Differential Equations - MATH246

Tom Mitchell

Conway -Fall 2024

Class Information

Grading

- Matlab assignments — 18% (6% each)
- Quizzes (drop two lowest) — 17%
- Two best in-class exams — 17% each
- Worst in-class exam — 8%
- Final exam — 23%

Office Hours

- Monday: 2:00 PM - 3:00 PM (in person, Kirwin 2400)
- Tuesday: 1:15 PM - 2:30 PM (in person, Kirwin 2400)
- TBA: Zoom (online)

Exams

- 3 midterms and a final exam

Lecture 1, Tuesday 8/27/2024

Course Overview: (Differential Equations)

Chapter 0:

A differential equation is an algebraic relation between functions, their derivatives, and independent variables.

Examples:

- $\left(\frac{dx}{dt}\right)^2 + x\sin(t) = \cos(x)$ *(Order = 1)*
- $y'' + ty' + y = \cos(t)$ (Note: $y' = \frac{dy}{dt}$)*(Order = 2)*
- $\frac{dy}{dt} \cdot \frac{dy}{ds} + y\frac{dz}{dt} = \sin(st)$ *(Order = 1)*

Order: The order of a differential equation is the order of the highest derivative that appears.

Notation: For $\frac{dy}{dx}$, we can write y' or $\dot{y}$ (dot notation).

An ordinary differential equation (ODE) involves no partial derivatives, as opposed to a partial differential equation (PDE).

Note: This course only deals with ODEs.

Linearity of ODEs

An ODE with function y and independent variable t is **linear** if it can be written as:

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = f(t)$$

where $y^{(n)}$ is the nth derivative of y.

Examples:

- $\left(\frac{dx}{dt}\right)^2 + x\sin(t) = \cos(x)$ *(Not linear: $\left(\frac{dx}{dt}\right)^2$ and $\cos(x)$)*
- $y'' + ty' + y = \cos(t)$ *(Linear)*
- $y^{(4)} + y^{(2)} = 2$ *(Linear)*

Systems of ODEs

A system of ODEs consists of multiple ordinary differential equations that are considered together:

$$\begin{cases} \text{ODE1} \\ \text{ODE2} \\ \vdots \\ \text{ODE}n \end{cases}$$

Chapter 1: Introduction

Section 1: First-Order ODEs

First-order ODEs can be complicated. We will focus on those that can be put into the standard form $\boxed{\frac{dy}{dt} = f(t,y)}$.

Example: Consider the equation $\frac{dw}{dz} = \frac{-z}{6w}$. This can be rewritten as:

$$\frac{dw}{dz} = \frac{-z}{6w}$$

A function $Y(t)$ is a solution to $y' = f(t, y)$ on the interval (a, b) if:

- $Y(t)$ and $Y'(t)$ exist on (a, b),
- $f(t, Y(t))$ exists on (a, b), and
- $Y'(t) = f(t, Y(t))$ on (a, b).

Example: Consider the equation $y'(t) = \frac{t}{y}$ with the solution $Y(t) = \sqrt{4 - t^2}$.

To check this, calculate:

$$Y'(t) = \frac{-t}{\sqrt{4 - t^2}}$$

$Y(t)$ is defined on the interval $[-2, 2]$, but $f(t, Y(t)) = \frac{t}{\sqrt{4-t^2}}$ is only defined for $(-2, 2)$, not at ± 2. Therefore, $Y(t)$ is a solution on $(-2, 2)$, not on $[-2, 2]$.

Explicit Equations

These are of the form $y' = f(t)$.

The general solution is:

$$y = \int f(t)\, dt = F(t) + C$$

where $F(t)$ is an antiderivative of $f(t)$ (i.e., $F'(t) = f(t)$) and C is a constant.

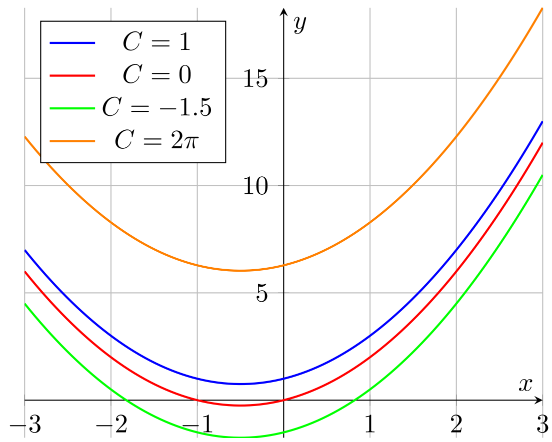
Example: Consider the ODE

$$\frac{dy}{dx} = 2x + 1$$

The general solution is:

$$y = x^2 + x + C$$

Graph for different values of C:



To select a specific solution from the general solution, we need an initial condition: $\boxed{y(t_I) = y_I}$.

The pair $y' = f(t)$ with $y(t_I) = y_I$ is called an Initial Value Problem (IVP).

