Wavelet Model

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1 Introduction

2 Wavelet Transform

2.1 Wavelets

Wavelets are, as the name suggests, mini-waves - they begin at zero, have a single oscillation (which increases, then decreases, which may go negative and increase again to zero), and end again at zero. In more formal languagem, wavelets are orthonormal families of (discretised) functions taken from $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, chosen to have desirable properties for signal analysis.

The advantage of wavelet based signal analysis over traditional Fourier analysis, is that it allows for signals to be analysed simultaneously in time as well as frequency (or in the case of images, space and scaling).

For example, Haar wavelets are defined by:

$$\psi(x) = \begin{cases} 1 & \text{if } \frac{1}{2} \leq x < 0 \\ -1 & \text{if } \frac{1}{2} < x \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.1.1}$$

and

$$\psi_{j,k}(x) = 2^{j/2}\psi\left(2^j x - k\right) \tag{2.1.2}$$

note that ψ and $\psi_{j,k}$ are orthogonal in $[0,1]$.

2.2 Wavelet Transform

The Wavelet transform of a function f is simply the convolution of the wavelet with the function:

$$\gamma_{j.k} = \int f\,\psi_{j,k}\,d\lambda \tag{2.2.3}$$

where $d\lambda$ is a suitable measure (e.g. Lebesgue, discrete etc). I.e the wavelet transform measures the overlap of the function f with the wavelet at scale j and position k.

The discrete Haar wavelet transform may be represented compactly as a unitary transform:

$$y_n = H_n x_n \tag{2.2.4}$$

where y_n is the wavelet transform, x_n is the original signal, and H_n is the Haar wavelet matrix defined recursively:

$$H_{2n} = \begin{pmatrix} H_n \otimes (1,1) \\ I_n \otimes (1,-1) \end{pmatrix} \tag{2.2.5}$$

where $\otimes$ is the Kronecker product, I_n is the identity matrix and:

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{2.2.6}$$

The inverse is:

$$x_n = H^t y_n \tag{2.2.7}$$

2.3 Implementation as a filter bank

All discrete wavelet transforms can be implemented as a cascaded series of low and high pass filters.

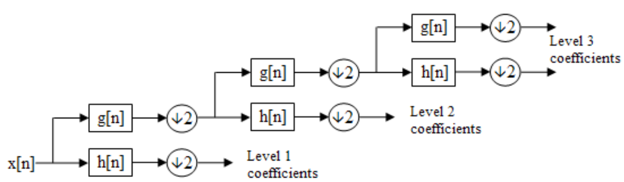

Figure 2.1: An example of a Wavelet Filter Bank