

Linear Algebra Notes

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1 Lecture 1, 1/24/2024

A system of linear equations is a collection of linear equations. For example,

$$3x_1 - 2x_2 + 4x_3 = -29$$

$$x_1 - x_2 + x_3 = 3$$

$$5x_1 - x_3 = -2$$

Definition: A solution to the system consists of variables x_1, \dots, x_n that satisfy all the equations of the system.

For instance, $x_1 = -1, x_2 = 7, x_3 = -3$ is a solution.

Theorem: A system of linear equations satisfies exactly one of the following:

1. A unique solution - consistent system
2. Infinite solutions - consistent system
3. No solution - inconsistent system

Cannot have exactly 2 answers.

Definition: An $m \times n$ matrix is an array of elements (often numbers) containing m rows and n columns.

We sometimes say "real matrix" to indicate that the entries are real numbers.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Consider the system

$$x_1 + 2x_2 - 3x_3 = 11$$

$$3x_1 + 6x_2 - 8x_3 = 32$$

$$-2x_1 - x_2 = -7$$

Definition: The augmented matrix of the system is:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 11 \\ 3 & 6 & -8 & 32 \\ -2 & -1 & 0 & -7 \end{array} \right]$$

The coefficient matrix is on the left of the vertical line.

We "reduce" the augmented matrix to find the solution.

We apply elementary row operations on an augmented matrix:

1. Interchange 2 rows (equations)
2. Multiply a row by a non-zero constant
3. Add a multiple of one equation to another equation

For example, consider the system:

$$3x + 4y = 10$$

$$6x - y = 3$$

Two augmented matrices are row equivalent if a sequence of row operations transforms one into the other. These two systems will be equivalent, meaning they have the same solution set.

Main Idea

$$\left[\begin{array}{ccc|c} - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array} \right]$$

This represents the system of equations:

$$x_1 = 7$$

$$x_2 = 3$$

$$x_3 = 4$$

So, the solution is $x_1 = 7$, $x_2 = 3$, and $x_3 = 4$.

Going back to our earlier example...

$$x_1 + 2x_2 - 3x_3 = 11$$

$$3x_1 + 6x_2 - 8x_3 = 32$$

$$-2x_1 - x_2 = -7$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 11 \\ 3 & 6 & -8 & 32 \\ -2 & -1 & 0 & -7 \end{array} \right]$$

The first row stays the same, and the second row changes during each operation.

Apply the operation $-3R_1 + R_2$.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 11 \\ 0 & 0 & 1 & -1 \\ -2 & -1 & 0 & -7 \end{array} \right]$$

Apply the operation $2R_1 + R_3$.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 11 \\ 0 & 0 & 1 & -1 \\ 0 & 3 & -6 & 15 \end{array} \right]$$

Apply the operation $\frac{1}{3}R_3$.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 11 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -2 & 5 \end{array} \right]$$

Apply the operation $R_2 \leftrightarrow R_3$.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 11 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

Apply the operation $-2R_2 + R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

Apply the operation $2R_3 + R_2$ and $-R_3 + R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

Unique Solution

We have found that the system of equations has a unique solution:

$$x_1 = 2$$

$$x_2 = 3$$

$$x_3 = -1$$

Now, let's plug in these values and check:

$$2 + 2(3) - 3(-1) = 11$$

$$3(2) + 6(3) - 8(-1) = 32$$

$$-2(2) - (3) = -7$$

If you say you checked and the answer is wrong (false equation), you will lose fewer points.

2 Lecture 2, 1/26/2024

Continuing from last lecture.

What are we “reducing” to

Definition: A matrix is in reduced row echelon form (**RREF**) if all the following hold:

1. Any rows of zeros are at the bottom

$$\begin{bmatrix} \vdots \\ \vdots \\ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \end{bmatrix}$$

2. The first non-zero entry (**leading entry**) is to the right of the preceding row's leading entry.

$$\begin{bmatrix} 0 & 0 & 3 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 7 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

3. All entries below a leading entry are 0.
4. The leading entry in each (non-zero) row is a 1.
5. Each leading 1 is the only non-zero value in the column.

(Note: 3 is implied by 2)

If only 1, 2, 3 hold, it is in echelon form.

Example:

a)

$$\begin{bmatrix} 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{Echelon form, not RREF})$$

b)

$$\begin{bmatrix} 1 & 8 & 8 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{RREF})$$

c)

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{Echelon form, not RREF})$$

Theorem: Any matrix has a unique RREF.

We apply row operations on an augmented matrix of a system of linear equations to get it in RREF.

Example:

$$\begin{aligned} 2x_3 - 2x_4 &= 2 \\ 3x_1 + 3x_2 - 3x_3 + 9x_4 &= 12 \\ 4x_1 + 4x_2 - 2x_3 + 11x_4 &= 12 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 0 & 0 & 2 & -2 & 2 \\ 3 & 3 & -3 & 9 & 12 \\ 4 & 4 & -2 & 11 & 12 \end{array} \right]$$

1. Interchange rows to have a non-zero element in the top row of the first non-zero column
 $R1 \leftrightarrow R2$

$$\left[\begin{array}{cccc|c} 3 & 3 & -3 & 9 & 12 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{array} \right]$$

2. Convert, if **efficient***, the first non-zero entry, called **pivot**, to a 1, and apply row operations to create zeros in all other entries in the column

$$\frac{1}{3}R1$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{array} \right]$$

$$-4R1+R3$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 2 & -1 & -4 \end{array} \right]$$

3. Cover the row that you worked with (and all rows above), and repeat steps 1 and 2

$$\frac{1}{2}R2$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 \end{array} \right]$$

$$R2+R1$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 \end{array} \right]$$

$$-2R2+R3$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -8 \end{array} \right]$$

$-2R3+R1$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 17 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -6 \end{array} \right]$$

$R3+R2$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 17 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -6 \end{array} \right]$$

RREF of original matrix

This is called Gauss-Jordan elimination.

efficient*

$$\left[\begin{array}{ccc|c} 3 & 7 & 5 & 1 \\ -6 & 2 & 3 & 0 \\ -3 & 1 & 4 & 0 \end{array} \right]$$

Using original rules gives us fractions $\frac{7}{3}, \frac{5}{3}, \frac{1}{3}$.

$2R1+R2$ and $R1+R3$

$$\left[\begin{array}{ccc|c} 3 & - & - & - \\ 0 & - & - & - \\ 0 & - & - & - \end{array} \right]$$

Do $\frac{1}{3}R1$ later; it's easier!

What does the RREF of an augmented matrix tell us?

a) If at any point during row operations you obtain a false equation, STOP.

The system has NO solution. Explain why!

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

In the second row, $0 = 4$ is a contradiction, which implies no solution since we would require $0 = 4$, which is false.

b) If the RREF yields a unique solution, simply write down the solution.

c) In the case of infinite solutions (not case (a) or (b)), we find a general solution in terms of

Last Example:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 17 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -6 \end{array} \right]$$

Write as equations:

$$x_1 + x_2 = 17$$

$$x_3 = -5$$

$$x_4 = -6$$

The columns corresponding to the **pivots** are the leading/basic variables (x_1, x_3, x_4) . The others are free variables (x_2) . The general solution expresses the leading variables in terms of the free variables (when applicable).

→ General solution is

$$\begin{cases} x_1 &= -x_2 + 17 \\ x_2 &\text{is free or } x_2 \in \mathbb{R} \text{ (where } \mathbb{R} \text{ represents the set of real numbers)} \\ x_3 &= -5 \\ x_4 &= -6 \end{cases}$$

3 Lecture 3, 1/29/2024

Last Time: General Solution

$$\begin{cases} x_1 &= 17 - x_2 \\ x_2 &\text{is free or } x_2 \in \mathbb{R} \text{ (where } \mathbb{R} \text{ represents the set of real numbers)} \\ x_3 &= -5 \\ x_4 &= -6 \end{cases}$$

prob 1 is general solution prob 2 is do row operations and find solution. check answer using substitution

Ex: Solve

$$\begin{aligned} 3x_1 + 6x_2 - 2x_3 &= 10 \\ -2x_1 - 4x_2 - 3x_3 &= -1 \\ x_1 + 2x_2 - x_3 &= 2 \end{aligned} \tag{1}$$

Solve

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 0 \\ -2x_1 - 4x_2 + 3x_3 - 2x_4 &= 0 \end{aligned} \tag{2}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 0 \\ -2 & -4 & 3 & -2 & 0 \end{array} \right]$$

Performing the operation $2R_1 + R_2$:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 3 & 6 & 0 \end{array} \right]$$

Performing the operation $\frac{1}{3}R_2$:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array} \right]$$

This leads to the system of equations:

$$\begin{cases} x_1 + 2x_2 + 4x_4 & = 0 \\ x_3 + 2x_4 & = 0 \end{cases}$$

Expressing x_1 and x_3 in terms of free variables x_2 and x_4 :

$$\begin{cases} x_1 & = -2x_2 - 4x_4 \\ x_3 & = -2x_4 \end{cases}$$

The general solution is given by:

$$\begin{cases} x_1 & = -2x_2 - 4x_4 \\ x_2 & \in \mathbb{R} \\ x_3 & = -2x_4 \\ x_4 & \in \mathbb{R} \end{cases}$$

Section 1.3: Vector Equations

Note: ":" means "such that"

Definition: Let $\mathbb{R} = \{(a_1, a_2, \dots, a_n) : a_1, a_2, \dots, a_n \in \mathbb{R}\}$

Where: a_1 is a real number, a_2 is a real number, \dots , a_n is a real number.