Example: Solve the IVP

$$\frac{dy}{dx} = 2x + 1 \quad \text{with} \quad y(0) = 2$$

Solution:

Start with the general solution:

$$y = x^2 + x + C$$

Using the initial condition $y(0) = 2$:

$$2 = 0^2 + 0 + C \quad \Rightarrow \quad C = 2$$

Thus, the specific solution is:

$$y = x^2 + x + 2$$

Interval of Definition/Existence

The interval of definition/existence of a solution to an IVP is the **largest** interval (a, b) where:

- $t_I \in (a, b)$
- $f(t)$ is continuous on (a, b)

Chapter 2: Linear Equations

These look like:

$p(t)y' + q(t)y = r(t)$ where $p(t) \neq 0$ for the values of t we are considering.

In standard form:

$$y' = -\frac{q(t)}{p(t)}y + \frac{r(t)}{p(t)}$$

Let:

$$a(t) = \frac{q(t)}{p(t)}, \quad f(t) = \frac{r(t)}{p(t)}$$

We write it as:

$$y' + a(t)y = f(t)$$

Here, $f(t)$ is called the forcing function.
If $f(t) = 0$, the ODE is called homogeneous; otherwise, it is non-homogeneous.

Recipe for Solving First-Order Linear ODEs

Given:

$$y' + a(t)y = f(t)$$

1. Choose an antiderivative $A(t)$ of $a(t)$. 2. Multiply both sides by $e^{A(t)}$:

$$e^{A(t)}y' + a(t)e^{A(t)}y = f(t)e^{A(t)}$$

Let:

$$f(t)e^{A(t)} = g(t)$$

This simplifies to:

$$\frac{d}{dt}\left(e^{A(t)}y\right) = g(t)$$

3. Integrate both sides:

$$e^{A(t)}y = G(t) + C \quad \Rightarrow \quad y = e^{-A(t)}G(t) + Ce^{-A(t)}$$

This is the general solution.
Example: Solve the ODE

$$\frac{dy}{dt} = -y$$

1. Rewrite as $y' + y = 0$. 2. Here, $a(t) = 1$, so choose $A(t) = t$. 3. Multiply both sides by e^t:

$$e^t y' + e^t y = 0 \quad \Rightarrow \quad \frac{d}{dt}(e^t y) = 0$$

4. Integrate:

$$e^t y = C \quad \Rightarrow \quad y = Ce^{-t}$$

This is the general solution.

Example: Consider the ODE

$$y' = -y + e^t$$

1. Rewrite as $y' + y = e^t$. 2. Here, $a(t) = 1$, so choose $A(t) = t$. 3. Multiply both sides by e^t:

$$e^t y' + e^t y = e^{2t} \quad \Rightarrow \quad \frac{d}{dt}(e^t y) = e^{2t}$$

4. Integrate:

$$e^t y = \frac{1}{2}e^{2t} + C \quad \Rightarrow \quad y = \frac{1}{2}e^t + Ce^{-t}$$

This is the general solution.

Example: Solve the IVP

$$\frac{dx}{dt} + \cos(t)x = \cos(t) \quad \text{with} \quad x\left(\frac{\pi}{2}\right) = 0$$

Solution:

1. Here, $a(t) = \cos(t)$, so choose $A(t) = \sin(t)$. 2. Multiply both sides by $e^{\sin(t)}$:

$$e^{\sin(t)}x' + \cos(t)e^{\sin(t)}x = \cos(t)e^{\sin(t)}$$

This simplifies to:

$$\frac{d}{dt}\left(e^{\sin(t)}x\right) = \cos(t)e^{\sin(t)}$$

3. Integrate:

$$e^{\sin(t)}x = \int \cos(t)e^{\sin(t)}\,dt = e^{\sin(t)} + C$$

Thus,

$$x = 1 + Ce^{-\sin(t)}$$

4. Apply the initial condition $x\left(\frac{\pi}{2}\right) = 0$:

$$0 = 1 + Ce^{-1} \quad \Rightarrow \quad C = -e$$

Thus, the specific solution is:

$$x = 1 - e^{1-\sin(t)}$$

Lecture 2, 8/29/2024

I.2 (continued)

Problem Statement

Consider the initial value problem (IVP):

$$y' + a(t)y = f(t), \quad y(t_I) = y_I$$

Theorem: If $a(t)$ and $f(t)$ are continuous over the interval (a, b) and $t_I \in (a, b)$, then there is a unique solution to the IVP that is continuous on (a, b), and it's given by our method.

Example

Consider the differential equation:

$$z' + \cot(t)z = \frac{1}{\ln(t^2)}, \quad z(4) = 3$$

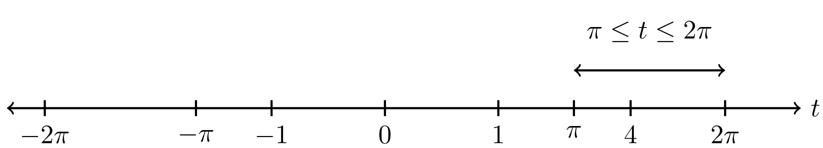
Find the largest interval on which we can guarantee a unique continuous solution to this IVP.

Solution

The function $\ln(t^2)$ is continuous on $(-\infty, 0)$ and $(0, \infty)$, but $\frac{1}{\ln(t^2)}$ is discontinuous at $t = 0$ and when $\ln(t^2) = 0$, i.e., $t = \pm 1$.

