

# Solution to Problem 12388

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**Problem 12388** *Proposed by A. Garcia(France)* Let  $\alpha$  be a real number. Evaluate

$$I(\alpha) = \int_0^\infty \frac{\ln^2(x) \arctan(x)}{x^2 - 2 \cos(\alpha)x + 1} dx$$

**Solution.**

$$\begin{aligned} I(\alpha) &= \int_0^1 \frac{\ln^2(x) \arctan(x)}{x^2 - 2 \cos(\alpha)x + 1} dx + \int_1^\infty \frac{\ln^2(x) \arctan(x)}{x^2 - 2 \cos(\alpha)x + 1} dx \\ &= \int_0^1 \frac{\ln^2(x) \arctan(x)}{x^2 - 2 \cos(\alpha)x + 1} dx + \int_0^1 \frac{\ln^2(x) \arctan(\frac{1}{x})}{x^2 - 2 \cos(\alpha)x + 1} dx \\ &= \int_0^1 \frac{\ln^2(x) \arctan(x)}{x^2 - 2 \cos(\alpha)x + 1} dx + \int_0^1 \frac{\ln^2(x) (\frac{\pi}{2} - \arctan(x))}{x^2 - 2 \cos(\alpha)x + 1} dx \\ &= \frac{\pi}{2} \int_0^1 \frac{\ln^2(x)}{x^2 - 2 \cos(\alpha)x + 1} dx. \end{aligned}$$

Using  $\cos(\alpha) = \frac{e^{i\alpha} + e^{-i\alpha}}{2}$ , we obtain

$$x^2 - 2 \cos(\alpha)x + 1 = x^2 - x(e^{i\alpha} + e^{-i\alpha}) + 1 = (x - e^{i\alpha})(x - e^{-i\alpha}).$$

Hence,

$$\begin{aligned} \frac{1}{x^2 - 2 \cos(\alpha)x + 1} &= \frac{1}{(x - e^{i\alpha})(x - e^{-i\alpha})} \\ &= \frac{1}{(1 - x e^{-i\alpha})(1 - x e^{i\alpha})} \\ &= \frac{1}{(e^{i\alpha} - e^{-i\alpha})x} \left( \frac{1}{1 - x e^{i\alpha}} - \frac{1}{1 - x e^{-i\alpha}} \right) \\ &= \frac{2i}{\sin(\alpha)x} \left( \sum_{n=0}^\infty x^n e^{ni\alpha} - \sum_{n=0}^\infty x^n e^{-ni\alpha} \right) \\ &= \frac{2i}{\sin(\alpha)} \sum_{n=0}^\infty x^{n-1} (e^{ni\alpha} - e^{-ni\alpha}) \\ &= \frac{2i}{\sin(\alpha)} \sum_{n=0}^\infty x^{n-1} \frac{\sin(n\alpha)}{2i} \\ &= \frac{1}{\sin(\alpha)} \sum_{n=1}^\infty x^{n-1} \sin(n\alpha). \end{aligned}$$

Therefore,

$$\begin{aligned} I(\alpha) &= \frac{\pi}{2} \frac{1}{\sin(\alpha)} \int_0^1 \sum_{n=1}^{\infty} x^{n-1} \sin(n\alpha) \ln^2(x) \, dx \\ &= \frac{\pi}{2} \frac{1}{\sin(\alpha)} \sum_{n=1}^{\infty} \sin(n\alpha) \int_0^1 x^{n-1} \ln^2(x) \, dx \\ &= \frac{\pi}{\sin(\alpha)} \sum_{n=1}^{\infty} \frac{\sin(n\alpha)}{n^3}. \end{aligned}$$

If  $\alpha = 0$ , then

$$\lim_{\alpha \rightarrow 0} I(\alpha) = \pi \sum_{n=1}^{\infty} \lim_{\alpha \rightarrow 0} \frac{\sin(n\alpha)}{\sin(\alpha)} \frac{1}{n^3} = \pi \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^3}{6}.$$

Assume  $\alpha \neq 0$ . Then, by using the Fourier series, we obtain the following

$$\sum_{n=1}^{\infty} \frac{\sin(n\alpha)}{n} = \frac{\pi - \alpha}{2}.$$

Integrate with respect to  $\alpha$ , and obtain

$$\begin{aligned} \int_0^{\alpha} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} \, dx &= \sum_{n=1}^{\infty} \int_0^{\alpha} \frac{\sin(nx)}{n} \, dx = \sum_{n=1}^{\infty} \frac{1 - \cos(\alpha n)}{n^2} \\ &= \int_0^{\alpha} \frac{\pi - x}{2} \, dx = \frac{\alpha\pi}{2} - \frac{\alpha^2}{4}. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^{\alpha} \sum_{n=1}^{\infty} \frac{1 - \cos(\alpha n)}{n^2} \, dx &= \sum_{n=1}^{\infty} \int_0^{\alpha} \frac{1 - \cos(nx)}{n^2} \, dx = \sum_{n=1}^{\infty} \left( \frac{\alpha}{n^2} - \frac{\sin(n\alpha)}{n^3} \right) \\ &= \int_0^{\alpha} \left( \frac{x\pi}{2} - \frac{x^2}{4} \right) \, dx = \frac{\alpha^2\pi}{4} - \frac{\alpha^3}{12} = \frac{\alpha^2(3\pi - \alpha)}{12}. \end{aligned}$$

Finally, isolating the sum,

$$\sum_{n=1}^{\infty} \frac{\sin(\alpha n)}{n^3} = \alpha \sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{\alpha^2(\alpha - 3\pi)}{12} = \alpha \left( \frac{\pi^2}{6} + \frac{\alpha(\alpha - 3\pi)}{12} \right) = \alpha \frac{\alpha^2 - 3\pi\alpha + 2\pi^2}{12},$$

and we conclude

$$I(\alpha) = \frac{\pi\alpha}{\sin(\alpha)} \frac{\alpha^2 - 3\pi\alpha + 2\pi^2}{12} = \frac{\pi}{12} \frac{\alpha(\alpha - \pi)(\alpha - 2\pi)}{\sin(\alpha)}.$$

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