This set consists of ordered elements (n-tuples) where each entry is a real number.

$\mathbb{R} \rightarrow (x, y) \rightarrow xy\text{-plane}$

\mathbb{R}^2

An $n \times 1$ $[1 \times n]$ matrix is a column [row] vector. In either case, these are elements of \mathbb{R}^n .

Example: (Column vector)

$$\begin{bmatrix} \sqrt{2} \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^3$$

(Row vector)

$$[4 \quad \pi \quad 7] \in \mathbb{R}^3$$

When the context is clear (usually column), we write $(\sqrt{2}, 0, 1)$ and $(4, \pi, 7)$.

For matrices (vectors) to be the same, the dimensions and entries must match.

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Note: "ε" means "in"

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ (let \vec{u} and \vec{v} be vectors in \mathbb{R}^n). Addition is done entry/component-wise.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

For a vector $\vec{u} \in \mathbb{R}^n$, and a scalar $c \in \mathbb{R}$, $c\vec{u}$ is the vector obtained by scaling each entry of \vec{u} by c .

$$7 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix}$$

Definition: The zero vector, denoted $\vec{0}$, is the vector whose entries are all 0.
 → Context determines dimensions

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \vec{0} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

So, $\vec{0}$ is

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

See page 27 in the textbook for basic properties.

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$

Definition: Let $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$ and $c_1, c_2, \dots, c_m \in \mathbb{R}$. Then the vector

$$\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m = \sum_{i=1}^m c_i\vec{v}_i$$

is a linear combination of $\vec{v}_1, \dots, \vec{v}_m$.

Given $\vec{v}_1, \dots, \vec{v}_m$ and some vector \vec{z} , is \vec{z} a linear combination of $\vec{v}_1, \dots, \vec{v}_m$?

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \vec{z} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Claim: It is impossible to write \vec{z} in terms of \vec{v}_1 and \vec{v}_2 as a linear combination.

Example: Can we write

$$\begin{bmatrix} 7 \\ 9 \\ 15 \end{bmatrix}$$

as a linear combination of

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}?$$

Answer: \vec{z} cannot be expressed as a linear combination of the given vectors.

Desired Expression: Find c_1, c_2, c_3 where

$$\begin{bmatrix} 7 \\ 9 \\ 15 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad c_3 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Resultant Matrix Equation:

$$\begin{bmatrix} 7 \\ 9 \\ 15 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + 2c_3 \\ c_1 + 2c_2 + 3c_3 \\ 2c_1 + c_2 + 4c_3 \end{bmatrix}$$

Solving a Linear System:

Performing row reduction:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 7 \\ 1 & 2 & 3 & 9 \\ 2 & 1 & 4 & 15 \end{array} \right]$$

Row reducing to echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

Thus, a combination is:

$$\begin{bmatrix} 7 \\ 9 \\ 15 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

More generally: If \vec{y} is a linear combination of $\vec{v}_1, \dots, \vec{v}_m$, say $\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m = \sum_{i=1}^m c_i\vec{v}_i$, then we row reduce the augmented matrix:

$$\left[\begin{array}{cccc|c} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m & \vec{y} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \right]$$

to obtain the constants c_1, c_2, \dots, c_m .

Definition: Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m \in \mathbb{R}^n$. The set of all linear combinations of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ is the span of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$, denoted $\text{span}(\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\})$.

4 Lecture 4, 1/31/2024

Recall from last lecture: $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ is the span of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$, denoted $\text{span}(\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\})$.

The span of vectors $\vec{v}_1, \dots, \vec{v}_m$ is denoted as:

$$\text{span}(\vec{v}_1, \dots, \vec{v}_m) = \left\{ \sum_{i=1}^m c_i \vec{v}_i \mid c_1, \dots, c_m \in \mathbb{R} \right\}$$

Remark: The vector $\vec{0}$ is always in the span.

Example:

$$\text{span}(\vec{v})$$

represents a line in the direction of \vec{v} .

If $\vec{r}(t) = t\vec{v}$, it represents a parameterization of the line.

Let's consider $\vec{v} = (1, 2, 3)$ and introduce $\vec{u} = (2, 4, 6)$.

$$\text{span}(\vec{v}, \vec{u})$$

Note: $\vec{u} = 2\vec{v}$, indicating that \vec{u} is a scalar multiple of \vec{v} , and therefore, it still represents the line in the direction of \vec{v} .

The question arises: What is the fewest number of vectors needed to span the same space?

Matrix Equation $A\vec{x} = \vec{b}$

Section 1.4: Matrix Equation $A\vec{x} = \vec{b}$

Given the system of equations:

$$7x_1 - 3x_2 + 4x_3 = 2$$

$$x_1 - x_2 = 4$$

$$-x_1 + 2x_2 - 3x_3 = 1$$

We can express this system as a matrix product in Chapter 2:

Coefficient matrix A :

$$A = \begin{bmatrix} 7 & -3 & 4 \\ 1 & -1 & 0 \\ -1 & 2 & -3 \end{bmatrix}$$

Column vector \vec{x} :

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

\vec{x} is the column vector of variables, and \vec{b} is the column vector of RHS (right-hand side) values of the equations.

$A\vec{x} = \vec{b}$ is the matrix equation of the system.

Equivalently, if $A =$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix}$$

and $\vec{x} =$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

then $A\vec{x} =$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_n\vec{v}_n = \vec{b}$$

Therefore, we can see $A\vec{x} = \vec{b}$ as:

1. Solving a system of linear equations. 2. Checking whether \vec{b} is a linear combination of $\vec{v}_1, \dots, \vec{v}_n$, i.e., whether \vec{b} is in the span $\text{span}(\vec{v}_1, \dots, \vec{v}_n)$.

Theorem: The following statements are equivalent (either all true or all false) for an $m \times n$ matrix A :

1. Equation $A\vec{x} = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^m$.
2. The columns (column vectors) of A span \mathbb{R}^m .
3. Every row in A has a pivot.

Example:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Performing row operations:

$$-2R_2 + R_1 \quad \text{and} \quad -R_2 + R_3$$

Results in the row-echelon form:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the row-echelon form does not have a pivot in every row, statement 3 is false. Therefore, statements 1 and 2 are also false.

Ex:

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

This is the augmented matrix of the system.

Solutions:

$$\begin{aligned}x_2 &= 3 \\x_3 &= 2 \\x_4 &= 4 \\x_1 &\text{ is free}\end{aligned}$$

The system has infinitely many solutions.

Does every row have a pivot? No.

This does NOT violate the theorem!

For some \vec{b} , there may still be solutions. However, if statement 3 fails (pivot in every row), there exists some \vec{b} where $A\vec{x} = \vec{b}$ has NO solutions.

$$\begin{aligned}A\vec{x} &= \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \\ \Rightarrow \vec{x} &= \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}\end{aligned}$$

Ex:

$$\begin{bmatrix} -1 & 2 \\ 4 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 14 \end{bmatrix}$$

Is $\begin{bmatrix} 5 \\ 0 \\ 14 \end{bmatrix}$ a linear combination of $\begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$? $\Rightarrow x_1 = 3, x_2 = 4$. Yes, but we cannot conclude whether the columns span \mathbb{R}^3 (Condition 1 is not satisfied yet).

5 Lecture 5, 2/2/2024

Last time:

$$\begin{aligned}\begin{bmatrix} -1 & 2 \\ 4 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 5 \\ 0 \\ 14 \end{bmatrix} \\ \begin{bmatrix} -1 & 2 & | & 5 \\ 4 & 3 & | & 0 \\ 2 & 2 & | & 14 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 0 & | & 3 \\ 0 & 1 & | & 4 \\ 0 & 0 & | & 0 \end{bmatrix}\end{aligned}$$

$$x_1 = 3, \quad x_2 = 4$$

Section 1.5: Solution Sets of Linear Equations

Definition: A system of linear equations $A\vec{x} = \vec{b}$ is homogeneous if $\vec{b} = \vec{0}$ (zero vector). Observe that $\vec{x} = \vec{0}$ is always a solution (called "trivial"). Does there exist non-trivial solutions?

