From the 840-Term Functional Equation to Goncharov's 22-Term Relation for the Trilogarithm

Tom Macauley

1 Motivation

In my reading I felt it was hard to understand the reduction from the 840-term trilogarithm functional equation to Goncharov's famous 22-term relation. Furthermore, the reduction is not given rigorously in any text I have come across. Therefore, for my own interest and to make the material surrounding this reduction more accessible to others I clearly explain the process.

I believe the content of this paper to be accessible to anyone who has finished their second year of mathematics at university.

The following references are how I learnt about this material outside of meetings with my supervisor; I believe that they are all useful resources for one trying to learn about functional equations involving polylogarithms. As I have simply read the below material, then written this paper separately, it would be impossible to reference in the usual way throughout the paper as there is a lot of crossover material between them. So instead, I present them here and explain a little bit about what each paper goes into:

[1] This paper is the first one I read on polylogarithms, it introduces them nicely and then touches on lots of different functional equations involving them. In particular, it looks in detail at Goncharov's 22-term relation and describes it using both algebraic and geometric arguments.

[2] This paper delves deep into many areas surrounding polylogarithms, functional equations and more. It is a much longer read and is at times tricky but the knowledge within is valuable.

2 The Setup

Firstly, we define the *tri-logarithm*, $\mathrm{Li}_3(z)$:

For $z \in \mathbb{C}$ with $|z| < 1$, we have:

$$\mathrm{Li}_3(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^3}.$$

Although not necessarily important here, $\mathrm{Li}_3(z)$ can be extended to a multivalued analytic function on $\mathbb{C}$ with branch points at $z = 0, 1, \infty$.

We now define the following one-valued continuous function $\mathcal{L}_3 : \mathbb{C} \to \mathbb{R}$ given by:

$$\mathcal{L}_3(z) = \mathrm{Re}\left(\sum_{r=0}^{3} \frac{2^r B_r}{r!} \big(\log|z| \big)^r \, \mathrm{Li}_{3-r}(z) \right), \quad |z| < 1, \; z \notin \{0, 1\},$$

where B_r is the r'th Bernoulli number. For $|z| > 1$, $\mathcal{L}_3(z)$ is given by $\mathcal{L}_3(z) = \mathcal{L}_3\left(\frac{1}{z}\right)$.

For 6 points $m_1, \ldots, m_6 \in \mathbb{P}^2$ write

$$\langle ijk \rangle = \det(m_i, m_j, m_k).$$

Now consider the triple ratio:

$$r'(m_1, m_2, \ldots, m_6) = \frac{\langle 124 \rangle \langle 235 \rangle \langle 316 \rangle}{\langle 125 \rangle \langle 236 \rangle \langle 314 \rangle},$$

we then have:

$$r_3 = \mathrm{Alt}_{S_6}(r') = \sum_{\sigma \in S_6} \mathrm{sgn}(\sigma)\, r'(m_{\sigma(1)}, \ldots, m_{\sigma(6)}).$$

With all of the above we can now finally state the 840 term relation:

$$\mathcal{L}_3[\sum_{i=1}^{7} (-1)^{i-1}\, r_3(l_1, \ldots, \hat{l_i}, \ldots, l_7)] = 0,$$

where $\hat{l_i}$ indicates missing out that point in the calculation of r_3.

This functional equation holds for any ordered 7-tuple of points in $\mathbb{P}^2$ in general position.

3 Defining our space: The Bloch Group

The functional equations we work with throughout this paper all live in the Bloch Group of weight 3. Formally, the group is:

$$B_3(F) \;=\; \frac{\mathbb{Z}[F \setminus \{0,1\}]}{R_3}$$

where:

- $\mathbb{Z}[F \setminus \{0,1\}]$ is the free abelian group on generators $\{x\}_3$, with $x \in F \setminus \{0,1\}$. In this paper we use $F = \mathbb{C}$.
- R_3 is the subgroup generated by a set of functional relations (to be described later).

Put more simply:

The Bloch group of weight 3 is an additive group generated by formal symbols $\{x\}_3$ (where $x \in F \setminus \{0,1\}$), modulo functional equations that the classical trilogarithm satisfies up to products of lower weight logarithms.

Modulo products of lower weight logarithms means we ignore terms like:

$$\log(x)\,\mathrm{Li}_2(y), \qquad \log^3(x), \quad \text{etc.}$$

It is also important to note that, in $B_3(F)$, constants such as $\{0\}_3$, $\{1\}_3$, and $\{\infty\}_3$ are usually declared zero.

Finally, in the Block group of weight 3, we take inverses to be 'equal', that is: $\{x\}_3 = \left\{\frac{1}{x}\right\}_3$. This is important in sections 4 and 7.

4 Why 840 terms?

$|S_6| = 6!$, hence r_3 has 6! terms in its sum. However, we must account for duplicates, we do this by considering the ways we can rewrite r' but leave it unchanged. Recall:

$$r'(m_1, m_2, \ldots, m_6) = \frac{\langle 124\rangle\langle 235\rangle\langle 316\rangle}{\langle 125\rangle\langle 236\rangle\langle 314\rangle},$$

There are 3 permutations in S_6 we can apply that leave r' unchanged, these are: $(id), (123)(456),$ and $(132)(465)$.

