Decoding bounds

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1 Typical Set decoding

The decoding strategy is to only decode defective sets if the defective set is typical. Write $\Sigma_{n,k}$, for the collection of sets of size$k$ out of n items and $\theta\left(S\right)$ for the vector of all test outcomes then:

$$A_y = \theta^{-1}\left(S\right) = \left\{S \in \Sigma_{n,k} \,:\, \theta\left(S\right) = \mathbf{y}\right\} \tag{1}$$

As the probabilities are not equal, we can do better than picking amongst the A_y with equal probability. Define:

$$\tilde{A}_y = A_y \cap \left\{\epsilon\text{-typical set}\right\} \tag{2}$$

for some ϵ.

Then decode the largest in probability. I.e.

$$\mathbb{P}\left(\text{succ}\right) = \frac{q_i}{\sum_{j=1}^{n} q_j} \tag{3}$$

Where

$$q_i \leq 2^{N(H-\epsilon)} \tag{4}$$

and:

$$q_j \geq 2^{-N(H+\epsilon)} \tag{5}$$

so,

$$\sum q_j \geq |\tilde{A}_y| 2^{-N(H+\epsilon)} \tag{6}$$

Putting these together:

$$\begin{aligned}
\mathbb{P}\left(\text{succ}\right) \leq \frac{2^{-N(H-\varepsilon)}}{|\tilde{A}_y| 2^{-N(H+\varepsilon)}} &= 2^{-N(H-\varepsilon)} 2^{N(H+\varepsilon)} && (7)\\
&= 2^{-N(H+\varepsilon)+N(H+\varepsilon)} && (8)\\
&= 2^{-NH+N\varepsilon+NH+N\varepsilon} && (9)\\
&= \frac{2^{2N\varepsilon}}{|\tilde{A}_y|} && (10)
\end{aligned}$$

2 AEP for non-iid sequences

Definition 2.1. *Given a sequence of n random variables $X_1 \dots X_n$ s.t. $\mathbb{P}(X_i = 1) = p_i$, and a random variable X s.t. $\mathbb{E}X = \frac{k}{n}$, let $S_n = n^{-1} \sum_{i=1}^{n} X_i$. Then the weak law of large numbers implies:*

$$S_n \to \mathbb{E}X \tag{11}$$

in probability

Define the random variable, $Z_i = \log p_i$ with expectation

$$\mathbb{E}Z_i = -p_i \log p_i - (1 - p_i) \log (1 - p_i) = h_2(p_i) \tag{12}$$

and variance:

$$\begin{align}
\mathbb{V}Z_i &= p_i z_i^2 - \mathbb{E}Z_i \tag{13}\\
&= p_i (-\log p_i)^2 + (1 - p_i)(-\log 1 - p_i)^2 - \mathbb{E}Z_i \tag{14}\\
&= (1 - p_i) p_i (\log 1 - p_i - \log p_i)^2 \tag{15}\\
&= g(p_i) \tag{16}
\end{align}$$

This isn't so surprising as the sequence $\{X_i\}$ has a Poisson limit (Le Cam's theorem).

Then the WLLN implies:

$$\frac{1}{n} \sum_{i=1}^{n} Z_i \to \sum_{i=1}^{n} h_2(p_i) \tag{17}$$

This result can be refined a little, note that for the sequence $X_1 \dots X_n$, $\mathbb{E}X_i = p_i$ and that $\mathbb{V}X_i = p_i(1 - p_i)$, where $\mathbb{V}$ denotes the variance. Let S_n be defined as above, and

$$\mathbb{E}S_n = \frac{1}{n}(p_1 \dots p_n) = \bar{p} \tag{18}$$

$$\mathbb{V}S_n = \frac{1}{n^2}(\mathbb{V}X_1 \dots \mathbb{X}_{\ltimes}) = \frac{1}{n^2} \sum_{i=1}^{n} p_i (1 - p_i) \tag{19}$$

Note that $\mathbb{V}X_i = p_i(1 - p_i) \leq 0.25$ for all i, and that $\mathbb{V}S_n \leq 0.25n$. Using Chebyshev's inequality we have:

$$\mathbb{P}\left(|S_n - \bar{p}| \leq \epsilon\right) \leq \frac{\mathbb{V}S_n}{\epsilon^2} \leq \frac{1}{4n\epsilon^2} \tag{20}$$

where $\epsilon \in (0,1)$.

Using the sequence $\{Z_i\}$, and defining $Y_n = n^{-1}\log S_n$ s.t. $\mathbb{E}Y_n = \sum_{i=1}^{n} h_2(p_i)$ and $\mathbb{V}Y_n = n^{-2}\sum \mathbb{V}Z_i = n^{-2}\sum g(p_i)$,, the AEP can be stated as:

$$\mathbb{P}\left(|Y_n - \sum_{i=1}^{n} h_2(p_i)| \leq \epsilon\right) \leq \frac{\mathbb{V}Y_n}{\epsilon^2} \tag{21}$$

$$\leq \frac{1}{n^2}\frac{\sum_{i=1}^{n} g(p_i)}{\epsilon^2} \tag{22}$$

Theorem 2.1 (Bernstein)**.** *Let $\{X_i\}_{i=1}^{n}$ be independent rvs with zero mean and finite variance s.t. $|X_i| \leq M\forall i$, write $L := \sum_{j=1}^{N}$ then for $t \geq 0$:*

$$\mathbb{P}\left(\sum_{i=1}^{N} X_i \geq t\right) \leq \exp\left(-\frac{t^2}{4L}\right) \tag{23}$$

This using this inequality and equation (16) on the rvs $\{Z_i\}$:

$$\mathbb{P}\left(\left(\sum_{i=1}^{N} Z_i - H(p)\right) \geq t\right) \leq \exp\left(-\frac{t^2}{4\sum_{i=1}^{N} g(p_i)}\right) \tag{24}$$

This implies that instead of conditioning over equiprobable sets of size $\binom{n}{k}$, we can bound the success probability by:

$$\mathbb{P}(\mathbf{succ}) = \frac{2^T}{2^{\sum \theta(p_i)}} \tag{25}$$

with the conjecture that

$$\theta(\circ) = h_2(\circ) \tag{26}$$

Putting this together with the results from the previous section (after marginalising out all possible sets A_y) we get:

$$\mathbb{P}(\mathbf{succ}) \leq 2^{T - \sum \theta(p_i) + 2N\varepsilon} \tag{27}$$

Choosing $\varepsilon = 1/2N$ gives us:

$$\mathbb{P}(\mathbf{succ}) \leq 2^{T - H(\mathbf{p}) + 1} \tag{28}$$