Module:	EE6452 Digital Control
Date:	25/03/2022
Assignment Number:	Assignment 2
Author / Student Name:	Tom Meehan
Student ID:	18220975

1 Question 1

A system (plant) has the following transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s(s+3.\text{xx})}$$

where xx represents the last 2 digits of your student ID. Determine a continuous-time *state-space* representation of the system, noting that a continuous-time *state-space* representation has the form

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} &= \mathbf{Cx} + \mathbf{Du} \end{aligned}$$

with appropriate matrix and vector dimensions.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 3.75s + 0}$$

$$Y(s)(s^2 + 3.75s + 0) = U(s)$$

$$\frac{d^2y}{dt} + 3.75\frac{dy}{dt} + 0y = u$$

$$\ddot{y} + 3.75\dot{y} + 0 = u$$

$$x1 \doteq y$$

$$x2 \doteq \frac{dy}{dt}$$

$$\dot{x}i \equiv \frac{dxi}{dt}$$

$$\ddot{x}i \equiv \frac{d^2xi}{dt}$$

$$\dot{x}1 = x2$$

$$\dot{x}2 = -0x1 - 3.75x2 + u$$

$$y = x1$$

$$\begin{bmatrix} \dot{x}1 \\ \dot{x}2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -3.75 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix} + [0]u$$

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{-3.75} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}, \boldsymbol{C} = [\mathbf{1} \quad \mathbf{0}], \boldsymbol{D} = [\mathbf{0}]$$

2 Question 2

Using the continuous-time state-space matrices, determine the discrete-time state and output equations. Assume a sampling period of T = 1 second.

$$x(k+1) = \varphi x(k) + \Gamma u(k)$$
$$y(k) = Cx(k) + Du(k)$$

$$\varphi = e^{AT} = L^{-1}(sI - A)^{-1}$$
$$\varphi = L^{-1}\left(s\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & 1\\0 & -3.75\end{bmatrix}\right)^{-1}$$
$$\varphi = L^{-1}\left(\begin{bmatrix}s & -1\\0 & s+3.75\end{bmatrix}\right)^{-1}$$
$$\varphi = L^{-1}\left(\frac{1}{s(s+3.75)}\begin{bmatrix}s+3.75 & 1\\0 & s\end{bmatrix}\right)$$
$$\varphi = L^{-1}\left(\begin{bmatrix}s & \dfrac{1}{s(s+3.75)}\\0 & \dfrac{1}{s+3.75}\end{bmatrix}\right)$$
$$\varphi = \begin{bmatrix}1 & \dfrac{1-e^{-3.75T}}{3.75}\\0 & e^{-3.75T}\end{bmatrix}$$

NOTE: T = 1

$$\boldsymbol{\varphi = \begin{bmatrix}1 & 0.2604\\0 & 0.02351\end{bmatrix}}$$

$$\Gamma = \left(\int_0^T e^{A\lambda}d\lambda\right)B$$
$$\Gamma = \begin{bmatrix}\int_0^T 1d\lambda & \int_0^T \dfrac{1-e^{-3.75T}}{3.75}d\lambda\\ \int_0^T 0d\lambda & \int_0^T e^{-3.75\lambda}d\lambda\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$$
$$\Gamma = \begin{bmatrix}\lambda|_0^T & \dfrac{\lambda}{3.75}+\dfrac{e^{-3.75\lambda}}{14.0625}|_0^T\\ 0 & -\dfrac{e^{-3.75\lambda}}{3.75}|_0^T\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$$

$$\Gamma = \begin{bmatrix}T & \dfrac{T}{3.75}+\dfrac{e^{-3.75T}}{14.0625}-\dfrac{1}{14.0625}\\ 0 & \dfrac{1-e^{-3.75T}}{3.75}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$$
$$\Gamma = \begin{bmatrix}\dfrac{T}{3.75}+\dfrac{e^{-3.75T}}{14.0625}-\dfrac{1}{14.0625}\\ \dfrac{1-e^{-3.75T}}{3.75}\end{bmatrix}$$
$$\boldsymbol{\Gamma = \begin{bmatrix}0.1972\\0.2604\end{bmatrix}}$$

$$\boldsymbol{x(k+1) = x(k)\begin{bmatrix} 1 & 0.2604 \\ 0 & 0.02351 \end{bmatrix} + \begin{bmatrix} 0.1972 \\ 0.2604 \end{bmatrix} u(k)}$$
$$\boldsymbol{y(k) = [1 \quad 0]x(k) + [0]u(k)}$$