The function $\cot(t)$ has discontinuities at multiples of π.

The largest interval of continuity that includes $t = 4$ is $(\pi, 2\pi)$.



I.3: Separable Equation

A first-order ordinary differential equation (ODE) is **separable** if it can be written in the form:

$$y' = f(t)g(y)$$

Example:
Consider the differential equation:

$$y' = 2ty^2 + 3t^2y^2$$

We can factor this as:

$$y' = (2t + 3t^2)y^2$$

Here, we have:

$$f(t) = 2t + 3t^2, \quad g(y) = y^2$$

An ODE of the form $y' = g(y)$ is called **autonomous**.

A solution is called **stationary** if it is constant. If $y = C$ is a stationary solution, then:

$$y' = 0 \Rightarrow \boxed{0 = g(C)}$$

Example:
Consider the equation:

$$y' = 4y - y^3$$

To find the stationary solutions, set:

$$4y - y^3 = 0 \Rightarrow y(4 - y^2) = 0 \Rightarrow y(2 - y)(2 + y) = 0$$

Thus, the stationary solutions are:

$$y = 0, \quad y = 2, \quad y = -2$$

Non-Stationary Solutions

To find non-stationary solutions of the equation $y' = g(y)$, we proceed as follows:

$$y' = g(y) \quad \Rightarrow \quad \frac{1}{g(y)}y' = 1$$

Taking the integral on both sides:

$$\int \frac{1}{g(y)}y'\,dt = \int 1\,dt$$

This simplifies to:

$$\int \frac{1}{g(y)}\,dy = t + C$$

The result is an implicit equation for our solution.

Why can we divide by $g(y)$? $g(y) = 0$ corresponds to stationary solutions, and we are looking for non-stationary solutions, i.e., $g(y) \neq 0$.

Example: Find All Solutions to $y' = y^2$

Stationary Solutions: Set $y^2 = 0$, which implies $y = 0$.

Non-Stationary Solutions:

Starting with the equation:

$$\frac{1}{y^2}y' = 1$$

Integrate both sides:

$$\int \frac{1}{y^2}y'\,dt = \int 1\,dt$$

This simplifies to:

$$\int \frac{1}{y^2}\,dy = t + C$$

Evaluating the integral:

$$-\frac{1}{y} = t + C$$

We can find an explicit solution:

$$-y = \frac{1}{t+C} \quad \Rightarrow \quad y = -\frac{1}{t+C}$$

Each solution $y = -\frac{1}{t+C}$ actually represents two solutions, one defined on $(-\infty, -C)$ and the other on $(-C, \infty)$.

Note: Our solution is discontinuous even though all functions in the original equation $y' = y^2$ are continuous.

General Separable Equations

Consider the general separable equation:

$$y' = f(t)g(y)$$

If $g(c) = 0$, then $y = c$ is a stationary solution (so set g(y) = 0).

For non-stationary solutions:

$$\frac{1}{g(y)}y' = f(t)$$

Taking the integral on both sides:

$$\int \frac{1}{g(y)}y'\,dt = \int f(t)\,dt$$

This simplifies to:

$$\int \frac{1}{g(y)}\,dy = F(t) + C$$

Example: Find All Solutions to $\frac{dz}{dx} = \frac{3x+xz^2}{z+x^2z}$

First, rewrite the equation:

$$\frac{dz}{dx} = \frac{x}{1+x^2} \cdot \frac{3+z^2}{z}$$

Thus, we identify:

$$f(x) = \frac{x}{1+x^2}, \quad g(z) = \frac{3+z^2}{z}$$

Stationary Solutions: Set $g(z) = 0$:

$$\frac{3+z^2}{z} = 0 \quad \Rightarrow \quad 3 + z^2 = 0$$

This equation has no real solution, so there are no stationary solutions.
Non-Stationary Solutions:
Start with:

$$\frac{1}{g(z)}\frac{dz}{dx} = f(x)$$

Which simplifies to:

$$\frac{z}{3+z^2} \cdot \frac{dz}{dx} = \frac{x}{1+x^2}$$

Integrate both sides:

$$\int \frac{z}{3+z^2}\frac{dz}{dx}\,dx = \int \frac{x}{1+x^2}\,dx$$

Use substitution:
- Let $u = 3 + z^2$, then $du = 2z\,dz$. - Let $v = 1 + x^2$, then $dv = 2x\,dx$.
The integrals become:

$$\int \frac{1}{2u} du = \int \frac{1}{2v} dv$$

This integrates to:

$$\frac{1}{2} \ln |u| = \frac{1}{2} \ln |v| + C$$

Substituting back u and v:

$$\frac{1}{2} \ln |3 + z^2| = \frac{1}{2} \ln |1 + x^2| + C$$

Initial Value Problems (IVPs)

Example: Solve the initial value problem:

$$y' = ty^2 - ty, \quad y(1) = 2$$

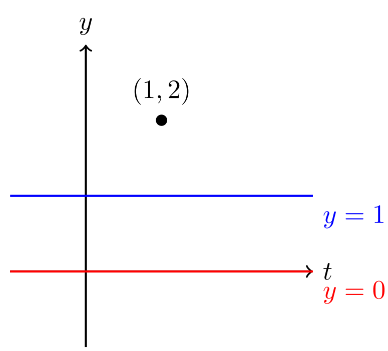
We can factor the equation as:

$$y' = t(y^2 - y)$$

Stationary Solutions: Set $y^2 - y = 0$:

$$y(y-1) = 0 \quad \Rightarrow \quad y = 0 \quad \text{or} \quad y = 1$$

Neither $y = 0$ nor $y = 1$ satisfies the initial condition $y(1) = 2$.



