How big should a group be?

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1 Introduction

GIven a series of probabilities q_i, $i = 1 \dots n$, $\sum_i = 1^n = K$, that each item is a set is defective, we need to choose the size of the set to be tested. That is, if we test a group of size S we get a payoff:

$$\theta\left(S\right) = \begin{cases} S & \text{if Group contains no defectives} \\ 0 & \text{if Group contains a defective} \end{cases} \tag{1.1}$$

i.e. in a group of size S the payoff is:

$$\theta\left(S\right) = S \prod_{i=1}^{S} (1 - q_i) \tag{1.2}$$

where $\prod_{i=1}^{S}(1 - q_i)$ is the probability that the group contains no defectives. How do we maximise θ?

2 Approximation

Assuming that the q_i are small (say of the order $\frac{1}{100}$ and have bounded variance, replace each q_i by the average:

$$\overline{q} = \frac{1}{n} \sum_{i=1}^{n} q_i \tag{2.1}$$

The payoff in this model is then:

$$\theta\left(S\right) = S(1 - \overline{q})^S \tag{2.2}$$

i.e. the model is geometric (if we were flipping coins with bias $\overline{q}$ then the number of steps until the first failure is distributed according to the geometric distribution).

The error can be computed:

$$(1-\overline{q})^S - \prod_{i=1}^{S}(1-q_i) \leq e^{-S\overline{q}} - e^{-\sum_{i=1}^{S} q_i} \tag{2.3}$$

Since the model is geometric, the expected number of steps until the first failure is:

$$\mathbb{E}Steps = \frac{1}{\overline{q}} \tag{2.4}$$

Because of the error, I think a reasonable choice of S is:

$$S = \lfloor \frac{1}{\overline{q}} \rfloor \tag{2.5}$$