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Sinewave Fit Algorithm Based on Total Least-Squares Method with Application to ADC Effective Bits Measurement

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***Abstract*—Sinewave fit is a fundamental task in many test and measurement systems. The characterizations of analog–digital converters and digital oscilloscopes are two examples. In this paper, we present a high-performance (i.e., high-precision and high-speed) algorithm to estimate the four parameters of a sinewave from a sample data record. By the use of trigonometric identity, we propose a frequency estimator that turns the nonlinear estimation problem into a linear one. Thus, the difficulty of the traditional nonlinear least-squares sinewave fit method is attenuated. The total least-squares method is used to estimate four parameters of a sinewave in order to minimize the estimation errors in the sense of l_2 norm. Simulation results exhibit that the proposed method gives superior performance over traditional ones and achieves excellent estimation of the true resolution of the simulated ideal ADC. This new algorithm is noniterative and gives swift and consistent results.**

I. INTRODUCTION

SINE-FIT routines are used extensively during the characterization of analog-to-digital converters (ADC's) and digital oscilloscopes [1]–[4]. These sine-fit algorithms estimate the four parameters (amplitude, frequency, phase, and offset) of the sinewave that best fits a given finite length record of discrete samples, which are assumed to be samples of a sinewave, possibly corrupted by noise and distortion. These approaches are basically a gradient search method which operates iteratively. Two common problems which this iterative approach has are that the convergence is not guaranteed and that the results from different runs may not be consistent due to possible trapping at a local minimum. If one wishes to guarantee the convergence by choosing a small step size in the iteration, experience indicates that it usually takes a long time to converge [6].

In order to reduce these difficulties, Jenq *et al*. proposed a method which used the weighted least-square and windowing technique to estimate sinewave parameters in [5]–[6]. To implement their method in a real life situation is, however, problematic because it is difficult how to maintain the monotonicity of the $x_i = \sin^{-1}[s(t_i)]$ (where $s(t_i)$ $(i = 1, 2, \ldots, N)$ are sampling data of a sinewave. More details about this equation can be found in [5]). In the simulated mode, one knows precisely the four parameters of the input sinewave; while in practice one has only the "digitized sinewave record" and does not know at which quadrant of the sinewave these data are located due to $x_i = \sin^{-1}[s(t_i)] = \pi - x_i$ and $2\pi - x_i = \sin^{-1}[s(t_i)] = \pi + x_i = -x_i$. Especially, one is unaware at which quadrant of the sinwave the original sampling point $s(t_1)$ of the digitized sinewave record is located.

In view of these difficulties and problems, we propose a new method, which is based on the trigonometric identity of sampling data of a sinewave function and the total least-squares method, to estimate the four parameters of a sinewave. First, the algorithm presented in this paper attenuates the difficulties of traditional nonlinear least-squares sine-fit algorithm about frequency estimation. Next, the proposed algorithm is optimal in the sense of l_2 norm because using total least-squares (TLS) are used to estimate four parameters of the sinewave.

This paper is organized as follows. In Section II, the principle of the TLS is briefly reviewed. The derivation of estimation algorithm of a four parameters sinewave is given in Section III. Simulation results and comparison between proposed and traditional algorithm are discussed in Section IV. Section V concludes the report.

II. THE TOTAL LEAST SQUARES METHOD

In this section we will briefly review the principle of TLS method. Major results will be stated without proofs. More details about this method can be found in [7]–[9].

Given an overdetermined set of L linear equation in $N \times 1$ unknowns

$$\mathbf{Ac} \approx \mathbf{b} \quad \mathbf{A} \in \mathbf{R}^{L\times N}, \quad \mathbf{b} \in \mathbf{R}^{L\times 1}, \quad \mathbf{c} \in \mathbf{R}^{N\times 1}. \tag{1}$$

A good way to motivate the TLS method is to formulate the ordinary least-squares (LS) problem as follows. The LS problem involves finding a solution vector $\mathbf{c}$ such that

$$||\mathbf{Ac} - \mathbf{b}||_2 = ||\mathbf{r}_2 \text{ minimum} \tag{2}$$

and

$$\mathbf{b} + \mathbf{r} \in \text{range}(\mathbf{A})$$

where $\mathbf{r}$ is a L vector which represents the observation noise record; $||\cdot||_2$ is the l_2 norm given by

$$||\mathbf{r}||_2 = \left(\sum_{i=1}^{L} r_i^2\right)^{1/2}. \tag{3}$$

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Thus, the LS problem amounts to perturbing the observation $\mathbf{b}$ by a minimum amount $\mathbf{r}$ so that $\mathbf{c}$ can be estimated by $\mathbf{A}$ and $\mathbf{b}+\mathbf{r}$ under minimum l_2 norm.