Example:

$$\begin{aligned}x_1 - x_2 + x_3 - 2x_4 &= 0 \\x_1 - 3x_3 + 2x_4 &= 0 \\2x_1 - x_2 - 2x_3 + 4x_4 &= 0\end{aligned}$$

Row reducing the augmented matrix yields the Reduced Row Echelon Form (RREF):

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{aligned}x_1 &= 3x_3 - 2x_4 \\x_2 &= 4x_3 \\x_3 &= x_3 \\x_4 &= x_4\end{aligned}$$

The solution is of the form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where x_3 and x_4 are real numbers ($x_3, x_4 \in \mathbb{R}$).

$$\text{span} \left(\left\{ \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right)$$

The above is called the parametric vector form of the solutions. What if the system is non-homogeneous?

Example:

$$\begin{aligned}-2x_1 + 3x_2 + x_3 &= 2 \\-x_1 + 3x_2 + 11x_3 &= 19 \\x_1 - x_2 + 3x_3 &= 5\end{aligned}$$

Row reducing the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 10 & 17 \\ 0 & 1 & 7 & 12 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The solution is given by:

$$x_1 = -10x_3 + 17x_2, \quad x_2 = -7x_3 + 12x_3, \quad x_3 = x_3$$

The solution in parametric vector form is:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -10 \\ -7 \\ 1 \end{bmatrix} + \begin{bmatrix} 17 \\ 12 \\ 0 \end{bmatrix}$$

where $x_3 \in \mathbb{R}$.

This can also be expressed as $x_3\vec{v} + \vec{p}$, which is equivalent to $\vec{p}_0 + t\vec{v}$, representing a line through \vec{p}_0 in the direction of \vec{v} .

$$\vec{x} = x_3\vec{v}$$

where $x_3 \in \mathbb{R}$. This can be interpreted as a line through the origin in the direction of \vec{v} .

Example: (two ex. ago, now NOT homogeneous)

$$\begin{aligned} x_1 - x_2 + x_3 - 2x_4 &= 4 \\ x_1 - 3x_3 + 2x_4 &= 1 \\ 2x_1 - x_2 - 2x_3 + 4x_4 &= 5 \end{aligned}$$

Solving this, we find:

$$\begin{aligned} x_1 &= 3x_3 - 2x_4 + 1 \\ x_2 &= 4x_3 - 3 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

where

$$\vec{x} = x_3 \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \end{bmatrix}$$

where $x_3, x_4 \in \mathbb{R}$.

Theorem: Suppose the system $A\vec{x} = \vec{b}$ is consistent. Let \vec{p} be a solution (may not be unique). Then the solution set to $A\vec{x} = \vec{b}$ is given by all vectors $\vec{x} = \vec{p} + \vec{v}_h$, where \vec{v}_h is ANY solution to the homogeneous system $A\vec{x} = \vec{0}$.

Section 1.7: Linear Independence

Definition: The set of vectors $\{\vec{v}_1, \dots, \vec{v}_m\}$ in \mathbb{R}^n is linearly independent if $a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_m\vec{v}_m = \vec{0}$ is true only if $a_1 = a_2 = \dots = a_m = 0$. Otherwise, the set is linearly dependent, meaning there exist constants $c_1, c_2, \dots, c_m \in \mathbb{R}$ not all zero, such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m = \vec{0}$.

$$2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \vec{0}$$

To check if $\{\vec{v}_1, \dots, \vec{v}_m\}$ is linearly independent, verify if $\vec{0}$ is a unique solution to the system $a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_m\vec{v}_m = \vec{0}$. Row reduce the augmented matrix:

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m & | & \vec{0} \\ \downarrow & \downarrow & \downarrow & \downarrow & | & \downarrow \end{bmatrix}$$

If the solution is unique, then the column vectors $\vec{v}_1, \dots, \vec{v}_m$ form a linearly independent set.

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Example: Is $\{(1,2,3), (-2,0,1), (4,-4,-9)\}$ linearly independent?

\Rightarrow Solve for constants c_1, c_2, c_3 where

$$c_1(1, 2, 3) + c_2(-2, 0, 1) + c_3(4, -4, -9) = (0, 0, 0)$$

\Rightarrow

$$\begin{cases} c_1 - 2c_2 + 4c_3 &= 0 \\ 2c_1 - 4c_3 &= 0 \\ 3c_1 + c_2 - 9c_3 &= 0 \end{cases}$$

\Rightarrow Row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -2 & 4 & | & 0 \\ 2 & 0 & -4 & | & 0 \\ 3 & 1 & -9 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 4 & | & 0 \\ 0 & 4 & -12 & | & 0 \\ 0 & 7 & -21 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 4 & | & 0 \\ 0 & 1 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 1 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

\Rightarrow The system is consistent, and it has infinitely many solutions, indicating that the vectors are linearly dependent.

Given:

$$\begin{cases} c_1 - 2c_3 &= 0 \\ c_2 - 3c_3 &= 0 \\ c_3 &\in \mathbb{R} \end{cases}$$

Find a linear dependent relationship.

Let $c_3 = 2$:

$$\begin{cases} c_1 - 2(2) &= 0 \implies c_1 = 4 \\ c_2 - 3(2) &= 0 \implies c_2 = 6 \end{cases}$$

So,

$$4(1, 2, 3) + 6(-2, 0, 1) + 2(4, -4, -9) = (0, 0, 0)$$

Thus, there exist constants c_1, c_2, c_3 (not all zero) forming a linear combination of $\vec{0}$.

\Rightarrow The set is linearly dependent.

Suppose we have a linear dependence: $c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{0}$ (c_1, \dots, c_m not all zero).

Suppose $c_1 \neq 0$:

$$c_2\vec{v}_2 + \dots + c_m\vec{v}_m = -c_1\vec{v}_1$$

$$\vec{v}_1 = -\frac{c_2}{c_1}\vec{v}_2 + \dots - \frac{c_m}{c_1}\vec{v}_m$$

\vec{v}_1 is a linear combination of the other vectors.

Linear dependence implies a vector is a linear combination of others.

Consider $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$\Rightarrow \vec{v}_3$ is NOT a linear combination of \vec{v}_1 and \vec{v}_2 , but \vec{v}_1 IS a linear combination of \vec{v}_2 and \vec{v}_3 .

Theorem: The set $\{\vec{v}_1, \dots, \vec{v}_m\}$ is linearly dependent if and only if at least one vector ("there exists") in the set is a linear combination of the others.

Example: Consider $\{\vec{v}_1, \vec{v}_2\}$, where $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$.

$\Rightarrow \vec{v}_1 = \frac{-c_2}{c_1}\vec{v}_2$, $c_1, c_2 \neq 0$.

Since there are multiples of each other, the set is linearly dependent.

Now, consider the set $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This implies the set must be linearly dependent. However, note that no vector in the set is a multiple of another.

Example: Consider the set $\{\vec{0}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$c_1(\vec{0} + 0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3) = \vec{0}$$

c_1 can be non-zero, implying that any set containing $\vec{0}$ is linearly dependent.

Example: Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$.

Are $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ linearly independent or linearly dependent?

Row reduce:

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 0 & 0 & -1 \\ 3 & 0 & 1 & -1 \end{bmatrix}$$

We know the system is consistent, and there can only be 3 pivots, but with 4 columns, $A\vec{x} = \vec{0}$ must have a free variable. So there are infinite ways to write $\vec{0}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$, implying a linearly dependent set.

Theorem: Let $\{\vec{v}_1, \dots, \vec{v}_m\}$ be vectors in \mathbb{R}^n . Suppose $m > n$. Then the set of vectors is linearly dependent.

Ex: Suppose $m \leq n$. Then no conclusion.

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Section 1.8: Introduction to Linear Transformation

Definition: A transformation T from \mathbb{R}^n to \mathbb{R}^m is a rule assigning each vector $\vec{x} \in \mathbb{R}^n$ to a unique vector $T(\vec{x})$ in \mathbb{R}^m . We define \mathbb{R}^n to be the domain of T , and \mathbb{R}^m as the codomain. The vector $\vec{y} = T(\vec{x})$ is the image of \vec{x} under T . The range is the set of all images $T(\vec{x})$, where $\vec{x} \in \mathbb{R}^n$, i.e., $\{T(\vec{x}) : \vec{x} \in \mathbb{R}^n\}$.

Example:

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad * \vec{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \vec{b} = \begin{bmatrix} 32 \\ 77 \end{bmatrix}$$

This is a "transformation" $\mathbb{R}^3 \rightarrow \mathbb{R}^2$.

Example: $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = k \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$

If $k > 1$, it is a dilation.

If $0 < k < 1$, it is a contraction.

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$$

Example: $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \end{bmatrix}$

Reflects along the x-axis. More generally, given A is $m \times n$, this can be seen as a transformation:

$$[- \dots -] [\mathbb{R}^n] = [\mathbb{R}^m] \text{ from } \mathbb{R}^n \text{ to } \mathbb{R}^m.$$

We write $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$. Alternatively, we write $\vec{x} \mapsto A\vec{x}$.

Ex: Let

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 0 & 2 \end{bmatrix}$$

$$a : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

a) What is the domain and range?

Domain is \mathbb{R}^2 .

Range? $A\vec{x}$?

$$\begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 0 & 2 \end{bmatrix} \vec{x}$$

$$\begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

$$\Rightarrow \text{range is span} \left(\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \right\} \right)$$

b) Is $\begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$ in the range?

Since the range is given by $\text{span} \left(\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \right\} \right)$, we need to check if the given vector can be

written as a linear combination of these vectors. We get No solution. Thus the columns do NOT span \mathbb{R}^3 .

Definition: Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. A transformation T is a linear transformation if:

1. $T(\vec{u} + \vec{v}) = T\vec{u} + T\vec{v}$

2. $cT(\vec{u}) = T(c\vec{u})$

Example: Let $T([x_1][x_2][x_3]) = [xy][z]$. Show it is NOT a linear transformation. To show it is NOT, find explicit vectors where a property fails. Let $c = 2$, $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Property 2? $2T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$.

(Determining if $2T(\vec{u}) = T(2\vec{u})$) Now $T\left(2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, which is not equal to $2T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$.

Hence, $T(2\vec{u}) \neq 2T(\vec{u})$, indicating that the transformation does not satisfy property 2, and therefore, it is not a linear transformation.

Now let's try property 1: $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. We will check if $T(\vec{u} + T(\vec{v})) = T(\vec{u} + \vec{v})$.

Properties: Let T be a linear transformation.

1. $T(\vec{0}) = \vec{0}$

2. $T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2)$

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Example:

Suppose $T(1, 4) = (7, 2, 3)$ (equivalent to $T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix}$). Given that $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ and T is a linear transformation, by inspection, find $T(4, 7)$.

Observe that $2(1, 4) + (-2, 1) = (4, 7)$, which is a linear combination of $(1, 4)$ and $(2, -1)$. According to the linear transformation property:

$$T(2(1, 4) + (-2, 1)) = 2T(1, 4) + 1T(2, -1) = 2(7, 2, 3) + 1(6, 1, 1) = (20, 5, 7) = T(4, 7)$$

→ We found $T(4, 7)$ without knowing the mapping T explicitly.

Section 1.9: The Matrix of a Linear Transformation

Definition: Let $\vec{e}_i \in \mathbb{R}^m$ denote the i^{th} standard basis vector in \mathbb{R}^m , the vector whose i^{th} entry is 1, and 0 elsewhere. For example, \vec{e}_2 is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Suppose $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$. Can we find a matrix A such that $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$?

Example: Suppose $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$.

Now let $\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = xT\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + yT\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$, by the property of a linear transformation.

$$= x\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 5 & 2 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \sim A\vec{x}$$

Matrix A is determined by how $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_m$ map (standard basis vectors).

Theorem: Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then there is a unique matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. Moreover, the matrix A is given by:

$$A = \begin{bmatrix} T\vec{e}_1 & T\vec{e}_2 & \dots & T\vec{e}_n \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix}$$

(the j th column is $T(\vec{e}_j)$), called the standard matrix of the linear transformation T .

Example: Counterclockwise rotation of θ degrees about the origin. We want to find the standard matrix A for the counterclockwise rotation of θ degrees.

We simply need $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} x' \\ y'' \end{bmatrix}$, where $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the standard basis vector along the x -axis, and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} x' \\ y'' \end{bmatrix}$.

$$\Rightarrow \cos \theta = \frac{x'}{1} \Rightarrow x' = \cos \theta \quad \text{and} \quad \sin \theta = \frac{y''}{1} \Rightarrow y'' = \sin \theta$$

$$\Rightarrow T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \rightarrow \text{first column of } A!$$

$$\Rightarrow T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} x' \\ y'' \end{bmatrix}$$

$$\sin \theta = -\frac{x'}{1} \Rightarrow x' = -\sin \theta$$

$$\text{Similarly, } \cos \theta = \frac{y''}{1} \Rightarrow y'' = \cos \theta \quad \text{so} \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$$

Thus, the standard matrix in \mathbb{R}^2 for a counterclockwise rotation of θ radians is:

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

What about clockwise? Use $-\theta$ on the c.c.w matrix, which becomes $\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$

Ex: 1) Reflection along $y=x$

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

(swaps x and y)

2) horizontal shear by K units

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ky \\ y \end{bmatrix}$$

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Definition: A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is one-to-one, or injective, if for each $\vec{b} \in \mathbb{R}^m$, there is at most one $\vec{x} \in \mathbb{R}^n$ such that $T(\vec{x}) = \vec{b}$.

Example: Consider $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ 0 \end{bmatrix}$.

Here, $T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $T\left(\begin{bmatrix} -2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Thus, there are multiple vectors $\vec{x} \in \mathbb{R}^n$ mapping to $\vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Hence, T is not one-to-one (injective).

Equivalently, T is not one-to-one (injective) if $T(\vec{x}_1) = T(\vec{x}_2)$ implies $\vec{x}_1 = \vec{x}_2$.

Definition: A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is onto, or surjective, if for each $\vec{b} \in \mathbb{R}^m$, there is at least one $\vec{x} \in \mathbb{R}^n$ such that $T(\vec{x}) = \vec{b}$.

To show **NOT one-to-one**: Show that there exist $\vec{x}_1, \vec{x}_2 \in \mathbb{R}^n$ such that $T(\vec{x}_1) = T(\vec{x}_2)$.

To show **NOT onto**: Show that there exists $\vec{z} \in \mathbb{R}^m$ such that no element maps to \vec{z} .

($A\vec{x} = \vec{b}$ has no solution)

Example: Is $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ injective with matrix representation

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \end{bmatrix}$$

“forces columns to be linearly dependent”

“Theorem: Let A be the matrix of a linear transformation. Then the transformation is one-to-one (injective) if and only if the columns of A form a linearly independent set, i.e., $A\vec{x} = \vec{0}$ has only the trivial solution. Equivalently, one-to-one if and only if there is a pivot in every column.”

“Theorem: Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation with matrix representation A . Then T is onto/surjective if and only if the columns of A span \mathbb{R}^m . Equivalently, T is onto/surjective if and only if A has a pivot in every row.”

Example: Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the linear transformation defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ -3x \\ 2x - 7y \end{bmatrix}$. We

want to determine if T is one-to-one or onto.

First, we find the standard matrix:

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ -7 \end{bmatrix}$$

So, the standard matrix is:

$$A = \begin{bmatrix} 1 & 1 \\ -3 & 0 \\ 2 & -7 \end{bmatrix}$$

Observing the two column vectors, we see that one vector is not a multiple of the other, indicating that the column vectors form a linearly independent set. Therefore, T is one-to-one.

To confirm, we row reduce A :

$$\begin{bmatrix} 1 & 1 \\ -3 & 0 \\ 2 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 0 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Since there is not a pivot in every row, the transformation is NOT onto.

Theorem: (One-to-One Condition): A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is one-to-one (injective) if and only if $T(\vec{x}) = \vec{0}$ holds true only when $\vec{x} = \vec{0}$.

Given a set of vectors check linear dependency

10 Lecture 10, 2/14/2024

Section 2.1: Matrix Operations

Given two matrices of the same dimensions (size), we define addition entry-wise and scalar multiplication by scaling each entry.

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -3 & 0 \\ -4 & 2 & 2 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 0 & -1 & 3 \\ 0 & 7 & 8 \end{bmatrix}, \quad 3A = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}$$

Properties: Let A and B be matrices of the same size, $c \in \mathbb{R}$:

1) $A + B = B + A$ 2) $A + 0 = A$ (where 0 is the zero matrix of appropriate size) 3) $k(A + B) = kA + kB = Ak + Bk$

Matrix Multiplication

Let A be an $m \times k$ matrix, and B be a $k \times n$ matrix. Then the product AB is an $m \times n$ matrix whose ij th entry is the dot product of row i of A (as a vector) with column j of B (as a vector).

