Applications of Analytic Continuation to PDE's, Topology, Theoretical Physics and More

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Contents

1 The aim of this paper

Analytic continuation is a cornerstone of complex analysis: the process of extending a holomorphic function beyond the region where it is initially defined. Classical examples include the extension of the geometric series $\sum_{n=0}^{\infty} z^n$ to the meromorphic function $\frac{1}{1-z}$, or the extension of the factorial function to the complex plane via the Gamma function. In each case, local data—a convergent series or integral representation—is extended into a global object, often with far richer structure.

The power of analytic continuation lies in this ability to propagate local information and uncover global properties. Although originally a tool of complex analysis, analytic continuation has found deep applications in many other branches of mathematics. In partial differential equations it allows boundary measurements on a subset of a domain to determine solutions everywhere; in theoretical physics it underpins Wick rotations that render oscillatory integrals tractable; in number theory it leads to the functional equation of the Riemann zeta function and the regularisation of divergent sums; and in topology and algebraic geometry it motivates the construction of Riemann surfaces to resolve multi-valuedness.

In this paper we explore these diverse applications. We begin with a simple example from PDEs, proceed to Wick rotations in physics, study the analytic continuation of the zeta function and its role in string theory, and conclude with the topological viewpoint through monodromy and Riemann surfaces. Our aim is to highlight analytic continuation not merely as a technique of complex analysis, but as a unifying idea across mathematics and physics.

Each application uses the same core principle: the Identity Theorem forces unique extension of analytic functions. This paper explores how this single idea unifies these diverse areas. Beyond its mathematical depth, analytic continuation is a unifying lens: it shows how problems that look unrelated—heat flow, quantum physics, divergent series, and topology—all share a common analytic backbone. This perspective not only simplifies technical arguments but also reveals surprising bridges between disciplines. In this sense, analytic continuation provides both a tool and a philosophy: local knowledge, when treated analytically, determines global truth.

2 Complex Analysis Revisited

In this section, our aim is to remind the reader of some of the main ideas from complex analysis that are necessary to read and understand the sections in this paper. We also provide diagrams and intuition to make the content of this section more accessible to those who find these ideas challenging. In this section and onward from this point, we assume the knowledge of someone who has studied first year university level courses in Calculus and Analysis as well as Topology.

Definition 2.1 (Holomorphic function). *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$. We say that f is* holomorphic *on U if it is complex differentiable at every point of U; that is, for each $z_0 \in U$ the limit*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists (as a complex limit).

Definition 2.2 (Domain). *A set $D \subset \mathbb{C}$ is called a* ***domain*** *if D is non-empty path connected and open.*

Remark 2.1. *Most of our functions we encounter are defined on domains. The fact a domain is* ***open*** *is crucial and is necessary for many of the theorems we have.*

Definition 2.3 (Open and Closed Balls). *Let $z_0 \in \mathbb{C}$ and $r > 0$.*

- *The* open ball *(or open disk) of radius r centered at z_0 is*

$$B(z_0, r) = \{\, z \in \mathbb{C} : |z - z_0| < r \,\}.$$

- *The* closed ball *(or closed disk) of radius r centered at z_0 is*

$$\overline{B}(z_0, r) = \{\, z \in \mathbb{C} : |z - z_0| \leq r \,\}.$$

Remark 2.2 (Intuition)**.** *Think of drawing a circle with a compass:*

- *The open ball is everything* ***inside*** *the circle, but not the boundary circle itself. It is like being inside a bubble, without being able to stand on the bubble's surface.*
- *The closed ball is everything inside* ***and including*** *the boundary circle. It is like having the bubble together with its thin soap film.*

Definition 2.4 (Isolated and Non-Isolated Points)**.** *Let $S \subseteq \mathbb{C}$ and $z_0 \in S$.*

- *The point z_0 is called an* isolated point *of S if there exists $\varepsilon > 0$ such that*

$$B(z_0, \varepsilon) \cap S = \{z_0\}.$$

In other words, z_0 is 'alone' in S within some neighborhood.

- *The point z_0 is called a* non-isolated point *(also called a* limit point *or* accumulation point*) of S if*

$$\forall \varepsilon > 0, \quad (B(z_0, \varepsilon) \setminus \{z_0\}) \cap S \neq \varnothing.$$

That is, every neighborhood of z_0 contains other points of S besides z_0 itself.

Remark 2.3 (Intuition)**.** *Think of points as* friends at a party*:*

- *An* ***isolated point*** *is someone standing alone in a corner with empty space around them. If you draw a small enough circle around them, nobody else is inside.*
- *A* ***non-isolated point*** *(limit point) is someone in the middle of a crowd. No matter how small a circle you draw around them, there will always be other people inside.*

Theorem 2.1 (Identity Theorem)**.** *Let $D \subseteq \mathbb{C}$ be a domain and let $f, g : D \to \mathbb{C}$ be holomorphic functions. If the set*

$$\{z \in D : f(z) = g(z)\}$$

has a limit point in D, then

$$f(z) \equiv g(z) \quad \text{for all } z \in D.$$

Remark 2.4. *A good way of thinking about this:*

Imagine drawing a curve using a very precise stencil. If two artists use the same stencil and we see that their curves line up on a 'dense' set of dots, then in fact the whole curves must coincide.

In contrast, for real functions without holomorphicity, this fails: for example,

$$f(x) = 0, \quad g(x) = \begin{cases} 0, & x \in \mathbb{Q}, \\ 1, & x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

agree on the dense set $\mathbb{Q}$ but are not equal elsewhere.

Definition 2.5 (Analytic Continuation)**.** *Let $f : D \to \mathbb{C}$ be holomorphic on a domain $D \subset \mathbb{C}$, and let D' be a domain with $D \subset D'$. A holomorphic function $F : D' \to \mathbb{C}$ is called an* ***analytic continuation*** *of f to Ω if $F|_D = f$.*
By the Identity Theorem, if such a continuation exists, it is unique.

3 Heat in a Satellite

For a simple example, consider a thin circular metal (unit) plate such as part of a satellite's sensor array. We want to see how temperature evolves across the plate due to some heat source.

Assume:

- The plate lies in the plane.
- We model the steady-state temperature $u(x, y)$, which satisfies Laplace's equation:

  $$\nabla^2 u = 0$$

  in a 2D domain.
- Our solution to Laplace's equation is real and analytic.
- We don't know the boundary conditions around the whole plate (some parts are inaccessible).

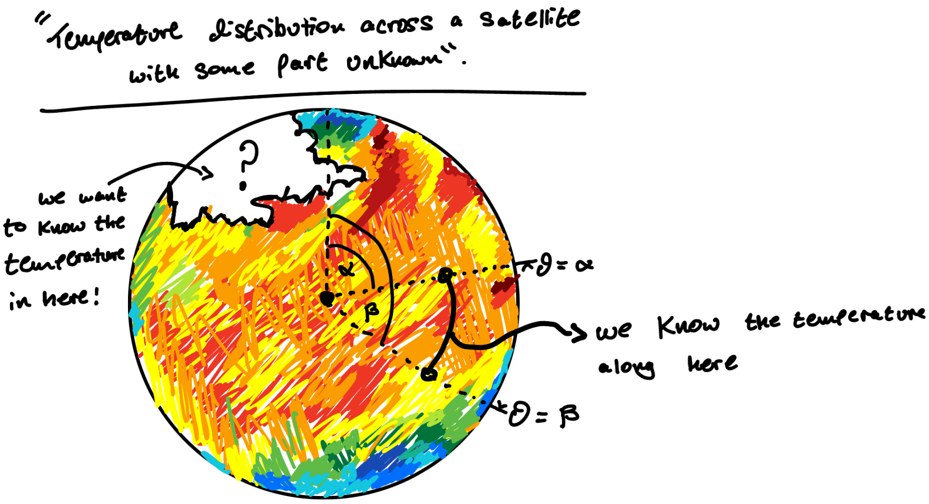
Then imagine:

- We can measure the temperature along some arc inside the disc, say for angles $\theta \in (\alpha, \beta)$ on the unit circle $\{z \in \mathbb{C} : |z| = r\}$ where $r < 1$.
- These measurements are precise, that is: $u(re^{i\theta})$ is known with certainty for $\theta \in (\alpha, \beta)$.

