New Basis

September 22, 2015

1 Preliminaries

Consider the basis defined by the function:

$$f_i(x) = \begin{cases} 1 & \text{if } x \leq i \\ 0 & \text{otherwise} \end{cases} \tag{1.0.1}$$

That is, f_i is a left-hand step function. We define the inner product between two vectors as follows:

Definition 1.1.

$$\langle x, y \rangle = x^T y = \sum_i x_i y_i \tag{1.0.2}$$

where x_i, y_i are the components of the vectors x, y in the i^{th} direction with respect to some basis vectors e_i.

We can define a matrix representation for the set of basis vectors f_i, by taking all inner products between all pairs basis vectors:

Definition 1.2.

$$F_{n,ij} = \langle f_i, f_j \rangle \tag{1.0.3}$$

This matrix has the representation:

$$F_{ij} = min(i, j) \tag{1.0.4}$$

An example of such a matrix is:

$$F_n = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \tag{1.0.5}$$

This matrix is invertible.

Theorem 1.3.

$$det(F_n) = 1 \tag{1.0.6}$$

Proof. Consider the matrix F^n. Subtract the $n-1$th column from the nth. We obtain a matrix with 0 on the final column except the entry $F^n_(n, n) = 1$. Since the top $(n-1) \times (n-1)$ is F^{n-1} we find that

$$det(F_n) = 1 \times det(F_{n-1}) \tag{1.0.7}$$

By recursion and $det(F_1) = 1$ we have $det(F_n) = 1$.

□

This matrix can be factorised as $F = LL^T$ where

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\dots0 \\ 1 & 1 & 0 & 0 & 0\dots0 \\ 1 & 1 & 1 & 0 & 0\dots0 \\ \dots \\ 1 & 1 & 1 & 1 & 1\dots1 \end{pmatrix} \tag{1.0.8}$$

From this it follows that

$$F^{-1} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0\dots0 \\ -1 & 2 & -1 & 0 & 0\dots0 \\ 0 & -1 & 2 & -1 & 0\dots0 \\ \dots \\ 0 & 0 & 0 & 0\dots-1 & 1 \end{pmatrix} \tag{1.0.9}$$

Another matrix we will use a lot of is:

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\dots0 \\ -1 & 1 & 0 & 0 & 0\dots0 \\ 0 & -1 & 0 & 0 & 0\dots0 \\ \dots \\ 0 & 0 & 0 & 0\dots-1 & 1 \end{pmatrix} \tag{1.0.10}$$

and by direct computation:

$$DL = I \tag{1.0.11}$$

2 Spectrum Sensing

We model our PSD signal g as a linear combination of the basis functions (1.0.1):

$$g(x) = \sum_i a_i f_i \tag{2.0.12}$$

To find the a_i, we correlate (take the inner product of) the signal against the basis (1.0.1).

Definition 2.1.

$$\begin{aligned} h_j &= \langle g, f_j \rangle && (2.0.13) \\ &= \sum_j g(x) f_j(x) && (2.0.14) \\ &= \sum_j a_i f_i(x) f_j(x) && (2.0.15) \\ &= a_i \langle f_i, f_j \rangle && (2.0.16) \\ &\left(= \sum_{x=1}^{j} g(x) \right) && (2.0.17) \end{aligned}$$

In matrix language $h = Fa^T$. This is the inner product between the signal g and the basis functions f_i.

Definition 2.2.

$$d_i(x) = \begin{cases} 1 & \text{if } x = i \\ -1 & \text{if } x = i - 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.0.18}$$

i.e. d_i is the i^{th} row of the matrix D.

Theorem 2.3.

$$D^T g = a \tag{2.0.19}$$

Proof. From the definition of g (2.0.12)

$$D^T g = D^T \sum_i a_i f_i \tag{2.0.20}$$

$$= \sum_i a_i d_j^T f_i \tag{2.0.21}$$

$$= a\langle d_j, f_i\rangle \tag{2.0.22}$$

$$= a \tag{2.0.23}$$

as

$$\langle d_j, f_i\rangle == \begin{cases} 1 & \text{if } i \leq j \\ 0 & \text{otherwise} \end{cases} \tag{2.0.24}$$