Now, simply put, the idea behind TLS is to consider perturbation of both $\mathbf{b}$ and $\mathbf{A}$, i.e.,

$$(\mathbf{A}+\mathbf{E})\mathbf{c}=\mathbf{b}+\mathbf{r} \tag{4}$$

or (4) can be put into the following form:

$$([\mathbf{A}\quad \mathbf{b}]+[\mathbf{E}\quad \mathbf{r}])\begin{bmatrix}\mathbf{c}\\ -1\end{bmatrix}=\mathbf{0}$$

or

$$(\mathbf{B}+\mathbf{D})\mathbf{Z}=\mathbf{0} \tag{5}$$

where $\mathbf{B}=[\mathbf{A}\quad \mathbf{b}]$, $\mathbf{D}=[\mathbf{E}\quad \mathbf{r}]$, $\mathbf{Z}=\begin{bmatrix}\mathbf{c}\\ -1\end{bmatrix}$.

The TLS solution to the above homogeneous equation (5) can be formulated as seeking a solution vector $\mathbf{c}$ such that

$$\|\mathbf{D}\|_F=\text{minimum} \tag{6}$$

and

$$(\mathbf{b}+\mathbf{r})\in\text{range}(\mathbf{A}+\mathbf{E})$$

where $\|\cdot\|_F$ denotes the Frobenius norm given by

$$\|\mathbf{D}\|_F=\left(\sum_i\sum_j d_{ij}^2\right)^{1/2}. \tag{7}$$

The TLS method minimizes the noise perturbation of effect from both $\mathbf{A}$ and $\mathbf{b}$. Equation (5) shows that the TLS problem involves finding a perturbation matrix $\mathbf{D}\in\mathbf{R}^{N\times(L+1)}$ having the minimum norm so that $\mathbf{B}+\mathbf{D}$ is rank deficient. The singular value decomposition (SVD) can be used for this purpose. The solution to (5) is obtained from a right singular vector $\mathbf{V}$ corresponding to the smallest singular value of the concatenated matrix $\mathbf{B}+\mathbf{D}$. The TLS estimate is [7]–[9]

$$\hat{\mathbf{c}}_{\text{TLS}}=-\frac{1}{(\nu_{L+1})_{L+1}}\begin{bmatrix}(\nu_{L+1})_1\\ \vdots\\ (\nu_{L+1})_L\end{bmatrix}\quad\text{provided}\quad(\nu_{L+1})_{L+1}\neq 0 \tag{8}$$

where $\{(\nu_{L+1})_i\}_{i=1}^{L+1}$ are elements of the $L+1$th singular vector. For methods of picking a total least-squares solution when the smallest singular value is repeated or when $(\nu_{L+1})_{L+1}$ is zero, the reader is referred to [8]. Since $\mathbf{B}+\mathbf{D}$ is not error-free, the total least-squares solution is preferable to the least-squares solution.

In Fig. 1, the LS and the TLS measures of goodness of fit are shown for a simple case when $L=1$. In the LS problem, it is the vertical distances that are important, whereas in the TLS problem, it is the perpendicular distances that are critical. So from this geometric interpretation, it follows that the TLS method is better than the LS method with respect to the residual error in the curve fitting.

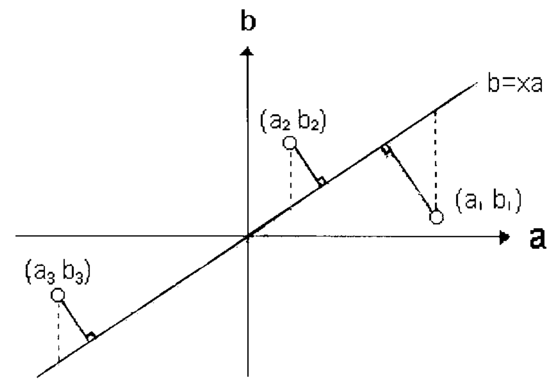

Fig. 1. Least-squared versus total squares.

III. The Estimation Algorithm of the Sinewave Parameters

Suppose a sine function $f(t)=D+A\sin(\omega t+\theta)$ is sampled at equal interval T_s, then the sampled data $f(n)$ can be written as

$$f(n)=D+A\sin(n\Omega+\theta)+e(n)\quad n=0,1,2,\ldots,N \tag{9}$$

where $\Omega=\omega T_s$ is the digital frequency corresponding to ω, $e(n)$ is the additive noise due to the sampling process (which includes the quantization noise of ADC's).

