Large Deviations

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1 Cramer's Theorem

Definition 1.1

$$\Lambda(t) = \log M_x(t) = \log \mathbb{E}e^{tX} \tag{1}$$

Definition 1.2 (LF Transform)

$$\Lambda^*(t) = \sup_{t \in \mathbb{R}} \{tx - \Lambda(t)\} \tag{2}$$

Lemma 1.1 (Properties of Rate Functions)

1. *Λ is convex. Λ^* is convex and a Rate Function:*
   - *(a)* *Λ^* is Lower Semi-continuous*
   - *(b)* *If all level sets are compact, Λ^* is a Good Rate Function*
2. *$D_\Lambda = \{t : \Lambda(t) < \infty\}$, D^*_Λ likewise.*
   - *(a)* $D_\Lambda = \{0\} \to \Lambda^* = 0$
   - *(b)* $\Lambda(t) < \infty$ *for some* $t > 0 \to \bar{x} < \infty$ *and* $\forall x \geq \bar{x}$

     $$\Lambda^*(x) = \sup_{t \geq 0} \{tx - \Lambda(t)\} \tag{3}$$

     and it is non-decreasing for $x \geq \bar{x}$
   - *(c)* $\Lambda(t) < \infty$ *for some* $t < 0 \to \bar{x} > -\infty$ *and* $\forall x \geq \bar{x}$

     $$\Lambda^*(x) = \sup_{t \leq 0} \{tx - \Lambda(t)\} \tag{4}$$

     and it is non-increasing for $x \leq \bar{x}$
   - *(d)* $\mid \bar{x} \mid < \infty \to \Lambda^*(\bar{x}) = 0$
   - *(e)* $\inf_{x \in \mathbb{R}} \Lambda^*(\bar{x}) = 0$

3. $\Lambda()$ *is differentiable in* $\dot{D_\Lambda}$ *with*

$$\Lambda^{'}(\eta) = \frac{1}{M(\eta)}\mathbb{E}X_1 e^{\eta X_1} \tag{5}$$

and

$$\Lambda^{'}(\eta) = y \to \Lambda^* = \eta y - \Lambda(\eta) \tag{6}$$

Proof.

1. We use Holder's inequality: given $p, q \in [1, \infty]$ s.t $\frac{1}{p} + \frac{1}{q} = 1$ $\forall f, g$ measurable:

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \tag{7}$$

$$\begin{aligned} \Lambda(t\theta + (1-\theta)s) \quad &= \log \mathbb{E}[e^{(tX_1)\theta} e^{(sX_1)(1-\theta)}] && (8)\\ &\leq \log \mathbb{E}[e^{(tX_1)\theta}]\mathbb{E}[e^{(sX_1)(1-\theta)}] && (9)\\ &= \theta \log \mathbb{E}[e^{tX_1}] + (1-\theta)\log \mathbb{E}[e^{sX_1}] && (10) \end{aligned}$$

$$\begin{aligned} \Lambda^*(x_1\theta + (1-\theta)x_2) \quad &= \sup_{\lambda \in \mathbb{R}}\{\lambda(x_1\theta + (1-\theta)x_2) - \Lambda(\lambda)\} && (11)\\ &\leq \sup_{\lambda \in \mathbb{R}}\{\lambda(x_1\theta - \theta\Lambda(\lambda)\} + && (12)\\ &\sup_{\lambda \in \mathbb{R}}\{(1-\theta)\lambda x_2) - (1-\theta)\Lambda(\lambda)\} && (13)\\ &= \theta\Lambda^*(x_1) + (1-\theta)\Lambda^*(x_2) && (14) \end{aligned}$$

To prove that Λ^* is a Rate Function:

   (a) (Non-negativity) $\Lambda(0) = \log \mathbb{E}[1] = 0 \to \Lambda^*(x) \geq 0x - \Lambda(0) = 0$

   (b) (Lower semi-continuity) FIx a sequence $x_n \to x$. Then $\forall \lambda \in \mathbb{R}$

$$\begin{aligned} \liminf_{x_n \to x} \Lambda^*(x_n) \quad &\geq \liminf_{x_n \to x} (\lambda x_n - \Lambda(\lambda)) && (15)\\ &= \lambda x - \Lambda(\lambda) && (16)\\ \implies \liminf_{x_n \to x} \Lambda^*(x_n) &\geq \sup_{\lambda \in \mathbb{R}} (\lambda x_n - \Lambda\lambda) && (17)\\ &= \Lambda^*(x) && (18) \end{aligned}$$

   which is the definition of lower-semicontinuity.

2. (a) $D_\Lambda := \{t : \Lambda(t) < \infty\}$ $D_\Lambda = \{0\} \implies \Lambda(\lambda) \begin{cases} 0 & \lambda = 0 \\ \infty & \text{otherwise} \end{cases}$
   $\implies \Lambda^*(x) = 0 \forall x$

   (b) Let $\Lambda(\lambda) = \log M(\lambda) < \infty$ for some $\lambda > 0$

$$\implies \int_0^\infty x d\mu \leq \frac{\log \int_{\mathbb{R}} e^{\lambda x} d\mu}{\lambda} \tag{19}$$

$$
\begin{aligned}
log\int_{\mathbb{R}} e^{\lambda x} d\mu &\geq \log \textstyle\int_0^\infty e^{\lambda x} d\mu && (20)\\
&\geq \textstyle\int_{\mathbb{R}} \log e^{\lambda x} d\mu && (21)\\
&= \lambda \textstyle\int_0^\infty x d\mu && (22)\\
&\implies \bar{x} < \infty && (23)
\end{aligned}
$$

$\square$