That is,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ik} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{i1} & b_{i2} & \cdots & b_{ij} & \cdots & b_{in} \end{bmatrix}$$

The ij th entry $= (AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$

Example:

$$\begin{bmatrix} -1 & 3 \\ 4 & 7 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} -7 & -10 \\ -10 & 2 \\ -4 & -4 \end{bmatrix}$$

What about

$$\begin{bmatrix} 1 & 4 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 4 & 7 \\ 0 & 2 \end{bmatrix}$$

Does not exist

Suppose A is $m \times k$, B is $k \times n$, and $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

$$\begin{aligned} \text{Then } AB\vec{x} &= A \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \\ \downarrow & \downarrow & \cdots & \downarrow \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ &= A(x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_n\vec{b}_n) \end{aligned}$$

$$\begin{aligned} AB\vec{x} &= A(x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_n\vec{b}_n) \\ &= x_1(A\vec{b}_1) + x_2(A\vec{b}_2) + \cdots + x_n(A\vec{b}_n) \\ &= \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_n \\ \downarrow & \downarrow & \cdots & \downarrow \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \end{aligned}$$

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -1 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$. Then $AB = \begin{bmatrix} 11 & -5 & 9 \\ 9 & 5 & 5 \end{bmatrix} =$

$$1 \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} + 2 \begin{bmatrix} 3 & 4 \\ 6 & 8 \\ 9 & 12 \end{bmatrix} \text{ (columns of } A\text{)}.$$

Example: Let $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \end{bmatrix}$. Since AB is defined (3×2) , $AB = \begin{bmatrix} 3 & 4 \\ 6 & 8 \\ 9 & 12 \end{bmatrix}$.

Definition: A matrix A is square if the number of rows equals the number of columns. For any non-negative integer k , $A^k = A * A * A * A * A$ (multiplied k times).

Moreover, $A^0 = I$, called the identity matrix.

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & 0 & 1 \end{bmatrix}$$

The identity matrix, denoted I , has ones on the diagonal and zeros elsewhere. It satisfies, for any square matrix A of the same size, $IA = AI = A$, and for a vector of appropriate size, say \vec{x} , $I\vec{x} = \vec{x}$.

Definition: Let A be an $m \times k$ matrix. The transpose of A , denoted A^T or A^t , is the $k \times m$ matrix where the ij th entry of A^T is $(A^T)_{ij} = A_{ji}$.

i.e., the rows

columns

of A^T are the columns

rows

of A .

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$. Then $A^T = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$.

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Properties of transpose:

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(kA)^T = kA^T$
4. $(AB)^T = B^T A^T$

Section 2.2: Matrix Inverse

Let $A\vec{x} = \vec{b}$ be a linear system, $3x = 5$. We want to isolate \vec{x} .

Def: Let A be $n \times n$. We say matrix A is invertible if there is a matrix C such that $AC = CA = I = I_n$, where

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & 0 & 1 \end{bmatrix}$$

The matrix C is the inverse of A , denoted A^{-1} .

We say A is invertible. If no inverse exists, we say A is singular. If it does exist, A^{-1} is unique. Now given $A\vec{x} = \vec{b}$, if A^{-1} exists, $A^{-1}(A\vec{x}) = A^{-1}\vec{b} \Rightarrow (A^{-1}A)\vec{x} = A^{-1}\vec{b} \Rightarrow I\vec{x} = \vec{x} = A^{-1}\vec{b}$.

Theorem: Let $A\vec{x} = \vec{b}$ be a system of linear equations. If A^{-1} exists, then the solution is unique, given by $\vec{x} = A^{-1}\vec{b}$.

Example: Suppose

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$$

which is A^{-1} :

$$A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{bmatrix}$$

or

$$A^{-1} = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Does $AA^{-1} = I$? Just check if AA^{-1} outputs the identity matrix.

$$AA^{-1} = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is A^{-1} .

Properties: Let A and B be invertible matrices and $k \neq 0$, both $n \times n$.

1. $(A^{-1})^{-1} = A$
2. $(kA)^{-1} = \frac{1}{k}(A)^{-1}$
3. $(A^m)^{-1} = (A^{-1})^m$ (where m is a positive integer)
4. $(A^T)^{-1} = (A^{-1})^T$
5. $(AB)^{-1} = B^{-1}A^{-1}$

How to find A^{-1} : To find A^{-1} , solve the equation $A \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ \downarrow & \downarrow & \cdots & \downarrow \end{bmatrix} = I$.

We need $A\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $A\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, and so on, up to $A\vec{x}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$.

The identity matrix simultaneously solves all these systems.

Algorithm for A^{-1} :

Let A be an $n \times n$ matrix.

1. Create the augmented matrix $[A|I_n]$.

2. Row reduce to row-echelon form (REF).
3. If we reduce it to $[I_n|C]$, where C is the inverse of A , then C is A^{-1} .

Example: Find A^{-1} if $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 & -1 & | & 1 & 0 & 0 \\ 2 & -2 & -1 & | & 0 & 1 & 0 \\ -1 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 1 & | & 0 & 0 & 1 \\ 2 & -2 & -1 & | & 0 & 1 & 0 \\ 0 & 1 & -1 & | & 1 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 1 & | & 0 & 0 & 1 \\ 0 & 0 & 1 & | & 0 & 1 & 2 \\ 0 & 1 & -1 & | & 1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 1 & | & 0 & 0 & 1 \\ 0 & 1 & -1 & | & 1 & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 1 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 1 & | & 0 & 0 & 1 \\ 0 & 1 & -1 & | & 1 & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 0 & 2 & | & -1 & 0 & 1 \\ 0 & 1 & 0 & | & 1 & 1 & 2 \\ 0 & 0 & 1 & | & 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & | & 1 & 2 & 3 \\ 0 & 1 & 0 & | & 1 & 1 & 2 \\ 0 & 0 & 1 & | & 0 & 1 & 2 \end{bmatrix}$$

Therefore, $A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$.

Example: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Section 2.3: Characterizations of Invertible Matrices

Theorem (Invertible Matrix): Let A be $n \times n$. The following are equivalent:

1. A is invertible
2. A is row equivalent to I_n (identity matrix)
3. A has n pivots
4. System $A\vec{x} = \vec{b}$ has a unique solution
5. The columns of A are linearly independent and span \mathbb{R}^n
6. The linear transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one AND onto (injective and surjective).

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Ex: True or False:

If A is the matrix representation of a linear transformation $\mathbb{R}^n \rightarrow \mathbb{R}^n$, then A has n pivots.

FALSE: This statement is incorrect as it doesn't specify how many pivots A has. A simple counterexample is the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which represents a linear transformation from \mathbb{R}^2 to \mathbb{R}^2 but has no pivots.

Example: Let A be a 2×3 matrix, and assume A has 2 pivots. Then $A\vec{x} = \vec{b}$ has a unique solution for any $\vec{b} \in \mathbb{R}^2$. However, this statement is false.

Counterexample: Consider the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 1 & 4 & 6 \end{array} \right]$$

The system $A\vec{x} = \vec{b}$ represented by this matrix has infinite solutions. Thus, the statement is false.

Inverses and Linear Transformation:

Let A be the standard matrix of a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

We say T is invertible if there is a transformation S such that $S(T(\vec{x})) = \vec{x} = T(S(\vec{x}))$ for all $\vec{x} \in \mathbb{R}^n$, where $S = T^{-1}$. Also, A^{-1} will be the standard matrix for T^{-1} .

Section 3.1: Introduction to Determinants

Definition: The determinant of $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is defined as $\det(A) = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$ (a number).

For a 3×3 matrix, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, we have

$$\det(A) = a_{11} \cdot (a_{22} \cdot a_{33} - a_{23} \cdot a_{32}) - a_{12} \cdot (a_{21} \cdot a_{33} - a_{23} \cdot a_{31}) + a_{13} \cdot (a_{21} \cdot a_{32} - a_{22} \cdot a_{31}).$$

Definition: The minor of element a_{ij} in matrix A , denoted M_{ij} , is the determinant of the remaining matrix after deleting row i and column j . The cofactor of a_{ij} is $c_{ij} = (-1)^{i+j} \cdot M_{ij}$.

From the definition, we have $\det(A) = a_{11} \cdot c_{11} + a_{12} \cdot c_{12} + a_{13} \cdot c_{13}$.

Example: For matrix $A = \begin{bmatrix} 6 & 3 & -2 \\ 7 & 0 & 4 \\ 1 & 3 & 3 \end{bmatrix}$:

$$M_{12} = 7 \cdot 3 - 4 \cdot 1 = 21 - 4 = 17$$

$$c_{12} = (-1)^{1+2}(7 \cdot 3 - 4 \cdot 1) = -17$$

Definition: The determinant of an $n \times n$ matrix is given by $|A| = \det(A) = a_{11} \cdot c_{11} + a_{12} \cdot c_{12} + \dots + a_{1n} \cdot c_{1n}$.

