PROBLEM 12142

Proposed by J. A. Scott (UK).

Let $f\colon [a,b] \to \mathbb{R}$ be a twice continously differential function satysfying $\int_a^b f(x)\,\mathrm{d}x = 0$. Prove

$$\int_a^b (f''(x))^2\,\mathrm{d}x \geq \frac{980}{\left(8\sqrt{2}-1\right)^2}\frac{(f(a)+f(b))^2}{(b-a)^3}.$$

Solution proposed by Tommaso Mannelli Mazzoli, Dipartimento di Matematica e informatica DIMAI, Università degli studi di Firenze. Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. By the Cauchy-Schwarz inequality

$$\left(\int_a^b f''(x)g(x)\,\mathrm{d}x\right)^2 \leq \int_a^b (f''(x))^2\,\mathrm{d}x \cdot \int_a^b g^2(x)\,\mathrm{d}x \quad \text{for every } g \in \mathrm{L}^2(a,b).$$

Hence

$$\int_a^b (f''(x))^2\,\mathrm{d}x \geq \frac{\left(\displaystyle\int_a^b f(x)''g(x)\,\mathrm{d}x\right)^2}{\displaystyle\int_a^b g^2(x)\,\mathrm{d}x}, \quad \text{for every } g \in \mathrm{L}^2(a,b).$$

By integrating by parts twice we have

$$\begin{aligned}\int_a^b f''(x)g(x)\,\mathrm{d}x &= g(x)f'(x)\Big|_a^b - \int_a^b g'(x)f'(x)\,\mathrm{d}x \\ &= g(x)f'(x)\Big|_a^b - g'(x)f(x)\Big|_a^b + \int_a^b g''(x)f(x)\,\mathrm{d}x.\end{aligned}$$

Let

$$g(x) = (x-b)(x-a) = x^2 - (a+b)x + ab.$$

We have that:

- $g(a) = g(b) = 0$;
- $g'(a) = a - b$;
- $g'(b) = b - a = -g'(a)$;
- $g''(x) = 1$ for all $x \in (a,b)$.

Hence

$$\int_a^b f''(x)g(x)\,\mathrm{d}x = (a-b)\left[f(a)+f(b)\right].$$

Moreover

$$
\begin{aligned}
\int_a^b g^2(x)\,\mathrm{d}x &= \int_a^b [(x-a)(x-b)]^2\,\mathrm{d}x && \left(t = x - \frac{a+b}{2}\right)\\
&= \int_{\frac{a-b}{2}}^{\frac{b-a}{2}} \left[\left(t+\frac{b-a}{2}\right)\left(t-\frac{b-a}{2}\right)\right]^2 \mathrm{d}t\\
&= \int_{\frac{a-b}{2}}^{\frac{b-a}{2}} \left[t^2-\left(\frac{b-a}{2}\right)^2\right]^2 \mathrm{d}t && \left(\alpha = \frac{b-a}{2}\right)\\
&= \int_{-\alpha}^{\alpha} \left(t^2-\alpha^2\right)^2 \mathrm{d}t\\
&= 2\int_0^{\alpha} (t^2-\alpha^2)^2\,\mathrm{d}t\\
&= 2\int_0^{\alpha} t^4\,\mathrm{d}t - 4\alpha^2\int_0^{\alpha} t^2\,\mathrm{d}t + 2\alpha^4\int_0^{\alpha} 1\,\mathrm{d}t\\
&= \frac{2}{5}\alpha^5 - \frac{4}{3}\alpha^5 + 2\alpha^5\\
&= \frac{16}{15}\alpha^5 = \frac{16}{15}\frac{(b-a)^5}{2^5} = \frac{(b-a)^5}{30}.
\end{aligned}
$$

Finally we get

$$
\int_a^b (f''(x))^2\,\mathrm{d}x \geq \frac{\left(\displaystyle\int_a^b f(x)''g(x)\,\mathrm{d}x\right)^2}{\displaystyle\int_a^b g^2(x)\,\mathrm{d}x} = 30\cdot\frac{\cancel{(b-a)^2}\,(f(a)+f(b))^2}{(b-a)^{\cancel{5}}} = 30\cdot\frac{(f(a)+f(b))^2}{(b-a)^3},
$$

which is a stronger inequality, since

$$
30 > \frac{980}{(8\sqrt{2}-1)^2} \approx 9.212.
$$

□