□

This means

$$g = \left(D^T\right)^{-1} a = \left(D^{-1}\right)^T a \tag{2.0.25}$$

so using (1.0.11),

$$g = L^T a \tag{2.0.26}$$

To recover $\hat{a}$, we minimise

$$||h - Fa||_2^2 \tag{2.0.27}$$

in the noiseless case, and

$$||h_\varepsilon - Fa||_2^2 + \lambda\,||a||_1 \tag{2.0.28}$$

in the noisy case, where

$$\left(h_\varepsilon\right)_j = \langle (g + \varepsilon)\,, f_j\rangle \tag{2.0.29}$$

and $\varepsilon \sim \mathcal{N}(0, \sigma^2\ I)$.
From $\hat{a}$ we can recover $\hat{g}$ from the following relation:

$$\hat{g} = L^T \hat{a} \tag{2.0.30}$$

3 Results

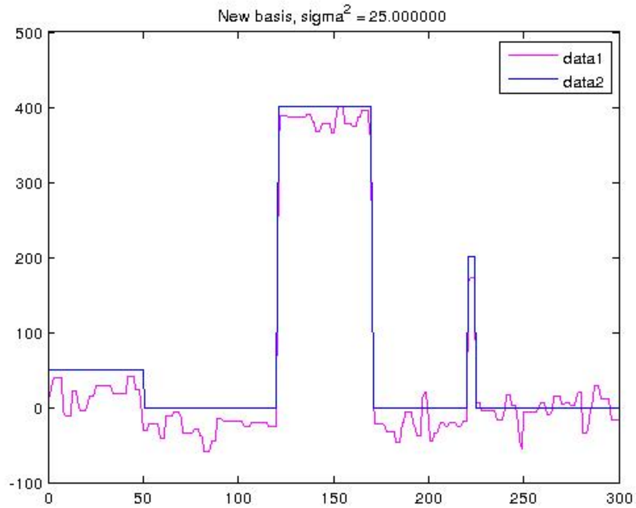


Figure 3.1:

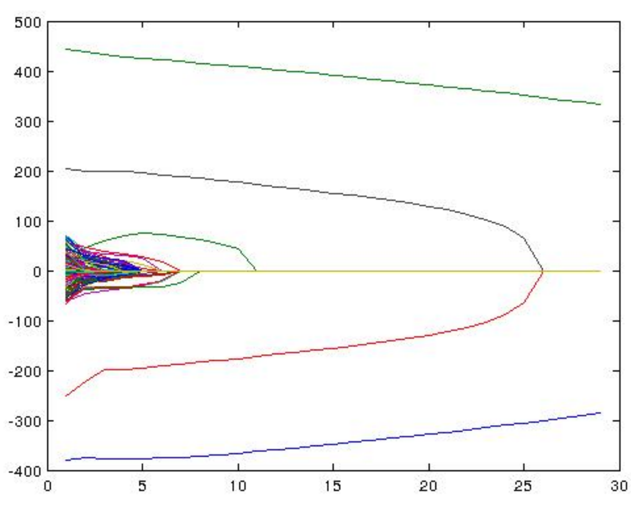


Figure 3.2:

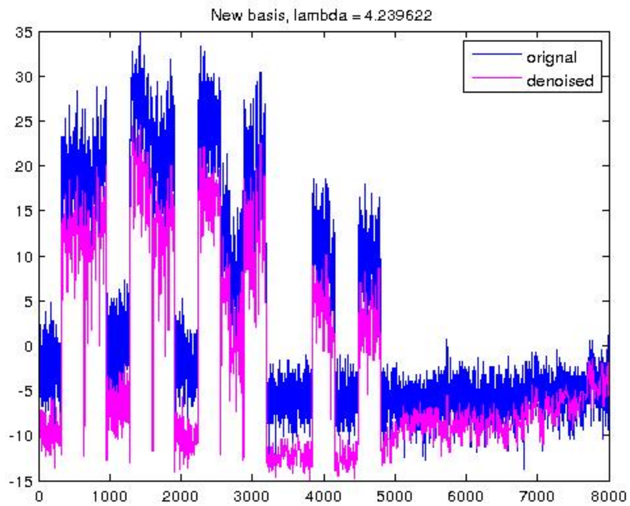


Figure 3.3:

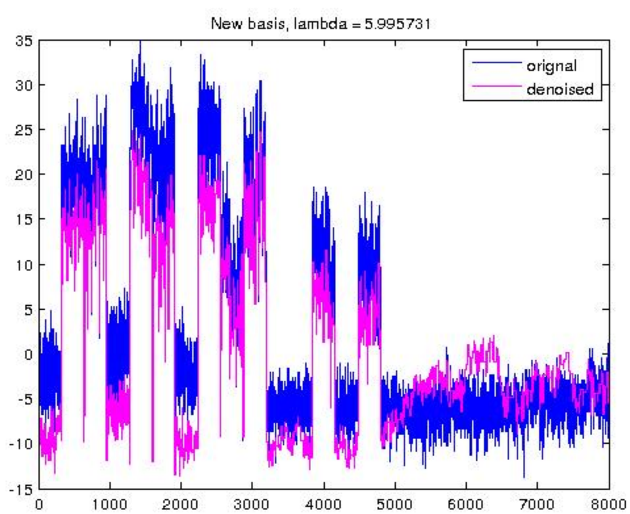
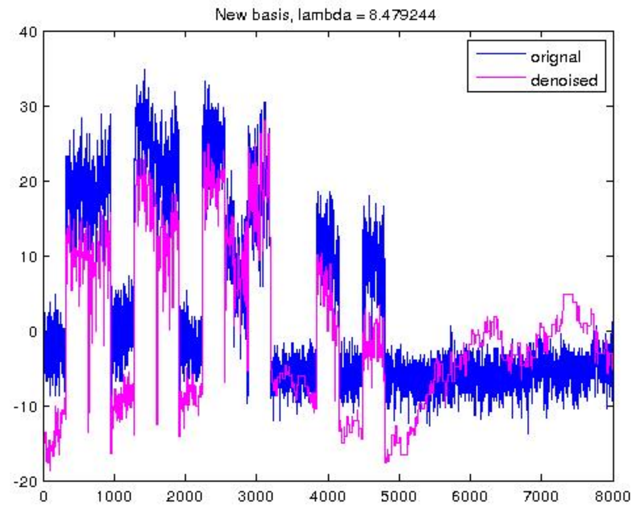


Figure 3.4:



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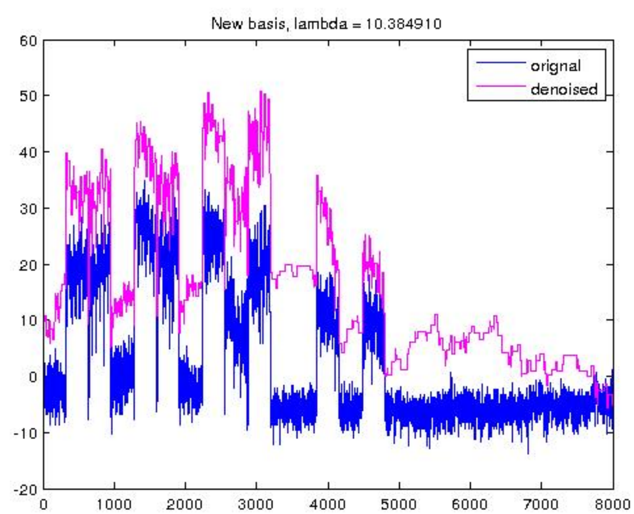

Figure 3.6:

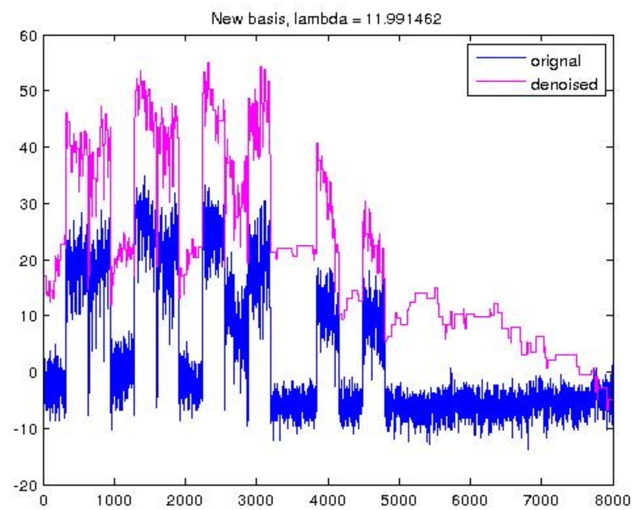
New basis, lambda = 11.991462
orignal
denoised
60
50
40
30
20
10
0
-10
-20
0
1000
2000
3000
4000
5000
6000
7000
8000

Figure 3.7: