PROBLEM 12229 (AMERICAN MATHEMATICAL MONTHLY)

Proposed by M. Omarjee (France).
Let $f\colon [0,1] \to \mathbb{R}$ be a function that has continuous derivatives and that satisfies $f(0) = f(1)$ and $\int_0^1 f(x)\,\mathrm{d}x = 0$. Prove

$$30240 \left(\int_0^1 x f(x)\,\mathrm{d}x \right)^2 \leq \int_0^1 \left(f''(x)\right)^2 \mathrm{d}x.$$

Solution proposed by Tommaso Mannelli Mazzoli, TU Wien, 1100 Vienna, Austria.

By the Cauchy-Schwarz inequality

$$\left(\int_0^1 f''(x) g(x)\,\mathrm{d}x \right)^2 \leq \int_0^1 (f''(x))^2\,\mathrm{d}x \cdot \int_0^1 g^2(x)\,\mathrm{d}x \quad \text{for every } g \in \mathrm{L}^2(0,1).$$

After applying integration by parts twice, we find

$$\begin{aligned} \int_0^1 f''(x)g(x)\,\mathrm{d}x &= g(x)f'(x)\Big|_0^1 - \int_0^1 g'(x)f'(x)\,\mathrm{d}x \\ &= g(x)f'(x)\Big|_0^1 - g'(x)f(x)\Big|_0^1 + \int_0^1 g''(x)f(x)\,\mathrm{d}x \\ &= g(1)f'(1) - g(0)f'(0) - f(0)[g'(1) - g'(0)] + \int_0^1 g''(x)f(x)\,\mathrm{d}x. \end{aligned}$$

Let

$$g(x) = \frac{1}{6}x(x-1)\left(x - \frac{1}{2}\right) = \frac{1}{12}\left(2x^3 - 3x^2 + x\right).$$

We have that:

- $g(0) = g(1) = 0$;
- $g'(0) = g'(1)$;
- $g''(x) = x$ for all $x \in [0,1]$.

Thus,

$$\int_0^1 f''(x)g(x)\,\mathrm{d}x = \int_0^1 x f(x)\,\mathrm{d}x.$$

Then

$$\begin{aligned} \int_0^1 g^2(x)\,\mathrm{d}x &= \frac{1}{36}\int_0^1 x^2(x-1)^2\left(x-\frac{1}{2}\right)^2 \mathrm{d}x = \\ &= \frac{1}{36}\int_0^1 \left(x^6 - 3x^5 + \frac{13x^4}{4} - \frac{3x^3}{2} + \frac{x^2}{4}\right)\mathrm{d}x = \frac{1}{36}\cdot\frac{1}{840} = \frac{1}{30240}. \end{aligned}$$

Therefore,

$$30240 \left(\int_0^1 x f(x)\,\mathrm{d}x \right)^2 \leq \int_0^1 (f''(x))^2\,\mathrm{d}x.$$