Definition: Call the cofactor expansion "row 1 entries".

Theorem: The cofactor expansion can be applied to any row or any column. That is,

$$\det(A) = \det(A) = a_{i1} \cdot c_{i1} + a_{i2} \cdot c_{i2} + \dots + a_{in} \cdot c_{in} = a_{1j} \cdot c_{1j} + a_{2j} \cdot c_{2j} + \dots + a_{nj} \cdot c_{nj}$$

Intuition: Expand on a row/column with lots of zeros.

How to know what to do with $(-1)^{i+j}c_{ij}$:

$$\begin{bmatrix} + & - & + & - & + & \dots \\ - & + & - & + & - & \dots \\ + & - & + & - & + & \dots \end{bmatrix}$$

Signs always alternate in cofactor expansion.

Example:

$$\begin{bmatrix} 7 & -1 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ -5 & 0 & -4 & 2 \\ 1 & 0 & 5 & 0 \end{bmatrix} = A, \text{ expand on row 2.}$$

$$\begin{aligned} \det(A) &= 0 + (-1)^{2+2} \cdot 1 \cdot \det \begin{pmatrix} 7 & 3 & 0 \\ -5 & -4 & 1 \\ 1 & 5 & 0 \end{pmatrix} + 0 + (-1)^{2+4} \cdot 3 \cdot \det \begin{pmatrix} 7 & -1 & 3 \\ -5 & 0 & -4 \\ 1 & 0 & 5 \end{pmatrix} \\ &= 1 \cdot (-2 \cdot \det \begin{pmatrix} 7 & 3 \\ 1 & 5 \end{pmatrix}) + 3(-(-1) \cdot \det \begin{pmatrix} -5 & -4 \\ 1 & 5 \end{pmatrix}) \\ &= -2 \cdot (35 - 3) + 3 \cdot (-25 - (-4)) = -127 \end{aligned}$$

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Section 3.2: Properties of Determinants

How do row operations affect the determinant?

Theorem: Let A be $n \times n$, k be a scalar.

Then:

- a) If matrix B is obtained from A by interchanging two rows [columns], then $\det(B) = -\det(A)$
i.e $|B| = -|A|$
- b) If B is obtained from A by multiplying the elements of a row [column] by k , then $\det(B) = k \det(A)$
- c) If B is obtained from A by adding a multiple of one row [column] to another row [column], then $\det(B) = \det(A)$

Note: row [column] means row or column but must be consistent

Example: Evaluate $\begin{vmatrix} 4 & 9 & -3 \\ 5 & -3 & 1 \\ 1 & 2 & 4 \end{vmatrix}$

Multiply column 3 by 3, add to column 2

Result: $\begin{vmatrix} 4 & 0 & -3 \\ 5 & 0 & 1 \\ 1 & 14 & 4 \end{vmatrix}$

Now, perform cofactor expansion on column 2.

$$\begin{aligned} &= -14 \cdot \begin{vmatrix} 4 & -3 \\ 5 & 1 \end{vmatrix} + 0 + 0 \\ &= -14(4 - (-15)) = -14 \cdot 19 = -266 \end{aligned}$$

Example:

$$\text{Evaluate } \begin{vmatrix} 3 & 3 & -9 & 9 \\ -1 & 2 & 4 & -6 \\ 2 & 2 & -8 & 10 \\ 1 & 1 & -3 & 8 \end{vmatrix} = 3 \cdot \begin{vmatrix} 1 & 1 & -3 & 3 \\ -1 & 2 & 4 & -6 \\ 2 & 2 & -8 & 10 \\ 1 & 1 & -3 & 8 \end{vmatrix}$$

$$\text{Row operations? Applying row operations, } = 3 \cdot \begin{vmatrix} 1 & 1 & -3 & 3 \\ 0 & 3 & 1 & -3 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 5 \end{vmatrix}$$

$$= 3 \cdot (1) \cdot (3) \cdot ((-2)(5) - (4)(0)) = 90$$

This is the product of the diagonal entries. We multiply by the diagonal because the matrix is in REF so we will always be subtracting by 0 in the determinant.

Definition: An upper (or lower) triangular matrix is a square matrix whose entries below (or above) the main diagonal are zero.

Example:

$$\begin{bmatrix} 1 & 4 & 5 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

Theorem: The determinant of an upper (or lower) triangular matrix is the product of the diagonal entries.

Given matrix A , if we apply row operations and get a zero on the diagonal (assuming it's upper triangular), then the determinant will be zero.

- If we apply the inverse algorithm, we cannot reach $[I|0|??]$ (zero on diagonal).
- Therefore, it won't be invertible, and the determinant is the product of diagonal entries, thus the determinant is zero.

Theorem: Matrix A is invertible if and only if $\det(A) \neq 0$.

Given that $\det(A) = 5$, if A is the matrix of a linear transformation, the transformation is one-to-one and onto by Section 2.3.

Properties: Let A and B be $n \times n$ matrices, and $c \in \mathbb{R}$. Then:

1) $\det(cA) = c^n \cdot \det(A)$. 2) $\det(AB) = \det(A)\det(B)$. 3) $\det(A) = \det(A^T)$. 4) $\det(A^{-1}) = \frac{1}{\det(A)}$ (provided A^{-1} exists).

Ex: Suppose $\det(A) = 3$, $\det(B) = 12$. Let A, B be the same size. Find $\det((A^2B^{-1})^T)$.

$$\begin{aligned}
\det((A^2 B^{-1})^T) &= \det(A^2 B^{-1}) \\
&= \det(A^2) \det(B^{-1}) \\
&= (\det(A))^2 \cdot \frac{1}{\det(B)} \\
&= (3)^2 \cdot \frac{1}{12} \\
&= \frac{9}{12} \\
&= \frac{3}{4}
\end{aligned}$$

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Ex: Suppose there exists a positive integer k such that $A^k = 0$ matrix. Show $\det(A) = 0 \Rightarrow \det(A^k) = (\det(A))^k$.

We have:

$$\begin{aligned}
\det(A^k) &= (\det(A))^k \\
&= (0)^k \\
&= 0
\end{aligned}$$

Thus, $\det(A^k) = (\det(A))^k = 0^k = 0$. Taking the k -th root of both sides, we get $\det(A) = \sqrt[k]{0} = 0$, as claimed.

Generally, $\det(A^k) = (\det(A))^k$.

Section 3.3: Volume and Linear Transformations

Suppose \vec{v}_1 and \vec{v}_2 form the sides of a parallelogram. We can represent this as a matrix $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \\ \downarrow & \downarrow \end{bmatrix}$ where:

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

We can then row reduce and add multiples of one row to another row.

The area is given by $a \cdot b = \det(A) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$.

Determinant stays the same with this row operation.

Theorem: If a parallelogram is determined by \vec{v}_1 and \vec{v}_2 (not multiples of each other), then the area is $|\det(A)|$ (absolute value), where

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \\ \downarrow & \downarrow \end{bmatrix}$$

Similarly, in \mathbb{R}^3 , if a parallelepiped is determined by $\vec{v}_1, \vec{v}_2, \vec{v}_3$ (assuming the vectors are linearly independent), then the volume is $|\det(A)|$, where

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

Example: Find the area of the parallelogram with vertices $(-1, 0)$, $(3, 0)$, $(2, 2)$, and $(6, 2)$.

Vectors formed: $(2 - (-1), 2 - 0) = (3, 2)$ and $(6 - 2, 2 - 2) = (4, 0)$.

The parallelogram is formed by the vectors:

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 0 \end{bmatrix}$$

The area is given by $|\det(A)| = |(3)(0) - (2)(4)| = |-8| = 8$.

Let $S = \{a_1\vec{v}_1 + a_2\vec{v}_2 : 0 \leq a_1, a_2 \leq 1\}$ be the points on or inside a parallelogram formed by \vec{v}_1 and \vec{v}_2 .

Let T be a linear transformation with standard matrix A . Then $T(A)$ (which means transforming all points of a parallelogram) will be vectors of the form:

$$T(a_1\vec{v}_1 + a_2\vec{v}_2) = T(a_1\vec{v}_1) + T(a_2\vec{v}_2) = a_1T(\vec{v}_1) + a_2T(\vec{v}_2) = a_1(A\vec{v}_1) + a_2(A\vec{v}_2)$$

\Rightarrow parallelogram formed by $A\vec{v}_1$ and $A\vec{v}_2 \Rightarrow$ Area of $T(S)$ = area of parallelogram formed by $A\vec{v}_1, A\vec{v}_2$

$$= \left| \det \begin{pmatrix} A\vec{v}_1 & A\vec{v}_2 \\ \downarrow & \downarrow \end{pmatrix} \right| = \left| \det(A) \cdot \det \begin{pmatrix} \vec{v}_1 & \vec{v}_2 \\ \downarrow & \downarrow \end{pmatrix} \right| = |\det(A \cdot B)| = |\det(A) \cdot \det(B)| = |\det(A)| \cdot (\text{area of } S) |$$

Theorem: Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation with standard matrix A . If S is a parallelogram, then the area of $T(S)$ is given by $|\det(A)|$ times the area of S . In \mathbb{R}^3 , the volume of $T(S)$ is $|\det(A)|$ times the volume of S .