3 Question 3

Using the discrete-time state and output equations, obtain the pulse transfer function, G(z), of the system.

$$G(z) = C(zI - \varphi)^{-1}\Gamma + D$$
$$G(z) = [1 \quad 0]\left(z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0.2604 \\ 0 & 0.02351 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0.1972 \\ 0.2604 \end{bmatrix}$$
$$G(z) = \left(\begin{bmatrix} z-1 & -0.2604 \\ 0 & z-0.02351 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0.1972 \\ 0.2604 \end{bmatrix}[1 \quad 0]$$
$$G(z) = \frac{1}{(z-1)(z-0.2604)}\begin{bmatrix} z-0.02351 & 0.2604 \\ 0 & z-1 \end{bmatrix}\begin{bmatrix} 0.1972 \\ 0.2604 \end{bmatrix}[1 \quad 0]$$
$$G(z) = \frac{1}{(z-1)(z-0.2604)}\begin{bmatrix} 0.1972(z-0.02351) + (0.2604)(0.2604) \\ 0.2604(z-1) \end{bmatrix}[1 \quad 0]$$
$$G(z) = \frac{1}{(z-1)(z-0.2604)}[0.1972(z-0.02351) + (0.2604)(0.2604)]$$
$$\boldsymbol{G(z) = \frac{0.1972z + 0.06317}{z^2 - 1.02352 + 0.02352}}$$

4 Question 4

Show that the pulse transfer function, G(z), of the system derived using the discrete-time state and output equations is the same as that derived using the zero-order hold (ZOH) method. Again assume a sampling period of T = 1 second.

$$G_{ZoH}(S) = \frac{1-e^{-sT}}{s}, G(S) = \frac{1}{s(s+3.75)}$$
$$G(Z) = Z[G_{ZoH}(S)\,G(S)] = Z\left[\left(\frac{1-e^{-sT}}{s}\right)\left(\frac{1}{s(s+3.75)}\right)\right]$$

NOTE: $\boldsymbol{e^{-sT} = z^{-1}}$

$$\therefore G(Z) = (1-z^{-1})Z\left[\left(\frac{1}{s^2(s+3.75)}\right)\right]$$

Using Partial Fractions:

$$\frac{1}{s^2(s+3.75)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+3.75}$$
$$A = \left.\frac{1}{(s+3.75)}\right|_{s=0} = \frac{1}{3.75}$$
$$C = \left.\frac{1}{s^2}\right|_{s=-3.75} = \frac{1}{(-3.75)^2} = \frac{16}{225}$$
$$\frac{1}{s^2(s+3.75)} = \frac{\left(\frac{1}{3.75}\right)}{s^2} + \frac{B}{s} + \frac{\left(\frac{16}{225}\right)}{s+3.75}$$

Let s = 1

$$\frac{1}{1^2(1+3.75)} = \frac{\left(\frac{1}{3.75}\right)}{1^2} + \frac{B}{1} + \frac{\left(\frac{16}{225}\right)}{1+3.75}$$

$$\frac{1}{4.75} - \frac{1}{3.75} - \frac{225}{76} = B$$

$$B = -\frac{16}{225}$$

Insert Values for A,B,C

$$G(Z) = (1-z^{-1})Z\left[\frac{\left(\frac{1}{3.75}\right)}{s^2} - \frac{\frac{16}{225}}{s} + \frac{\frac{16}{225}}{s+3.75}\right]$$

Using the Z-Transform Tables

$$G(Z) = (1-z^{-1})\left[\left(\frac{\left(\frac{1}{3.75}\right)Tz^{-1}}{(1-z^{-1})^2}\right) - \frac{\frac{16}{225}}{1-z^{-1}} + \frac{\frac{16}{225}}{1-e^{-3.75T}z^{-1}}\right]$$

$$G(Z) = \left[\left(\frac{\left(\frac{1}{3.75}\right)Tz^{-1}\cancel{(1-z^{-1})}}{(1-z^{-1})^{\cancel{2}}}\right) - \frac{\frac{16}{225}\cancel{(1-z^{-1})}}{\cancel{1-z^{-1}}} + \frac{\frac{16}{225}(1-z^{-1})}{1-e^{-3.75T}z^{-1}}\right]$$