A. The Sine-Wave Frequency Estimator

According to (9), we define

$$\begin{aligned}x(n)&\hat{=}A\sin(n\Omega+\theta)\\ &=f(n)-e(n)-D\\ &\hat{=}y(n)-D\end{aligned} \tag{10}$$

where $\hat{=}$ denotes "definition." By means of trigonometric identity, we can deduce

$$\begin{aligned}x(n)+x(n-2)&=(2\cos\Omega)x(n-1)\\ &\hat{=}cx(n-1).\end{aligned} \tag{11}$$

When (10) is substituted into (11), we get

$$y(n)-D+y(n-2)-D=cy(n-1)-cD \tag{12}$$

and

$$y(n+1)-D+y(n-1)-D=cy(n)-cD. \tag{13}$$

Equation (13) minus (12) is

$$y(n+1)-y(n)=c[y(n)-y(n-1)]+y(n-2)-y(n-1). \tag{14}$$

In (14) let $n=2,3,\ldots,N$ respectively, we can formulate overdetermined equation as (1) where

$$\mathbf{b}=[y(3)-y(2),y(4)-y(3),\ldots,y(N)-y(N-1)]^T$$

$$\mathbf{A}=\begin{bmatrix}y(2)-y(1), & y(3)-y(2),\ldots,y(N-1)-y(N-2)\\ y(0)-y(1), & y(2)-y(3),\ldots,y(N-2)-y(N-3)\end{bmatrix}^T$$

(T for transpose)

