Analysis of the WKB Eigenfunction Approximation for the Linearised Swift–Hohenberg Operator

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Contents

1 Introduction

The Swift–Hohenberg equation was first introduced as a model for pattern formation in non-linear systems such as convection and reaction–diffusion processes. Even in its linearised form, the equation exhibits rich spectral behaviour governed by a fourth-order differential operator with a small parameter. Understanding the eigenfunctions and eigenvalues of this operator is central to studying the stability and emergence of spatial structures.

In this paper, we focus on the linearised Swift–Hohenberg operator with spatially varying growth rate $r(x)$ and Neumann boundary conditions. The operator is of high order, and obtaining exact eigenfunctions is not possible analytically. Instead, we turn to WKB analysis, a classical asymptotic method for approximating solutions to differential equations with small parameters.

First, we construct WKB approximations of eigenfunctions and eigenvalues for the Swift–Hohenberg operator. Second, we place these approximations in a rigorous spectral-theoretic framework: showing that the operator is self-adjoint, bounded below, has compact resolvent, and that the WKB eigenfunctions act as quasimodes. Using this, we justify that the WKB eigenfunctions are asymptotically orthonormal and approximate a basis for the solution space as $\varepsilon \to 0$.

This approach not only connects asymptotic methods with operator theory, but also illustrates how tools from functional analysis can rigorously underpin approximations widely used in applied mathematics.

2 The Eigen-Equation

We consider the linearised Swift–Hohenberg system (N(u)=0)

$$\frac{\partial u}{\partial t} = r(x)u - \left(1 + \varepsilon^2 \frac{d^2}{dx^2}\right)^2 u, \quad x \in [0,1],$$

with Neumann boundary conditions:

$$u'(0) = u'(1) = u'''(0) = u'''(1) = 0.$$

We begin by expanding the operator:

$$L_\varepsilon = r(x) - \left(1 + \varepsilon^2 \frac{d^2}{dx^2}\right)^2.$$

Expanding the square gives:

$$L_\varepsilon = r(x) - \left(1 + 2\varepsilon^2 \frac{d^2}{dx^2} + \varepsilon^4 \frac{d^4}{dx^4}\right).$$

Thus, acting on $\phi(x)$, the eigenvalue problem becomes:

$$\begin{aligned} L_\varepsilon \phi(x) &= \left[r(x) - 1 - 2\varepsilon^2 \frac{d^2}{dx^2} - \varepsilon^4 \frac{d^4}{dx^4} \right] \phi(x) = \lambda \phi(x), \\ \text{where:} \quad & \phi'(0) = \phi'(1) = \phi^{(3)}(0) = \phi^{(3)}(1) = 0. \end{aligned}$$

Our next goal is to construct approximate eigenfunctions and eigenvalues of L_ε in the small ε limit using the WKB method.

3 The WKB Ansatz

We now assume a solution of the form:

$$\phi(x) = A(x) e^{iS(x)/\varepsilon}.$$

For simplicity later on we assume that $S'(x) > 0$ and is an infinitely differentiable function on [0,1].

Using Mathematica, we find the 2nd and 4th derivatives, then multiply through by ε^2 and ε^4 respectively:

We expand the second derivative:

$$\phi''(x) = \left(\frac{iS''(x)}{\varepsilon} - \frac{S'(x)^2}{\varepsilon^2} \right) \phi(x) + \mathcal{O}(1)$$

Multiplying by ε^2 gives:

$$\varepsilon^2 \phi''(x) = -S'(x)^2 \phi(x) + i\varepsilon S''(x) \phi(x) + \mathcal{O}(\varepsilon^2).$$

Doing the same for the fourth derivative:

$$\varepsilon^4 \phi^{(4)}(x) = S'(x)^4 \phi(x) + \mathcal{O}(\varepsilon).$$

Substituting back into the operator:

$$L_\varepsilon \phi = \left[r(x) - \left(1 + \varepsilon^2 \phi''(x)\right)^2 \right] \phi(x) = \left[r(x) - \left(1 - S'(x)^2\right)^2 \right] \phi(x) + \mathcal{O}(\varepsilon)$$

So the eigenvalue equation becomes:

$$L_\varepsilon \phi(x) = \lambda^{\text{WKB}} \phi(x) \quad \Rightarrow \quad \left[r(x) - \left(1 - S'(x)^2\right)^2 \right] \phi(x) = \lambda^{\text{WKB}} \phi(x),$$

to order ε.

Hence, at leading order the WKB ansatz yields the *eikonal relation*

$$\left(1 - (S'(x))^2\right)^2 \;=\; r(x) - \lambda^{\text{WKB}}, \tag{1}$$

where λ^{WKB} is a constant eigenvalue (independent of x).

In the next section, we will then see that boundary conditions impose quantisation conditions which determine the (assumably discrete) set of eigenvalues λ_n^{WKB}.

