Applied Probability and Statistics I Discussion - STAT400

Tom Mitchell

Yi - Fall 2024

Syllabus

Grading

- Homework — 28% (7% each)
- R Projects — 12% (4% each)
- Two exams — 30% (15% each)
- Final exam — 30%

Office Hours

- Tuesday: 1:00 PM - 1:50 PM (in person, MTH 4106)
- Wednesday: 11:00 AM - 11:50 AM (online)

Exams

- 2 midterms and a final exam

Discussion 1: Monday 9/5/2024

Notation and Key Concepts

- Cardinality: For a finite set A, the number of elements is denoted as $|A|$.
- Example: If A = {1, 2, 3, 4, 5}, then $|A| = 5$.

Probability Axioms

The total probability of all outcomes in a sample space must sum to 1:

1. For discrete probability: $\sum_i P_i = 1$
2. For continuous probability: $\int p(x)\, dx = 1$

Homework 1

1. Biased Coin Experiment

A biased coin comes up heads with probability $\frac{1}{3}$. An experiment consists of tossing this coin until heads is seen for the first time, at which point the experiment ends. The probability of seeing heads on the ith toss is $\frac{1}{3}\left(\frac{2}{3}\right)^{i-1}$.

Probability table:

Outcome	H	T
Probability	$\frac{1}{3}$	$\frac{2}{3}$

Compute the probability that you will see heads for the following scenarios, clearly identifying the pairwise disjoint events used in the calculations:

a) H (on first toss) b) TH (on second toss) c) TTTH (on fourth toss) d) TTT...TH (on ith toss)

Note: The ith toss ending the game corresponds to seeing heads on the ith toss.

Derive the formula for the probability of seeing heads on the ith toss:

$P(\text{H on } i\text{th toss}) = \frac{1}{3}\left(\frac{2}{3}\right)^{i-1}$

This formula represents the probability of getting tails $(i-1)$ times followed by heads on the ith toss. This is already given in the problem statement.

(a) Probability of heads on the 1st, 3rd, or 7th toss

Let H_i denote the event that heads occurs on the ith toss. The events H_1, H_3, and H_7 are pairwise disjoint.

$$P(\text{heads on 1st, 3rd, or 7th toss}) = P(H_1 \cup H_3 \cup H_7)$$

Since the events are pairwise disjoint:

$$P(H_1 \cup H_3 \cup H_7) = P(H_1) + P(H_3) + P(H_7)$$

Using the given probabilities:

$$P(H_1) = \frac{1}{3}, \quad P(H_3) = \frac{1}{3}\left(\frac{2}{3}\right)^2, \quad P(H_7) = \frac{1}{3}\left(\frac{2}{3}\right)^6$$

Thus:

$$P(H_1 \cup H_3 \cup H_7) = \frac{1}{3} + \frac{1}{3}\cdot\frac{4}{9} + \frac{1}{3}\cdot\frac{64}{729} = \boxed{\frac{1117}{2187}}$$

(b) Probability of heads on an odd-numbered toss

Let H_{odd} denote the event that heads occurs on an odd-numbered toss. The set of odd-numbered tosses is the union of pairwise disjoint events:

$$P(H_{\text{odd}}) = P(H_1 \cup H_3 \cup H_5 \cup \dots)$$

Using the geometric series:

$$P(H_{\text{odd}}) = \sum_{k=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{2k}$$

This sum can be simplified as:

$$P(H_{\text{odd}}) = \frac{\frac{1}{3}}{1-\left(\frac{2}{3}\right)^2} = \frac{\frac{1}{3}}{1-\frac{4}{9}} = \frac{1}{3}\cdot\frac{9}{5} = \boxed{\frac{3}{5}}$$

Problem 5

Suppose that $\Omega = \{\omega_1, \ldots, \omega_n\}$ be a finite sample space with $n > 2$ elements. Suppose $P(\{\omega_1\}) = \frac{1}{2}$ and $P(\{\omega_i\}) = P(\{\omega_j\})$ for all $i, j \neq 1$.

ω_1	ω_2	$\cdots$	ω_n
$\frac{1}{2}$	x	$\cdots$	x

Table 1: Probability distribution of sample space Ω

Where:

1. $P(\{\omega_1\}) = \frac{1}{2}$
2. $P(\{\omega_i\}) = P(\{\omega_j\})$ for all $i, j \neq 1$

Note on disjointness:

- Pairwise disjoint: $A_i \cap A_j = \emptyset$ for all $i \neq j$
- Disjoint: $\bigcap_{i=1}^n A_i = \emptyset$ for all i, j such that $i \neq j$

(a) Show that $P(\{\omega_i\}) = \frac{1}{2(n-1)}$ for $i = 2, \ldots, n$

Let $P(\{\omega_i\}) = x$ for $i = 2, \ldots, n$. We know:

$$P(\Omega) = 1 \quad \text{and} \quad P(\{\omega_1\}) + (n-1)x = 1$$

Substituting $P(\{\omega_1\}) = \frac{1}{2}$:

$$\frac{1}{2} + (n-1)x = 1 \quad \Rightarrow \quad (n-1)x = \frac{1}{2} \quad \Rightarrow \quad x = \frac{1}{2(n-1)}$$

(b) Construct a formula for computing $P(A)$ for any $A \subseteq \Omega$

We can consider two cases (x is defined as found in part a):

Case 1: $\omega_1 \in A$
If $\omega_1 \in A$, then:

$$P(A) = \frac{1}{2} + |A \cap \{\omega_2, \ldots, \omega_n\}| \cdot x = \frac{1}{2} + \frac{|A \cap \{\omega_2, \ldots, \omega_n\}|}{2(n-1)}.$$

Case 2: $\omega_1 \notin A$
If $\omega_1 \notin A$, then:

$$P(A) = |A \cap \{\omega_2, \ldots, \omega_n\}| \cdot x = \frac{|A|}{2(n-1)} = \frac{|A \cap \{\omega_2, \ldots, \omega_n\}|}{2(n-1)}.$$

Final Formula

Thus, the formula for $P(A)$ is:

$$P(A) = \begin{cases} \frac{1}{2} + \frac{|A \cap \{\omega_2, \ldots, \omega_n\}|}{2(n-1)} & \text{if } \omega_1 \in A, \\ \frac{|A \cap \{\omega_2, \ldots, \omega_n\}|}{2(n-1)} & \text{if } \omega_1 \notin A. \end{cases}$$