=1}^{m} w_{t,j} \exp(-\alpha_t y_i h_t(x_j)) \tag{18}$$

We may categorize the data points into two groups, one group is correctly classified by the predictor and antoher is mistakenly classified by the predictor, and thus we have:

$$Z_t = \sum_{h_t(x_j)=y_j} w_{t,j} \exp(-\alpha_t y_i h_t(x_j)) + \sum_{h_t(x_j)\neq y_j} w_{t,j} \exp(-\alpha_t y_i h_t(x_j)) \tag{19}$$

By the definition of the weighted error, we then have:

$$\epsilon_t = \sum_{i=1}^{m} w_{t,i} \mathbb{1}[h_t(x_i) \neq y_i] = \sum_{h_t(x_j)\neq y_j} w_{t,j} \tag{20}$$

and the sum of weight of all the correctly classified data points is $1 - \epsilon_t$, along with $y_i h_t(x_i) = 1$ whet the data point x_i is classfieid correctly abd $y_i h_t(x_i) = -1$ when it is wrong, and thus we have:

$$Z_t = (1 - \epsilon_t) \exp(-\alpha_t) + \epsilon_t \exp(\alpha_t) \tag{21}$$

Proved.

1.4 From the statement proved in problem 1.3, we have:

$$Z_t = (1 - \epsilon_t) \exp(-\alpha_t) + \epsilon_t \exp(\alpha_t) \tag{22}$$

To find the minized value of Z_t, we can take the derivative of Z_t with respect to α_t and set it to 0:

$$\begin{aligned}\frac{dZ_t}{d\alpha_t} &= -(1-\epsilon_t)\exp(-\alpha_t) + \epsilon_t \exp(\alpha_t) = 0 \\ \epsilon_t \exp(\alpha_t) &= (1-\epsilon_t)\exp(-\alpha_t) \\ \epsilon_t \exp(2\alpha_t) &= (1-\epsilon_t) \\ \exp(2\alpha_t) &= \frac{1-\epsilon_t}{\epsilon_t}\end{aligned} \tag{23}$$

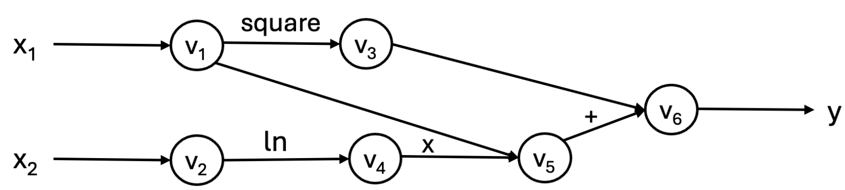
Taking the logarithm of both sides, we have:

$$\alpha_t = \frac{1}{2}\log(\frac{1-\epsilon_t}{\epsilon_t}) \tag{24}$$

Proved.

Problem 2: Auto-Differentiation

2.1 The following is the computation graph for the function $f(x, y) = x_1 \cdot \ln(x_2) + x_1^2$:



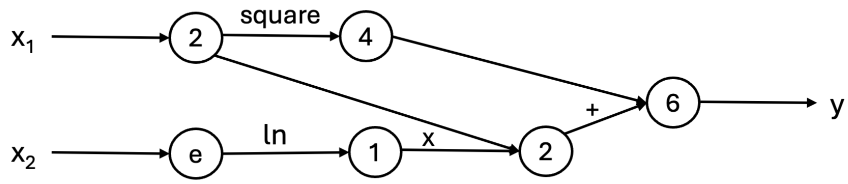
2.2 For the conputation graph in problem 2.1, substituting $x_1 = 2$ and $x_2 = e$, we then have:

$$\begin{aligned}f(x_1, x_2) &= x_1 \cdot \ln(x_2) + x_1^2 \\ &= 2 \cdot \ln(e) + 2^2 \\ &= 2 \cdot 1 + 4 \\ &= 6\end{aligned} \tag{25}$$

