Module:	EE6452 Digital Control
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Assignment Number:	Assignment 2
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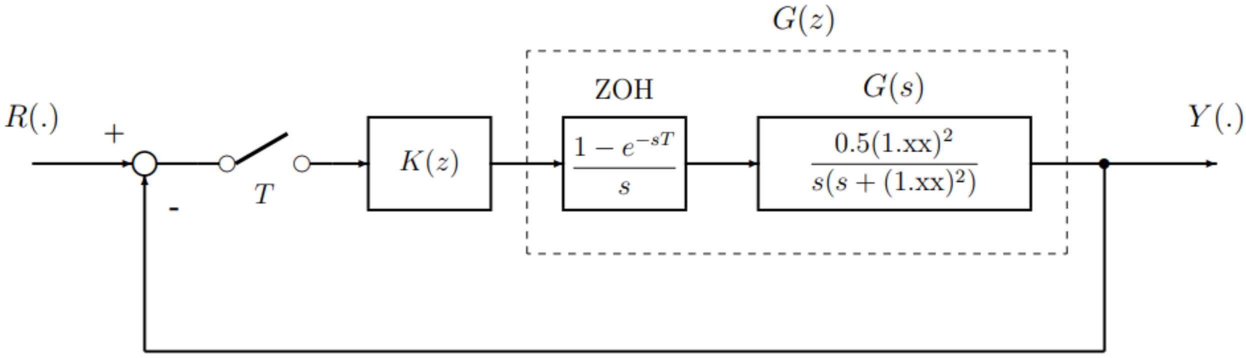

Figure 1: Digital Control Loop

1 Question 1

With reference to Figure 1 and using the last 2 digits of your student ID for the plant model, G(s), determine the discrete-time equivalent model, G(z), via ZOH for a sampling period of T = 0.1 seconds. Write a short Matlab script to validate your derivation.

$$G_{ZOH}(S) = \frac{1-e^{-sT}}{s}, G(S) = \frac{1.53125}{s(s+3.0625)}$$

$$G(Z) = Z[G_{ZOH}(S)\,G(S)] = Z\left[\left(\frac{1-e^{-sT}}{s}\right)\left(\frac{1.53125}{s(s+3.0625)}\right)\right]$$

NOTE: $e^{-sT} = z^{-1}$

$$\therefore G(Z) = 1.53125(1-z^{-1})Z\left[\left(\frac{1}{s^2(s+3.0625)}\right)\right]$$

Using Partial Fractions:

$$\frac{1}{s^2(s+3.0625)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+3.0625}$$

$$A = \frac{1}{(s+3.0625)}\bigg|_{s=0} = \frac{1}{3.0625} = 0.3265$$

$$C = \frac{1}{s^2}\bigg|_{s=-3.0265} = \frac{1}{(-3.0265)^2} = 0.1066$$

$$\frac{1}{s^2(s+3.0265)} = \frac{0.3265}{s^2} + \frac{B}{s} + \frac{0.1066}{s+3.0625}$$

Let s = 1

$$\frac{1}{1^2(1+3.0265)} = \frac{0.3265}{1^2} + \frac{B}{1} + \frac{0.1066}{1+3.0265}$$

$$0.2483 = 0.3265 + B + 0.0265$$

$$B = -0.1066$$

Insert Values for A,B,C

$$G(Z) = 1.53125(1-z^{-1})Z\left[\frac{0.3265}{s^2} - \frac{0.1066}{s} + \frac{0.1066}{s+3.0625}\right]$$

Using the Z-Transform Tables

$$G(Z) = 1.53125(1-z^{-1})\left[\left(\frac{0.3265Tz^{-1}}{(1-z^{-1})^2}\right) - \frac{0.1066}{1-z^{-1}} + \frac{0.1066}{1-e^{-3.0625T}z^{-1}}\right]$$

$$G(Z) = \left[\left(\frac{0.5Tz^{-1}\cancel{(1-z^{-1})}}{(1-z^{-1})^{\cancel{2}}}\right) - \frac{0.1632\cancel{(1-z^{-1})}}{\cancel{1-z^{-1}}} + \frac{0.1632(1-z^{-1})}{1-e^{-3.0625T}z^{-1}}\right]$$

$$G(Z) = \frac{0.5Tz^{-1}(1-e^{-3.0625T}z^{-1}) - 0.1632(1-z^{-1})(1-e^{-3.0625T}z^{-1}) + 0.1632(1-z^{-1})^2}{(1-z^{-1})(1-e^{-3.0625T}z^{-1})}$$