Assuming that the WKB ansatz has a discrete set of eigenvalues, we obtain the discretised WKB phase function $S_n(x)$:

$$S_n(x) = \int_a^x \sqrt{1 \pm \sqrt{r(\theta) - \lambda_n^{\text{WKB}}}}\, d\theta.$$

Although we're only interested in the small ε limit here, it is still important to remember that the WKB eigenvalues, eigenfunctions and phase function are all approximations. That being said, these approximations get increasingly more accurate as ε approaches zero and therefore are appropriate for computation in this context.

4 Neumann Boundary Conditions

To enforce Neumann BCs, take the real part of the solution:

$$\phi_n^{\text{WKB}}(x) = A_n(x) \cos\left(\frac{S_n(x)}{\varepsilon}\right) + \mathcal{O}(\varepsilon)$$

For convenience, from here on we drop the $\mathcal{O}(\varepsilon)$.

I should state here that there do exist certain $A_n(x) and S_n(x)$ such that, after enforcing Nuemann BCs, the sine term also survives in the WKB eigenfunction. However, as these are somewhat of a 'special case', we omit them here and continue with just the cosine.

Boundary conditions are satisfied if $\phi_n^{\text{WKB}\prime}(0) = \phi_n^{\text{WKB}\prime}(1) = 0$, this tells us that we need:

$$\Rightarrow A_n(x) \cos\left(\frac{S_n(x)}{\varepsilon}\right)\bigg|_{x=0,1} - A_n(x) \sin\left(\frac{S_n(x)}{\varepsilon}\right) \cdot \frac{S_n'(x)}{\varepsilon}\bigg|_{x=0,1} = 0$$

For small ε:

$$\Rightarrow A_n(x) \frac{S_n'(x)}{\varepsilon} \sin\left(\frac{S_n(x)}{\varepsilon}\right)\bigg|_{x=0,1} = 0$$

$$\Rightarrow A_n(0) S_n'(0) \sin\left(\frac{S_n(0)}{\varepsilon}\right) = A_n(1) S_n'(1) \sin\left(\frac{S_n(1)}{\varepsilon}\right) = 0$$

Firstly, we have $S_n'(x) > 0$, hence $S_n'(0), S_n'(1) \neq 0$. Secondly, we assume $A_n(0), A_n(1) \neq 0$, we note that this condition does not always hold and may require boundary layer analysis to address. However, since this is not our main focus, the assumption is reasonable in certain contexts.

$$\sin\left(\frac{S_n(0)}{\varepsilon}\right) = \sin\left(\frac{S_n(1)}{\varepsilon}\right) = 0$$
$$\Rightarrow S_n(1) - S_n(0) = n\pi\varepsilon$$

This gives the quantisation condition on the WKB eigenvalues:

$$\int_0^1 S_n'(x)\,dx = n\pi\varepsilon \ , n \in \mathbb{N}$$

If we wanted to, we could now determine approximate values of the WKB eigenvalues for various values of n.

In this section we have skipped over many details, for one who wants to address turning points, boundary layer analysis and more I suggest reading [4].

5 Solving the Full PDE

Firstly, from here onwards we assume that $u \in \mathcal{D}$ where:

$$\mathcal{D} \ = \ \left\{ u \in C^4([0,1]) \;\middle|\; u'(0) = u'(1) = u'''(0) = u'''(1) = 0 \right\}.$$

We now consider the full PDE:

$$u_t = L_\varepsilon u.$$

We assume a separated WKB-style solution of the form:

$$u(x,t) = \sum_n [A_n(x)\cos\left(\frac{S_n(x)}{\varepsilon}\right) T_n(t)]$$

Where $S_n(x)$ is as before, $T_n \in C^1([0,1])$, and $n \in \mathbb{N}$.

Substituting into the PDE gives:

$$\sum_n [A_n(x)\cos\left(\frac{S_n(x)}{\varepsilon}\right)\frac{\mathrm{d}T_n}{\mathrm{d}t}] = L_\varepsilon u(x,t).$$

Where:

$$L_\varepsilon u(x,t) = L_\varepsilon(\sum_n [A_n(x)\cos\left(\frac{S_n(x)}{\varepsilon}\right) T_n(t)]) = \sum_n (L_\varepsilon [A_n(x)\cos\left(\frac{S_n(x)}{\varepsilon}\right)] T_n(t))$$

The second equality is justified as L_ε is a linear operator.