We can now use the numerical data on the arc of the disc to fit a candidate analytic function — for example, a Taylor or Fourier series — that matches the measured values of u along that arc. Because solutions of Laplace's equation are real-analytic, any such function can, in principle, be extended smoothly across the whole disc.

This gives us a way to construct an analytic continuation of the measured data: starting from reliable values on a part of the domain, we can extend them to obtain a consistent steady-state solution across the plate. In practice, this continuation is delicate - small errors in the data can grow when extended — but mathematically it shows that knowledge of the solution on part of the region can constrain its behavior elsewhere.

Returning to our satellite example: If a rock in space damages one of the temperature sensors, we might lose direct readings at that location. However, if we can still measure the temperature along part of the disc (our circular arc) with certainty, then by analytic continuation, we can extend these values into the rest of the disc. This way, we can still recover information about the steady-state temperature profile of the satellite plate, even when only partial measurements are available.



4 Minkowski Space, Wick rotations and Oscillatory Integrals

4.1 Distances in some useful spaces:

Minkowski Space in $\mathbb{R}^4$:
Distance is:

$$S^2 = -t^2 + x^2 + y^2 + z^2,$$

where t is time and (x, y, z) are our usual coordinates in $\mathbb{R}^3$.

Euclidean space in $\mathbb{R}^4$:
Distance is:

$$r^2 = \tau^2 + x^2 + y^2 + z^2,$$

where τ is time and (x, y, z) are our usual coordinates in $\mathbb{R}^3$.

4.2 Wick Rotation

Definition:

$$t \quad \rightarrow \quad i\tau$$

Effect on Geometry:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \quad \longrightarrow \quad dr^2 = d\tau^2 + dx^2 + dy^2 + dz^2$$

$$\text{(Minkowski)} \quad \longrightarrow \quad \text{(Euclidean)}$$

4.3 Effect on Oscillations

$$e^{i\omega t} \quad \longrightarrow \quad e^{-\omega\tau}$$

This is useful for integrals because it takes highly oscillatory exponential functions to easy to deal with rapidly decaying functions.

4.4 Evaluating Integrals

We evaluate the real-time oscillatory integral:

$$I(t) = \int_{-\infty}^{\infty} e^{\frac{i}{2}\, tx^2}\, dx.$$

For real $t > 0$, $I(t)$ is highly oscillatory and only converges in the distributional sense. Physicists make sense of it by doing a Wick rotation.

Set $t := i\tau$ with $\tau > 0$. Then

$$I(t) = I(i\tau) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}\, \tau x^2}\, dx.$$

Now the exponent is real and negative; this is a Gaussian integral we can compute!

We know that:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\, \tau x^2}\, dx = \sqrt{\frac{2\pi}{\tau}}.$$

Hence,

$$I(i\tau) = \sqrt{\frac{2\pi}{\tau}} \qquad \Rightarrow \qquad F(t) = \sqrt{\frac{2\pi}{i\, t}}\,.$$

Where F is the analytic continuation (closed form) of I. So a Wick rotation has given us an actual value to a not always convergent, oscillatory integral. But where is it valid?

4.5 Where have we used analytic continuation?

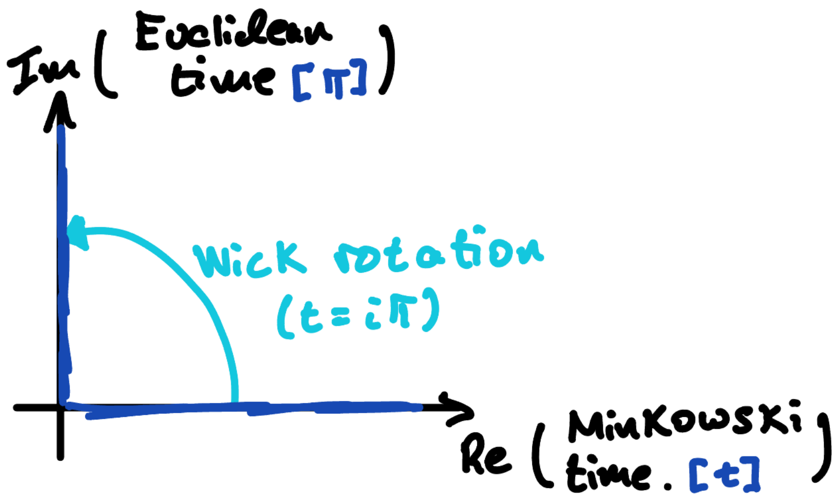
- One can easily check that $I(t)$ is holomorphic on the upper half plane.
- Clearly $F(t)$ is holomorphic on $\mathbb{C} \setminus \{0\}$.
- $I(t) = F(t)$ on the positive imaginary axis.
- The set

  $$\{\xi \in \mathbb{C} \mid I(\xi) = F(\xi)\} = \{\xi \in \mathbb{C} \mid \Re(\xi) = 0,\ \Im(\xi) > 0\}$$

  contains a non-isolated point.

Hence by the Identity Theorem, $I(t) = F(t)$ on the upper half plane.

So, to recap what we have done. When dealing with a problem in real (Minkowski) time, we can 'move' from one space to another by applying a Wick transformation to our time. Once in imaginary (Euclidean) time, certain problems (integrals in our case) become much easier to deal with. Then, by using analytic continuation, our solution in Euclidean space can be 'matched up' to a solution in our original Minkowski space that is valid in some region (the upper half plane in our case).



5 Infinite sums, The Riemann Zeta Function and String Theory

5.1 We begin by "summing" some divergent series - an informal treatment

$$S_1 = 1 - 1 + 1 - 1 + \cdots \ = \ \tfrac{1}{2} \quad \text{(Abel sum of an alternating series)}$$

There are many proofs of the above result that can be found online, we will not go into more detail on this here as it is not necessary. We now consider:

$$S_2 = 1 - 2 + 3 - 4 + \cdots$$

Shifting S_2 one term to the right:

$$\begin{aligned} 2S_2 &= (1 - 2 + 3 - 4 + \cdots) \\ &\quad + (0 + 1 - 2 + 3 - 4 + \cdots) \\ &= 1 - 1 + 1 - 1 + \cdots \\ &= S_1 \\ &= \tfrac{1}{2} \end{aligned}$$

Hence:

$$S_2 = \tfrac{1}{4}.$$

Finally we consider:

$$S_3 = 1 + 2 + 3 + 4 + \cdots.$$

Then

$$S_3 - S_2 = (1 + 2 + 3 + 4 + \cdots) \ - \ (1 - 2 + 3 - 4 + \cdots) = 4 + 8 + 12 + \cdots = 4\,S_3,$$

So: $-3S_3 = S_2$ and thus:

$$S_3 = -\tfrac{1}{12}, \quad \text{hence} \quad \sum_{n=1}^{\infty} n = -\tfrac{1}{12}.$$

5.2 A more rigorous approach:

Firstly, some standard definitions and formulae:

The Riemann Zeta Function:

For $\Re(s) > 1$ we define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \;=\; \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx,$$

where

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x}\, dx \text{ is the Gamma function.}$$

Fourier Transform pairs:

$$\widehat{g}(t) = \int_{-\infty}^{\infty} g(x) e^{-2\pi i t x}\, dx, \qquad g(x) = \int_{-\infty}^{\infty} \widehat{g}(t) e^{2\pi i t x}\, dt.$$

Then g and $\widehat{g}$ are called a Fourier-transform pair.

Poisson Summation Formula:

If g and $\widehat{g}$ are Fourier Transform pairs, then:

$$\sum_{n=-\infty}^{\infty} g(n) = \sum_{k=-\infty}^{\infty} \widehat{g}(k).$$

Euler's Reflection formula:

$$\Gamma(z)\,\Gamma(1-z) \;=\; \frac{\pi}{\sin(\pi z)} \qquad \forall\, z \in \mathbb{C} \setminus \mathbb{Z}$$

Duplication formula:

$$\Gamma(z)\,\Gamma\Big(z + \tfrac{1}{2}\Big) = 2^{\,1-2z}\, \sqrt{\pi}\, \Gamma(2z).$$

The Functional Equation: The Riemann zeta-function $\zeta(s)$, originally defined for $\Re(s) > 1$, can be analytically continued to $\mathbb{C} \setminus \{1\}$.