T = 0.1

$$G(Z) = \frac{0.05z^{-1} - 0.03681z^{-2} - 0.1632 + 0.2833z^{-1} - 0.12015z^{-2} + 0.1632 - 0.3264z^{-1} + 0.1632z^{-2}}{1 - 1.7362z^{-1} + 0.7362z^{-2}}$$

$$\therefore G(Z) = \frac{0.0069z^{-1} + 0.00624z^{-2}}{1 - 1.7362z^{-1} + 0.7362z^{-2}} x \left(\frac{z^2}{z^2}\right)$$

$$\boldsymbol{G(Z) = \frac{0.0069z + 0.00624}{z^2 - 1.7362z + 0.7362}}$$

Below shows the MATLAB code used to verify the transfer function G(Z) using zero order hold.

```
%% Question 1
% Determine the ZOH
T = 0.1; % sampling period, T
Gs = tf(1.53125,[1 3.0625 0]); % G(s)
Gz = c2d(Gs,T); % ZOH equivalent, G(z)

% Verify the partial fractions

num = [1];
den = [1 3.0625 0 0];

[r,p,k] = residue(num,den) % Calculate the residues A,B,C
```

2 Question 2

Using the Jury stability criterion, determine the range of stability for the closed-loop system as the proportional controller, KP , is varied from 0 → ∞.

Find the Closed Loop Response

$$\frac{Y(z)}{R(z)} = \frac{L(z)}{1+L(z)} = \frac{K(z)B(z)}{A(z)+kB(z)}$$

$$Gcl(z) = \frac{K(z)(0.0069z + 0.00624)}{z^2 - 1.7362z + 0.7362 + K(z)(0.0069z + 0.00624)}$$

$$P(z) = z^2 + (0.0069K - 1.7362)z + (0.00624K + 0.7362)$$

$$a_0 = 1,\ a_2 = 0.0069K - 1.7362,\ a_3 = 0.00624K + 0.7362$$

Apply the Jury test

1. $|a_n| < a_0$

$$0.00624k + 0.7362 < 1$$
$$0.00624K < 0.2638$$
$$-42.27 < K < 42.27$$

2. $P(1) > 1$

$$1 + 0.0069K - 1.7362 + 0.00624K + 0.7362 > 0$$
$$0.01314K > 0$$
$$K > 0$$

3. $(-1)^n P(-1) > 0$

$$(-1)^2(-1 - 0.0069K + 1.7362 + 0.00624K + 0.7362) > 0$$
$$-6.6x10^{-4}K > -1.4724$$
$$-K > -2230.9$$
$$K < 2230.9$$

4. Step 4 is not needed as the denominator is a quadratic equation.

$$\therefore \mathbf{0 < Kp < 42.27}$$
$$Ku = 42.27$$

3 Question 3

1. **Starting Location @ Kp = 0**

   When Kp = 0,

$$P(z) = A(z) = 0$$
$$z^2 - 1.7362z + 0.7362 = 0$$
$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$z = \frac{1.7362 \pm \sqrt{(1.7362)^2 - 4(0.7362)}}{2}$$
$$z = 1, z = 0.7362$$

2. **Finish Location @ Kp → ∞**

   When Kp → ∞,

$$P(z) = B(z) = 0$$
$$0.0069z + 0.00624 = 0$$
$$z = -0.9043$$

3. **Breakaway and Break in Points**

$$Kp = \frac{-A(z)}{B(z)}$$
$$\frac{dKp}{dz} = -\frac{B(z)A'(z) - A(z)B'(z)}{(B(z)^2)} = 0$$
$$A'(z) = 2z - 1.7362, B'(z) = 0.0069$$
$$(0.0069z + 0.00624)(2z - 1.7362) - (z^2 - 1.7362z + 0.7362)(0.0069z + 0.00624) = 0$$
$$0.0138z^2 + 0.00500225z - 0.010833 - 0.0069z^2 + 0.01198z - 0.00507978 = 0$$
$$0.0069z^2 + 0.01248z - 0.01519 = 0$$
$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$z = \frac{-0.01248 \pm \sqrt{(0.01248)^2 - 4(0.0069)(-0.01519)}}{2(0.0069)}$$
$$z = 0.863, z = -2.6717$$

