The Proximity Operator

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1 The Proximity Operator

The Proximity Operator for a closed, convex, and proper function f is defined as:

$$\text{Prox}_f(x) := \arg\min_y f(y) + \frac{1}{2}||y-x||^2 \tag{1.0.1}$$

Intuitively the Proximity Operator approximates a point x by another point y, that is close in the mean-square sense under the penalty f.

The Prox $(\circ)$ operator exists for closed and convex f as $(y)+\frac{1}{2}||y-x||^2$ is closed with compact level sets and is unique as $(y)+\frac{1}{2}||y-x||^2$ is strictly convex.

2 Examples

2.1 Indicator

From the definition

$$\text{Prox}_I(x) := \arg\min_y I_C(y) + \frac{1}{2}||y-x||^2 \tag{2.1.2}$$

$$= \arg\min_{y\in C} \frac{1}{2}||y-x||^2 \tag{2.1.3}$$

$$= P_C(x) \tag{2.1.4}$$

where $I_C(y)$ is the indicator of some set C and P_C is the projection operator onto that set.

2.2 l_2 norm

For $f(y)=\frac{\mu}{2}||y||^2$ the Prox operator is:

$$\text{Prox}_f(x) := \arg\min_y \frac{\mu}{2}||y||^2 + \frac{1}{2}||y-x||^2 \tag{2.2.5}$$

$$= \frac{1}{1+\mu}x \tag{2.2.6}$$

2.3 l_1 norm

$f=\|x\|_1$

$$\text{Prox}_f(x) := \text{sign}(x_i)\left(|x_i|-\gamma\right)^+ = S_\gamma(x)_i \tag{2.3.7}$$

2.4 Elastic Net

Consider

$$f(x) = \lambda \|x\|_1 + \mu \, ||x|| \tag{2.4.8}$$

$$\mathrm{Prox}_f \, (x) := \frac{\lambda}{1+\mu} S_\gamma \, (x)_i \tag{2.4.9}$$

2.5 Fused Lasso

Consider

$$f(x) = \|x\|_1 + \sum_{i=1}^{d-1} (x_i - x_{i-1}) \tag{2.5.10}$$

i.e the sum of the l_1 and TV norms

$$\mathrm{Prox}_f \, (x) := \mathrm{Prox}_{l_1} \circ \mathrm{Prox}_{TV} = S_\gamma \, (\mathrm{Prox}_{TV})_i \tag{2.5.11}$$