And the corresponding forward pass trace is:

$$\begin{aligned}v_1 &= x_1 = 2 \\ v_2 &= x_2 = e \\ v_3 &= x1^2 = 2^2 = 4 \\ v_4 &= \ln(v_2) = \ln(e) = 1 \\ v_5 &= v_1 \times v_4 = 2 \cdot 1 = 2 \\ v_6 &= v_3 + v_5 = 2 + 4 = 6 \\ y &= v_6 = 6\end{aligned} \tag{26}$$

The corresponding forward pass image is shown below:



2.3 First, to calculate the derivate of the function with respect to x_1 with the forward-mode automatic differenta=iation, we can set its seed value to 1 and the seed value of x_2 to 0, and thus we have:

$$
\begin{aligned}
\dot{v_1} &= 1 \\
\dot{v_2} &= 0 \\
\dot{v_3} &= 2 \cdot v_1 \cdot \dot{v_1} = 2 \cdot 2 \cdot 1 = 4 \\
\dot{v_4} &= \frac{1}{v_2} \cdot \dot{v_2} = \frac{1}{e} \cdot 0 = 0 \\
\dot{v_5} &= v_1 \cdot \dot{v_4} + \ln(v_2) \cdot \dot{v_1} = 2 \cdot 0 + 1 \cdot 1 = 1 \\
\dot{v_6} &= \dot{v_3} + \dot{v_5} = 4 + 1 = 5 \\
\frac{\partial y}{\partial x_1} &= \dot{v_6} = 5
\end{aligned}
\tag{27}
$$

For the partial derivative of the function with respect to x_2, we can set its seed value to 1 and the seed value of x_1 to 0, and thus we have:

$$
\begin{aligned}
\dot{v_1} &= 0 \\
\dot{v_2} &= 1 \\
\dot{v_3} &= 2 \cdot v_1 \cdot \dot{v_1} = 2 \cdot 2 \cdot 0 = 0 \\
\dot{v_4} &= \frac{1}{v_2} \cdot \dot{v_2} = \frac{1}{e} \cdot 1 = \frac{1}{e} \\
\dot{v_5} &= v_1 \cdot \dot{v_4} + \ln(v_2) \cdot \dot{v_1} = 2 \cdot \frac{1}{e} + 1 \cdot 0 = \frac{2}{e} \\
\dot{v_6} &= \dot{v_3} + \dot{v_5} = 0 + \frac{2}{e} = \frac{2}{e} \\
\frac{\partial y}{\partial x_2} &= \dot{v_6} = \frac{2}{e}
\end{aligned}
\tag{28}
$$

2.4 For the reverse-mode automatic differentiation, with $\overline{y} = \frac{\partial y}{\partial y} = 1$, we have:

$$\begin{aligned}
\overline{v_6} &= \overline{y} = 1 \\
\overline{v_5} &= \overline{v_6}\frac{\partial v_6}{\partial v_5} = \overline{v_6} = 1 \\
\overline{v_4} &= \overline{v_5}\frac{\partial v_5}{\partial v_4} = \overline{v_5} \cdot v_1 = 1 \cdot 2 = 2 \\
\overline{v_3} &= \overline{v_6}\frac{\partial v_6}{\partial v_3} = \overline{v_6} = 1 \\
\overline{v_2} &= \overline{v_4}\frac{\partial v_4}{\partial v_2} = \overline{v_4} \cdot \frac{1}{v_2} = 2 \cdot \frac{1}{e} = \frac{2}{e} \\
\overline{v_1} &= \overline{v_5}\frac{\partial v_5}{\partial v_1} + \overline{v_3}\frac{\partial v_3}{\partial v_1} = \overline{v_5} + \overline{v_3} \cdot 2v_1 = 1 + 1 \cdot 2 \cdot 2 = 5
\end{aligned} \tag{29}$$

Thus, the final value of the partial derivative of y with respect to x_1 is:

$$\frac{\partial y}{\partial x_1} = \overline{v_1} = 5 \tag{30}$$

For the partial derivative of y with respect to x_2, we have:

$$\frac{\partial y}{\partial x_2} = \overline{v_2} = \frac{2}{e} \tag{31}$$