Module:	EE6452 Digital Control
Date:	24/02/2022
Assignment Number:	Assignment 1
Author / Student Name:	Tom Meehan
Student ID:	18220975

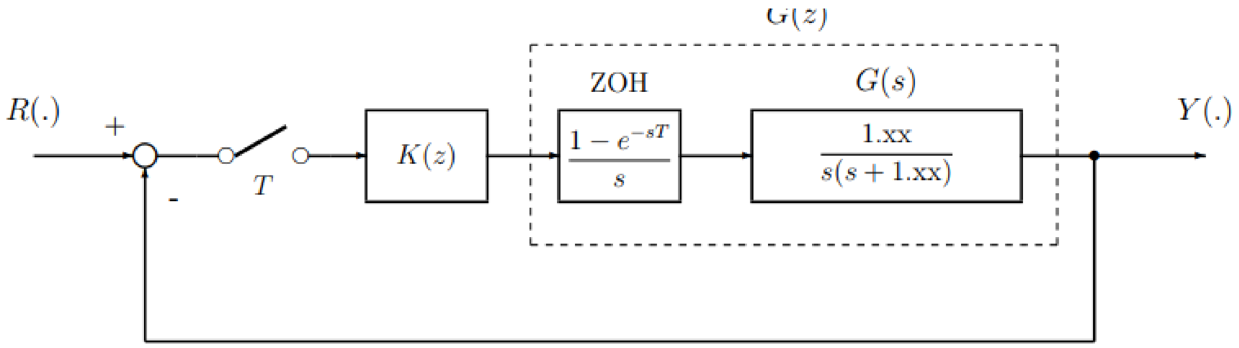

Figure 1: Digital Control Loop

1 Question 1

With reference to Figure 1 and using the last 2 digits of your student ID for the plant model, G(s), determine the discrete-time equivalent model, G(z), via ZOH for a sampling period of T = 0.1 seconds. Write a short MATLAB script to validate your derivation.

$$G_{ZoH}(S) = \frac{1-e^{-sT}}{s}\,, G(S) = \frac{1.75}{s(s+1.75)}$$

$$G(Z) = Z[G_{ZoH}(S)\; G(S)] = Z\left[\left(\frac{1-e^{-sT}}{s}\right)\left(\frac{1.75}{s(s+1.75)}\right)\right]$$

NOTE: $e^{-sT} = z^{-1}$

$$\therefore G(Z) = 1.75(1-z^{-1})Z\left[\left(\frac{1}{s^2(s+1.75)}\right)\right]$$

Using Partial Fractions:

$$\frac{1}{s^2(s+1.75)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1.75}$$

$$A = \left.\frac{1}{(s+1.75)}\right|_{s=0} = \frac{1}{1.75} = 0.57$$

$$C = \left.\frac{1}{s^2}\right|_{s=-1.75} = \frac{1}{(-1.75)^2} = 0.327$$

$$\frac{1}{s^2(s+1.75)} = \frac{0.57}{s^2} + \frac{B}{s} + \frac{0.327}{s+1.75}$$

Let s = 1

$$\frac{1}{1^2(1+1.75)} = \frac{0.57}{1^2} + \frac{B}{1} + \frac{0.327}{1+1.75}$$

$$0.36 = 0.57 + B + 0.12$$

$$B = -0.327$$

Insert Values for A,B,C

$$G(Z) = 1.75(1-z^{-1})Z\left[\frac{0.57}{s^2} - \frac{0.327}{s} + \frac{0.327}{s+1.75}\right]$$

Using the Z-Transform Tables

$$G(Z) = 1.75(1-z^{-1})\left[\left(\frac{0.57Tz^{-1}}{(1-z^{-1})^2}\right) - \frac{0.327}{1-z^{-1}} + \frac{0.327}{1-e^{-1.75T}z^{-1}}\right]$$

$$G(Z) = \left[\left(\frac{Tz^{-1}\cancel{(1-z^{-1})}}{(1-z^{-1})^{\cancel{2}}}\right) - \frac{0.57\cancel{(1-z^{-1})}}{\cancel{1-z^{-1}}} + \frac{0.57(1-z^{-1})}{1-e^{-1.75T}z^{-1}}\right]$$

$$G(Z) = \frac{Tz^{-1}(1-e^{-1.75T}z^{-1}) - 0.57(1-z^{-1})(1-e^{-1.75T}z^{-1}) + 0.57(1-z^{-1})^2}{(1-z^{-1})(1-e^{-1.75T}z^{-1})}$$

$$G(Z) = \frac{Tz^{-1} - Te^{-1.75T}z^{-2} - 0.57 + 1.048z^{-1} - 0.4785z^{-2} + 0.57 - 1.14z^{-1} + 0.57z^{-2}}{1 - e^{-1.75T}z^{-1} - z^{-1} + e^{-1.75T}z^{-2}}$$

T = 0.1

$$\therefore G(Z) = \frac{0.008z^{-1} + 0.00768z^{-2}}{1 - 1.839z^{-1} + 0.8395z^{-2}} x \left(\frac{z^2}{z^2}\right)$$

$$\boldsymbol{G(Z) = \frac{0.008z + 0.00768}{z^2 - 1.839z + 0.8395}}$$

Below shows the MATLAB script used to verify the response as well figure 2 which is the actual value for G(Z) calculated in MATLAB taken from the command window.

```
%% Question 1
% Create a variable for s
syms s

% Verify the partial fractions

num = [1.75];
den = [1 1.75 0 0];

[r,p,k] = residue(num,den) % Calculate the residues A,B,C

x = partfrac(1/(s^3+1.75*s^2)); % Calculate the partial fraction expansion
pretty(x) % Pretty print

% Determine the ZOH
T = 0.1; % sampling period, T
Gs = tf(1.75,[1 1.75 0]); % G(s)
Gz = c2d(Gs,T) % ZOH equivalent, G(z)
```

Figure 2: MATLAB Script to Verify the Above Calculations

```
Gz =

  0.008261 z + 0.007793
  ----------------------
  z^2 - 1.839 z + 0.8395

Sample time: 0.1 seconds
Discrete-time transfer function.
```

Figure 2: MATLAB Calculation

2 Question 2

Determine the poles and zeros of the plant model, G(z). Show how the poles and zeros may be determined using Matlab. Is the plant model stable? Justify your answer.

$$G(Z) = \frac{0.008z + 0.00768}{z^2 - 1.839z + 0.8395}$$

Factorise the Numerator and Denominator of G(Z)

$$G(Z) = \frac{0.008(z + 0.96)}{(z - 1)(z - 0.84)}$$

Zeros

$$0.008(z + 0.96) = 0$$

$$\boldsymbol{z = -0.96}$$

Poles

$$z - 1 = 0$$

$$\boldsymbol{z = 1}$$

$$z - 0.84$$

$$\boldsymbol{z = 0.84}$$

The above system has a zero at z = -0.96 and 2 poles at z = 1 and z = 0.84 respectively.

Below shows the MATLAB script used to calculate the pole and zero locations for the open loop system as well as figure 3 and figure 4 which show the output of the poles and zeros in the command window and a pole zero map plotting the respective poles and zeros around the unit circle.

```
%% Question 2
figure(1)
pzmap(Gz) % Pole Zero Plot of Gz

Zg = zero(Gz) % Calculate the zeros for GZ
Pg = pole(Gz) % Calculate the poles for GZ
```

```
Zg =

   -0.9433


Pg =

    1.0000
    0.8395
```

Figure 3: MATLAB Command Window Showing Pole and Zero Locations for G(Z)