By the above:

$$L_\varepsilon[A_n(x)\cos\left(\frac{S_n(x)}{\varepsilon}\right)] = \lambda_n^{\mathrm{WKB}} A_n(x)\cos\left(\frac{S_n(x)}{\varepsilon}\right)$$

So we get:

$$\sum_n A_n(x)\cos\left(\frac{S_n(x)}{\varepsilon}\right)\left(\frac{\mathrm{d}T_n}{\mathrm{d}t} - \lambda_n^{\mathrm{WKB}} T_n(t)\right) = 0.$$

Since the functions $A_n(x)\cos\left(\frac{S_n(x)}{\varepsilon}\right)$ are not identically zero, the only way for the sum to vanish for all x is for each time-dependent factor to vanish, hence:

$$\frac{\mathrm{d}T_n}{\mathrm{d}t} = \lambda_n^{\mathrm{WKB}} T_n(t).$$

Solving this gives:

$$T_n(t) = M_n e^{\lambda_n^{\mathrm{WKB}} t}.$$

This then gives us the general solution:

$$u(x,t) = \sum_n A_n(x)\cos\left(\frac{S_n(x)}{\varepsilon}\right) e^{\lambda_n^{\mathrm{WKB}} t}.$$

6 Our main Goal

We want to show that:

> ”WKB eigenfunctions approximate an orthonormal basis for the solution space of the linearised Swift–Hohenberg equation with Neumann boundary conditions, in the limit as $\varepsilon \to 0$.’

It should be noted that in this section we work only with the true eigenvalues and eigenfunctions of L_ε, not with the WKB ones, we come to those later.

From here on we work in $L^2([0,1])$ and we assume that $u, w \in C^4([0,1])$.

6.1 Self-Adjointness

$$\begin{aligned}
\langle L_\varepsilon u, \omega\rangle_{L^2([0,1])} &= \int_0^1 \left[r(x)u - \left(1+\varepsilon^2\frac{d^2}{dx^2}\right)^2 u\right]\omega\, dx \\
&= \int_0^1 r(x)u\omega - u\omega - 2\varepsilon^2\frac{d^2u}{dx^2}\omega - \varepsilon^4\frac{d^4u}{dx^4}\omega\, dx \\
&= \int_0^1 u(r(x)-1)\omega\, dx - 2\varepsilon^2\int_0^1 u''\omega\, dx - \varepsilon^4\int_0^1 u^{(4)}\omega\, dx
\end{aligned}$$

Integrating by parts:

$$
\begin{aligned}
\int_0^1 u''\omega\,dx &= [u'\omega]_0^1 - \int_0^1 u'\omega'\,dx \\
&= -\int_0^1 u'\omega'\,dx \quad (\text{as } u'(0) = u'(1) = 0) \\
&= -[u\omega']_0^1 + \int_0^1 u\omega''\,dx
\end{aligned}
$$

Similarly:

$$
\int_0^1 u^{(4)}\omega\,dx = [-u\omega^{(3)} - u''\omega']_0^1 + \int_0^1 u\omega^{(4)}\,dx \qquad (\text{as } u'(0) = u'(1) = u^{(3)}(0) = u^{(3)}(1) = 0)
$$

Substituting back:

$$
\begin{aligned}
\langle L_\varepsilon u, \omega\rangle_{L^2([0,1])} = \int_0^1 u(r(x)-1)\omega\,dx - 2\varepsilon^2 \int_0^1 u\omega''\,dx - \varepsilon^4 \int_0^1 u\omega^{(4)}\,dx \\
- 2\varepsilon^2[-\omega' u]_0^1 - \varepsilon^4[-\omega' u'' - \omega^{(3)}u]_0^1
\end{aligned}
$$

Now rewriting:

$$
\begin{aligned}
\langle L_\varepsilon u, \omega\rangle_{L^2([0,1])} &= \left[2\varepsilon^2\omega' u + \varepsilon^4\omega' u'' + \varepsilon^4\omega^{(3)}u\right]_0^1 + \int_0^1 u\left[(r(x)-1)\omega - 2\varepsilon^2\omega'' - \varepsilon^4\omega^{(4)}\right] dx \\
&= \left[2\varepsilon^2\omega' u + \varepsilon^4\omega' u'' + \varepsilon^4\omega^{(3)}u\right]_0^1 + \langle u, L_\varepsilon\omega\rangle_{L^2([0,1])}
\end{aligned}
$$

Now analysing the boundary conditions:

$$
\begin{aligned}
&2\varepsilon^2\omega'(1)u(1) + \varepsilon^4\omega'(1)u''(1) + \varepsilon^4\omega^{(3)}(1)u(1) - 2\varepsilon^2\omega'(0)u(0) - \varepsilon^4\omega'(0)u''(0) - \varepsilon^4\omega^{(3)}(0)u(0) = 0 \\
&\Rightarrow \omega'(0) = \omega'(1) = 0 \\
&\Rightarrow \omega^{(3)}(0) = \omega^{(3)}(1) = 0
\end{aligned}
$$

Hence we have Neumann boundary conditions:

$\Rightarrow \langle L_\varepsilon u, \omega\rangle_{L^2([0,1])} = \langle u, L_\varepsilon\omega\rangle_{L^2([0,1])}$, so the operator L_ε with Neumann BCs is fully self-adjoint.

