Channel Dispersion Thoughts

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Define the information density etc as per the Verdu, Polyanski paper. We have an input alphabet $\mathscr{X}$ with n symbols, and an output alphabet, $\mathscr{Y}$ with 2 symbols. There is also some discrete spaces where the probability distributions live, $S(\mathscr{X})$ and $S(\mathscr{Y})$.

In particular the channel dispersion is:

$$V = \sum_{x,y \in \mathscr{X},\mathscr{Y}} p(x,y) \left(\log \frac{p(y \mid x)}{p(x)} \right)^2 - I(\mathscr{X};\mathscr{Y})^2 \tag{1}$$

This can be rewritten as:

$$V = \sum_{x,y \in \mathscr{X},\mathscr{Y}} I(\mathscr{X};\mathscr{Y}) \log \frac{p(y \mid x)}{p(x)} - I(\mathscr{X};\mathscr{Y})^2 \tag{2}$$

Define Q as:

$$Q = \sum_{x,y \in \mathscr{X},\mathscr{Y}} I(\mathscr{X};\mathscr{Y}) \log \frac{p(y \mid x)}{p(x)} \tag{3}$$

Now Q^2 is:

$$Q^2 = \left(\sum_{x,y \in \mathscr{X},\mathscr{Y}} I(\mathscr{X};\mathscr{Y}) \log \frac{p(y \mid x)}{p(x)} \right)^2 \tag{4}$$

By Cauchy-Swartz:

$$Q^2 \leq \left(\sum_{x,y \in \mathscr{X},\mathscr{Y}} I(\mathscr{X};\mathscr{Y}) \right)^2 \left(\sum_{x,y \in \mathscr{X},\mathscr{Y}} \log \frac{p(y \mid x)}{p(x)} \right)^2 \tag{5}$$

Using the bound on the mutual information:

$$I\left(\mathscr{X};\mathscr{Y}\right)\leq 1 \tag{6}$$

So,

$$Q^2 \leq \left(\sum_{x,y\in\mathscr{X},\mathscr{Y}} 1\right)^2 \left(\sum_{x,y\in\mathscr{X},\mathscr{Y}} \log\frac{p\left(y\mid x\right)}{p\left(x\right)}\right)^2 \tag{7}$$

The first term is at most n, so:

$$Q^2 \leq n\left(\sum_{x,y\in\mathscr{X},\mathscr{Y}} \log\frac{p\left(y\mid x\right)}{p\left(x\right)}\right)^2 \tag{8}$$

So

$$Q \leq \sqrt{n\left(\sum_{x,y\in\mathscr{X},\mathscr{Y}} \log\frac{p\left(y\mid x\right)}{p\left(x\right)}\right)^2} \tag{9}$$

$$Q \leq \sqrt{n}\left(\sum_{x,y\in\mathscr{X},\mathscr{Y}} \log\frac{p\left(y\mid x\right)}{p\left(x\right)}\right) \tag{10}$$

Then we are left with:

$$V \leq \sqrt{n}\left(\sum_{x,y\in\mathscr{X},\mathscr{Y}} \log\frac{p\left(y\mid x\right)}{p\left(x\right)}\right) - 1 \tag{11}$$

So I don't think the channel dispersion can be bounded in anything better than $\mathrm{O}\left(\sqrt{n}\log n\right)$.