It's analytic continuation satisfies the functional equation:

$$\boxed{\zeta(s) = 2^s\, \pi^{\,s-1}\, \sin\!\big(\tfrac{\pi s}{2}\big)\, \Gamma(1-s)\, \zeta(1-s).}$$

I haven't come across a proof of this functional equation in enough detail such that one can follow along and actually understand every step, so I provide one here:

Proof. Firstly, for $\Re(s) > 0$ and $n \geq 1$,

$$\int_0^{\infty} x^{\frac{s}{2}-1} e^{-n^2 \pi x}\, dx = \frac{\Gamma(\frac{s}{2})}{n^s\, \pi^{\frac{s}{2}}}.$$

We can see this by using the substitution $t = n^2\pi x$, so $x = t/(n^2\pi)$ and $dx = dt/(n^2\pi)$. Then:

$$\int_0^\infty x^{\frac{s}{2}-1} e^{-n^2\pi x}\, dx = \int_0^\infty \left(\tfrac{t}{n^2\pi}\right)^{\frac{s}{2}-1} e^{-t}\, \frac{dt}{n^2\pi} = \frac{1}{(n^2\pi)^{\frac{s}{2}}} \int_0^\infty t^{\frac{s}{2}-1} e^{-t}\, dt = \frac{\Gamma(\frac{s}{2})}{n^s\, \pi^{\frac{s}{2}}}.$$

For convenience, we define the Theta-Series:

$$\psi(x) = \sum_{n=1}^\infty e^{-n^2\pi x}.$$

Then for $\Re(s) > 1$, by uniform convergence:

$$\frac{\Gamma(\frac{s}{2})\, \zeta(s)}{\pi^{\frac{s}{2}}} = \sum_{n=1}^\infty \int_0^\infty x^{\frac{s}{2}-1} e^{-n^2\pi x}\, dx = \int_0^\infty x^{\frac{s}{2}-1}\, \psi(x)\, dx.$$

By the Poisson summation formula:

$$\sum_{n=-\infty}^\infty e^{-n^2\pi x} = \frac{1}{\sqrt{x}} \sum_{n=-\infty}^\infty e^{-n^2\pi/x},$$

So:

$$2\,\psi(x) + 1 = \frac{1}{\sqrt{x}}\big(2\,\psi(1/x) + 1\big).$$

Split

$$\int_0^\infty x^{\frac{s}{2}-1}\psi(x)\, dx = \int_1^\infty x^{\frac{s}{2}-1}\psi(x)\, dx + \int_0^1 x^{\frac{s}{2}-1}\psi(x)\, dx.$$

In the second integral substitute the identity for $2\psi(x) + 1$ to obtain:

$$\int_0^1 x^{\frac{s}{2}-1}\psi(x)\, dx = \frac{1}{2}\int_0^1 x^{\frac{s}{2}-1}\Big(\tfrac{1}{\sqrt{x}}(2\psi(1/x) + 1) - 1\Big)\, dx.$$

$$\begin{aligned}
\int_0^1 x^{\frac{s}{2}-1}\, \psi(x)\, dx &= \frac{1}{2}\int_0^1 x^{\frac{s}{2}-1}\Big(x^{-1/2}\big(2\psi(1/x) + 1\big) - 1\Big)\, dx \\
&= \frac{1}{2}\int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\, \big(2\psi(1/x) + 1\big)\, dx \;-\; \frac{1}{2}\int_0^1 x^{\frac{s}{2}-1}\, dx \\
&= \int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\, \psi(1/x)\, dx \;+\; \frac{1}{2}\int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\, dx \;-\; \frac{1}{2}\int_0^1 x^{\frac{s}{2}-1}\, dx \\
&= \int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\, \psi(1/x)\, dx + \frac{1}{2}\left[\frac{x^{\frac{s}{2}-\frac{1}{2}}}{\frac{s}{2}-\frac{1}{2}}\right]_0^1 - \frac{1}{2}\left[\frac{x^{\frac{s}{2}}}{\frac{s}{2}}\right]_0^1 \\
&= \int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\, \psi(1/x)\, dx \;+\; \frac{1}{s-1} \;-\; \frac{1}{s} \\
&= \int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\, \psi(1/x)\, dx \;+\; \frac{1}{s(s-1)}.
\end{aligned}$$

Now make the substitution $x = 1/u$, so $dx = -u^{-2}du$. As x goes from 0 to 1, u goes from ∞ to 1. Hence:

$$
\begin{aligned}
\int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\,\psi(1/x)\,dx &= \int_\infty^1 \left(\tfrac{1}{u}\right)^{\frac{s}{2}-\frac{3}{2}}\,\psi(u)\,\left(-u^{-2}du\right)\\
&= \int_1^\infty u^{-\frac{s}{2}+\frac{3}{2}-2}\,\psi(u)\,du\\
&= \int_1^\infty u^{-\frac{s}{2}-\frac{1}{2}}\,\psi(u)\,du.
\end{aligned}
$$

Renaming $u \to x$ gives the final form:

$$
\int_0^1 x^{\frac{s}{2}-1}\,\psi(x)\,dx = \frac{1}{s(s-1)} + \int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}}\,\psi(x)\,dx.
$$

Putting both pieces together yields the formula:

$$
\frac{\Gamma(\frac{s}{2})\,\zeta(s)}{\pi^{\frac{s}{2}}} = \frac{1}{s(s-1)} + \int_1^\infty \left(x^{\frac{s}{2}-\frac{1}{2}} + x^{-\frac{s}{2}-\frac{1}{2}}\right)\psi(x)\,dx,
$$

We notice that the right hand side is invariant under $s \mapsto 1-s$. Hence, we have that:

$$
\frac{\Gamma(\frac{s}{2})\,\zeta(s)}{\pi^{\frac{s}{2}}} = \frac{\Gamma(\frac{1-s}{2})\,\zeta(1-s)}{\pi^{\frac{1-s}{2}}}.
$$

$$
\begin{aligned}
&\frac{\Gamma\left(\frac{s}{2}\right)\,\zeta(s)}{\pi^{s/2}} = \frac{\Gamma\left(\frac{1-s}{2}\right)\,\zeta(1-s)}{\pi^{(1-s)/2}},\\
\implies\quad &\Gamma\left(\tfrac{s}{2}\right)\,\zeta(s) = \Gamma\left(\tfrac{1-s}{2}\right)\,\zeta(1-s)\,\pi^{\,s-\frac{1}{2}},
\end{aligned}
$$

Multiply both sides by $\Gamma\left(1-\frac{s}{2}\right)$:

$$
\Gamma\left(\tfrac{s}{2}\right)\Gamma\left(1-\tfrac{s}{2}\right)\zeta(s) = \Gamma\left(\tfrac{1-s}{2}\right)\Gamma\left(1-\tfrac{s}{2}\right)\zeta(1-s)\,\pi^{\,s-\frac{1}{2}},
$$

Apply Euler's reflection formula $\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ with $z = \frac{s}{2}$:

$$
\begin{aligned}
&\frac{\pi}{\sin\left(\frac{\pi s}{2}\right)}\,\zeta(s) = \Gamma\left(\tfrac{1-s}{2}\right)\Gamma\left(1-\tfrac{s}{2}\right)\zeta(1-s)\,\pi^{\,s-\frac{1}{2}},\\
\implies\quad &\zeta(s) = \Gamma\left(\tfrac{1-s}{2}\right)\Gamma\left(1-\tfrac{s}{2}\right)\zeta(1-s)\,\sin\left(\tfrac{\pi s}{2}\right)\,\pi^{\,s-\frac{3}{2}},
\end{aligned}
$$

Then use the duplication formula $\Gamma(z)\,\Gamma(z+\frac{1}{2}) = 2^{\,1-2z}\sqrt{\pi}\,\Gamma(2z)$ with $z = \frac{1}{2}-\frac{s}{2}$:

$$
\begin{aligned}
\Gamma\left(\tfrac{1-s}{2}\right)\Gamma\left(1-\tfrac{s}{2}\right) &= 2^s\,\sqrt{\pi}\,\Gamma(1-s),\\
\therefore\quad \zeta(s) &= 2^s\,\pi^{\,s-1}\,\sin\left(\tfrac{\pi s}{2}\right)\Gamma(1-s)\,\zeta(1-s).
\end{aligned}
$$

□

Where have we used analytic continuation here?