6.2 Boundedness

Definition: For a self-adjoint operator A on a Hilbert space $\mathcal{H}$, we say A is *bounded below* if:

$$
\langle Au, u\rangle \geq C\|u\|^2 \quad \forall u \in D(A),
$$

for some constant $C \in \mathbb{R}$, where $D(A)$ denotes the domain of the operator A.

We work in the Hilbert space $L^2([0,1])$, and the domain of the operator is:

$$D(L_\varepsilon) = \left\{ u \in C^4([0,1]) \;\middle|\; u'(0) = u'(1) = u'''(0) = u'''(1) = 0 \right\}.$$

We compute the inner product $\langle L_\varepsilon u, u \rangle_{L^2([0,1])}$ directly. Expanding the operator:

$$L_\varepsilon = r(x) - \left(1 + \varepsilon^2 \frac{d^2}{dx^2}\right)^2 = r(x) - \left(1 + 2\varepsilon^2 \frac{d^2}{dx^2} + \varepsilon^4 \frac{d^4}{dx^4}\right).$$

Then acting on u, we have:

$$L_\varepsilon u = r(x)u - 2\varepsilon^2 u'' - \varepsilon^4 u^{(4)} - u.$$

Taking the inner product:

$$\langle L_\varepsilon u, u \rangle_{L^2([0,1])} = \int_0^1 \left[r(x)|u|^2 - 2\varepsilon^2 u'' u - \varepsilon^4 u^{(4)} u - |u|^2 \right] dx.$$

We now integrate by parts: - For the second derivative term:

$$\int_0^1 u'' u \, dx = -\int_0^1 |u'|^2 \, dx \quad (\text{since } u'(0) = u'(1) = 0),$$

- For the fourth derivative term:

$$\int_0^1 u^{(4)} u \, dx = \int_0^1 |u''|^2 \, dx \quad (\text{since } u'''(0) = u'''(1) = 0).$$

Putting this together:

$$\langle L_\varepsilon u, u \rangle_{L^2([0,1])} = \int_0^1 \left[r(x)|u|^2 + 2\varepsilon^2 |u'|^2 - \varepsilon^4 |u''|^2 - |u|^2 \right] dx.$$

Hence:

$$\langle L_\varepsilon u, u \rangle_{L^2([0,1])} = \int_0^1 \left[(r(x) - 1)|u|^2 + 2\varepsilon^2 |u'|^2 - \varepsilon^4 |u''|^2 \right] dx.$$

We assume that $r \in C^1([0,1])$. By the Weierstrass Extreme Value Theorem, this guarantees the existence of a point $x_0 \in [0,1]$ such that:

$$r_{\min} = r(x_0) = \min_{x \in [0,1]} r(x).$$

Therefore, we can now say:

$$\langle L_\varepsilon u, u \rangle_{L^2([0,1])} \geq (r_{\min} - 1)(\|u\|^2_{L^2([0,1])}) + 2\varepsilon^2 \|u'\|^2_{L^2([0,1])} - \varepsilon^4 \|u''\|^2_{L^2([0,1])}.$$

We now state and prove the following Poincare style inequality:

$$\|u''\|_{L^2([0,1])} \leq C \left(\|u'\|_{L^2([0,1])} + \|u\|_{L^2([0,1])}\right)$$

Proof:

By contradiction: Suppose the inequality is false, then for each $k \in \mathbb{N}$ there exists a function:

$$u_k \in C^4\big([0,1]\big)$$

such that

$$\|u_k''\|_{L^2([0,1])} > k\Big(\|u_k'\|^2_{L^2([0,1])} + \|u_k\|^2_{L^2([0,1])}\Big).$$

Now define

$$v_k \ := \ \frac{u_k}{\|u_k''\|_{L^2([0,1])}}.$$

It follows that

$$v_k'' \ = \ \frac{u_k''}{\|u_k''\|_{L^2([0,1])}},$$

And hence

$$\|v_k''\|_{L^2([0,1])} \ = \ 1.$$

Since

$$\|u_k''\|_{L^2([0,1])} > k\big(\|u_k'\|_{L^2([0,1])} + \|u_k\|_{L^2([0,1])}\big),$$

It follows that

$$\|u_k''\|_{L^2([0,1])} > k\|u_k'\|_{L^2([0,1])} \quad \text{and} \quad \|u_k''\|_{L^2([0,1])} > k\|u_k\|_{L^2([0,1])}.$$

Now going back to

$$v_k := \frac{u_k}{\|u_k''\|_{L^2([0,1])}} \quad \text{and therefore} \quad v_k' := \frac{u_k'}{\|u_k''\|_{L^2([0,1])}}.$$

$$\implies \quad \begin{aligned} &\|v_k\|_{L^2([0,1])}\,\|u_k''\|_{L^2([0,1])} = \|u_k\|_{L^2([0,1])}, \\ &\|v_k'\|_{L^2([0,1])}\,\|u_k''\|_{L^2([0,1])} = \|u_k'\|_{L^2([0,1])}. \end{aligned}$$