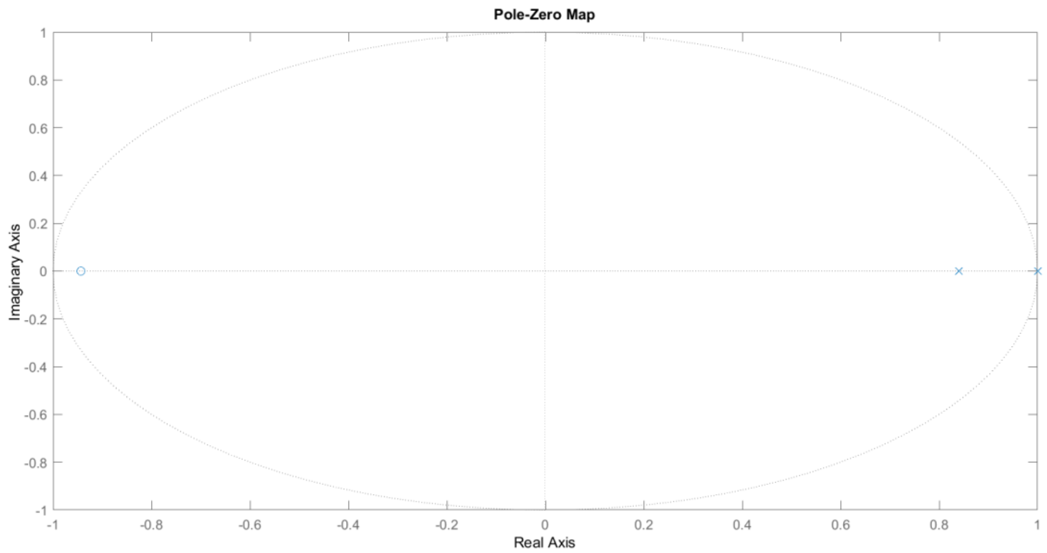

Figure 4: Pole Zero Plot Around the Unit Circle Plotted in MATLAB for G(Z)

From the pole zero plot above as well as from the relevant calculations it can be determined that this system is marginally stable as there is one pole located on the unit circle while the other pole lies within the limits of the unit circle.

3 Question 3

Derive the general closed-loop transfer function, Gcl(z), for the feedback configuration in Figure 1. For K(z) = 1.xx, where xx are the last 2 digits of your student ID, determine the characteristic equation for the closed-loop system. Is the closed-loop system stable? Justify your answer. Show how the closed-loop transfer function may be determined using Matlab.

$$Gcl(z) = \frac{K(z)G(z)}{1 + K(z)G(z)}$$

$$Gcl(Z) = \frac{1.75\left(\frac{0.008z + 0.00768}{z^2 - 1.839z + 0.8395}\right)}{1 + 1.75\left(\frac{0.008z + 0.00768}{z^2 - 1.839z + 0.8395}\right)}$$

$$Gcl(Z) = \frac{\left(\frac{0.014z + 0.01344}{z^2 - 1.839z + 0.8395}\right)}{\left(\frac{z^2 - 1.839z + 0.8395 + 0.0014z + 0.01344}{z^2 - 1.839z + 0.8395}\right)}$$

$$Gcl(Z) = \frac{\left(\frac{0.014z + 0.01344}{\cancel{z^2 - 1.839z + 0.8395}}\right)}{\left(\frac{z^2 - 1.825z + 0.85}{\cancel{z^2 - 1.839z + 0.8395}}\right)}$$

$$\boldsymbol{Gcl(Z) = \frac{(0.014z + 0.01344)}{(z^2 - 1.825z + 0.85)}}$$

Factorise the Numerator and Denominator of G(Z)

$$Gcl(Z) = \frac{(0.014z + 0.01344)}{(z^2 - 1.825z + 0.85)}$$

$$Gcl(Z) = \frac{(0.014(z + 0.96))}{((z - 0.9125 + 0.143j)(z - 0.9125 - 0.143j))}$$

Zeros

$$0.014(z + 0.96)$$
$$\mathbf{z = -0.96}$$

Poles

$$(z - 0.9125 + 0.143j)(z - 0.9125 - 0.143j)$$
$$\mathbf{z = 0.9125 + 0.143j}$$
$$\mathbf{z = 0.9125 - 0.143j}$$

Stability

$$|p1| = |p2| = \sqrt{(0.9125)^2 + (0.143j)^2}$$
$$|p1| = |p2| = 0.923$$

The above closed loop model has a zero located at z = 0.96 and 2 complex poles located at z = 0.9125 + 0.143j and z = 0.9125 – 0.143j.