Earlier in the proof we saw:

$$
\frac{\Gamma\left(\frac{s}{2}\right)\zeta(s)}{\pi^{\,s/2}} = \frac{1}{s(s-1)} + \int_1^\infty \left(x^{\frac{s}{2}-\frac{1}{2}} + x^{-\frac{s}{2}-\frac{1}{2}}\right)\psi(x)\,dx.
$$

By construction, the left–hand side is holomorphic for $\Re(s) > 1$, while the right–hand side converges (absolutely) and hence defines a holomorphic function on $\mathbb{C} \setminus \{0, 1\}$. Since these two expressions agree on $\{\Re(s) > 1\}$, and this set contains a non isolated point, the Identity Theorem forces them to coincide on the entire connected domain $\mathbb{C} \setminus \{0, 1\}$. This extension of from $\Re(s) > 1$ to all of $\mathbb{C} \setminus \{0, 1\}$ is precisely the analytic continuation of $\zeta(s)$. We can in fact use certain arguments relating to poles and zeros of the Zeta and Gamma function to further extend the Riemann Zeta function to a holomorphic function on $\mathbb{C} \setminus \{1\}$. However, we omit this as it isn't necessary for our purposes.

How this links with our Infinite Sum:

Since we have analytically continued $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s)$ to $\mathbb{C} \setminus \{0, 1\}$, and shown the functional equation:

$$\zeta(s) = 2^s \, \pi^{s-1} \, \sin\!\left(\tfrac{\pi s}{2}\right) \Gamma(1-s) \, \zeta(1-s),$$

We may substitute $s = -1$, this gives:

$$\begin{aligned}
\sum_{n=1}^{\infty} n = \sum_{n=1}^{\infty} \frac{1}{n^{-1}} &= \zeta(-1) = 2^{-1} \, \pi^{-2} \, \sin\!\left(-\tfrac{\pi}{2}\right) \Gamma(2) \, \zeta(2) \\
&= 2^{-1} \, \pi^{-2} \, (-1) \, \Gamma(2) \, \zeta(2) = -\frac{1}{2\pi^2} \, \Gamma(2) \, \zeta(2).
\end{aligned}$$

We compute:

$$\Gamma(2) = \int_0^{\infty} x \, e^{-x} \, dx = \big[-x \, e^{-x}\big]_0^{\infty} + \int_0^{\infty} e^{-x} \, dx = 1,$$

And, we of course have the famous result: $\zeta(2) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Putting this all together gives:

$$\sum_{n=1}^{\infty} n = -\frac{1}{2\pi^2} \, (1) \, \frac{\pi^2}{6} = -\frac{1}{12}.$$

This is the result we obtained earlier, but done rigorously. Although this seems somewhat like wizardry, and clearly the sum of the positive integers diverges, this result does show up in theoretical physics.

5.3 An application to Bosonic String Theory:

Bosonic String Theory is the earliest version of string theory. Although it is not realistic (it contains tachyons, has no fermions, and requires 26 spacetime dimensions), it remains extremely useful as a toy model for understanding the general structure of superstring theory.

In Bosonic String Theory, the basic things in the universe aren't points, they're tiny strings. These strings can vibrate in many different ways. Each vibration corresponds to a different particle (like notes on a guitar string correspond to different sounds).

These different vibrations are called modes of vibration: first mode, second mode, third mode and so on...

A quantized string has an infinite tower of normal modes. Each mode behaves like an independent quantum harmonic oscillator with energy levels

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega, \qquad n = 0, 1, 2, \ldots \tag{1}$$

so even in the ground state there is a zero–point contribution $\frac{1}{2}\hbar\omega$.
Summing the zero–point energies of *all* modes gives the (regularized) vacuum energy of the string, commonly called its Casimir energy:

$$E_{\text{Casimir}} = \sum_{\text{modes}} \frac{1}{2}\hbar\omega_{\text{mode}}. \tag{2}$$

For equally spaced frequencies (after the standard rescalings for a free bosonic string), we have that:

$$E_{\text{Casimir}} \;\propto\; \frac{1}{2}\sum_{n=1}^{\infty} n, \tag{3}$$

which is formally divergent.
To make sense of this sum we use our analytically continued Zeta-function, from before we saw:

$$\sum_{n=1}^{\infty} n \;\equiv\; \zeta(-1) \;=\; -\frac{1}{12},$$

This allows us to obtain the Casimir energy:

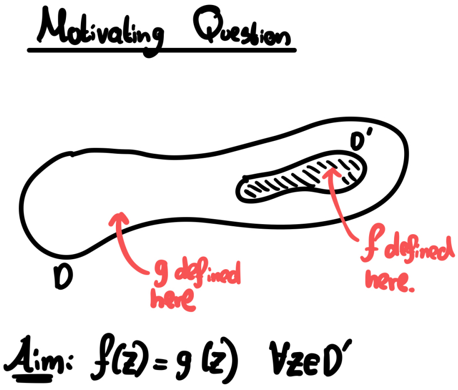
$$E_{\text{Casimir}} \;\propto\; \frac{1}{2}\left(-\frac{1}{12}\right) \;=\; -\frac{1}{24}. \tag{4}$$

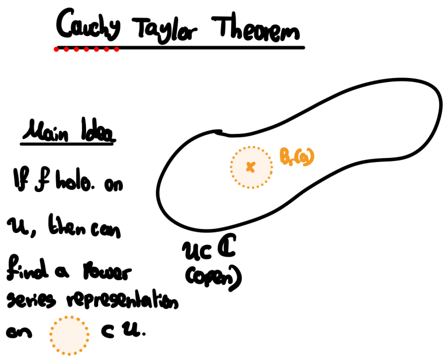
This negative energy plays a crucial role in making bosonic string theory consistent (for example, it explains why the theory only works in 26 dimensions). In simple terms, the Casimir energy is just the total energy from the zero–point vibrations of all the different modes of the string, added up in a way that gives a finite, meaningful result.

6 Analytic Continuation along Paths

6.1 Introduction - Basic Definitions

We begin by recalling the hopefully familiar notion of Analytic Continuation. To begin with we ask the following question:
Suppose we have two domains D and D' with $D' \subset D$ then consider a ***holomorphic*** *function $f : D' \to \mathbb{C}$, can we find another holomorphic function $g : D \to \mathbb{C}$ such that $f(z) = g(z)$ $\forall z \in D'$?*
Yes we can! Such a function is called the **Analytic Continuation of** f which we will explore more here. We will consider the notion of such a continuation **along paths** and what we mean by this. We will then see that analytic continuations may depend on the choice of path, we will explore when this dependence exists. Further we will see the importance of singularities in this realm, and look to develop our initial understanding of Riemann Surfaces. To do this, we need some basic definitions and ideas which we will develop both formally and also intuitively through diagrams - these offer particularly useful insight in Complex Analysis.





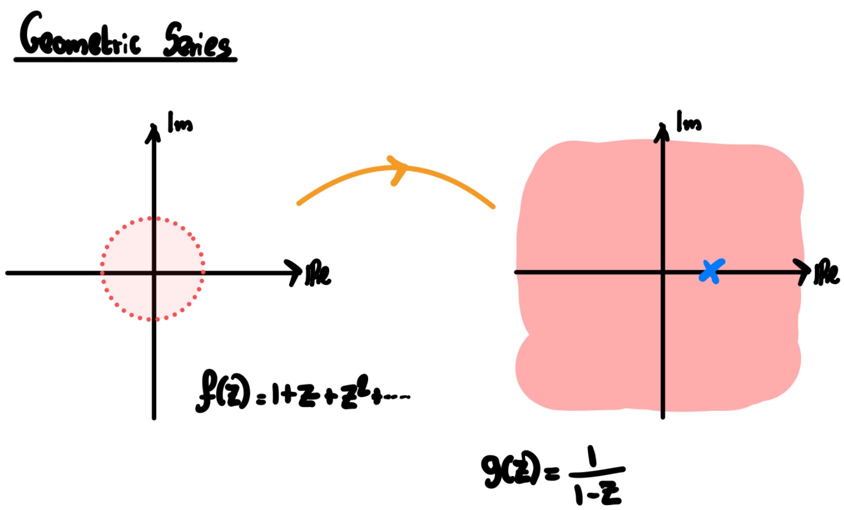
We firstly recall the following very important theorem:

Theorem 6.1 (Cauchy-Taylor Theorem)**.** *Let $f : U \to \mathbb{C}$ be holomorphic on some open set U. Then for any $r > 0$, such that $B_r(a) \subset U$, f is given by a power series that converges on $B_r(a)$.*

Proof. See Complex Analysis II. □

This theorem will be very helpful and we now consider a hopefully familiar example.