As shown in the graph, neither $y = 0$ nor $y = 1$ passes through the point $(1, 2)$.

Other Solutions: We solve the differential equation for non-stationary solutions:

$$\frac{1}{y^2 - y} \frac{dy}{dt} = t \quad \Rightarrow \quad \frac{1}{y^2 - y} \, dy = t \, dt$$

Integrate both sides:

$$\int \frac{1}{y^2 - y}\,dy = \int t\,dt$$

Using partial fractions:

$$\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1}$$

This leads to:

$$1 = A(y-1) + B(y)$$

By guessing $A = -1$ and $B = 1$, we get:

$$\int \left(-\frac{1}{y} + \frac{1}{y-1}\right) dy = \int t\,dt$$

Integrating both sides:

$$-\ln|y| + \ln|y-1| = \frac{t^2}{2} + C$$

Using the logarithm property $\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)$, this simplifies to:

$$\ln\left|\frac{y-1}{y}\right| = \frac{t^2}{2} + C$$

Applying the Initial Condition: Given $y(1) = 2$:

$$\ln\left|\frac{2-1}{2}\right| = \frac{1^2}{2} + C$$

$$\ln\left(\frac{1}{2}\right) = \frac{1}{2} + C \quad \Rightarrow \quad C = \ln\left(\frac{1}{2}\right) - \frac{1}{2}$$

Substituting C back into the equation:

$$\ln\left|\frac{y-1}{y}\right| = \frac{t^2}{2} + \ln\left(\frac{1}{2}\right) - \frac{1}{2}$$

Uniqueness and Existence Theorem

If $f(t)$ is continuous on (a,b) and $g(y)$ is continuous and differentiable on (c,d), then for every $t_I \in (a,b)$ and $y_I \in (c,d)$, there exists a unique continuous solution to the equation

$$y' = f(t)g(y)$$

with the initial condition $y(t_I) = y_I$, defined on some interval around t_I. The solution is determined by our method.

Example

Consider the differential equation:

$$\frac{dy}{dt} = 3y^{2/3}, \quad y(0) = 0$$

Stationary Solution: $y = 0$ is a stationary solution, and it solves our initial value problem (IVP).

However, $g(y) = 3y^{2/3}$ is not differentiable at $y = 0$, so we might have other solutions.

Finding Other Solutions:

$$\frac{1}{3y^{2/3}}\frac{dy}{dt} = 1$$

Integrating both sides:

$$\int \frac{1}{3y^{2/3}}\,dy = \int 1\,dt$$

This simplifies to:

$$y^{1/3} = t + C$$

Raising both sides to the power of 3:

$$y = (t + C)^3$$

Applying the Initial Condition: For $y(0) = 0$, we get $C = 0$, so:

$$y = t^3$$

Thus, $y = t^3$ also solves our IVP.

Lecture 3, Tuesday 9/3/2024

Quiz tomorrow: Up to Section I.3

I.4. Theory

Consider Initial Value Problems (IVPs) of the form:

$$y' = f(t, y), \quad y(t_i) = y_i$$

We say the problem is *well-posed* if:

1. There exists a solution
2. The solution is unique
3. The solution depends continuously on the initial conditions

Existence and Uniqueness

Consider a set S of points in the (t, y) plane.

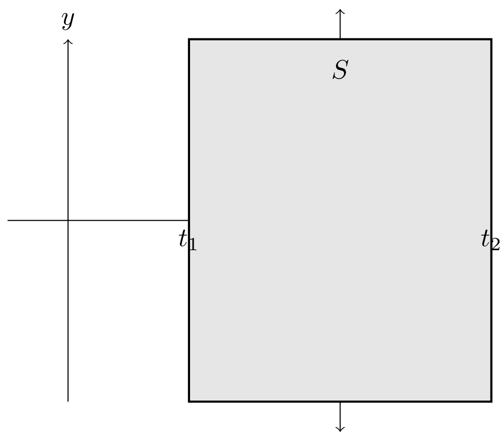


Figure 1: The set S in the (t, y) plane

If $f(t, y)$ is continuous on S and $\frac{\partial f}{\partial y}$ is continuous on S, then for any $(t_i, y_i) \in S$, there exists a unique continuous solution $Y(t)$ to the initial value problem:

$$\begin{cases} y' = f(t, y) \\ Y(t_i) = y_i \end{cases}$$

defined over some interval (a, b) containing t_i.

Moreover, the interval (a, b) can be extended as long as $(t, Y(t))$ remains inside S.

Example: Consider $y' = \frac{\sin(t+ty^2)}{1+t^2}$, $y(0) = 1$

Let's show that there is a unique solution defined on $(-1, 1)$.