   - Breakaway point = 0.863
   - Break In Point = -2.6717

4. **Intersection Points**

$$Kp = Ku$$
$$P(z) = z^2 + (0.0069Ku - 1.7362)z + (0.00624Ku + 0.7362) = 0$$

$$Ku = 42.27$$
$$P(z) = z^2 - 1.444537 + 1 = 0$$
$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$z = \frac{1.444537 \pm \sqrt{(1.444537)^2 - 4}}{2}$$
$$z = 0.7225 \pm j0.6916$$

By using the following pole locations for the starting location, the finish location, the breakaway and break in points and the intersection point, the root locus can be drawn by hand as seen in figure 2 below.

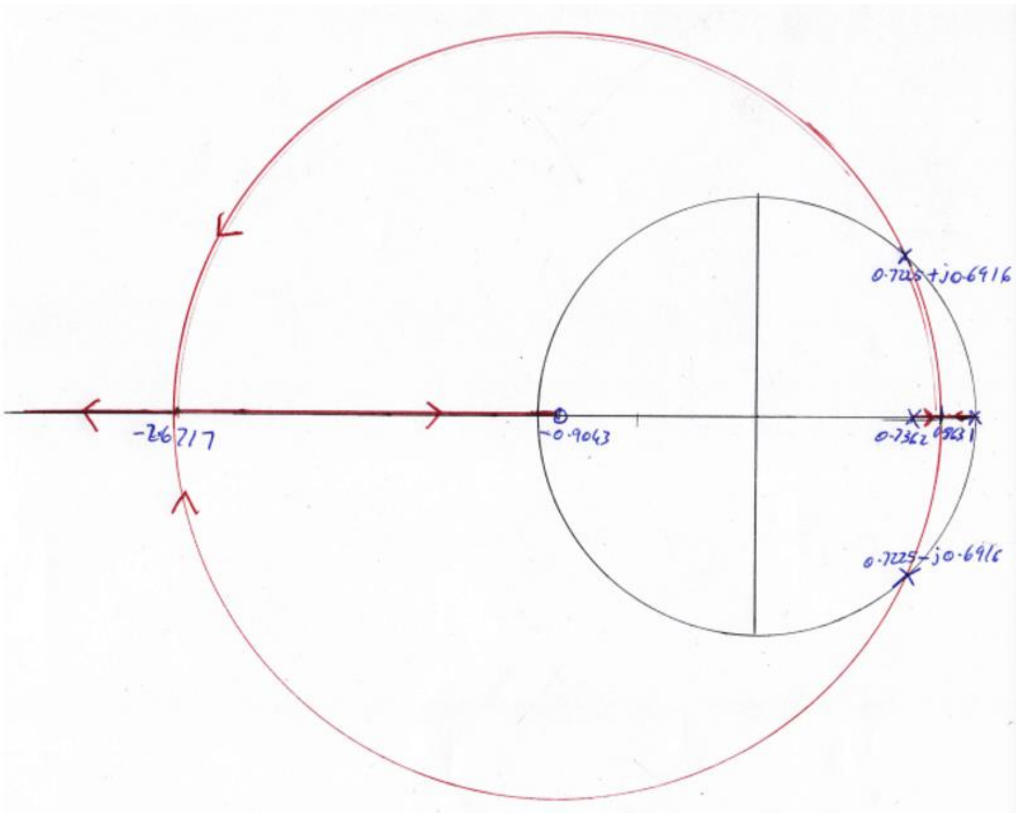

Figure 2: Root Locus for the Closed-Loop Transfer Function

Below shows the MATLAB code used to plot the root locus around the unit circle for the closed-loop transfer function.

```
%% Question 3
% plot the root locus for Gz which is the start point when Kp = 0
figure(2)
rlocus(Gz)
```

Figure 3 shows the root locus of the system plotted in MATLAB

From the root locus plotted in MATLAB the estimations made above are an accurate representation of the sketched root locus seen above.