Example 6.1 (Geometric Series)**.** *We consider the function: $f(z) = \sum_{n=0}^{\infty} z^n$ which is convergent only inside the unit disc (its radius of convergence is one). We observe the function $g(z) = \frac{1}{1-z}$ is exactly matching this on the unit disc with a singularity. Indeed our idea of stitching together local power series is applicable here - we start a chain of continuations within the disc and then continue to a disc with $|z| > 1$ with the discs overlapping and continue from there to any point in $\mathbb{C} \setminus \{1\}$.*



We should be comfortable with the notion of a Complex Logarithm. An issue which arises is this has a multi-valued nature, which is a problem if we want the Complex Logarithm to be a well defined function. We hence define:

Definition 6.1 (Branch of a Function, Branch Cut)**.** *A **Branch** of a multi-valued function f is a single-valued function g analytic in some domain D such that for any point in this domain $z \in D$ we have $g(z)$ being **one** of the values of f. A **Branch Cut** is a portion of a line/curve used to make a single valued branch of a multi-valued function.*

Recalling that we have $log(z) = log(|z|) + i(\theta + 2k\pi)$ we see how we can make this single valued - hence resolving our problem. Write $\theta' = \theta + 2k\pi$. Restricting $\theta' \in (\alpha, \alpha + 2k\pi]$ we can ensure we now have a single valued function. The choice of α is a freedom we have, and there is a common such choice we have, giving our notion of the **Principal Branch of Log**

Definition 6.2. *By taking the branch cut at $\theta' = -\pi$ we get the Principal Branch of Log:*

$$Log(z) = log(|z|) + i\theta', \theta \in (-\pi, \pi]$$

One final, hopefully familiar definition we will use:

Definition 6.3. *(Analytic Function) We say a function $f(z)$ is Analytic if it is locally given by a Convergent Power Series. Note every analytic function is holomorphic (think Cauchy Taylor). The converse is also true though this is less obvious.*

6.2 Analytic Continuation Along Paths

Our goal in this section is to define the notion of Analytic Continuation **along** a path. Before I formally define this let's develop some intuition.

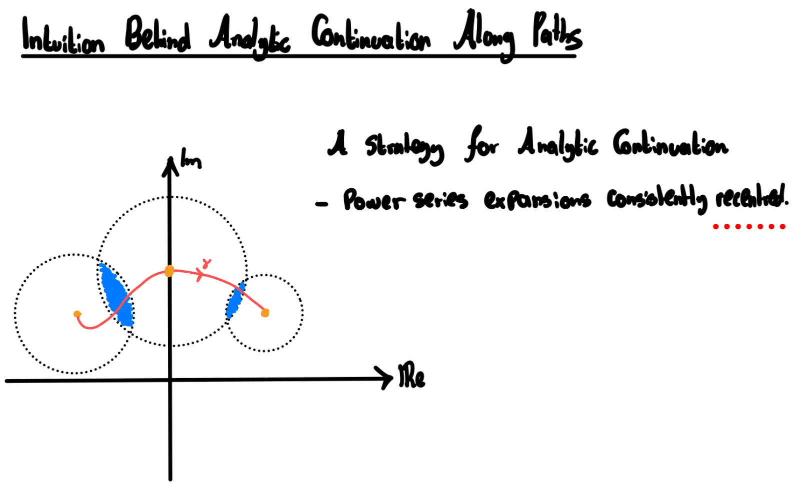
Analytic Continuation Along a Path

Given a holomorphic function at $z_0 \in \mathbb{C}$ with a convergent power series and disc of convergence, we can *recentre* around a new point within this disc. When discs overlap, function values in the intersection uniquely determine the new power series coefficients.

Continuing this process along a sequence of points forming a path, with successive overlapping discs, we reach endpoint z_1. Each re-centering step is forced by agreement in overlapping regions, making continuation well-defined along the specific path.

However, continuation is fundamentally *path-dependent*: different paths may yield different values at z_1, or fail if encountering singularities. Overlapping discs create a "chain of validity" transporting analytic structure along the route, but changing routes can change destinations.

This process breaks down at natural boundaries or certain singularities, which makes a careful choice of route essential.



Perhaps this feels a little imprecise. We now define this notion formally.

Definition 6.4 (Analytic Continuation Along a Path)**.** *Suppose $\gamma : [0, 1] \to \mathbb{C}$ is a C^1 curve, and f is an Analytic function on an open disc D centred at $\gamma(0)$. The Analytic Continuation of (f, U)* ***along*** *γ is a collection of pairs (f_t, D_t) such that:*

- ▷ ***Initial Conditions:*** *The initial function and disc of convergence agree — that is, our collection starts with the same disc of convergence and function:*

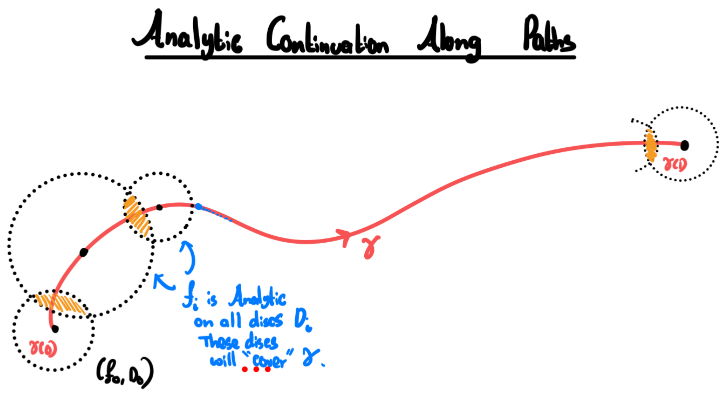
$$f_0 = f \quad \textit{and} \quad D_0 = D$$

- ▷ ***Regularity:*** *At every point on the curve, we maintain analytic structure within open discs:*

$$\forall\, t \in [0,1]: \quad f_t \textit{ is analytic on } D_t \textit{ and } D_t \textit{ is an open disc}$$

- ▷ ***Continuity Property:*** *Consecutive discs of convergence have intersections on which their functions agree.*

> *Formal Statement: For all $t \in [0,1]$, there exists $\varepsilon > 0$ such that for all $t' \in [0,1]$ with $|t - t'| < \varepsilon$:*
>
> $$\gamma(t') \in D_t \quad \textit{and} \quad f_t = f_{t'} \textit{ on } D_t \cap D_{t'}$$



6.3 Path Choice Matters

We will now see how our choice of path matters when considering analytic continuation along a path via two examples.

6.3.1 Case Study I - $log(z)$

Consider $log(z)$ defined at the initial point $z_0 = 1$ we define this via a power series, convergent on the right hand side of our Complex Plane $\mathbb{R}e \geq 0$ - so this defines a holomorphic function $log(z) = ln(|z|) + i\theta'$ with branch cut $\theta' \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, and we have the local power series expansion about $z_0 = 1$:

$$log(z) = (z-1) - \frac{(z-1)^2}{2} - .....$$

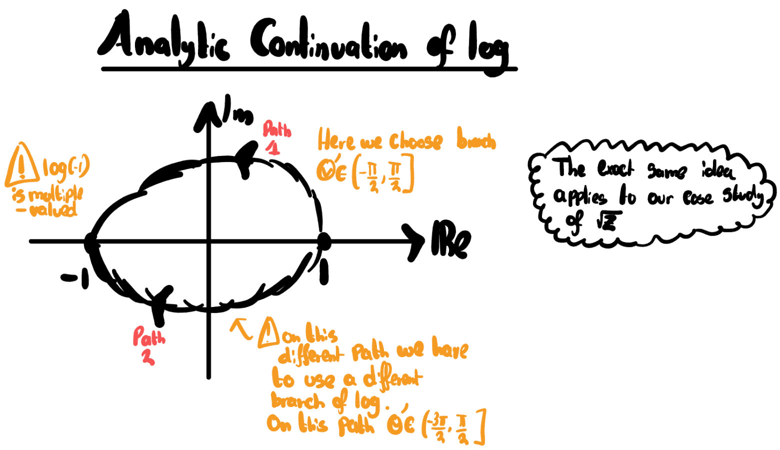
this our function f_0 convergent on $D_0 = \{z \in \mathbb{C} : |z-1| < 1\}$ Now using our previous strategy for Analytic Continuation, we stitch together discs of convergence along a semi circular path γ_1 in the upper half plane. In order for our overlapping discs to agree on the intersection we take branch cut $\theta' \in (-\frac{\pi}{2}, \frac{3\pi}{2})$ and hence see:

$$\boxed{\text{log(-1)=i}\pi}$$

Now consider the lower semi-circular path γ_2 we see that now because log has a different branch cut to be defined at $z = -1$ namely $\theta' \in (-\frac{3\pi}{2}, \frac{\pi}{2})$:

$$log(-1) = -i\pi$$

Both are valid Analytic Continuations along two different paths (satisfying our previous definition) but give different results - our continuations are local and NOT global like our nice example of a geometric series.



