Queueing Theory summing Tricks

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1 Introduction

This is a compendium of summing tricks used in Ganesh's IQN course.

2 Geometric sums

This is a standard summation technique, which is complicated by considering derivatives of x^n.

2.1 Infinite Geometric Sums

The standard geometric sum is below:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ , } |x| < 1 \tag{2.1.1}$$

2.2 Change of Variables

Sums like:

$$\sum_{n=1}^{\infty} x^n \tag{2.2.2}$$

can be solved by making the change of variables $m = n + 1$

$$\sum_{n=1}^{\infty} x^n = \sum_{m=0}^{\infty} x^{m+1} = x \sum_{m=0}^{\infty} x^n = \frac{x}{1-x} \text{ , } |x| < 1 \tag{2.2.3}$$

More generally, for sums starting at a make the change of variables $m = n + a$

$$\sum_{n=a}^{\infty} x^n = \sum_{m=0}^{\infty} x^{m+a} = x^a \sum_{m=0}^{\infty} x^n = \frac{x^a}{1-x} \text{ , } |x| < 1 \tag{2.2.4}$$

For example, this type of sum occurred in problem sheet 2 where we had to evaluate the generating function of a random number of random variables, we came across a sum which looked like:

$$\sum_{k=1}^{\infty} p(1-p)^{k-1}z^k \tag{2.2.5}$$

(This is the probability generating function of a geometric random variable). This is solved by removing anything without a dependence on k, and then applying a change of variables $l = k - 1$:

$$\sum_{k=1}^{\infty} p(1-p)^{k-1}z^k = pz\sum_{k=1}^{\infty}(1-p)^{k-1}z^{k-1} = pz\sum_{l=0}^{\infty}[(1-p)z]^l = \frac{pz}{1-(1-p)z} \tag{2.2.6}$$

2.3 Geometric Series and Derivatives

Sometimes sums like:

$$\sum_{n=0}^{\infty} nx^n \text{ or } \sum_{n=0}^{\infty} nx^{n-1} \tag{2.3.7}$$

The way to solve problems with these sums is either to find the appropriate generating function of the quantity you are given (for example a geometric random variable), or to note that:

$$\frac{d\left(x^n\right)}{dx} = nx^{n-1} \text{ and } x\frac{d\left(x^n\right)}{dx} = nx^n \tag{2.3.8}$$

and that the order of summation and differentiation can be exchanged because they are linear operators. So,

$$\sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}\frac{d\left(x^n\right)}{dx} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2} \tag{2.3.9}$$

and

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2} \tag{2.3.10}$$

by a similar argument.

3 Taylor Series

Taylor series are another summing type which appears in the solution of birth death processes (for example the M/M/infinity queue and queues with state dependent transitions - see problem sheet 6 questions 3 and 4).

3.1 Taylor expansion

A Taylor series is an expansion of a function around a point a. They are extremely useful in maths generally as you can convert a sum into a function.

$$f\left(a\right) = \sum_{i=0}^{\infty}\frac{(x-a)}{i!}f^{(i)}\left(a\right) \tag{3.1.11}$$

3.2 Common Taylor Series

The two series encountered in the course are the series for the exponential (in the M/M/infinty queue and PS6Q3) and the logarithm (PS6Q4 and last year's exam).

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots = \sum_{i=0}^{\infty} \frac{x^i}{i!} \tag{3.2.12}$$

A useful variation on this theme is:

$$e^x - 1 = x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots = \sum_{i=1}^{\infty} \frac{x^i}{i!} \tag{3.2.13}$$

The Taylor series for the logarithm is:

$$-\log(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots = \sum_{i=0}^{\infty} \frac{x^i}{i} \tag{3.2.14}$$

It appears in problems when we need to sum things like:

$$1 + \frac{1}{2}x + \frac{1}{3}x^2 + \ldots = \frac{-\log(1-x)}{x} \tag{3.2.15}$$