Solution: For $f(t, y) = \frac{\sin(t+ty^2)}{1+t^2}$, f is continuous except at $t = \pm 1$.

$\frac{\partial f}{\partial y} = \frac{2ty\cos(t+ty^2)}{1+t^2}$, which is also continuous except at $t = \pm 1$.

By the Existence and Uniqueness Theorem, since f and $\frac{\partial f}{\partial y}$ are continuous on S, there exists a unique solution $y(t)$ to the initial value problem, defined on some interval containing $t = 0$. This interval can be extended as long as $(t, y(t))$ remains inside S, which in this case is the entire interval $(-1, 1)$.

Since $t_i = 0$, $y_i = 1 \implies (0, 1) \in S$, the theorem tells us we have a unique solution $Y(t)$ defined on a larger (a, b) such that $y(t)$ remains inside S.

Since any solution will not leave S as long as $-1 < t < 1$, we get $(a, b) = (-1, 1)$.

So, we choose S as follows:

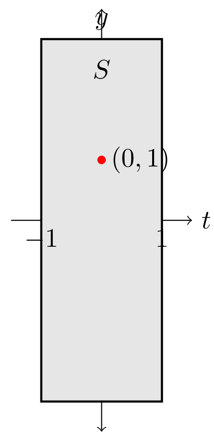


Figure 2: Set S for the given differential equation

Example: Consider the differential equation $y' = \frac{1}{t^2+y^2-1}$, with initial condition $y(0) = 0$. Find the set S where the solution is guaranteed to exist and be unique.

Solution: The function $f(t, y) = \frac{1}{t^2+y^2-1}$ is discontinuous when $t^2+y^2 = 1$. This equation describes a circle with radius 1 centered at the origin.

The set S where the solution exists and is unique should be the interior of this circle, excluding the circle itself. We can represent this as:

$$S = \{(t, y) : t^2 + y^2 < 1\}$$

Let's visualize this set:

The solution is guaranteed to exist and be unique within this circular region S, but not on or outside the boundary where $t^2 + y^2 = 1$.

Example: Consider the differential equation $y' = \frac{1}{t^2+y^2-1}$, with initial condition $y(0) = 3$. Let's visualize the set S where the solution is guaranteed to exist and be unique.

Solution: The function $f(t, y) = \frac{1}{t^2+y^2-1}$ is discontinuous when $t^2+y^2 = 1$. This equation describes a circle with radius 1 centered at the origin.

The set S where the solution exists and is unique should be the region outside this circle, including the initial point $(0, 3)$. We can represent this as:

$$S = \{(t, y) : t^2 + y^2 > 1\}$$

Let's visualize this set:

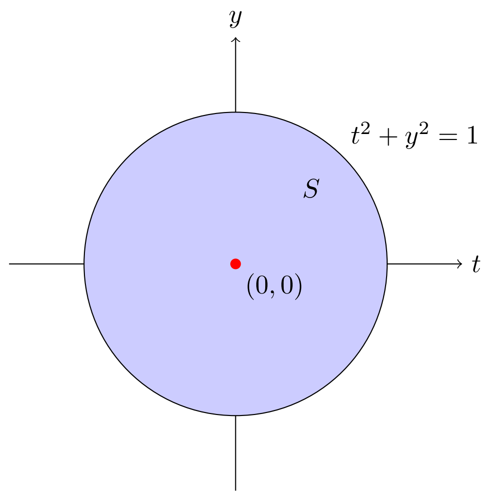


Figure 3: Set S for the given differential equation

The solution is guaranteed to exist and be unique within the region S, which is the area outside the circle $t^2 + y^2 = 1$, including the initial point $(0, 3)$.

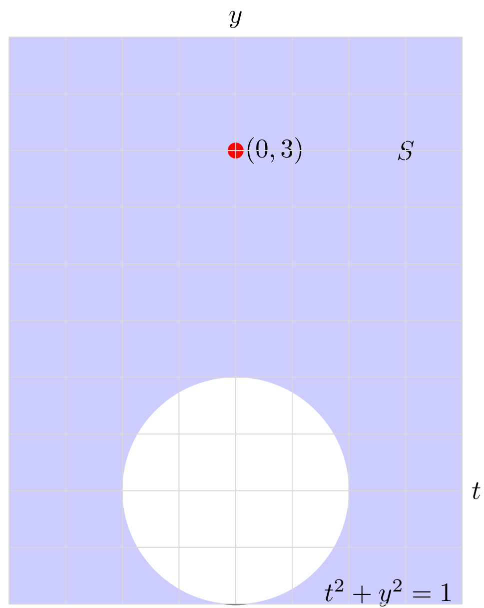


Figure 4: Set S for the given differential equation with $y(0) = 3$

1 I.5 - Graphical Methods

1.1 Phase Portraits for Autonomous Equations

Consider the autonomous differential equation:

$$\frac{dy}{dt} = g(y)$$

Our goal is to describe the qualitative behavior of solutions without explicitly solving the equation.

- When $g(y) = 0$:
  - $y' = 0$
  - We have a stationary solution
- When $g(y) > 0$:
  - $y' > 0$
  - The solution is increasing
- When $g(y) < 0$:
  - $y' < 0$
  - The solution is decreasing

Example: $y' = 4y - y^3$

Let's analyze the differential equation $y' = 4y - y^3$.