So this gives

$$\|u_k''\|_{L^2([0,1])} > k\,\|v_k'\|_{L^2([0,1])}\,\|u_k''\|_{L^2([0,1])} \quad \text{and} \quad \|u_k''\|_{L^2([0,1])} > k\,\|v_k\|_{L^2([0,1])}\,\|u_k''\|_{L^2([0,1])}.$$

$$\implies \quad \|v_k'\|_{L^2([0,1])} < \frac{1}{k}, \qquad \|v_k\|_{L^2([0,1])} < \frac{1}{k}.$$

Hence the sequence $\{v_k\} \subset C^4([0,1])$ satisfies

- $\|v_k''\|_{L^2([0,1])} = 1$,
- $v_k' \to 0$ in $L^2([0,1])$ as $k \to \infty$,

- $v_k \to 0$ in $L^2([0,1])$ as $k \to \infty$.

Now let $\varphi \in C_0^\infty([0,1]) \subset L^2([0,1])$. Then

$$\langle v_k'', \varphi \rangle_{L^2([0,1])} = \int_0^1 v_k''(x)\,\varphi(x)\,dx = \big[\varphi(x)\,v_k'(x)\big]_0^1 - \int_0^1 \varphi'(x)\,v_k'(x)\,dx = -\int_0^1 \varphi'(x)\,v_k'(x)\,dx.$$

Hence

$$\big|\langle v_k'', \varphi \rangle_{L^2([0,1])}\big| = \Big|\int_0^1 \varphi'(x)\,v_k'(x)\,dx\Big| \le \|\varphi'\|_{L^2([0,1])}\,\|v_k'\|_{L^2([0,1])} \;\to\; 0 \text{ as } k \to \infty \quad \text{(by Cauchy-Schwarz).}$$

Therefore, for every $\varphi \in C_0^\infty([0,1])$,

$$\langle v_k'', \varphi \rangle_{L^2([0,1])} = \int_0^1 v_k''(x)\,\varphi(x)\,dx \;\to\; 0 \text{ as } k \to \infty$$

So

$$v_k'' \;\rightharpoonup\; 0 \quad \text{weakly in } L^2([0,1]).$$

But

$$\|v_k''\|_{L^2([0,1])} = 1,$$

a contradiction.

Hence our assumption was false and the desired inequality must hold. $\square$

Squaring and multiplying by ε^4, we get:

$$-\varepsilon^4\|u''\|^2_{L^2([0,1])} \ge -\varepsilon^4 C^2\left(\|u'\|^2_{L^2([0,1])} + \|u\|^2_{L^2([0,1])} + 2\|u'\|_{L^2([0,1])}\|u\|_{L^2([0,1])}\right)$$

So our operator satisfies:

$$\langle L_\varepsilon u, u\rangle_{L^2([0,1])} \ge (r_{\min}-1)\,\|u\|^2_{L^2([0,1])} + 2\varepsilon^2\|u'\|^2_{L^2([0,1])} - \varepsilon^4 C^2\left(\|u'\|^2_{L^2([0,1])} + \|u\|^2_{L^2([0,1])} + 2\|u'\|_{L^2([0,1])}\|u\|_{L^2([0,1])}\right)$$

Using Young's inequality: $ab \le \frac{a^2}{2} + \frac{b^2}{2}$ for any $a, b > 0$, with $a = \|u'\|_{L^2([0,1])}$, $b = \|u\|_{L^2([0,1])}$, we get:

$$2\|u'\|_{L^2([0,1])}\|u\|_{L^2([0,1])} \le \|u'\|^2_{L^2([0,1])} + \|u\|^2_{L^2([0,1])}$$

Substituting back:

$$\langle L_\varepsilon u, u\rangle_{L^2([0,1])} \ge (r_{\min}-1)\;\|u\|^2_{L^2([0,1])} + 2\varepsilon^2\|u'\|^2_{L^2([0,1])} - \varepsilon^4 C^2\left(3\|u'\|^2_{L^2([0,1])} + 3\|u\|^2_{L^2([0,1])}\right)$$

Group terms:

$$\langle L_\varepsilon u, u\rangle_{L^2([0,1])} \ge (r_{\min} - 1 - 3\varepsilon^4 C^2)\|u\|^2_{L^2([0,1])} + \left(2\varepsilon^2 - 3\varepsilon^4 C^2\right)\|u'\|^2_{L^2([0,1])}$$

Let

$$M = (r_{\min} - 1 - 3\varepsilon^4 C^2), \quad N = 2\varepsilon^2 - 3\varepsilon^4 C^2.$$

Provided ε is sufficiently small, we have $N \ge 0$, and hence:

$$\langle L_\varepsilon u, u\rangle_{L^2([0,1])} \ge M\|u\|^2_{L^2([0,1])} + N\|u'\|^2_{L^2([0,1])}, \quad \text{where } M \in \mathbb{R},\; N \in \mathbb{R}_{\ge 0},$$

$$\Rightarrow \langle L_\varepsilon u, u\rangle_{L^2([0,1])} \ge M\|u\|^2_{L^2([0,1])} \quad \text{for some } M \in \mathbb{R}.$$

$$\Rightarrow L_\varepsilon \text{ is bounded below.}$$

6.3 Compact Resolvent

Theorem 1 (Rellich–Kondrachov). *[2] The embedding*

$$H^n(0,1) \hookrightarrow L^2([0,1])$$

is compact for any integer $n \geq 1$. In particular,

$$H^4([0,1]) \hookrightarrow L^2([0,1])$$

compactly.