$$\mathbf{c}=\begin{bmatrix}c\\ 1\end{bmatrix}.$$

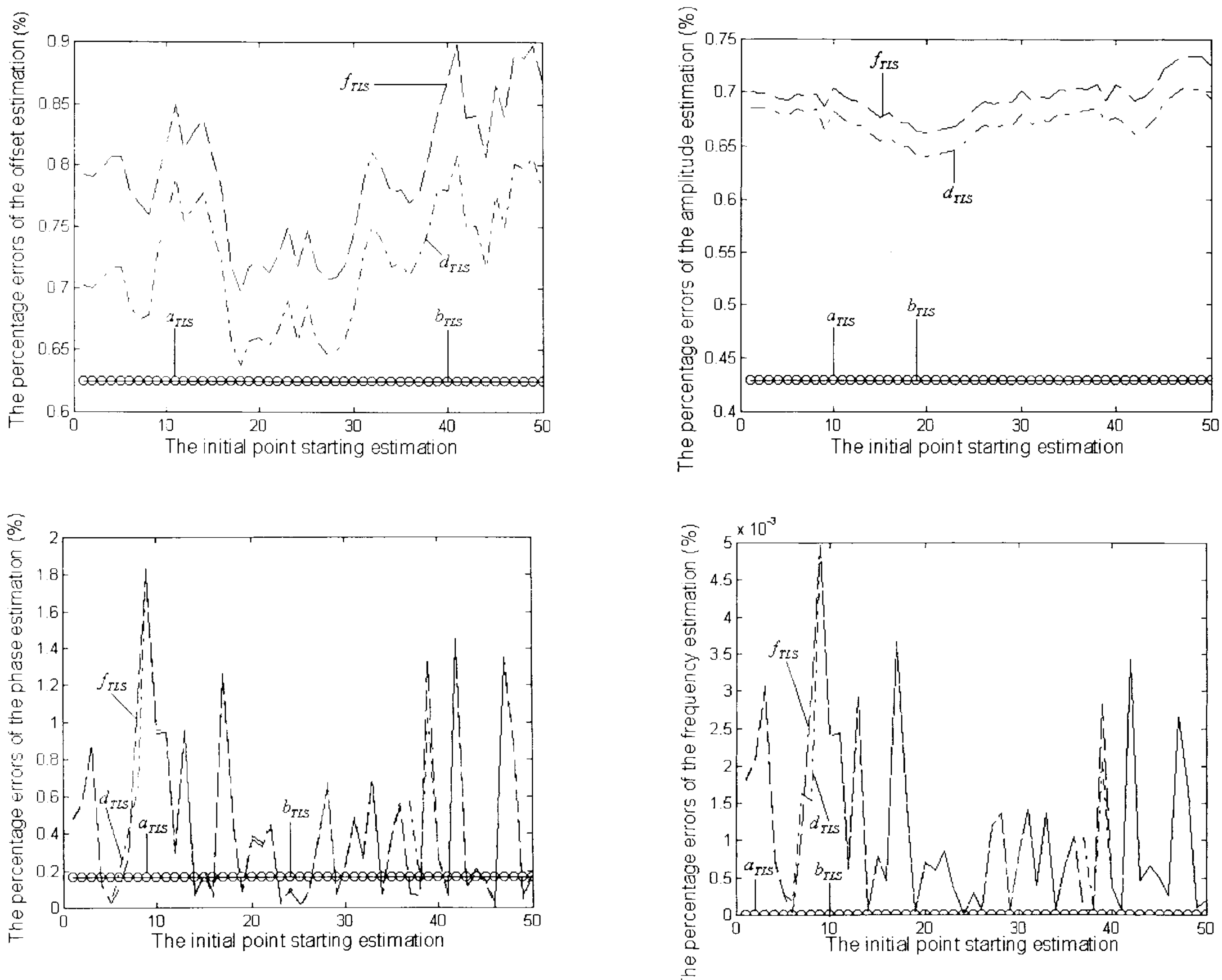

Fig. 2. The percentage errors of offset, amplitude, phase, and frequency estimation, of the new algorithm.

Obviously, when $y(n)\,(n = 0, 1, 2, \ldots, N)$ are replaced by sampling data $f(n) = y(n) + e(n) \quad (n = 0, 1, 2, \ldots, N)$ which from (10), both $\mathbf{b}$ and $\mathbf{A}$ are perturbed by noise $e(n)$. Thus, the total least-squares method, which is discussed in Section II, can be utilized to obtain the optimal estimation $\mathbf{c}_{\mathrm{TLS}}$ of vector $\mathbf{c}$ in the sense of l_2 norm. Once $\mathbf{c}_{\mathrm{TLS}}$ has been estimated from (8), the estimation $\hat{\Omega}$ of sinewave frequency Ω can be written as[1]

$$\hat{\Omega} = \cos^{-1} \frac{c_{\mathrm{TLS}}}{2}. \tag{15}$$

B. Amplitude, Phase, and Offset Estimator

When the frequency Ω is replaced by the estimation $\hat{\Omega}$, and let $n = 0, 1, 2, \ldots, N$ in (9), we can also formulate the overdetermined equation as follows:

$$\underbrace{\begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N) \end{bmatrix}}_{\mathbf{b}} \approx \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ \sin\hat{\Omega} & \cos\hat{\Omega} & 1 \\ \vdots & \vdots & \vdots \\ \sin(N\hat{\Omega}) & \cos(N\hat{\Omega}) & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} A\cos\theta \\ A\sin\theta \\ D \end{bmatrix}}_{\mathbf{c}.} \tag{16}$$

Similar to the frequency estimator discussed above, both $\mathbf{A}$ and $\mathbf{b}$ in (16) are perturbed by noise. Again, the TLS method can be used to solve (16). Consequently, we can obtain the estimation $\mathbf{c}_{\mathrm{TLS}}$ of vector $\mathbf{c}$. Finally, the amplitude, phase, and offset estimation can be obtained, respectively, as follows:

$$\begin{aligned} \hat{A} &= \sqrt{(c_1)^2_{\mathrm{TLS}} + (c_2)^2_{\mathrm{TLS}}} \\ \hat{\theta} &= tg^{-1} \frac{(c_2)_{\mathrm{TLS}}}{(c_1)_{\mathrm{TLS}}} \\ \hat{D} &= (c_3)_{\mathrm{TLS}} \end{aligned} \tag{17}$$

where $\mathbf{c} = [c_1 \quad c_2 \quad c_3]^T = [A\cos\theta \quad A\sin\theta \quad D]^T$.

[1]When the sampling rate is higher than Nyquist rate, the estimation result of frequency is unique.

IV. Computer Simulation

Consider an example whose digitizer output is

$$f(n) = D + y(n) + h(n) + e(n) + q(n) \quad n = 0, 1, 2, \ldots N \tag{18}$$

where

$y(n) =$ $2\sin(2\pi f \cdot n + \theta) =$ simulated digitizer input;
$h(n) =$ $0.001\sin(6\pi f \cdot n) =$ simulated harmonic distortion;
$D =$ simulated offset;
$e(n) =$ simulated normal random noise;
$q(n) =$ simulated quantization error.