$$G(Z) = \frac{\left(\frac{1}{3.75}\right)Tz^{-1}(1-e^{-3.75T}z^{-1}) - \frac{16}{225}(1-z^{-1})(1-e^{-3.75T}z^{-1}) + \frac{16}{225}(1-z^{-1})^2}{(1-z^{-1})(1-e^{-3.75T}z^{-1})}$$

$$G(Z) = \frac{\left(\frac{1}{3.75}\right)Tz^{-1}(1-e^{-3.75T}z^{-1}) - \frac{16}{225}(1-z^{-1}(e^{-3.75T}+1)+e^{-3.75T}z^{-2}) + \frac{16}{225}(1-2z^{-1}+z^{-2})}{1-1.02352z^{-1}+0.02352z^{-2}}$$

T = 0.1

$$\therefore G(Z) = \frac{0.1972z^{-1}+0.06317z^{-2}}{1-1.02352z^{-1}+0.02352z^{-2}}x\left(\frac{z^2}{z^2}\right)$$

$$\boldsymbol{G(Z) = \frac{0.1972z+0.06317}{z^2-1.7362z+0.7362}}$$

5 Question 5

$$G(Z) = \frac{Y(z)}{U(z)} = \frac{0.1972z^{-1}+0.06317z^{-2}}{1-1.02352z^{-1}+0.02352z^{-2}}$$

$$Y(z)(1-1.02352z^{-1}+0.02352z^{-2}) = U(z)(0.1972z^{-1}+0.06317z^{-2})$$

- The corresponding difference equation can be directly obtained.

$$y(k) - 1.02352y(k-1) + 0.02352y(k-2) = 0.1972u(k-1) + 0.06317u(k-2)$$

$$\boldsymbol{y(k) = 1.02352y(k-1) - 0.02352y(k-2) + 0.1972u(k-1) + 0.06317u(k-2)}$$

6 Question 6

Plot and compare the impulse and step responses of the continuous-time plant model, G(s), and the discrete-time plant model, G(z), using Matlab.

- Recall the following equations for the continuous and discrete time domains:

$$G(s) = \frac{1}{s^2 + 3.75s}, G(z) = \frac{0.1972z + 0.06317}{z^2 - 1.7362z + 0.7362}$$

Impulse Response:

Below shows the MATLAB code used to plot and compare the impulse response of both the continuous time system and the discrete time system.

```
%% Question 5

n = 21; % samples
k = 0:n-1;
u = [1 zeros(1,n-1)]; % input u(k) = impulse

% Determine the impulse response for the discrete time system
num_GZ = [0.1972 0.06317 0]; % numerator for Gz
den_GZ = [1 -1.02352 0.02352]; % denominator for Gz
yk = filter(num_GZ,den_GZ,u); % impulse using the filter command


figure(1)
subplot(2,1,1)
hold on
stairs(k,yk,'b') % stairs function plots the impulse
title('Impulse Response for the Discrete Time System')
xlabel('Sample (y(k))')
ylabel('Amplitude')


% Use impulse command to verify results
figure(1)
subplot(2,1,2)
hold on
impulse(Gss,'b')
title('Impulse Response for the Continious Time System')
```

Figure 1 below shows the impulse response for both the continuous time system and the discrete time system taken from MATLAB.