The closed loop system is stable as all poles lie within the unit circle.

Figure 5 and figure 6 show the poles and zeros calculated in MATLAB as well as a pole zero plot around the unit circle for the poles and zeros of the closed loop model.

```
%% Question 3
KZ = 1.75; % Assign value for gain Kz

GclZ = feedback(Gz*KZ,1) % Calculate the closed loop transfer function
figure(2)
pzmap(GclZ) % Pole Zero Plot of GclZ

Zgcl = zero(GclZ) % Calculate the zeros for GclZ
Pgcl = pole(GclZ) % Calculate the poles for GclZ

Sgcl = abs(Pgcl) % Determine if GclZ is stable
```

```
Zgcl =

   -0.9433


Pgcl =

   0.9125 + 0.1430i
   0.9125 - 0.1430i


Sgcl =

     0.9236
     0.9236
```

Figure 5: MATLAB Command Window Showing Pole and Zero Locations for Gcl(Z)

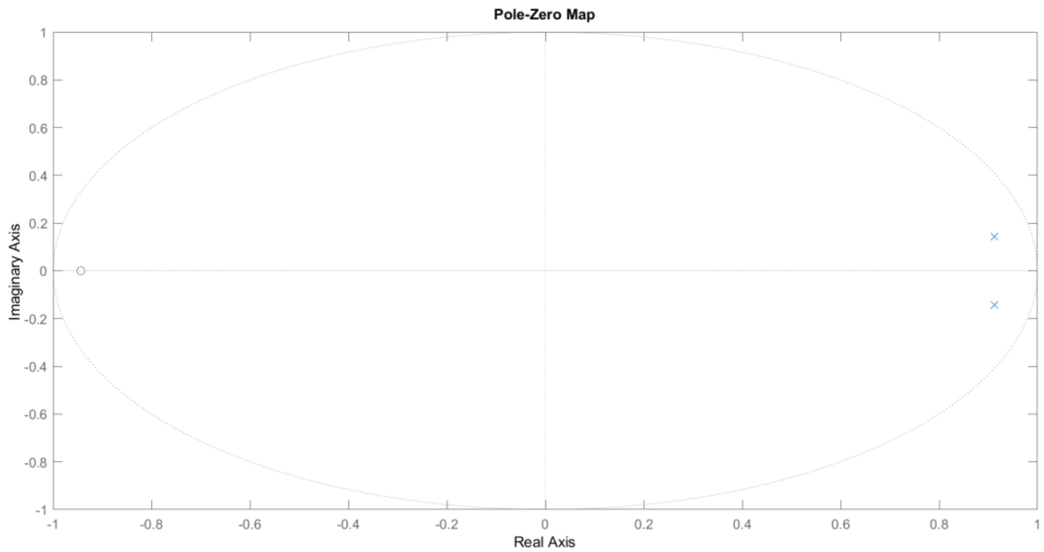

Figure 6: Pole Zero Plot Around the Unit Circle Plotted in MATLAB for Gcl(Z)

4 Question 4

Derive the respective plant model and closed-loop system difference equations where for the latter use K(z) = Kp for 0 < Kp ∈ R. Briefly comment on the influence (or weighting) of the digital controller, Kp, on closed-loop system response.