So what is going on? We will soon see this is due to singularities in our domain, and explore when our Continuation is global, all of this will be explained by the **Monodromy Theorem**.

6.3.2 Case Study II - $\sqrt{z}$

Recall our definition for Complex Powers, we have $w \in \mathbb{C}$ fixed and then define:

$$z^w = exp(wlogz)$$

Initially note that the logarithm here, as discussed earlier is dependent on our choice of branch and thus the according to our definition the complex power is. So we consider now $\sqrt{z}$:

$$\sqrt{z} = exp\left(\frac{1}{2}logz\right) = exp\left(\frac{1}{2}log|z|\right) exp\left(\frac{i\theta'}{2}\right) = |z|^{\frac{1}{2}} e^{\frac{i\theta'}{2}}$$

Again, θ' here is as defined for log - in a given interval such as $\theta' \in [-\pi, \pi)$. We now consider a similar setup as before for log - with two different paths and an analytic continuation to -1. Again, we see our issue arises again, with our square root taking two different values - differing by a factor of -1.

Why does topology enter here? The examples of $\log(z)$ and $\sqrt{z}$ highlight a new phenomenon: analytic continuation can depend on the *path* we choose. But why should the path matter at all? The answer is not purely analytic: it is *topological*.

When we continue along different paths, the obstruction arises because the domain of definition is not simply connected. In the case of $\log(z)$ and $\sqrt{z}$, the singularity at $z = 0$ creates a "hole" in the complex plane. Loops encircling this hole cannot be contracted to a point without crossing the singularity, and this topological obstruction manifests as multi-valuedness of the function.

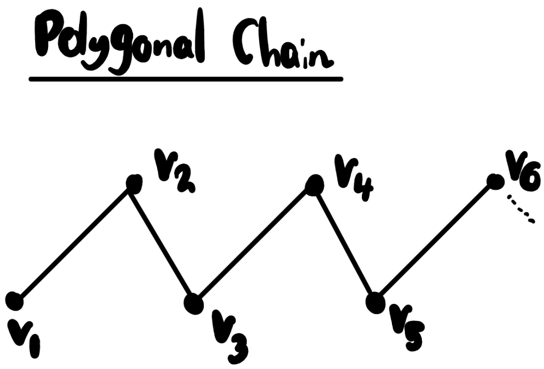
Thus, to decide whether analytic continuation along paths yields a single-valued global function, we must look beyond analysis and examine the topology of the domain. This motivates the introduction of the *Monodromy Theorem*, which precisely connects these topological properties (such as simple connectedness) with the analytic behavior of continuation.

6.4 The Monodromy Theorem

We define two important notions:

Definition 6.5 (Simply Connected)**.** *A Topological Space X (see Topology II)is Simply Connected if it is Path Connected and any path between two points can be continuously deformed into any other such part while preserving the two endpoints. Intuitively this means there are no holes or loops.*

Definition 6.6 (Polygonal Chain)**.** *A Polygonal Chain is a connected Series of Line Segments.*

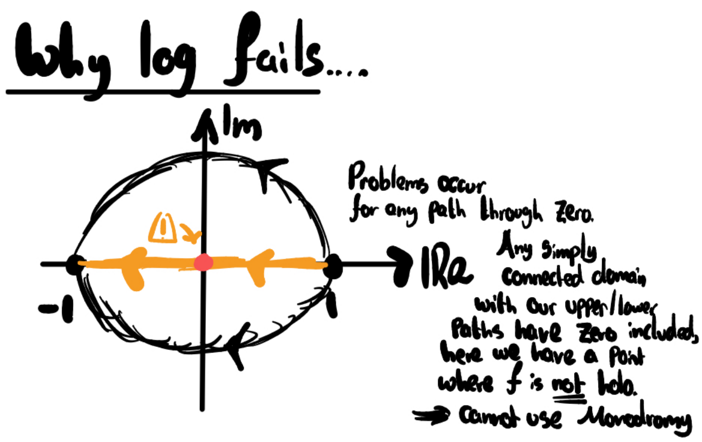


So now we turn to the Monodromy Theorem. The motivation behind this, is to understand when an Analytic Continuation is/isn't Path Independent.

Theorem 6.2 (The Monodromy Theorem)**.** *If a Complex function f is* ***holomorphic*** *in a* ***Simply Connected Domain*** *$D \subset \mathbb{C}$ and f can be analytically continued along every* ***polygonal arc*** *in D. Then f can be Analytically Continued to a single valued Analytic Function on all of D.*

Proof. See Complex Analysis I (Bishop) □

The Monodromy theorem (Theorem 6.2) explains why our log(z) example behaved differently on different paths: $\mathbb{C} \setminus \{0\}$ is not simply connected (Definition 6.5), so the theorem's hypotheses fail. Notice that this path dependence occurs precisely because we encounter the singularity at z = 0 when going from z = 1 to z = -1. This connects to our discussion of the Identity Theorem: analytic continuation is unique only within domains where the function remains holomorphic.



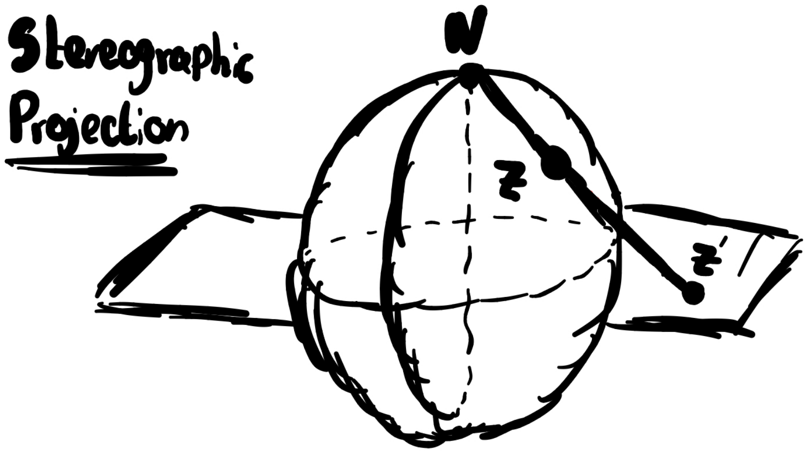
We have now solved our issue with Path Dependence by at least understanding **when** these issues occur. It should be clear now, a large source of our problems is the need to specify branches and not having a natural domain where these functions are holomorphic. This motivates our next section - the study of **Riemann Surfaces** a way of dealing with the issues of Multi-valuedness and the other problems we have faced so far. We will study what these are Topologically, consider constructions of these surfaces for our two case studies so far and then begin to look at the connections to Algebraic Geometry.

7 Riemann Surfaces

We now study our means to resolving the issues we have faced so far. To begin with we need some basic Topological definitions. We will formally define a Riemann Surface, but then move on quickly to specific constructions, and briefly look into the relations with Algebraic Geometry.