Furthermore, there are 3 permutations in S_6 that take $r' \to 1/r'$, these are: $(12)(45), (13)(46),$ and $(23)(56)$.

So, recalling that in $B_3(F)$, inverses are equivalent. We now have a total of 6 permutations in S_6 that leave r' 'unchanged'.

Therefore, r_3 actually has $\frac{6!}{6} = 120$ distinct terms in its sum after accounting for duplicate terms.

Hence, as we sum over 7 (r_3) terms, the total number of distinct terms in the sum:

$$[\sum_{i=1}^{7}(-1)^{i-1}\, r_3(l_1, \ldots, \hat{l_i}, \ldots, l_7)]$$

is $(7)(120) = 840$.

Then, as $\mathcal{L}_3$ can be applied linearly to each term, the total number of terms in the sum does not change. Therefore, the functional equation:

$$\mathcal{L}_3[\sum_{i=1}^{7}(-1)^{i-1}\, r_3(l_1, \ldots, \hat{l_i}, \ldots, l_7)] = 0,$$

has 840 terms in total.

5 Our degenerate set of points

In order to arrive at Goncharovs 22-term relation we cannot just use 7 general points, we must use a more degenerate case in order to reduce the number of variables in the equation down to 3. A convenient choice, and the one I will use is:

$$l_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad l_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad l_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}, \quad l_4 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad l_5 = \begin{pmatrix}1\\a\\0\end{pmatrix}, \quad l_6 = \begin{pmatrix}0\\1\\b\end{pmatrix}, \quad l_7 = \begin{pmatrix}c\\0\\1\end{pmatrix}.$$

Other authors may use a different set of points; this is the choice I find to be most convenient.

6 Our first and biggest reduction

Due to the degenerate set of points we have chosen, lots of the terms in our 840 term relation now contain a 0 in the numerator or the denominator; these terms can now be removed as in the Bloch Group (where Goncharov's 22-term relation lives), these terms are trivial. Think of it like adding zero to a number in the integers.

We are now left with only the terms that contain expressions of a, b and c in both the numerator and denominator.

7 Using some relations to reduce again

Now we are left with all non-trivial terms after applying the 840-term functional equation to our degenerate set of points, we can use some relations that we have in the Bloch group of weight 3. These make up some of the relations in $R_3(F)$. They are:

$$\{x\}_3 = \left\{\tfrac{1}{x}\right\}_3, \quad \{x\}_3 + \left\{1 - \tfrac{1}{x}\right\}_3 + \left\{\frac{1}{1-x}\right\}_3 = \{1\}_3, \quad \{x^2\}_3 = 4(\{x\}_3 + \{-x\}_3),$$

After applying these everywhere possible, no further reductions can be made and our number of terms in our equation is as small as possible.

8 Extracting Goncharov's 22-term relation

From this point, we have then gotten rid of all terms necessary and can reduce no further, we are now left with 22-terms. These make up Goncharov's

22-term relation. It can be written in many different ways, I now provide the one that I was first shown which also happens to be my favourite:

$$\gamma(a,b,c) = \bigoplus_{\text{cycle}} \left(\{ca-a+1\}_3 + \left\{\tfrac{ca-a+1}{ca}\right\}_3 + \{c\}_3 + \left\{\tfrac{bc-c+1}{(ca-a+1)b}\right\}_3 - \left\{\tfrac{ca-a+1}{c}\right\}_3 \right.$$
$$\left. + \left\{\tfrac{(bc-c+1)a}{ca-a+1}\right\}_3 - \left\{\tfrac{bc-c+1}{(ca-a+1)bc}\right\}_3 \right) + \{-abc\}_3 \, .$$

Where $\bigoplus_{\text{cycle}} f(a,b,c) := f(a,b,c) + f(c,a,b) + f(b,c,a)$.

$\gamma(a,b,c)$ are the leftover terms in the sum: $\sum_{i=1}^{7} (-1)^{i-1} r_3(l_1, \ldots, \widehat{l_i}, \ldots, l_7)$.

Goncharov's 22-term relation is therefore:

$$\mathcal{L}_3\big(\gamma(a,b,c)\big) = 0.$$

9 Conclusion

In theory, it would be possible to do this reduction by hand; however, due to the vast number of terms and complex calculations, I see it to be nearly impossible. One could alternatively calculate all terms in mathematica to make this process easier, even still, when trying to then perform the reduction in Section 7, spotting all simplifications would be very tricky even to the most trained eye. Therefore, this paper serves as an explanation of how to derive the Goncharov's 22-term relation from the 840-term trilogarithm relation, not a term-by-term proof of the reduction. I hope that this paper helps those interested understand this process better.

References

[1] Herbert Gangl. "Functional Equations For Higher Polylogarithms". In: *Sel. math., New ser. 9, 361–377* (2003).

[2] A.B. Goncharov. "Geometry of configurations, polylogarithms and motivic cohomology". In: *Advances in Mathematics 114 (1995), no. 2, 197–318* (1995).