Plant Model:

$$G(Z) = \frac{0.008z^{-1} + 0.00768z^{-2}}{1 - 1.839z^{-1} + 0.8395z^{-2}} = \frac{Y(Z)}{U(Z)}$$

$$Y(Z)(1 - 1.839z^{-1} + 0.8395z^{-2}) = U(Z)(0.008z^{-1} + 0.00768z^{-2})$$

Inverse Z-Transform

$$y(k) - 1.839y(k-1) + 0.8395y(k-2) = 0.008\delta(k-1) + 0.00768\delta(k-2)$$

$$\boldsymbol{y(k) = 1.839y(k-1) - 0.8395y(k-2) + 0.008\delta(k-1) + 0.00768\delta(k-2)}$$

Closed Loop Model:

$$Gcl(Z) = \frac{(0.014z^{-1} + 0.01344z^{-2})}{(1 - 1.825z^{-1} + 0.85z^{-2})} = \frac{Y(Z)}{U(Z)}$$

$$Ycl(Z)(1 - 1.825z^{-1} + 0.85z^{-2}) = Ucl(Z)(0.014z^{-1} + 0.01344z^{-2})$$

Inverse Z-Transform

$$ycl(k) - 1.825ycl(k-1) + 0.85ycl(k-2) = 0.014\delta cl(k-1) + 0.01344\delta cl(k-2)$$

$$\boldsymbol{ycl(k) = 1.825ycl(k-1) - 0.85ycl(k-2) + 0.014\delta cl(k-1) + 0.01344\delta cl(k-2)}$$

From comparing the difference equations for both models along with the other results calculated above such as the pole and zero locations it can be determined the due to the weighting factor caused by the introduction of the K(Z) term in closed loop model, the original open loop system is marginally stable but by introducing a closed loop with unity feedback the system is now stable.

5 Question 5

Using the Matlab filter function, plot on one graph the plant model (G(z)) and closed-loop system (Gcl(z)) impulse responses and on a separate graph plot the respective unit step responses. In terms of system response, briefly comment on any observations you may have in each case.

Impulse Response:

Below shows the MATLAB code used to calculate and plot the impulse response for both the plant model G(z) and the closed loop model Gcl(Z).

```
%% Question 5
n = 120; % samples
k = 0:n-1;
u = [1 zeros(1,n-1)]; % input u(k) = impulse

num_GZ = [0 0.008 0.00768]; % numerator for Gz
den_GZ = [1 -1.839 0.8395]; % denominator for Gz
yk = filter(num_GZ,den_GZ,u); % impulse using the filter command

figure(3)
hold on
stairs(k,yk,'b') % stairs function plots the impulse

num_GclZ = [0 0.014 0.01344]; % numerator for Gclz
den_GclZ = [1 -1.825 0.8531]; % denominator for Gclz
yk_cl = filter(num_GclZ,den_GclZ,u); % impulse using the filter comman

stairs(k,yk_cl,'r') % stairs function plots the impulse
xlabel('Sample Period (k)')
ylabel("Amplitude")
title("Impulse Response")
legend('G(Z) Plant Model','Gcl(Z) Closed Loop Model')
```

The corresponding impulse response can be seen plotted in figure 7 below.