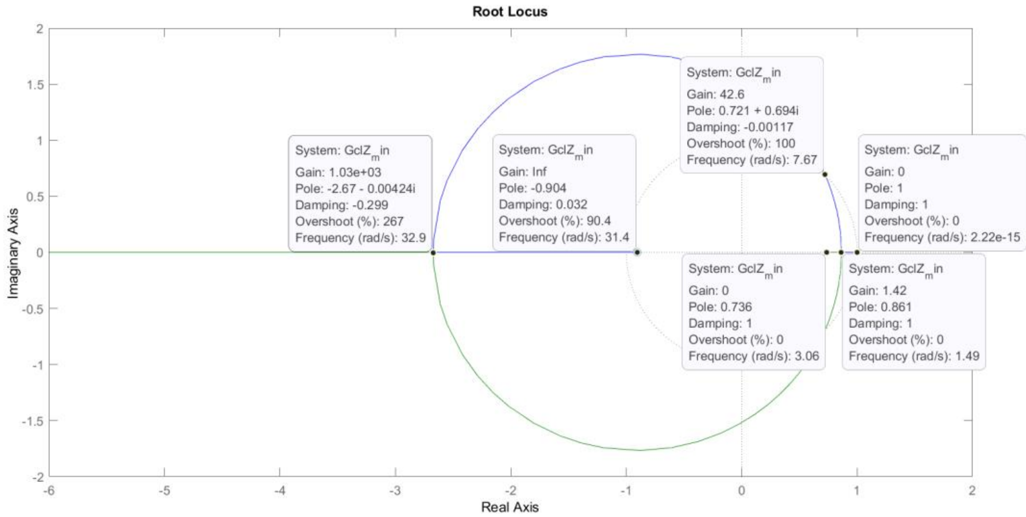

Figure 3: Root Locus Plotted in MATLAB

4 Question 4

If a unit-step is applied to this closed-loop system, what will the percentage overshoot, %OS, and the 2% settling time be for KP = KU/8, where KU is the critical gain. Plot the unit-step input and the unit-step response using Matlab.

$$Gcl(z) = \frac{K(z)(0.0069z + 0.00624)}{z^2 + (0.0069Kp - 1.7362)z + (0.00642Kp + 0.7362)}$$

$$Kp = \frac{Ku}{2} = 5.28375$$

$$Gcl(z) = \frac{0.03646z + 0.03297}{z^2 - 1.6997z + 0.7692}$$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$z = \frac{1.6997 \pm \sqrt{(-1.6997)^2 - 4(0.7692)}}{2}$$

$$z = 0.8498 \pm j0.2167$$

$$r = |z| = \sqrt{(0.8498)^2 + (0.2167)^2} = 0.877$$

$$\emptyset = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{0.2167}{0.8498}\right) = 0.25rad$$

$$\hat{\zeta} = \left(-\frac{\ln(r)}{\sqrt{\emptyset^2 + \ln^2(r)}}\right) = \left(-\frac{\ln(0.877)}{\sqrt{(0.25)^2 + \ln^2(0.877)}}\right) = 0.465$$

$$\widehat{\omega_n} = \frac{1}{T}\sqrt{\emptyset^2 + \ln^2(r)} = \frac{1}{0.1}\sqrt{(0.25)^2 + \ln^2(0.877)} = 2.82rad/s$$

$$t_s = \frac{4}{\hat{\zeta}\widehat{\omega_n}} = \frac{4}{(0.465)(2.82)}$$

$$\boldsymbol{t_s = 3.05s}$$

$$\%OS = e^{\left(\frac{-\hat{\zeta}\pi}{\sqrt{1-\hat{\zeta}^2}}\right)}x100 = e^{\left(\frac{-0.465\pi}{\sqrt{1-(0.465)^2}}\right)}x100$$

$$\boldsymbol{\%OS = 19.2\%}$$

Before the unit step response can be plotted, the transfer function must first be calculated. Below shows the method used to calculate the transfer function for the unit step response.

$$\boldsymbol{For\ Kp = \frac{Ku}{8} = 5.28375}$$

$$Gcl(z) = \frac{0.03646z + 0.03297}{z^2 - 1.6997z + 0.7692}$$

Multiply by the unit step function

$$Gcl(z)\_step = \frac{0.03646z + 0.03297}{z^2 - 1.6997z + 0.7692}\left(\frac{z}{z-1}\right)$$

$$Gcl(z)\_step = \frac{0.03646z^2 + 0.03297z}{z^3 - 2.6997z^2 + 2.4689z - 0.7692}$$