1. First, we find the stationary solutions:

$$4y - y^3 = y(4 - y^2) = y(2 - y)(2 + y) = 0$$

   Thus, the stationary solutions are $y = 0, \pm 2$.

2. Next, we determine the sign of $g(y) = 4y - y^3$ between these zeros:

$$\begin{aligned} g(1) &= 4(1) - (1)^3 = 3 > 0 \\ g(3) &= 4(3) - (3)^3 = 12 - 27 = -15 < 0 \\ g(-3) &= 4(-3) - (-3)^3 = -12 + 27 = 15 > 0 \\ g(-1) &= 4(-1) - (-1)^3 = -4 + 1 = -3 < 0 \end{aligned}$$

Based on this analysis, we can create a phase portrait:

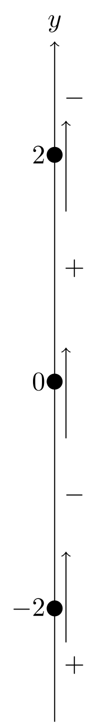


Figure 5: Phase portrait for $y' = 4y - y^3$

The phase portrait shows:

- Solutions increase when $y \in (-\infty, -2) \cup (0, 2)$

- Solutions decrease when $y \in (-2, 0) \cup (2, \infty)$
- Stationary solutions at $y = 0, \pm 2$

This diagram is called a phase portrait or phase line. It provides valuable information about the behavior of solutions $y(t)$ for different initial conditions:

- For $y(t)$ starting in $(-\infty, -2)$:
  - $y(t)$ increases as t increases
  - $y(t) \to -2$ as $t \to \infty$ (asymptotically approaching -2)
- For $y(t)$ starting in $(-2, 0)$:
  - $y(t)$ is decreasing
  - $y(t) \to -2$ as $t \to \infty$
- For $y(t)$ starting in $(0, 2)$:
  - $y(t)$ is increasing
  - $y(t) \to 2$ as $t \to \infty$ (asymptotically approaching 2)
- For $y(t)$ starting in $(2, \infty)$:
  - $y(t)$ is decreasing
  - $y(t) \to 2$ as $t \to \infty$ (asymptotically approaching 2)

Sketch solutions:
This figure illustrates:

- Constant solutions at $y = 2$, $y = 0$, and $y = -2$ (blue lines)
- Solutions approaching $y = 2$ from above and below
- Solutions approaching $y = -2$ from above and below
- Solutions approaching $y = 0$ from above and below

The red curves represent various solutions to the differential equation, showing how they behave over time depending on their initial conditions.

We classify stationary solutions as follows:

- **Stable or attracting:** All nearby solutions move towards it as $t \to \infty$. (In this case: $y = \pm 2$)
- **Unstable or repelling:** All nearby solutions move away from it as $t \to \infty$.
  (In this case: $y = 0$)
- **Semi-stable:** Some solutions move towards it and some move away from it as $t \to \infty$.

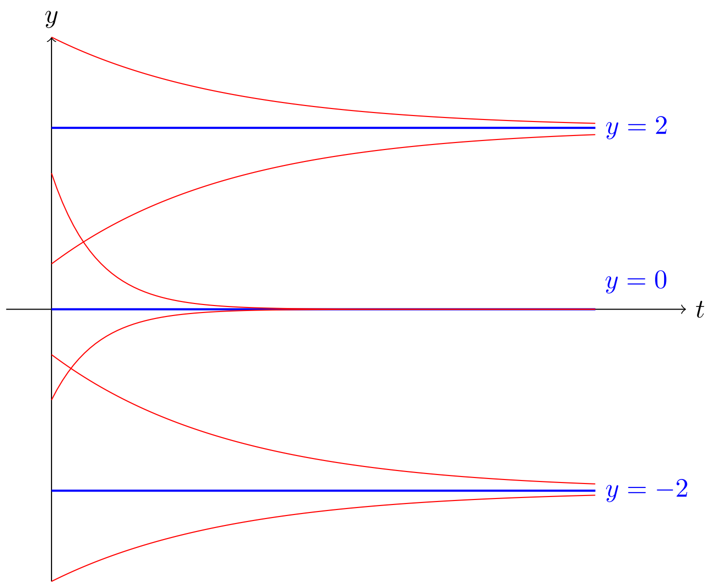


Figure 6: Sketch of solutions for $y' = 4y - y^3$

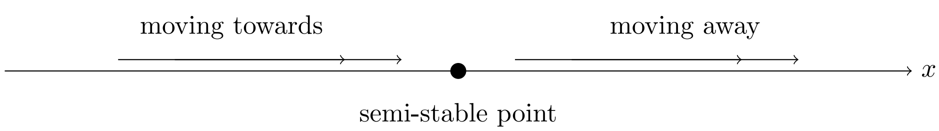


Figure 7: Behavior near a semi-stable point

Example: Consider the differential equation $y' = y^2$

- Stationary solution: $y = 0$
- For $y \neq 0$: $y^2 > 0$, implying solutions move away from $y = 0$

This example demonstrates an unstable stationary solution at $y = 0$.

