Hoeffding inequality for 2d random walk

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In the plane, consider the sum of n 2-dimensional vectors. Let θ_i be a phase chosen uniformly at random from $[0, 2\pi)$, and let $r_i = 1\ \forall i$.The position of the random walk after n steps will be:

$$z = \sum_{j=1}^{n} e^{i\theta_j} \tag{0.0.1}$$

The absolute value of this distance is:

$$\begin{aligned} \mid z \mid^2 &= \sum_{k}^{n} e^{i\theta_k} \sum_{j=1}^{n} e^{-i\theta_j} \\ &= N + \sum_{j,k=1 j\neq k}^{n} e^{i(\theta_k - \theta_j)} \end{aligned}$$

note that the successive differences $|z_{j=1} - z_j| \leq 1$ a.s. as the most the walk can differ by is 1. Then we can apply Hoeffding's inequality:

$$\mathbb{P}\left(|X_n - X_0| \geq t\right) \leq \exp - \frac{t^2}{2\sum_{i=1}^{n} c_i} \tag{0.0.2}$$

where X_i is a martingale and $|X_{k+1} - X_k| \leq c_k\ \forall k$. So we have:

$$\mathbb{P}\left(|z_n|^2 \geq t\right) \leq \exp - \frac{t^2}{2n} \tag{0.0.3}$$

by defining the martingale $X_n = |z_n|^2$