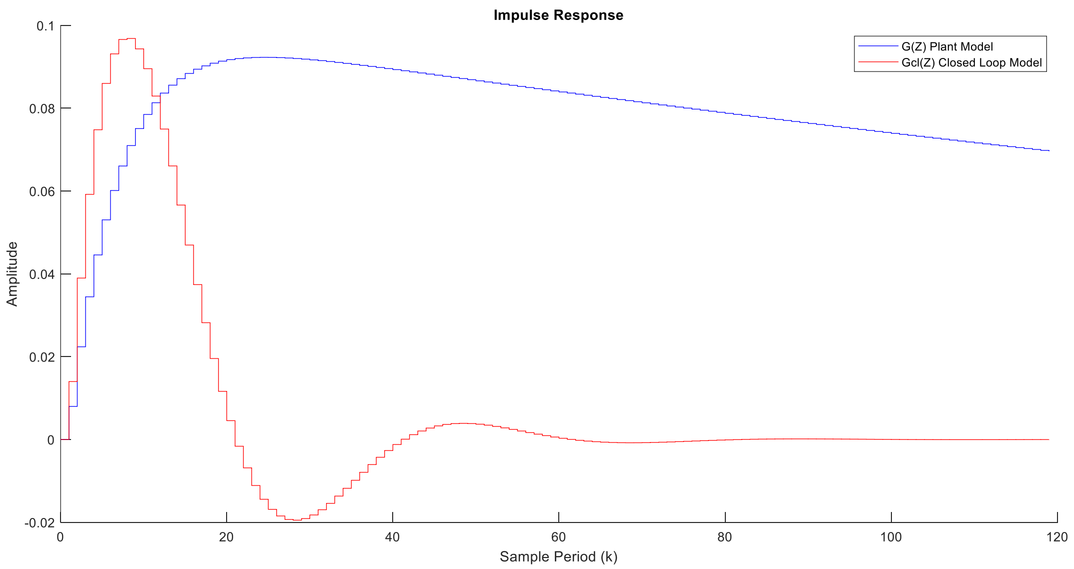

Figure 7: Impulse Response for G(Z) (blue) and Gcl(Z) Red Plotted for 120 Samples.

From the impulse response it can be observed that the plant model will eventually go towards 0 after a large number of samples as it is marginally stable. The point at which G(z) reaches 0 can be observed by simply increasing the number of samples in the above MATLAB code.

The closed loop model experiences a small amount of oscillation before reaching a final value of 0 after approximately 60 samples.

Step Response:

Before writing a MATLAB script, the unit step response for both the plant model and closed loop model must first be calculated by multiplying each transfer function by the unit step function seen below.

Plant Model:

$$S(Z) = \frac{0.008z + 0.00768}{z^2 - 1.839z + 0.8395}\left(\frac{z}{z-1}\right)$$

$$S(Z) = \frac{0.008z^2 + 0.00768z}{z^3 - 2.839z^2 + 2.6785z - 0.8395}$$

Closed Loop Model:

$$Scl(Z) = \frac{(0.014z + 0.01344)}{(z^2 - 1.825z + 0.85)}\left(\frac{z}{z-1}\right)$$

$$Scl(Z) = \frac{0.014z^2 + 0.01344z}{z^3 - 2.825z^2 + 2.6781z - 0.8531}$$

Using the above calculations, the MATLAB script below was used to calculate and plot the unit step function for both the plant model and the closed loop model for 120 samples and the corresponding step response for each system can be seen in figure 8.

```
figure(4)
hold on
num_Sz = [0 0.008 0.00768 0]; % numerator for Gz step function
den_Sz = [1 -2.839 2.6785 -0.8395]; % denominator for Gz step function

sk = filter(num_Sz,den_Sz,u); % step function for Gz
stairs(k,sk,'b')

num_SclZ = [0 0.014 0.01344 0]; % numerator for Gclz step function
den_SclZ = [1 -2.825 2.6781 -0.8531]; % denominator for Gclz step function

sk_cl = filter(num_SclZ,den_SclZ,u); % step function for Gclz
stairs(k,sk_cl,'r')
xlabel('Sample Period (k)')
ylabel("Amplitude")
title("Impulse Response")
legend('G(Z) Plant Model','Gcl(Z) Closed Loop Model')
```