Remark: This generalizes to any regions, e.g., circles.

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Example: Let S be the parallelogram formed by $\vec{v}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation with the standard matrix $A = \begin{bmatrix} 2 & 2 \\ 5 & 2 \end{bmatrix}$. What is the area of $T(S)$?

$$\Rightarrow \text{area of } T(S) = |\det(A)| \cdot (\text{area of } S) = |4 - 10| \left| \det \begin{pmatrix} 3 & -1 \\ 5 & 1 \end{pmatrix} \right| = |-6| \cdot |3 + 5| = 48$$

Alternate solution:

First map vectors:

$$A\vec{v}_1 = \begin{bmatrix} 22 \\ 20 \end{bmatrix}, \quad A\vec{v}_2 = \begin{bmatrix} -2 \\ -4 \end{bmatrix} \Rightarrow \text{parallelogram formed by } \begin{bmatrix} 22 \\ 20 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$

$$\Rightarrow \text{Area of } T(S) = \left| \det \begin{pmatrix} 22 & -2 \\ 20 & -4 \end{pmatrix} \right| = |-88 + 40| = 48$$

Section 4.1: Vector Space and Subspaces

Definition: A vector space is a set V of elements called vectors, defined by two operations, addition and scalar multiplication, such that the following axioms hold for any $\vec{u}, \vec{v}, \vec{w} \in V$, and scalars c and d :

1. $\vec{u} + \vec{v} \in V$ ("closure under addition")
2. $c\vec{u} \in V$ ("closure under scalar multiplication")
3. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
4. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
5. There is a zero vector, denoted $\vec{0}$, such that $\vec{u} + \vec{0} = \vec{u}$.
6. For every $\vec{v} \in V$, there is a vector $-\vec{v} \in V$ such that $(\vec{v} + (-\vec{v})) = \vec{0}$
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
8. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
9. $c(d\vec{u}) = (cd)\vec{u}$
10. There is a 1, such that $1\vec{u} = \vec{u}$

Examples:

1. $\mathbb{R}^n = \{(a_1, a_2, \dots, a_n) : a_1, a_2, \dots, a_n \in \mathbb{R}\}$
2. M_n = space of all $n \times n$ matrices with real entries. Remark: This can generalize to $m \times n$ matrices.
3. \mathbb{P}_n : The space of polynomials of degree at most n , where addition and scalar multiplication are given by $(f + g)(x) = f(x) + g(x)$ and $cf(x) = c \cdot f(x)$. Example: $7x^5 - 3x + 1 \in \mathbb{P}_{10} \rightarrow$ all of degree at most 10. In \mathbb{P}_n , the zero vector is $0'(x) = 0$ ("zero function").
4. Let V be the set of all real-valued functions $f : D \rightarrow \mathbb{R}$ (D is often \mathbb{R}). Example: $\sin^5(7x^2) \in V$.

Subspaces Definition: A subspace of vector space V is a subset U of V such that

1. $\vec{0} \in U$
2. Closed under addition i.e. if $\vec{u}_1, \vec{u}_2 \in U$, then $\vec{u}_1 + \vec{u}_2 \in U$
3. Closed under scalar multiplication i.e. if $\vec{u}_1 \in U$ and c is a scalar, then $c\vec{u}_1 \in U$.

Note: 3 implies 1 already by choosing $c=0$.

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Remarks:

1. All other properties of a vector space will be inherited from V (see page 195).
2. All subspaces are vector spaces.

Examples:

1. $U = \{\vec{0}\}$ is always a subspace of any vector space called the zero subspace or trivial subspace.
2. Consider $U = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R} \right\}$, the set of diagonal matrices. Then U is a subspace of M_2 , the space of 2×2 matrices.

Closure: How to show this? Let $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \in U$. Then $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$, which is an element of U .

Why does this not prove property 2? This may fail for other matrices! Instead, we need to show it holds for any diagonal matrices. Let's show all three properties:

1. Let $a = b = 0$. Then $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in U$, showing the zero matrix is in U .
2. Let $\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}$ and $\begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix} \in U$, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Then $\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix} + \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & 0 \\ 0 & b_1 + b_2 \end{bmatrix} \in U$, showing closure under addition.
3. Let $\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix} \in U$ and $c \in \mathbb{R}$. Then $c \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix} = \begin{bmatrix} ca_1 & 0 \\ 0 & cb_1 \end{bmatrix} \in U$, showing closure under scalar multiplication.

Example: Is $U = \left\{ \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} : a \in \mathbb{R} \right\}$ a subspace of M_2 ?

No, the zero matrix is not in U ! Here the first entry must be 1. To disprove, a counterexample is enough:

Let $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in U$, $c = 2$. Then $2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \notin U$, failing scalar multiplication.

Let $\begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} \in U$, $c \in \mathbb{R}$. Then $c \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} c & ac \\ 0 & 0 \end{bmatrix}$, $c = \pi$.

Example: Is $U = \{(a, b, a - b) : a, b \in \mathbb{R}\}$ a subspace of \mathbb{R}^3 ?

1. Let $a = b = 0 \Rightarrow (0, 0, 0 - 0) = \vec{0} \in U$.
2. Let $\vec{u}_1 = (a_1, b_1, a_1 - b_1)$, $\vec{u}_2 = (a_2, b_2, a_2 - b_2) \in U$. Then $\vec{u}_1 + \vec{u}_2 = (a_1 + a_2, b_1 + b_2, a_1 - b_1 + a_2 - b_2) = (a_1 + a_2, b_1 + b_2, (a_1 + a_2) - (b_1 + b_2))$, which is in U ; hence, U is closed under addition.
3. $\vec{u} = (a_1, b_1, a_1 - b_1)$, $c \in \mathbb{R} \Rightarrow c\vec{u}_1 = (ca_1, cb_1, c(a_1 - b_1)) = (ca_1, cb_1, ca_1 - cb_1)$, which is in $U \Rightarrow$ closed under scalar multiplication

Example: Show \mathbb{P}_1 is a subspace of \mathbb{P}_2 (polynomials of degree at most n for \mathbb{P}_n).

1. Let $a_1 = b_1 = 0$, which means the zero polynomial is in \mathbb{P}_1 .
2. Let $f_1(x) = a_1x + b_1$ and $f_2(x) = a_2x + b_2 \in \mathbb{P}_1$. Then $f_1(x) + f_2(x) = a_1x + b_1 + a_2x + b_2 = (a_1 + a_2)x + (b_1 + b_2) \in \mathbb{P}_1$, showing closure under addition.
3. Let $f_1(x) = a_1x + b_1$ and $c \in \mathbb{R}$. Then $cf_1(x) = c(a_1x + b_1) = (ca_1)x + (cb_1) \in \mathbb{P}_1$, showing closure under scalar multiplication.

Let $\vec{v}_1, \vec{v}_2 \in V$, where V is a vector space. Let $U = \text{span}(\{\vec{v}_1, \vec{v}_2\})$.

Take $\vec{y}_1, \vec{y}_2 \in U$. Then, $\vec{y}_1 = c_1\vec{v}_1 + c_2\vec{v}_2$ and $\vec{y}_2 = d_1\vec{v}_1 + d_2\vec{v}_2$.

Therefore, $\vec{y}_1 + \vec{y}_2 = (c_1 + d_1)\vec{v}_1 + (c_2 + d_2)\vec{v}_2 \in U$, showing closure under addition.

Let $k \in \mathbb{R}$ and $\vec{y}_1 \in U$. Then, $\vec{y}_1 = c_1\vec{v}_1 + c_2\vec{v}_2$.

Therefore, $k\vec{y}_1 = (kc_1)\vec{v}_1 + (kc_2)\vec{v}_2 \in U$, showing closure under scalar multiplication.

Theorem: Let $\vec{v}_1, \dots, \vec{v}_m$ be vectors in a vector space V . Then $\text{span}(\{\vec{v}_1, \dots, \vec{v}_m\})$ is a subspace of V .

(i.e. just show $u = \text{span}(??)$)

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To show U is a subspace of U , we simply need to define a set A such that $U = \text{span}(A)$, and we are done!

