Generating Functions

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1 The functions themselves

A generating of a sequence, a_n, is the function G_a defined by:

$$G_a(s) = \sum_{i=0}^{\infty} a_i s^i \tag{1.0.1}$$

The sequence may be reconstructed by setting $a_i = \frac{G_a^{(i)}(s)}{i!}$ where (i) is the ith derivative.

These functions are extremely useful when the a_i are probabilities.

Probability Generating Function (pgf): of a random variable X is defined as the generating function $G(s) := \mathbb{E}\left(s^X\right)$ of the probability mass function of X. ie.

Moment Generating Function (mgf): of a random variable X is defined as:

$$M_X(t) := G_X\left(e^t\right) = \mathbb{E}\left(e^{tX}\right) \tag{1.0.2}$$

i.e. the moment generating function is a change of variables.

$$G(s) = \mathbb{E}\left(s^X\right) = \sum_i s^i \mathbb{P}(X = i) = \sum_i s^i f(i) \tag{1.0.3}$$

1.1 Examples

1.1.1 Constant

$$\mathbb{P}(X = c) = 1 \ , \ G(s) = \mathbb{E}\left(s^X\right) = s^c \tag{1.1.4}$$

There is no moment generating function for this distribution as it has no moments!

1.1.2 Bernoulli

$$\mathbb{P}(X = 1) = p \ , \ \mathbb{P}(X = 0) = 1 - p \ , \ G(s) = (1 - p) + ps \tag{1.1.5}$$

$$M(t) = (1 - p) + pe^t \tag{1.1.6}$$

1.1.3 Binomial

The probability generating function is just the n-fold product of the pgf of the Bernoulli pgf, and the mgf is obtained from the change of variables. No proof is given here.

$$G(s) = [(1-p) + ps]^n \tag{1.1.7}$$

$$M(t) = \left[(1-p) + pe^t\right]^n \tag{1.1.8}$$

1.1.4 Geometric

$$\mathbb{P}(X = k) = p(1-p)^{k-1}\ ,\ k \geq 1 \tag{1.1.9}$$

$$G(s) = \sum_{k=1}^{\infty} s^k p(1-p)^{k-1} = sp\sum_{m=0}^{\infty}[s(1-p)]^m = \frac{sp}{1-s(1-p)} \tag{1.1.10}$$

$$M(t) = \frac{pe^t}{1-(1-p)e^t} \tag{1.1.11}$$

1.1.5 Poisson

$$G(s) = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(s\lambda)^k}{k!} = e^{-\lambda}e^{s\lambda} = e^{\lambda(s-1)} \tag{1.1.12}$$

$$M(t) = e^{\lambda\left(e^t-1\right)} \tag{1.1.13}$$

where the Taylor series expansion has been used to sum in the second stage.

1.1.6 Exponential

There is only the Moment generating function for this distribution as it is continuous.

$$M(t) = \frac{\lambda}{\lambda - t} \tag{1.1.14}$$

2 Some Problems

2.1 How to find an arbitrary generating function

1. Multiply the sequence/recurrence you have by s^k
2. Sum both sides
3. Identify a generating function
4. Solve for the generating function

2.1.1 Example

$$g_{i+1} = 2g_i + 1 \tag{2.1.15}$$

Multiplying and summing (steps 1 and 2):

$$\sum_{i=0}^{\infty} g_{i+1}x^i = 2\sum_{i=0}^{\infty} g_i x^i + \sum_{i=0}^{\infty} x^i \tag{2.1.16}$$

Write $\sum_{i=0}^{\infty} g_i x^i = G(x)$ and solve for G:

$$\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i=0}^{\infty} x^i \tag{2.1.17}$$

Simplifying and solving for G and then expanding the partial fractions yields:

$$G(x) = \frac{x}{(1-x)(1-2x)} = \sum_{i=0}^{\infty} \left(2^{i+1} - 1\right) x^{i+1} \tag{2.1.18}$$

so $g_i = 2^i - 1$

2.2 Examples from problem sheets

This example is taken from PS6 Q4 (which was also the first question from last year's exam). First you solve the detailed balance equations to find:

$$\pi_n = \frac{1}{-\log(1-\rho)} \frac{\rho^{n+1}}{n+1} \tag{2.2.19}$$

The generating function is:

$$G(z) = \sum_{i=0}^{\infty} \pi_n z^n = \frac{1}{-\log(1-\rho)} \sum_{i=0}^{\infty} \frac{(\rho z)^{n+1}}{n+1} \tag{2.2.20}$$

This sum can be done by using the Taylor expansion for the logarithm (see the 'Summing tricks' sheet). So:

$$G(z) = \frac{\log(1-\rho z)}{z\log(1-\rho)} \ , \ for|z| < 1/\rho \tag{2.2.21}$$

The mean queue length can be obtained from this by differentiating and evaluating the resulting function at $z = 1$.