Theorem 2. *Let T be a self-adjoint operator in a Hilbert space H such that $(T - \lambda I)^{-1}$ maps into a space that embeds compactly into H. Then the resolvent is compact. [5]*

Since

$$(L_\varepsilon - \lambda_k I)^{-1} : L^2([0,1]) \to H^4([0,1]) \hookrightarrow L^2([0,1]),$$

the resolvent is compact. Therefore, L_ε has compact resolvent.

Spectral Theorem Application

Theorem 3 (Spectral Theorem for Compact Self-Adjoint Operators). *[2] Let A be a self-adjoint operator with compact resolvent on a Hilbert space $\mathcal{H}$. Then:*

- *The eigenvalues of A tend to infinity.*
- *The corresponding eigenfunctions form an orthonormal basis for $\mathcal{H}$.*

In particular, if A is also bounded below (i.e., $\langle Au, u\rangle \geq C\|u\|^2$), then all eigenvalues are bounded from below, and the eigenfunctions still form an orthonormal basis.

Since L_ε is self-adjoint, bounded below, and has compact resolvent:

The true eigenfunctions of L_ε form an orthonormal basis in $L^2([0,1])$ and the true eigenvalues of L_ε are bounded from below.

It now seems reasonable to try and justify that:

$$\left\|\phi_n^{\mathrm{WKB}} - \phi_n^{\mathrm{True}}\right\|_{L^2([0,1])} \to 0 \quad \text{as } \varepsilon \to 0$$

To do this, we construct WKB *quasimodes.*

7 Quasimodes

Definition: Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$, and let $\lambda \in \mathbb{R}$. We say that $\varphi \in \mathcal{H}$ is a *quasimode* for A with approximate eigenvalue λ if:

$$\|(A - \lambda)\varphi\|_{\mathcal{H}} \ll \|\varphi\|_{\mathcal{H}}.$$

A good way of thinking about Quasimodes is that they're just approximate eigenvector and eigenvalue pairs, hence they almost satisfy the standard eigen-equation.

We want our WKB eigenfunction ϕ_n^{WKB} to satisfy:

$$\left\|(L_\varepsilon - \lambda_n^{\mathrm{WKB}})\phi_n^{\mathrm{WKB}}\right\|_{L^2([0,1])} = \mathcal{O}(\varepsilon) \quad \text{and} \quad \left\|\phi_n^{\mathrm{WKB}}\right\|_{L^2([0,1])} = 1 + \mathcal{O}(\varepsilon),$$

As this will tell us that ϕ_n^{WKB} is a quasimode for small epsilon.

Furthermore, the second of the conditions tells us that the WKB eigenfunctions are/ can be normalised in the epsilon limit.

From before, we know:

$$L_\varepsilon \phi_n^{\mathrm{WKB}} = \left[r(x) - (1 - k(x)^2)^2\right] A(x) \cos\left(\tfrac{1}{\varepsilon} S(x)\right) + \mathcal{O}(\varepsilon),$$

and:

$$\lambda_n^{\mathrm{WKB}} := r(x) - (1 - k(x)^2)^2.$$

Hence, clearly:

$$\|(L_\varepsilon - \lambda_n^{\mathrm{WKB}})\phi_n^{\mathrm{WKB}}\|_{L^2([0,1])} = \mathcal{O}(\varepsilon) \quad \text{is satisfied.}$$

Next, we show

$$\left\|\phi_n^{\mathrm{WKB}}\right\|_{L^2([0,1])} = 1 + \mathcal{O}(\varepsilon).$$

We want to compute the L^2 norm-squared of the WKB eigenfunction:

$$\left\|\phi_n^{\mathrm{WKB}}\right\|^2_{L^2([0,1])} = \int_0^1 \left|A_n(x) \cos\left(\tfrac{S_n(x)}{\varepsilon}\right)\right|^2 dx.$$

Expanding this using the double-angle formula:

$$= \int_0^1 A_n(x)^2 \cos^2\left(\tfrac{S_n(x)}{\varepsilon}\right) dx = \tfrac{1}{2}\int_0^1 A_n(x)^2 dx + \tfrac{1}{2}\int_0^1 A_n(x)^2 \cos\left(\tfrac{2S_n(x)}{\varepsilon}\right) dx.$$

For the WKB method, we assume that $A_n(x)$ (the amplitude) is square-integrable.