7.1 A First Example - The Riemann Sphere

We begin with an important study of the Riemann sphere. Our motivation for studying this concept stems from the need to handle functions that become unbounded at certain points—a situation that frequently arises with rational functions. To address this challenge, we require a notion of infinity, and the Riemann sphere provides an elegant solution.
To construct the Riemann sphere, we first consider the unit sphere $S^2 \subset \mathbb{R}^3$ and employ stereographic projection to map points from S^2 onto the complex plane $\mathbb{C}$. This mapping establishes a clear bijection between S^2 minus the north pole and $\mathbb{C}$. We then identify the north pole with the point at infinity, denoted ∞.



This construction gives us the extended complex plane, denoted $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$, which provides a natural domain for defining functions that were previously problematic due to their unbounded behavior. The Riemann sphere represents our simplest yet most fundamental example of a Riemann surface, a concept that we will explore in greater depth throughout this study.
Having established our motivation and construction, we now proceed to lay the foundations with precise mathematical definitions.

7.2 Riemann Surfaces, Manifolds, Charts, Transition Functions

Loosely speaking, a **Riemann Surface** is a domain in which we can define a given function f such that it is holomorphic. Rather than just defining a function on $\mathbb{C}$ we define f on a subset of $\mathbb{C}$ - some *many sheeted region* which allows us to get rid of our multi-valuedness. To formally define this however, we will need to understand what is meant by a Manifold. Note that knowledge of Topology II is assumed.

Definition 7.1 (Topological Manifold)**.** *Let U_p denote an open neighbourhood containing the point p in some topological space. A topological space $(\mathcal{M}, \mathcal{O})$ is called a d-**dimensional topological manifold** if*

$$\forall p \in \mathcal{M} \quad \exists U_p \in \mathcal{O} : \exists x : U_p \to x(U_p) \subseteq \mathbb{R}^d$$

*such that the map x (called a **chart** or **coordinate map**) satisfies:*

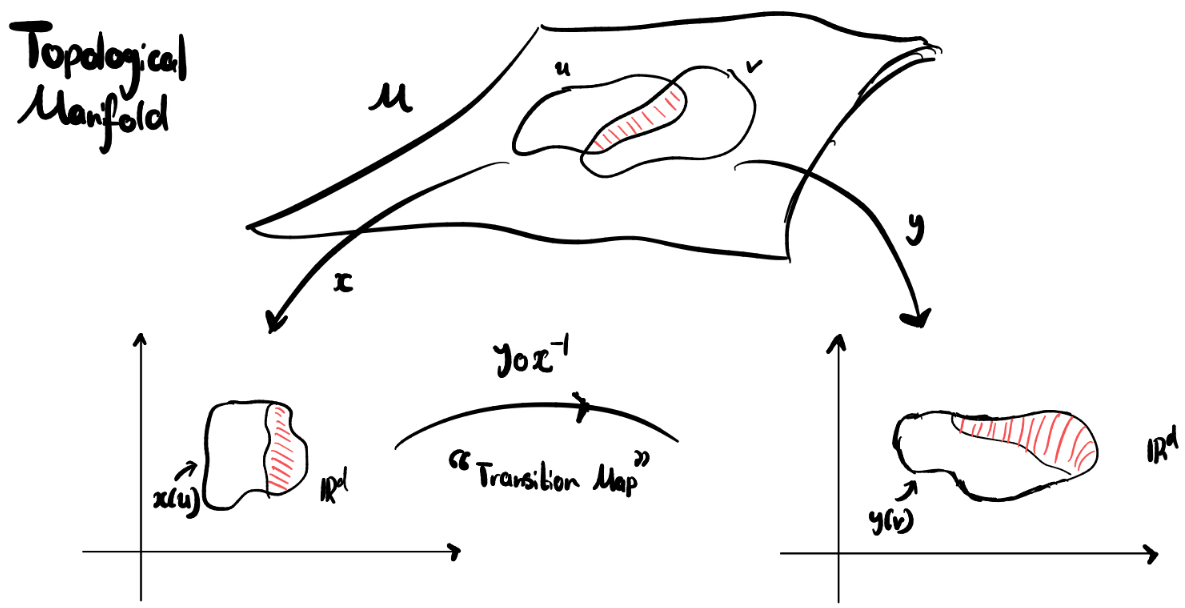
1. *x is **invertible**: $x^{-1} : x(U_p) \to U_p$;*
2. *x is **continuous**;*
3. *x^{-1} is **continuous**.*

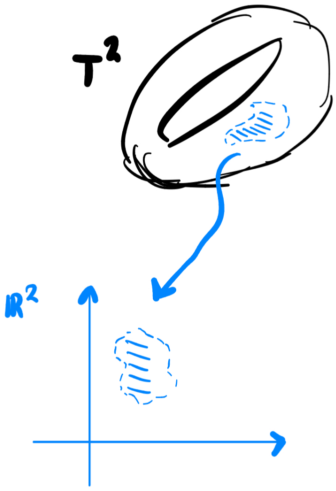
Remark 7.1. *Conditions (i), (ii), and (iii) together just are that x is a homeomorphism between the open set U_p and its image $x(U_p)$ in $\mathbb{R}^d$. The pair (U_p, x) is called a **coordinate chart** or simply a **chart**.*

Remark 7.2. *If we have two charts (U, x) (V, y) with an overlapping region $U \cap V \neq \phi$ we must have that they are compatible. This just means the map between the two images on $\mathbb{R}^n$ is C^∞. This map*

$$y \circ x^{-1} : x(U \cap V) \to y(U \cap V)$$

*is called our **Transition Map** - a map that takes us between our two coordinates. This is depicted below alongside a common example of a topological 2-manifold - the Torus.*





This is an important definition to understand since Riemann Surfaces are in fact **Complex Manifolds**, we now explore this.

Definition 7.2 (Complex Manifold)**.** *A **complex manifold** of complex dimension n is a topological manifold $\mathcal{M}$ of real dimension $2n$ together with a collection of charts $\{(U_\alpha, z_\alpha)\}_{\alpha \in A}$ such that:*

1. *Each chart map $z_\alpha : U_\alpha \to \mathbb{C}^n$ is a homeomorphism onto an open subset of $\mathbb{C}^n$;*
2. *The charts cover $\mathcal{M}$: $\bigcup_{\alpha \in A} U_\alpha = \mathcal{M}$;*
3. *For any two overlapping charts (U_α, z_α) and (U_β, z_β) with $U_\alpha \cap U_\beta \neq \emptyset$, the **transition map***

$$z_\beta \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U_\beta) \to z_\beta(U_\alpha \cap U_\beta)$$

   *is **holomorphic** (i.e., complex analytic).*

*Such a collection of charts is called a **complex atlas** or **holomorphic atlas**.*

Remark 7.3. *The key difference from a real manifold is condition (iii): the transition maps must be holomorphic rather than merely continuous. This additional structure allows us to define holomorphic functions on the manifold and develop complex analysis in the manifold setting.*

Example 7.1. *The Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is a complex manifold of complex dimension 1. It can be covered by two charts:*

- *(U_1, z_1) where $U_1 = \mathbb{C}$ and $z_1(z) = z$*
- *(U_2, z_2) where $U_2 = \hat{\mathbb{C}} \setminus \{0\}$ and $z_2(z) = 1/z$ (with $z_2(\infty) = 0$)*

The transition map on $U_1 \cap U_2 = \mathbb{C} \setminus \{0\}$ is $z_2 \circ z_1^{-1}(w) = 1/w$, which is holomorphic.

Before we formally define the Riemann Surface recall that intuitively a Riemann surface is just a surface where we can do complex analysis.

Definition 7.3 (Riemann Surface)**.** *A **Riemann surface** is a connected complex manifold of complex dimension 1. More explicitly, a Riemann surface is a connected Hausdorff topological space S together with a holomorphic atlas $\{(U_\alpha, z_\alpha)\}_{\alpha \in A}$ such that:*

1. *Each U_α is an open subset of S;*

2. *$\bigcup_{\alpha \in A} U_\alpha = S$ (the charts cover S);*

3. *Each chart map $z_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{C}$ is a homeomorphism onto an open subset V_α of $\mathbb{C}$;*

4. *For overlapping charts with $U_\alpha \cap U_\beta \neq \emptyset$, the transition map*

$$z_\beta \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U_\beta) \to z_\beta(U_\alpha \cap U_\beta)$$

is holomorphic.