Once the unit step transfer function is calculated the following MATLAB code can be used to plot both the step response and the impulse response of the closed loop system.

```
%% Question 4
n = 51; % samples
k = 0:n-1;
u = [1 zeros(1,n-1)]; % input u(k) = impulse

% Determine the impulse response
num_GZ = [0.03646 0.03297 0]; % numerator for Gz
den_GZ = [1 -1.6997 0.7692]; % denominator for Gz
yk = filter(num_GZ,den_GZ,u); % impulse using the filter command

figure(3)
hold on
stairs(k,yk*10,'b') % stairs function plots the impulse

% Use impulse command to verify results
figure(4)
hold on
impulse(GclZ)

% Determine the step response
hold on
num_Sz = [0 0.03646 0.03297 0]; % numerator for Gz step function
den_Sz = [1 -2.6997 2.4689 -0.7692]; % denominator for Gz step function

sk = filter(num_Sz,den_Sz,u); % step function for Gz
figure(3)
hold on
stairs(k,sk,'r')
xlabel('Sample Period (k)')
ylabel("Amplitude")
title("Impulse Response & Step Response")
legend('Impulse Response','Step Response')
axis([0 50 -0.4 1.6])

% Use step command to verify results
figure(4)
hold on
step(GclZ)
title("Impulse Response & Step Response")
legend('Impulse Response','Step Response')
```

Figure 4 below shows the unit step response and impulse response plotted on the same graph using MATLAB for the above closed loop system.

From the graph, the overshoot can be measure at 19% and the settling time can be measured at 3.05s which is as expected from the above calculations.

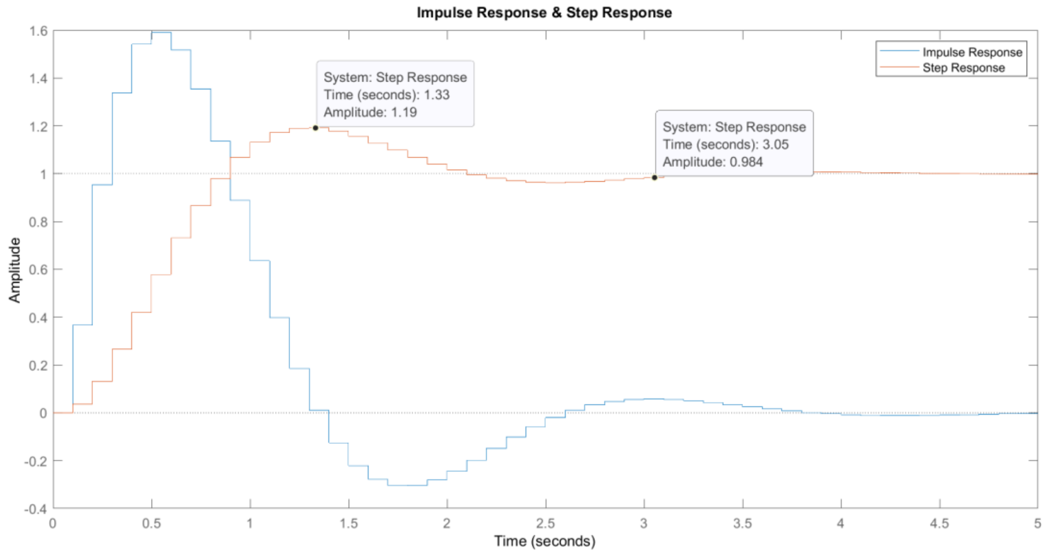

Figure 4: Unit Step vs Impulse Response Plotted in MATLAB

5 Question 5

Using the static velocity error constant, Kv, determine the steady-state error if a unit-ramp is applied to this closed-loop system for 0 < KP < KU . Plot the unit-ramp input and the unit-ramp response using Matlab.