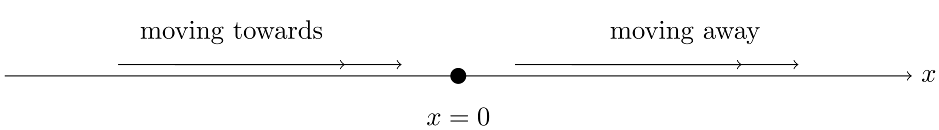


Figure 8: Behavior on x-axis for $y' = y^2$

This figure illustrates the behavior of solutions to $y' = y^2$ near the stationary solution $y = 0$. The blue curves represent solutions moving away from $y = 0$,

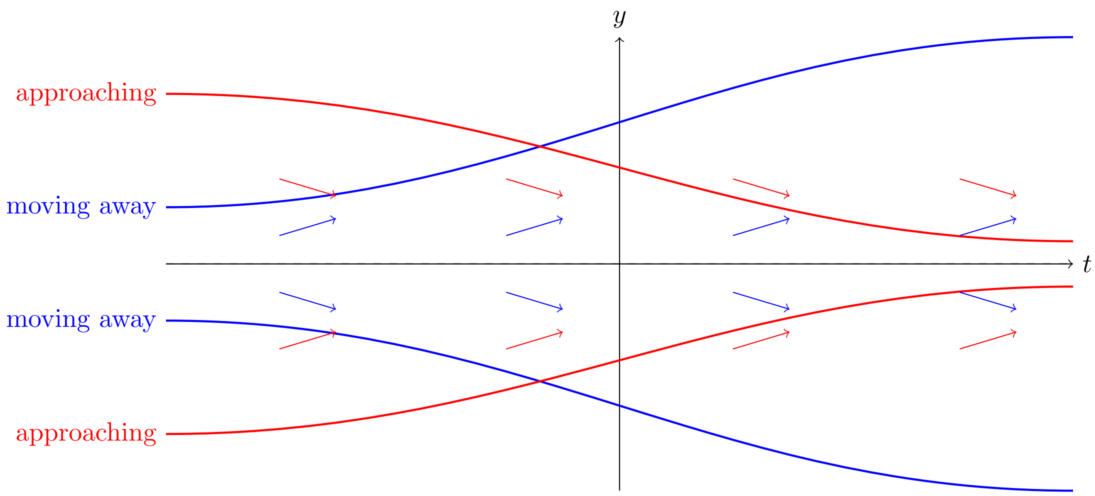


Figure 9: Behavior of solutions in t-y plane for $y' = y^2$

while the red curves represent solutions approaching $y = 0$. This demonstrates that $y = 0$ is an unstable or repelling stationary solution.

For stability think of a pendulum

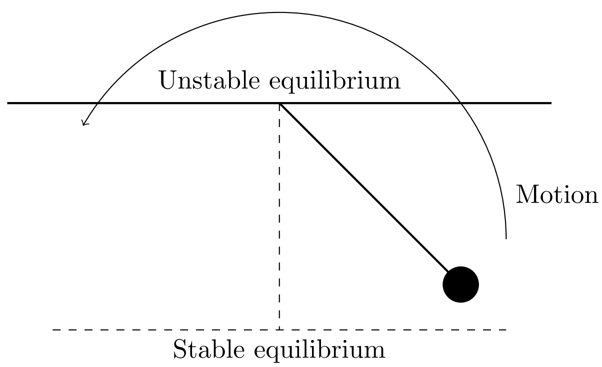


Figure 10: Pendulum illustrating stable and unstable equilibrium points

This figure illustrates a pendulum moving from right to left. The top position (vertically upright) represents an unstable equilibrium, while the bottom position represents a stable equilibrium. These correspond to the stationary solutions of the pendulum's differential equation.

unstable solution is pendulum at the top and stable is pendulum at the bottom

Example: Phase Line Analysis

Consider the differential equation:

$$y' = \frac{(y^2-1)(y-3)^2}{(y+3)^2}$$

Stationary Solutions:

- $y = -1$
- $y = 1$
- $y = 3$

Note: $y = -3$ is undefined in the equation.

We will now draw the phase line for this differential equation to analyze its behavior.

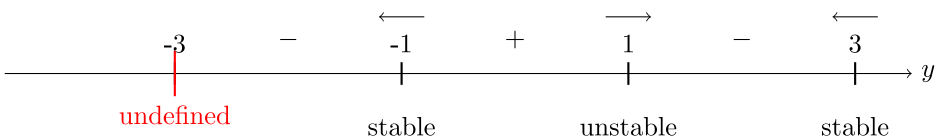


Figure 11: Phase line for $y' = (y^2-1)(y-3)^2/(y+3)^2$

Note: stability only applies to stationary solutions, not to undefined points.
Sketch

