Cramer Rao bound for CS

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1 Introduction

This document is a short derivation of theCramer Rao bound for the Compressive sensing.

2 Model and Derivation

We capture a deterministic sparse vector $x \in \mathbb{R}^n$ through a sensing matrix $A \in \mathbb{R}^{m\times n}$, giving us compressive measurements $y \in \mathbb{R}^m$.

$$y = Ax \tag{2.0.1}$$

We can model the signal with:

$$p\left(y \mid x,\sigma^2\right) = \left(2\pi\sigma^2\right)^{-K/2} \exp\left(-\frac{1}{2\sigma^2}\left|\left|y - Ax\right|\right|_2^2\right) \exp\left(-\lambda\left|\left|x\right|\right|_1\right) \tag{2.0.2}$$

Up to a constant we find that":

$$-\log p\left(y \mid x,\sigma^2\right) = \frac{1}{2\sigma^2}\left|\left|Ax - y\right|\right|_2^2 + \lambda\left|\left|x\right|\right|_1 \tag{2.0.3}$$

and we calculate that:

$$\frac{\partial^2}{\partial^2 x} - \log p\left(y \mid x,\sigma^2\right) = \frac{1}{\sigma^2}A^T A \tag{2.0.4}$$

Remark 2.1. *The second term in* (2) *doesn't add anything to this as the first derivative is either* λ*,* $-\lambda$ *or* 0*, depending on the sign of* x*.*

Taking the expectation of (2) we find that:

$$\mathbb{E}\frac{\partial^2}{\partial^2 x} - \log p\left(y \mid x,\sigma^2\right) = \frac{1}{\sigma^2 n} \tag{2.0.5}$$

where we have used:

Theorem 2.2 (Expected Value of Wishart Matrices)**.** *Given a matrix* $W \in \mathbb{R}^{r\times r}$

$$\mathbb{E}\left(W\right) = rI \tag{2.0.6}$$

So we find that the Cramer Rao bound is:

$$\mathbb{E}||\hat{x} - x|| \geq \frac{\sigma^2}{n} \tag{2.0.7}$$

where $\hat{x}$ is any unbiased estimator of x.