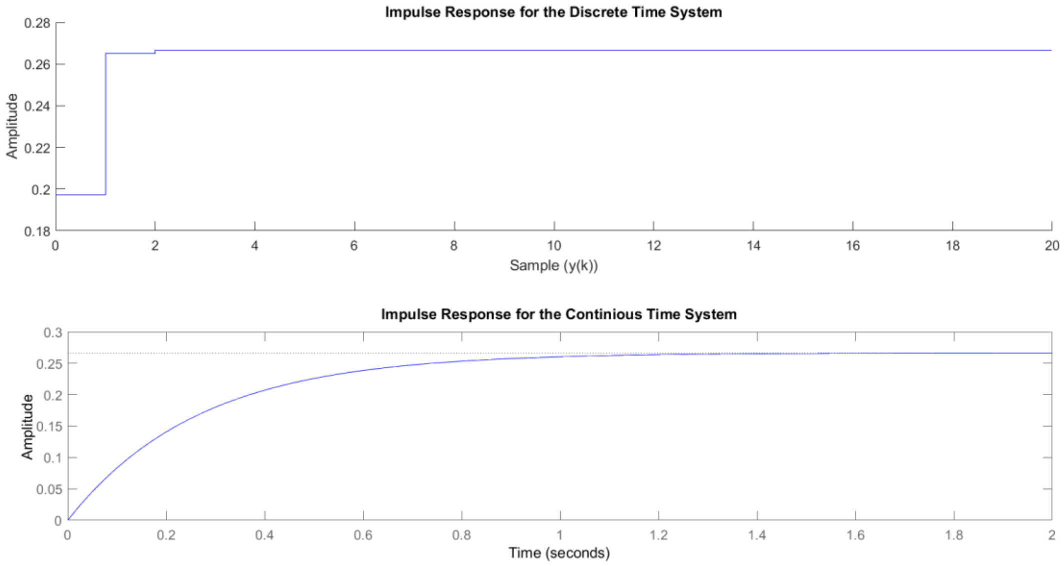

Figure 1: Impulse Response of the Discrete System (Top) and the Continuous System (Bottom)

From the impulse response it can be observed that both systems impulse response settles at a final value of approximately 0.27 which would suggest that the system is marginally stable as the impulse response doesn't go to 0 but it does have a final value.

Step Response

Below shows the MATLAB code used to plot and compare the step response of both the continuous time system and the discrete time system.

```
% Determine the step response for the discrete time system
num_GZ_Step = [0.1972 0.06317 0 0]; % numerator for Gz
den_GZ_Step = [1 -2.02352 1.04704 -0.02352]; % denominator for Gz
yk_step = filter(num_GZ_Step,den_GZ_Step,u); % impulse using the filter command

figure(2)
subplot(2,1,1)
hold on
stairs(k,yk_step,'b') % stairs function plots the impulse
title('Step Response for the Discrete Time System')
xlabel('Sample (y(k))')
ylabel('Amplitude')

figure(2)
subplot(2,1,2)
hold on
step(Gss,'b')
title('Step Response for the Continious Time System')
```

Figure 2 below shows the step response for both the continuous time system and the discrete time system taken from MATLAB.

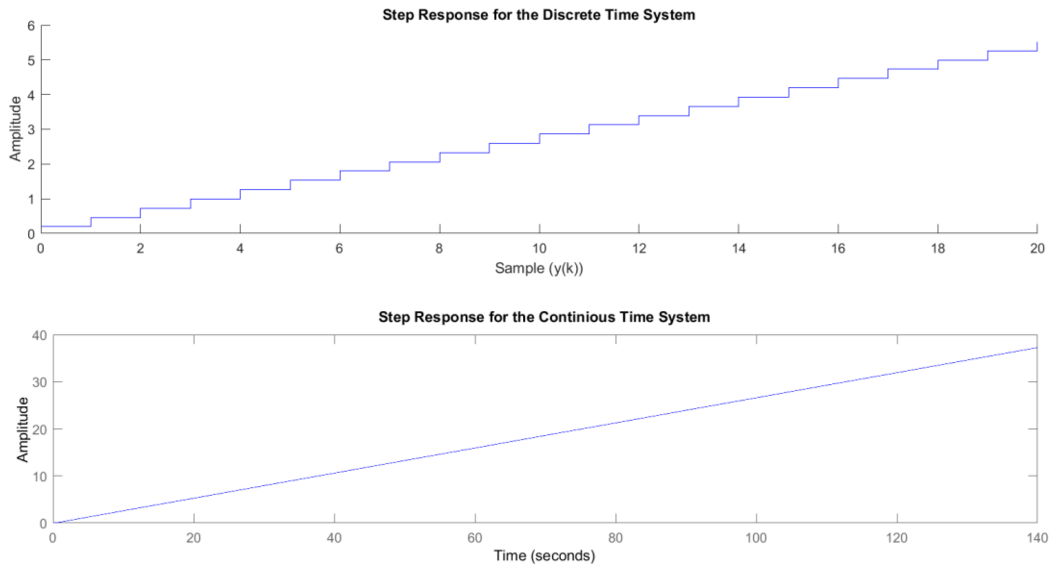

Figure 2: Impulse Response of the Discrete System (Top) and the Continuous System (Bottom)

From the step response it can again be observed that both systems are marginally stable as instead of reaching a final value of 1 for the step response, both systems will go off to infinity.