Because of this, we can then choose $A_n(x)$ so that:

$$\int_0^1 A_n(x)^2 dx = 2 \quad \Rightarrow \quad \tfrac{1}{2}\int_0^1 A_n(x)^2 dx = 1.$$

Now consider the oscillatory integral:

$$\tfrac{1}{2}\int_0^1 A_n(x)^2 \cos\left(\tfrac{2S_n(x)}{\varepsilon}\right) dx.$$

Lemma (Riemann–Lebesgue) [1]: If $g \in C^1([a,b])$ and $g'(x) \geq c > 0$ $\forall x \in [a,b]$, then:

$$\int_a^b f(x) e^{ig(x)/\varepsilon} dx = \mathcal{O}(\varepsilon), \quad \text{as } \varepsilon \to 0.$$

Since $A_n(x)^2 \in C^1([0,1])$, we can write:

$$\tfrac{1}{2} \int_0^1 A_n(x)^2 \cos\left(\tfrac{2S_n(x)}{\varepsilon}\right) dx = \mathrm{Re}\left[\int_0^1 A_n(x)^2 e^{\frac{2i}{\varepsilon} S_n(x)} dx\right],$$

and since the derivative of the phase is:

$$\frac{d}{dx}\left(2S_n(x)\right) = 2S_n'(x) > 0 \quad \forall x \in [0,1] \quad \text{(via our earlier assumption)},$$

the lemma applies and we conclude:

$$\int_0^1 A_n(x)^2 \cos\left(\tfrac{2S_n(x)}{\varepsilon}\right) dx = \mathcal{O}(\varepsilon).$$

Hence:

$$\left\|\phi_n^{\mathrm{WKB}}\right\|_{L^2([0,1])} = 1 + \mathcal{O}(\varepsilon).$$

So $(\phi_n^{\mathrm{WKB}}, \lambda_n^{\mathrm{WKB}})$ is a **quasimode**.

Theorem 4 (Quasimode Spectral Approximation Theorem): [3] Let A be a self-adjoint operator with compact resolvent on a Hilbert space $\mathcal{H}$. Suppose you have a quasimode — that is, a pair (φ_n, λ_n) such that:

$$\|(A - \lambda_n)\varphi_n\|_{\mathcal{H}} = \mathcal{O}(\varepsilon), \quad \text{and} \quad \|\varphi_n\|_{\mathcal{H}} = 1 + \mathcal{O}(\varepsilon),$$

then:

- There exists a true eigenvalue $\lambda_n^{\mathrm{True}} \in \mathrm{Spec}(A)$ such that

$$|\lambda_n - \lambda_n^{\mathrm{True}}| = \mathcal{O}(\varepsilon),$$

- and there exists a corresponding eigenfunction $\varphi_n^{\mathrm{True}}$ such that

$$\|\varphi_n - \varphi_n^{\mathrm{True}}\|_{\mathcal{H}} = \mathcal{O}(\varepsilon).$$

So in our case, as our $(\phi_n^{\mathrm{WKB}}, \lambda_n^{\mathrm{WKB}})$ are quasimodes, L_ε is self-adjoint and has compact resolvent on $L^2[0,1]$ (a Hilbert space), we have that:

$$|\lambda_n^{\mathrm{WKB}} - \lambda_n^{\mathrm{True}}| = \mathcal{O}(\varepsilon) \quad \text{and} \quad \left\|\phi_n^{\mathrm{WKB}} - \phi_n^{\mathrm{True}}\right\|_{L^2([0,1])} = \mathcal{O}(\varepsilon)$$

for some Eigenvalue-Eigenfuction pair $(\phi_n^{\mathrm{True}}, \lambda_n^{\mathrm{True}})$ of L_ε.

8 Asymptotic Orthonormality

As $\left\|\phi_k^{\text{WKB}} - \phi_k^{\text{True}}\right\|_{L^2([0,1])} = \mathcal{O}(\varepsilon)$, we know that:

$\|\phi_n^{\text{WKB}} - \phi_n^{\text{True}}\|_{L^2([0,1])} \to 0$ as $\varepsilon \to 0$, this is the same as saying:

$\phi_n^{\text{WKB}} = \phi_n^{\text{True}} + e_n \quad \text{where } \|e_n\|_{L^2([0,1])} \to 0 \quad \text{as } \varepsilon \to 0.$

