$$\int_0^{\theta_0} \frac{d\theta}{(\sin^2\frac{\theta_0}{2} - \sin^2\theta/2)^{1/2}} = 2\sqrt{\frac{g}{l}}\int_0^{a} dt = 2\sqrt{\frac{g}{l}}\,\frac{T}{4}$$

$$\int_0^{\theta_0} \frac{d\theta}{(\sin^2\theta_0/2 - \sin^2\theta/2)^{1/2}} = \frac{1}{2}\sqrt{\frac{g}{l}}\,T$$

let $\sin(\theta/2) = \sin(\theta_0/2)\sin(\phi)$, by substitution.

$$\sin(\theta/2) = k\sin(\phi)$$

Implicitly,

$$\tfrac{1}{2}\cos(\theta/2)\,d\theta = k\cos(\phi)\,d\phi$$

$$\tfrac{1}{2}(1-\sin^2\theta/2)^{1/2}d\theta = k\cos(\phi)\,d\phi$$

$$\Rightarrow d\theta = \frac{2k\cos(\phi)}{\sqrt{1-\sin^2\theta/2}}\,d\phi$$

$$= \frac{2k\cos(\phi)\,d\phi}{(1-k^2\sin^2(\phi))^{1/2}}$$

$$\int \frac{k \cos\phi \, d\phi}{\sqrt{1-k^2\sin^2\phi}\,(k\cos\phi)} = \int \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$

A complete elliptic integral of the first kind.

Conventionally, $$K(k) = \int_0^{\phi} \frac{d\phi}{\sqrt{1-k^2\sin^2(\phi)}}$$

Or, $K(k) = \int_0^{\phi_0} \frac{d\phi}{(1-k^2\sin^2\phi)^{1/2}}$, consider expansion of $(1-k^2\sin^2\phi)^{-1/2}$

$$= \int_0^{\phi_0} \sum_{n\geq 0} {}^{-1/2}C_n (k^2\sin^2\phi)^n \, d\phi \quad \text{(Binomial expansion)}$$

$$= \int_0^{\phi} \sum_{n\geq 0} k^{2n} \underbrace{\left(\prod_{i=1}^{n} \frac{1/2 - i}{i}\right)}_{\frac{(-1/2)(-1/2-1)\cdots}{n(n-1)\cdots 1}} (-1)^n \sin^{2n}(\phi)\, d\phi$$

, Binomial coeff

$$= \sum k^{2n} \left(\prod_{i=1}^{n} \frac{1/2 - i}{i}\right)(-1)^n \underbrace{\int_0^{\theta_0} \sin^{2n}(\phi)\, d\phi}$$

Reduction formula for Integral of Power of sine, $n\in\mathbb{N}$

Reduction Formula.., George F. Simmons

$$\int \sin^n x\, dx = \left(\frac{n-1}{n}\right)\int \sin^{n-2} x\, dx - \frac{\sin^{n-1} x \cos x}{n}$$

$$\Rightarrow \int_0^{\phi_1} \sin^{2n}\phi\, d\phi = \frac{2n-1}{2n}\int_0^{\phi} \sin^{2n-2}\phi\, d\phi - \frac{\sin^{2n-1}\phi\cos\phi}{2n}$$

Recall substitution : $\sin(\theta_0/2) = \sin\left(\frac{\theta_0}{2}\right)\sin(\phi)$, $\sin(\phi) = 1$

$\Rightarrow \phi = \pi/2$, limit of integration.

$$\sin^{2n}(0) = 0,\ \cos(\pi/2) = 0 \Rightarrow \left.\frac{\sin^{2n-1}\phi\cos\phi}{2n}\right|_0^{\pi/2} = 0$$

$$\int_0^{\pi/2} \sin^{2n}\phi\, d\phi = \frac{2n-1}{2n}\int_0^{\pi/2} \sin^{2n-2}\phi\, d\phi$$, Reduction again,

$$= \prod_{i=1}^{n} \frac{2i-1}{2i} \int_0^{\pi/2} d\phi$$

$$= \frac{\pi}{2} \prod_{i=1}^{n} \frac{2i-1}{2i}$$

$$\Rightarrow K(k) = \sum_{n\geq 0} k^{2n} \left(\prod_{i=1}^{n} \frac{1/2 - i}{i} \right) (-1)^{i} \left(\frac{\pi}{2} \prod_{i=1}^{n} \frac{2i-1}{2i} \right)$$

$$= \frac{\pi}{2} \sum_{n\geq 0} \left(\prod_{i=1}^{n} \frac{2i-1}{2i} \right)^2 k^{2n}$$

$$K(k) = \frac{1}{2} \sqrt{g/l}\; T$$

$$T = \frac{2}{\omega} K(k)$$

$$= \frac{\pi}{\omega} \sum_{n\geq 0} \left(\prod_{i=1}^{n} \frac{2i-1}{2i} \right)^2 k^{2n}$$