Remark 7.4. *The key points distinguishing Riemann surfaces from general complex manifolds are:*

- ***Dimension:*** *Complex dimension 1 (real dimension 2)*
- ***Connectivity:*** *We require the surface to be connected (see Topology II)*
- ***Hausdorff property:*** *A common assumption for separation properties.*

The holomorphic structure allows us to define holomorphic functions, meromorphic functions, and other key objects on the surface.

Example 7.2 (Classical Examples of Riemann Surfaces)**.** ***1. The Riemann Sphere*** $\hat{\mathbb{C}}$***:*** *As constructed earlier, this is a one dimensional Complex Manifold. Further this is connected - should be easy to see that this is path connected (and hence connected), hence this is a Riemann Surface.*
2. The Complex Plane $\mathbb{C}$***:*** *With the single chart $(U, z) = (\mathbb{C}, id)$, this is the simplest non-compact Riemann surface.*
3. The Punctured Plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$***:*** *Using the single chart $(\mathbb{C}^*, id)$, this is a non-compact Riemann surface that is not simply connected.*
4. Algebraic Curves (Compact Case): *Let $P(z, w) \in \mathbb{C}[z, w]$ be an irreducible polynomial. The Riemann surface associated to the algebraic curve $P(z, w) = 0$ can be constructed by resolving singularities and compactifying. For instance:*

- *The curve $w^2 = z(z-1)(z-\lambda)$ for $\lambda \neq 0, 1$ gives an elliptic curve*
- *The curve $w^2 = z^n - 1$ gives a hyper-elliptic Riemann surface. This is where we note the links to Algebraic Geometry, and how we can study this example as a variety. This is something I wish we could dive more into but it requires further prerequisites.*

7.3 Constructing Riemann Surfaces

7.3.1 Reminder of why we care

In complex analysis, many familiar functions are *multi-valued.* For example our two previous case studies:

- The square root function $f(z) = \sqrt{z}$ has two possible values for each $z \neq 0$.
- The logarithm $f(z) = \log(z)$ has infinitely many values, differing by integer multiples of $2\pi i$.

On the complex plane $\mathbb{C}$, there is no way to make these functions *single-valued everywhere* without introducing arbitrary branch cuts. But Riemann's insight was that instead of "forcing" the complex plane to behave, we could build a *new surface* — a *Riemann surface* — on which the function is naturally single-valued.
Thus, a Riemann surface can be viewed as the *natural domain* of a multi-valued analytic function.

7.3.2 First Example: The Square Root

Consider $f(z) = \sqrt{z}$. Locally, near any point $z \neq 0$, we can define a branch of the square root holomorphically. However, globally, if we go once around the origin, we switch to the other value of the square root.
To resolve this, we construct the *two-sheeted Riemann surface* of $\sqrt{z}$:

1. Take two copies of the complex plane, each cut along the negative real axis.
2. On one sheet, define the 'principal' branch of $\sqrt{z}$. On the other sheet, define the other branch.
3. Glue the top edge of the cut from the first sheet to the bottom edge of the cut from the second sheet, and vice versa.

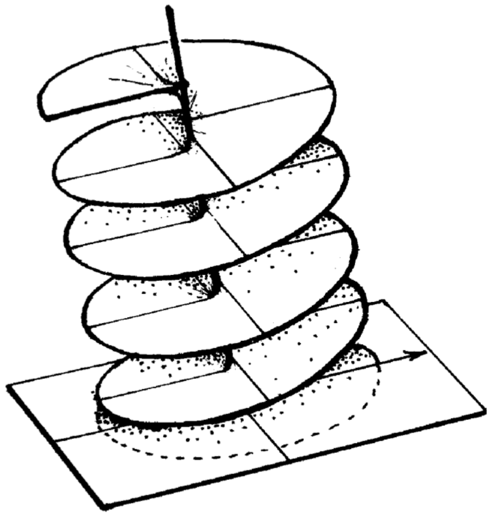
The result is a single connected surface on which $\sqrt{z}$ is holomorphic and single-valued. On this two-sheeted surface, the path dependence we saw in Section 6.3.2 disappears: there is now only one path from any point to any other point (up to homotopy), so z becomes single-valued

7.3.3 Second Example: The Logarithm

The logarithm $f(z) = \log(z)$ behaves differently:
Every time we loop around the origin, the value increases by $2\pi i$. Unlike $\sqrt{z}$, there is no finite number of sheets: instead, we need *infinitely many sheets.*
We construct its Riemann surface by:

1. Taking infinitely many copies of the cut plane (say, cut along the negative real axis).
2. Label them by integers $k \in \mathbb{Z}$, corresponding to the branch where $\mathrm{Im}(\log(z)) \in (2\pi k, 2\pi(k+1))$.
3. Gluing the top edge of sheet k to the bottom edge of sheet $k+1$.

The resulting surface spirals infinitely upward like a *helicoid.* On this surface, $\log(z)$ is a perfectly good single-valued holomorphic function. This is our solution to those frustrating Multiple Valued functions we encountered earlier!



7.3.4 General Construction from Branch Points

The examples above illustrate the general method:

1. **Local branches:** Every multi-valued function can be defined holomorphically in small neighborhoods (away from branch points).
2. **Sheets:** Each possible analytic continuation defines a "sheet" of the surface.
3. **Gluing:** Where two neighborhoods overlap, we glue the corresponding sheets according to how the branches continue into one another.
4. **Branch points:** At special points (like $z = 0$ for $\sqrt{z}$), the local structure of the surface is not simply a copy of the plane but multiple sheets meeting in a controlled way.

Formally:

- Start with charts $(U_\alpha, \varphi_\alpha)$, where $\varphi_\alpha : U_\alpha \to \mathbb{C}$ maps a neighborhood of the surface to the complex plane.
- Glue these charts along overlaps using holomorphic transition maps.
- The result is a topological surface with a holomorphic atlas: a *Riemann surface.*

7.3.5 Riemann Surfaces from Algebraic Functions

Another systematic way to construct Riemann surfaces comes from algebraic functions defined by polynomial equations.

Example 7.3. *Consider the algebraic relation*

$$w^2 = z^3 - z.$$

For each z, this defines up to two values of w. By analysing the equation, we see branch points at $z = -1, 0, 1, \infty$. By gluing two sheets of the complex plane along appropriate cuts, we obtain a compact Riemann surface of genus 1 — a torus.

This shows that Riemann surfaces naturally arise from algebraic curves over $\mathbb{C}$.

7.3.6 Why This Matters

Constructing Riemann surfaces gives us:

- **Natural domains** for multi-valued functions, making them single-valued and holomorphic.
- **Bridges to algebraic geometry**, since every complex smooth algebraic curve corresponds to a Riemann surface.
- **Insights into topology**, since surfaces constructed this way have genus, fundamental groups, and covering space structures that reflect the analytic behavior of the function.

7.4 Conclusion

So we have now seen the topological nature of a Riemann Surface, looking at the core definitions, examples and then also seeing why the study of Algebraic Geometry would help us understand these surfaces better. Despite all these deep, abstract definitions we return to why we care. We care because these surfaces are a natural domain for our multi-valued functions we encountered earlier. These arise as a natural domain by seeing each layer or sheet of our Riemann Surface as corresponding to a different branch of the multi-valued function. In fact, this is the essential point: Riemann surfaces take a function that appears multi-valued on the complex plane and provide it with a proper home where it becomes single-valued and holomorphic. The square root, logarithm, and inverse trigonometric functions are classic examples. By cutting and gluing together copies of the complex plane, we build surfaces where these functions are well-defined. More generally, we construct Riemann surfaces by piecing together local neighborhoods where a branch is valid, and the resulting global surface encodes the branching structure of the function. This construction not only clarifies the analytic behavior of multi-valued functions but also explains why algebraic geometry and Riemann surfaces are inseparable: every smooth complex algebraic curve can be viewed as a compact Riemann surface. Thus, Riemann surfaces sit at the intersection of topology, analysis, and algebra, providing the natural domain for complex functions and the unifying language for their study. Stepping back, we see that analytic continuation acts as a universal principle: it extends local definitions into global truths, connects disparate fields, and motivates deep structures such as Riemann surfaces. Whether in the heat equation, Wick rotations, zeta regularisation, or topology, the same idea resurfaces in different guises. This unifying role is precisely why analytic continuation occupies such a central place in modern mathematics and theoretical physics.