Using this, we consider the inner product of WKB eigenfunctions:

$$\langle \phi_n^{\text{WKB}}, \phi_m^{\text{WKB}} \rangle_{L^2([0,1])} = \langle \phi_n^{\text{True}} + e_n, \phi_m^{\text{True}} + e_m \rangle_{L^2([0,1])}.$$

$$= \int_0^1 (\phi_n^{\text{True}} + e_n)(\phi_m^{\text{True}} + e_m)\, dx$$

$$= \int_0^1 \phi_n^{\text{True}}\phi_m^{\text{True}} + e_n\phi_m^{\text{True}} + e_m\phi_n^{\text{True}} + e_n e_m\, dx$$

$$= \langle \phi_n^{\text{True}}, \phi_m^{\text{True}} \rangle_{L^2([0,1])} + \langle e_n, \phi_m^{\text{True}} \rangle_{L^2([0,1])} + \langle e_m, \phi_n^{\text{True}} \rangle_{L^2([0,1])} + \langle e_n, e_m \rangle_{L^2([0,1])}.$$

Now consider:

$$\left|\langle \phi_n^{\text{WKB}}, \phi_m^{\text{WKB}} \rangle_{L^2([0,1])} - \langle \phi_n^{\text{True}}, \phi_m^{\text{True}} \rangle_{L^2([0,1])}\right| = \left|\langle e_n, \phi_m^{\text{True}} \rangle_{L^2([0,1])} + \langle e_m, \phi_n^{\text{True}} \rangle_{L^2([0,1])} + \langle e_n, e_m \rangle_{L^2([0,1])}\right|$$

$$\leq |\langle e_n, \phi_m^{\text{True}} \rangle_{L^2([0,1])}| + |\langle e_m, \phi_n^{\text{True}} \rangle_{L^2([0,1])}| + |\langle e_n, e_m \rangle_{L^2([0,1])}|$$

$$\leq \|e_n\|_{L^2([0,1])}\|\phi_m^{\text{True}}\|_{L^2([0,1])} + \|e_m\|_{L^2([0,1])}\|\phi_n^{\text{True}}\|_{L^2([0,1])} + \|e_n\|_{L^2([0,1])}\|e_m\|_{L^2([0,1])}.$$

From before we know: $\|\phi_n^{\text{True}}\|_{L^2([0,1])}$ can be normalized so set

$$\|\phi_n^{\text{True}}\|_{L^2([0,1])} = 1.$$

This gives:

$$\left|\langle \phi_n^{\text{WKB}}, \phi_m^{\text{WKB}} \rangle_{L^2([0,1])} - \langle \phi_n^{\text{True}}, \phi_m^{\text{True}} \rangle_{L^2([0,1])}\right| \leq \|e_n\|_{L^2([0,1])} + \|e_m\|_{L^2([0,1])} + \|e_n\|_{L^2([0,1])}\|e_m\|_{L^2([0,1])}.$$

We know $\|e_n\|_{L^2([0,1])} \to 0$ as $\varepsilon \to 0$, hence:

$$\left|\langle \phi_n^{\text{WKB}}, \phi_m^{\text{WKB}} \rangle_{L^2([0,1])} - \langle \phi_n^{\text{True}}, \phi_m^{\text{True}} \rangle_{L^2([0,1])}\right| \to 0 \quad \text{as } \varepsilon \to 0.$$

So:

$$\langle \phi_n^{\text{WKB}}, \phi_m^{\text{WKB}} \rangle_{L^2([0,1])} \to \langle \phi_n^{\text{True}}, \phi_m^{\text{True}} \rangle_{L^2([0,1])} \quad \text{as } \varepsilon \to 0.$$

But, from earlier we know:

$$\langle \phi_n^{\text{True}}, \phi_m^{\text{True}} \rangle_{L^2([0,1])} = \delta_{nm},$$

So:

$$\langle \phi_n^{\text{WKB}}, \phi_m^{\text{WKB}} \rangle_{L^2([0,1])} \to \delta_{nm} \quad \text{as } \varepsilon \to 0.$$

So, the WKB eigenfunctions for the Linearised Swift-Hohenberg Operator are asymptotically orthonormal.

9 Assumptions made

To finish this paper, I thought it may be a good idea to compile a list of the assumptions made throughout (in order). Now, a reader can see if they can extend the arguments I have made and reduce the assumptions to reach an even more complete conclusion.

Our assumptions:

- In the WKB ansatz we assumed throughout that $S'(x) > 0$ and that $S(x)$ is an infinitely differentiable function on [0,1]. We do this to avoid turning points, ensure expansions are valid, and guarantee a consistent definition of the phase function $S(x)$
- We assumed that the WKB ansatz has a discrete set of eigenvalues
- We assumed that the sine term doesn't survive in the WKB eigenfunction after applying Nuemann BCs
- In the WKB ansatz we also assumed that $A_n(0), A_n(1) \neq 0$
- We assumed that $u \in C^4[0,1]$
- We assumed that our solution to the full PDE was a separable one with $T_n(t) \in C^1[0,1]$
- Throughout section 6, we assumed that $u, w \in C^4[0,1]$
- We assumed that our spatially varying growth rate $r(x) \in C^1[0,1]$
- We assume that our WKB ansatz's amplitude $A_n(x)$ is square integrable

If any extensions are made, please email tom.macauley@durham.ac.uk to discuss.

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