A. Sinewave Fit

The digitizer has eight bits. The sinewave frequency is nearly one-third of the sampling rate of the digitizer. The total data points used are 100. The operation of quantizing was to round the sum of $D, y(n), h(n)$ and $e(n)$ to the nearest integer. In simulation, the four situations are considered as follows:

1) $h(n) = 0, e(n) = 0, (n = 0, 1, 2, \ldots, 99)$. Fifty estimations are performed with θ of the input sinewave uniformly distributed on, $(0, 2\pi)$, then the curves of percentage errors of four parameters estimation of $y(n)$ and D are shown in Fig. 2. and the percentage error is defined as

$$\text{percentage error} = \frac{|R - \hat{R}|}{R} \times 100\% \tag{19}$$

where R is the no error parameter, $\hat{R}$ is the estimation of R. a_{TLS} in Fig. 2 denotes curves of the percentage errors of phase, offset, amplitude and frequency respectively in this situation.

2) $e(n) = 0, (n = 0, 1, 2, \ldots, 99)$, that is, there is harmonic distortion in the digitizer data records. b_{TLS} in Fig. 2 denotes the percentage errors curves of phase, offset, amplitude, and frequency, respectively, in this situation.

3) $h(n) = 0, (n = 0, 1, 2, \ldots, 99)$, that is, digitizer data records are perturbed by random noise $e(n)$ of the Gaussian distribution with mean 0 and variance 0.01, which are produced by means of Monte Carlo method. In this situation the percentage errors curves of phase, offset, amplitude, and frequency, which are denoted as d_{TLS}, respectively, are shown in Fig. 2.

4) In (18) all the interference has been considered. In this situation, f_{TLS} denotes the percentage errors of phase, offset, amplitude, and frequency estimation, respectively, in Fig. 2.

B. Comparison Between TLS and Traditional Methods [1]

Making use of the same digitizing data record, we compare the performance between algorithm proposed in this paper and the traditional one in [1]. The specified comparison environment is $h(n) = 0.01\sin(6\pi f \cdot n)$, $e(n) = 0$, $(n = 0, 1, 2, \ldots, 202)$ in (18); in addition, the traditional algorithm has a requirement for a reasonably accurate initial guess for the sought parameters. The comparison results have been shown in Table I. From Table I one can find that the estimation errors of the frequency, phase and amplitude of the algorithm described in this paper are at least 10^{-1} less than those of the algorithm in [1], whereas the estimation errors of the offset is a little better than one of the algorithm in [1]. Furthermore, because this new algorithm is noniterative, it gives consistent results for various runs. It is also swift: for a 203-point data record, it only takes a couple of seconds to estimate the four parameters of a sinewave, whereas the traditional one [1] needs to take 1.5 h to obtain the results in Table I.

TABLE I
Comparative Results Between the TLS Method and Traditional Method

Fitting Method	Frequency(%)	Phase (%)	Amplitude (%)	Offset (%)	Computation Time
Traditional Method	3.06E-5	3.70	5.46E-1	1.86	1.5 Hours
TLS Method	3.51E-14	3.33E-1	4.45E-1	5.69E-1	7 Seconds

TABLE II
Simulated and Ideal Effective Bits

NUMBE OF BITS	SIMULATED EFFECTIVE BITS	IDEAL EFFECTIVE BITS*
20	20.08	20.02
18	18.06	18.01
16	16.10	16.07
14	14.07	14.03
12	11.97	11.99
10	9.97	10.00
8	8.01	8.03
6	6.03	6.01
5	5.02	4.99
4	3.93	3.98
3	2.87	3.02
2	1.45	2.18

•assuming no errors in parameter estimation

C. ADC Effective Bits Measurement

A commonly used formula for an N-bit ADC is given by [2]

$$B = N - \log_2\left(\frac{rmse}{\text{Ideal noise}}\right). \tag{20}$$

If one assumes that the quantization noise is uniformly distribution and that the quantization errors from sample to sample are statistically independent, then, $\text{Ideal Noise} = \frac{Q}{\sqrt{12}}$, where Q is the ideal code bin width. We use the sinewave fit algorithm described in this paper to estimate the four parameters of the input sinewave, and then use the estimated sinewave as if it were the actual sinewave to compute the effective bits of the ADC by using (20). The critical question is "How good is it?" A simulation to implement this proposal has been done. The results are shown in Table II. It is seen that this approach gives excellent results with resolution not less than three bits.

V. Conclusion

In this paper, we have presented a high-performance sinewave parameters estimation algorithm, which gives highly accurate and speedy estimations. The total least-squares method is utilized to estimate the four parameters of a sinewave in order to make the estimation errors be minimized in the sense of l_2 norm. Simulation results exhibit that the proposed method gives superior performance over traditional ones and produces excellent estimation of the true resolution of the simulated ideal ADC.

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