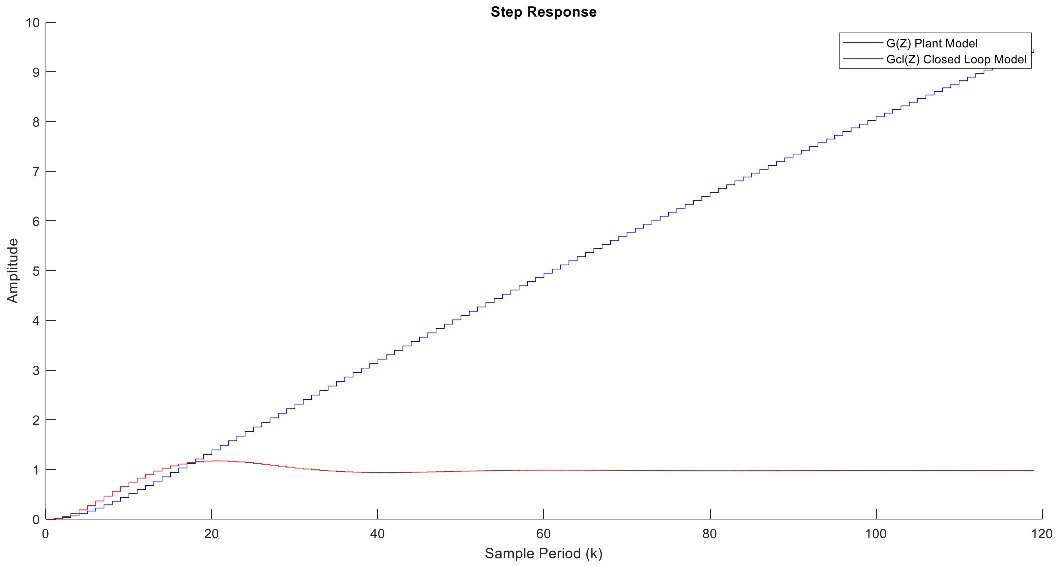

Figure 8: Step Response for G(Z) (blue) and Gcl(Z) Red Plotted for 120 Samples.

From the plot above it can be observed that as the plant model is marginally stable the step response will not go to 1 but instead will go towards a larger value that does not occur during the 120 samples. In order to see the plant models final value, the number of samples must be increased.

As the closed loop model is stable, the step response of the system reaches the ideal value of 1 after approximately 40 samples.

Figure 9 below shows the same step response but for 3200 samples.

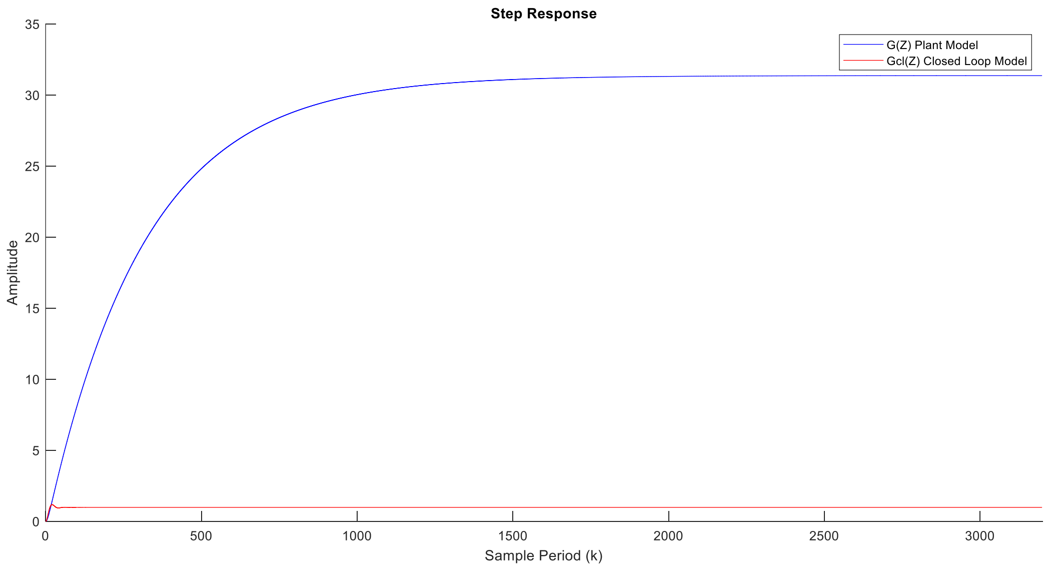

Figure 9: Step Response for G(Z) (blue) and Gcl(Z) Red Plotted for 3200 Samples.