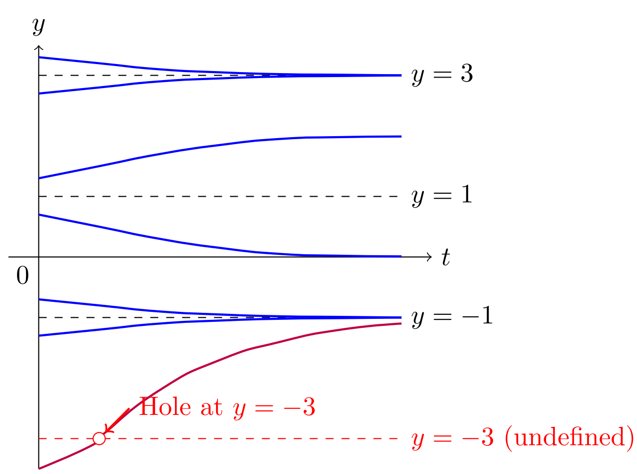


Figure 12: Sketch of solutions for $y' = (y^2-1)(y-3)^2/(y+3)^2$, including a solution crossing $y = -3$ with a clear hole

Example: Consider the differential equation $y' = t - y^2$
Procedure: To sketch the slope field:

1. Choose a representative selection of (t, y) points.
2. At each point (t, y), draw an arrow with slope $t - y^2$.
3. Connect the arrows to visualize solution curves.

Sample calculations:

$$\begin{aligned}
\text{At } (0,0): &\quad t - y^2 = 0 \\
\text{At } (1,0): &\quad t - y^2 = 1 \\
\text{At } (-1,0): &\quad t - y^2 = -1 \\
\text{At } (0,\pm 1): &\quad t - y^2 = -1
\end{aligned}$$

Continue this process for other points to build a comprehensive slope field.

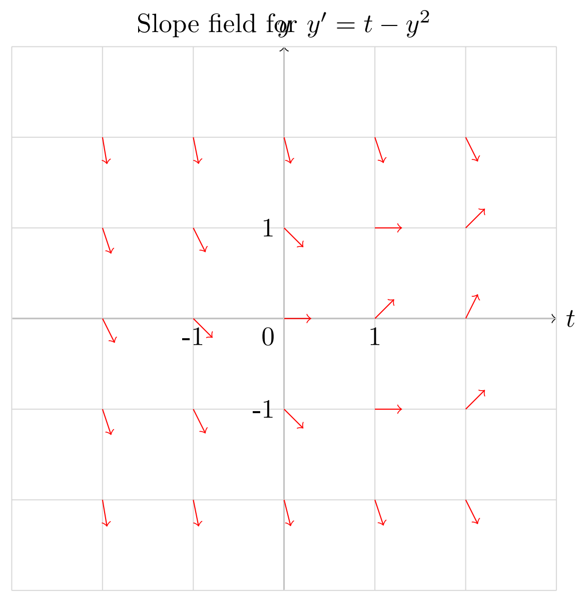


Figure 13: Slope field for the differential equation $y' = t - y^2$

if y(0) = 1,
then the point should follow the slope field

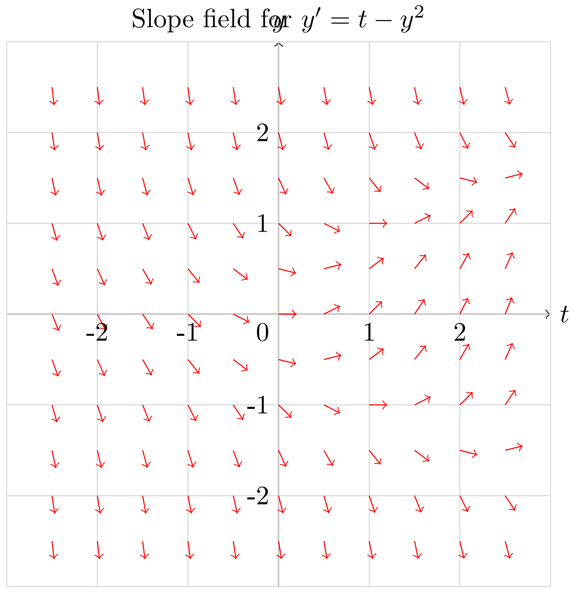


Figure 14: Detailed slope field for the differential equation $y' = t - y^2$

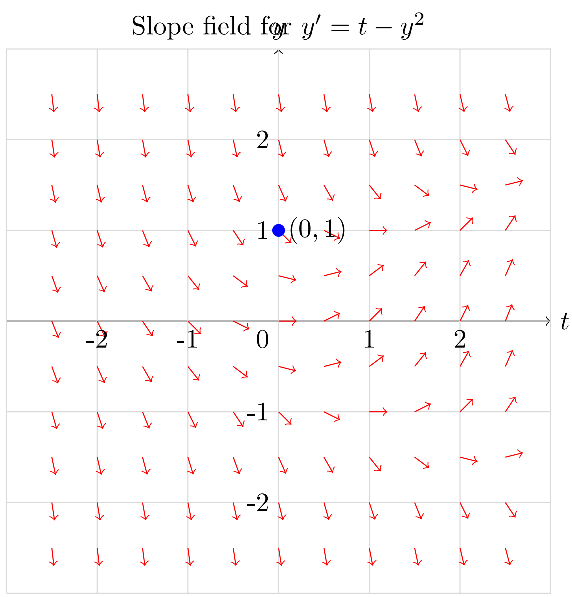


Figure 15: Slope field for $y' = t - y^2$ with initial point $y(0) = 1$