$$ess = \lim_{z\to 1}\left(\left(\cancel{(1-z^{-1})}S(z)\left(\frac{Tz^{-1}}{(1-z^{-1})^{\cancel{2}}}\right)\right)\right), S(z) = \frac{1}{1+L(z)}$$

$$ess = \lim_{z\to 1}\left(\frac{Tz^{-1}}{(1-z^{-1})\big(1+L(z)\big)}\right) \equiv \lim_{z\to 1}\left(\frac{T}{(1-z^{-1})L(z)}\right) = \frac{1}{Kv}$$

$$L(z) = (KP)G(z) = \frac{KP(0.0069z + 0.00624)}{z^2 - 1.7362z + 0.7362}$$

$$Kv = \lim_{z\to 1}\left(\frac{(1-z^{-1})\left(\frac{KP(0.0069z+0.00624)}{z^2-1.7362z+0.7362}\right)}{0.1}\right) = \lim_{z\to 1}\left(\frac{10(1-z^{-1})\big(KP(0.0069z+0.00624)\big)}{z^2-1.7362z+0.7362}\right)$$

$$Kv = \frac{10(1-1^{-1})\big(KP(0.0069(1)+0.00624)\big)}{(1)^2-1.7362(1)+0.7362} = \frac{10(1-1^{-1})\big(KP(0.0069(1)+0.00624)\big)}{0} = \infty$$

$$As\ Kv \to \infty, \boldsymbol{ess = 0}$$

The following MATLAB code can be used to plot the unit-ramp input and the unit ramp response on the same graph.

```
%% Question 5
num_GZ_ramp_Input = [0 T 0]; % numerator for Gz
den_GZ_ramp_Input = [1 -2 1]; % denominator for Gz
yk_ramp_In = filter(num_GZ_ramp_Input,den_GZ_ramp_Input,u); % impulse using the
filter command

figure(5)
hold on
stairs(k,yk_ramp_In,'b') % stairs function plots the impulse

num_GZ_ramp_Response = [0 0 0.03646*T 0.03297*T 0]; % numerator for Gz
den_GZ_ramp_Response = [1 -3.6992 5.1676 -3.2376 0.7692]; % denominator for Gz
yk_ramp = filter(num_GZ_ramp_Response,den_GZ_ramp_Response,u); % impulse using
the filter command

figure(5)
hold on
stairs(k,yk_ramp,'r') % stairs function plots the impulse
title("Ramp Input & Ramp Response")
legend('Ramp Input','Ramp Response')
```

Figure 5 shows the plot taken from MATLAB showing the unit step input versus the unit step response for the above transfer function.

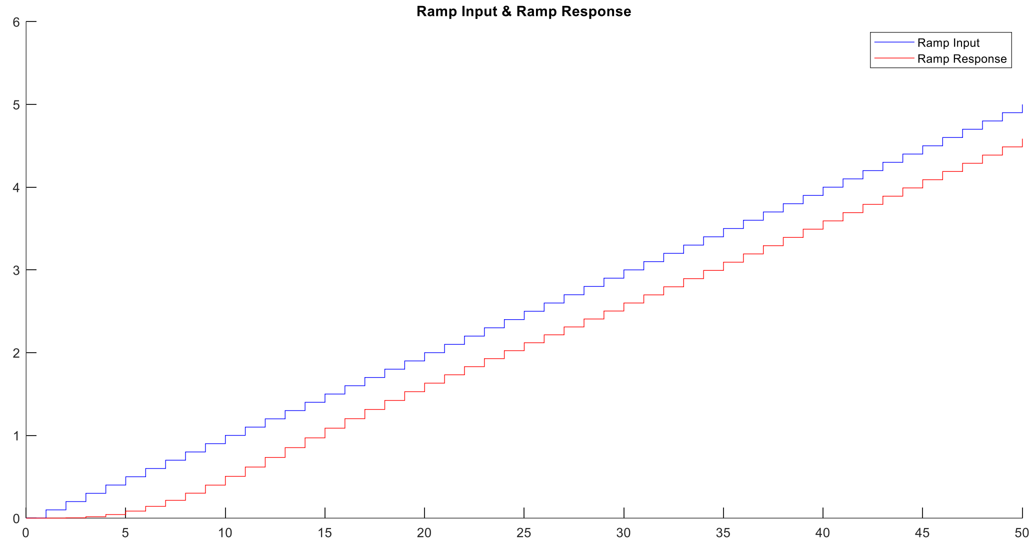

Figure 5: Ramp Input vs Ramp Response Plotted in MATLAB