From the step response for 3200 samples, it can observe that the marginally stable plan model reaches a final value of 30 at approximately 1500 samples.

From this it can be determined that the benefit of using a closed loop system is that the original plant model which was marginally stable is now stable with the addition of a unity feedback loop.

NOTE: The step response in figure 9 appears as a continuous signal however, due to the large number of samples, it's difficult to see the impulses.

6 Appendices

Below shows the full MATLAB code required for this assignment.

```
%--------------------------------------------------------------------------
% University of Limerick - Dept. of Electronic and Computer Engineering
%--------------------------------------------------------------------------
% filename: Assignment 1 - Z_Transforms_Assignment_1.m
%
% purpose: Use MATLAB to solve for a ZoH system
%
% created by: Tom Meehan
% created on: 24 Feburary 2022
%
%--------------------------------------------------------------------------
% Copyright 2021 University of Limerick
%--------------------------------------------------------------------------
clc
clear
clear all
close all

%% Question 1
% Create a variable for s
syms s

% Verify the partial fractions

num = [1.75];
den = [1 1.75 0 0];

[r,p,k] = residue(num,den) % Calculate the residues A,B,C

x = partfrac(1/(s^3+1.75*s^2)); % Calculate the partial fraction expansion
pretty(x) % Pretty print

% Determine the ZOH
T = 0.1; % sampling period, T
Gs = tf(1.75,[1 1.75 0]); % G(s)
Gz = c2d(Gs,T) % ZOH equivalent, G(z)

%% Question 2
figure(1)
pzmap(Gz) % Pole Zero Plot of Gz

Zg = zero(Gz) % Calculate the zeros for GZ
Pg = pole(Gz) % Calculate the poles for GZ

%% Question 3
KZ = 1.75; % Assign value for gain Kz

GclZ = feedback(Gz*KZ,1) % Calculate the closed loop transfer function
figure(2)
pzmap(GclZ) % Pole Zero Plot of GclZ

Zgcl = zero(GclZ) % Calculate the zeros for GclZ
Pgcl = pole(GclZ) % Calculate the poles for GclZ

Sgcl = abs(Pgcl) % Determine if GclZ is stable
```

```
%% Question 5
n = 120; % samples
k = 0:n-1;
u = [1 zeros(1,n-1)]; % input u(k) = impulse

num_GZ = [0 0.008 0.00768]; % numerator for Gz
den_GZ = [1 -1.839 0.8395]; % denominator for Gz
yk = filter(num_GZ,den_GZ,u); % impulse using the filter command

figure(3)
hold on
stairs(k,yk,'b') % stairs function plots the impulse

num_GclZ = [0 0.014 0.01344]; % numerator for Gclz
den_GclZ = [1 -1.825 0.8531]; % denominator for Gclz
yk_cl = filter(num_GclZ,den_GclZ,u); % impulse using the filter comman

stairs(k,yk_cl,'r') % stairs function plots the impulse
xlabel('Sample Period (k)')
ylabel("Amplitude")
title("Impulse Response")
legend('G(Z) Plant Model','Gcl(Z) Closed Loop Model')

figure(4)
hold on
num_Sz = [0 0.008 0.00768 0]; % numerator for Gz step function
den_Sz = [1 -2.839 2.6785 -0.8395]; % denominator for Gz step function

sk = filter(num_Sz,den_Sz,u); % step function for Gz
stairs(k,sk,'b')

num_SclZ = [0 0.014 0.01344 0]; % numerator for Gclz step function
den_SclZ = [1 -2.825 2.6781 -0.8531]; % denominator for Gclz step function

sk_cl = filter(num_SclZ,den_SclZ,u); % step function for Gclz
stairs(k,sk_cl,'r')
xlabel('Sample Period (k)')
ylabel("Amplitude")
title("Step Response")
legend('G(Z) Plant Model','Gcl(Z) Closed Loop Model')
```