Ex: We had $U = \{(a, b, a - b) : a, b \in \mathbb{R}\}$. Observe $(a, b, a - b) = a(1, 0, 1) + b(0, 1, -1)$, hence $U = \text{span}\{(1, 0, 1), (0, 1, -1)\}$, and we know the span of any set is a subspace, thus U is a subspace of \mathbb{R}^3 .

Ex: Show \mathbb{P}_1 is a subspace of \mathbb{P}_2 .

$\mathbb{P}_1 = \{ax + b : a, b \in \mathbb{R}\}$

Then $\mathbb{P}_1 = \text{span}\{1, x\}$. All linear combinations are of the form $b(1) + a(x)$, $a, b \in \mathbb{R}$. Hence \mathbb{P}_1 is the span of a set, so \mathbb{P}_1 is a subspace of \mathbb{P}_2 .

\mathbb{R}^2 as a subspace of \mathbb{R}^3 is false.

$\{(x, y) : x, y \in \mathbb{R}\}$ versus $\{(x, y, z) : x, y, z \in \mathbb{R}\}$ "looks like" $\{(x, y, 0) : x, y \in \mathbb{R}\}$.

$\mathbb{R}^2 = \text{span}\{(1, 0), (0, 1)\}$

$(1, 1) + (2, 2, 2)$

True or False:

$\mathbb{P}_2 = \text{span}(1, x, x^2, 3x^2 + 5) \rightarrow \text{span}(\{1, x, x^2\}) \Rightarrow$ all linear combinations of the form $a(1) + b(x) + c(x^2)$, $a, b, c \in \mathbb{R}$

True, the $3x^2 + 5$ is irrelevant since it is already a linear combination of $\{1, x, x^2\}$.

Note that $\{1, x, x^2, 3x^2 + 5\}$ is linearly dependent.

However, $\{1, x, x^2\}$ is linearly independent, and $\mathbb{P}_2 = \text{span}\{1, x, x^2\}$.

Section 4.2: Null space, column space, and linear transformations

Definition: The kernel, or null space of a linear transformation $T : V \rightarrow W$ is denoted as $\ker(T) = \text{nul}(T) = \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}$.

Definition: The range of the linear transformation $T : V \rightarrow W$ is denoted by $\text{range}(T)$ and defined as follows:

$$\text{range}(T) = \{T(\vec{v}) \in W : \vec{v} \in V\}$$

Definition: In matrix form, if $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$, then $A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_n \vec{v}_n$.

We define the column space of A as $\text{col}(A) = \text{span}(\{\vec{v}_1, \dots, \vec{v}_n\})$.

Thus, given the mapping $\vec{x} \mapsto A\vec{x}$, the range of the transformation is $\text{col}(A)$.

Example: Find $\ker(T)$ if $T(x, y) = (x + y, 3x, y)$ (want standard matrix)

We have:

$$T(1, 0) = (1, 3, 0) \quad \text{and} \quad T(0, 1) = (1, 0, 1)$$

So, the standard matrix A is given by:

$$A = \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 1 \end{bmatrix}$$

We want all $\vec{x} \in \mathbb{R}^2$ such that $A\vec{x} = \vec{0}$

Solve the homogeneous system

Row reduce

$$\left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_1 = 0$$

$$x_2 = 0$$

$$\text{So } \ker(T) = \{\vec{0}\}$$

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Given transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined as $T(x, y) = (x + y, 3x, y)$, we want to find the range $\text{range}(T)$.

The span of the given set $\{[1, 3, 0], [1, 0, 1]\}$.

For the set $S = \{[a, b, c, d] : a + b - 2c = d, 3a + 4c + d = 0\}$, where $a, b, c, d \in \mathbb{R}$:

We seek vectors satisfying $a + b - 2c - d = 0$ and $3a + 4c + d = 0$.

Treating S as solutions to a linear system:

$$\begin{bmatrix} 1 & 1 & -2 & -1 \\ 3 & 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, S is the kernel of the linear transformation with standard matrix $A = \begin{bmatrix} 1 & 1 & -2 & 1 \\ 3 & 0 & 4 & 1 \end{bmatrix}$.

Solving, the RREF of the augmented matrix is:

$$S = \ker(A) = \text{span} \left\{ \left[\frac{16}{3}, -\frac{10}{3}, 1, 0 \right], \left[\frac{7}{3}, -\frac{4}{3}, 0, 1 \right] \right\}$$

S is a subspace of \mathbb{R}^4 .

Theorem: Let $T : V \rightarrow W$ where A is $m \times n$. Then $\ker(T) = \text{nul}(T)$ (i.e., $\ker(A) = \text{nul}(A)$) is a subspace of V (or \mathbb{R}^n). Moreover, $\text{range}(T)$ is a subspace of W (or $\text{col}(A)$ is a subspace of \mathbb{R}^m).

Example 1: Is $S = \{(x, y, z) : x = 2y + z, x, y, z \in \mathbb{R}\}$ a subspace of \mathbb{R}^3 ?

Method 1: We want (x, y, z) where $x - 2y - z = 0 \Rightarrow [1 \ -2 \ 1][x \ y \ z] = [0]$. S is the kernel of $A = [1 \ -2 \ 1]$. The kernel is always a subspace $\Rightarrow S$ is a subspace.

Method 2: Vectors of the form $(2y + z, y, z)$. Then $\{(2y + z, y, z)\}$ subspace? $(2y + z, y, z) = y(2, 1, 0) + z(1, 0, 1) \Rightarrow S = \text{span}\{(2, 1, 0), (1, 0, 1)\}$, and we know the span of any set is a subspace, so S is a subspace.

Method 2: Vectors of the form $(2y + z, y, z)$. Then $\{(2y + z, y, z)\}$ subspace? $(2y + z, y, z) = y(2, 1, 0) + z(1, 0, 1) \Rightarrow S = \text{span}\{(2, 1, 0), (1, 0, 1)\}$, and we know the span of any set is a subspace, so S is a subspace.

Example 2: Let $T : \mathbb{P}_1 \rightarrow \mathbb{P}_1$, where $T(ax + b) = a$ (derivative). What is the kernel? Range? Is T one-to-one? Onto?

- **Kernel ($\text{Ker}(T)$):** $\text{Ker}(T) = \{c : c \in \mathbb{R}\}$ (constant polynomials).

- **One-to-one (T injective):**

$T(3) = 0$ and $T(2) = 0$, i.e., two different domain elements map to the same range value. Also, for T to be one-to-one, $\text{Ker}(T) = \{\vec{0}\}$.

- T is one-to-one (or injective) if and only if $\text{Ker}(T) = \{\vec{0}\}$.

If $\text{ker}(T) = \{\vec{0}\}$, why can't other values map to the same range value?

One-to-one:

$$T(\vec{u}) = T(\vec{v}) \Rightarrow \vec{u} = \vec{v}$$

$$T(\vec{u}) = T(\vec{v}) \Rightarrow T(\vec{u}) - T(\vec{v}) = \vec{0} \Rightarrow T(\vec{u} - \vec{v}) = \vec{0} \Rightarrow \vec{u} - \vec{v} = \vec{0} \text{ (trivial kernel)} \Rightarrow \vec{u} = \vec{v} \Rightarrow \text{one-to-one}$$

Onto:

$$T(ax + b) \neq x$$

No, since the codomain is \mathbb{P}_1 , $g(x) = x \in \mathbb{P}_1$, but no polynomial $ax + b$ will satisfy $T(ax + b) = x \Rightarrow$ not onto/surjective. Note that if $T : \mathbb{P}_1 \rightarrow \mathbb{P}_0$, then T is onto/surjective.

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Section 4.3: Linear Independent Sets and Bases

A set of vectors may span \mathbb{R}^n but may not be linearly independent.

A set may be linearly independent in \mathbb{R}^3 but may not span \mathbb{R}^3 .

Definition: A set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ is a basis of a vector space if the set spans the vector space and it is linearly independent.

Example: $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$ is the standard basis for \mathbb{R}^4 . We sometimes write $\vec{e}_i = (0, 0, \dots, 1, 0, \dots, 0)$ (1 is the i^{th} entry) to refer to the i^{th} standard basis vector.

Example: Basis for $\mathbb{P}_2 = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$ is $\{1, x, x^2\}$.

More generally, $\{1, x, x^2, \dots, x^n\}$ is the standard basis for \mathbb{P}_n .

Example: Is $\{1, x + x^2\}$ a basis of \mathbb{P}_2 ?

Span $\rightarrow a(1) + b(x + x^2) \rightarrow bx + bx^2$

The polynomial $2x + x^2$ is not in span $\{1, x + x^2\}$, so $\{1, x + x^2\}$ is not a basis.

Theorem: Let $\vec{v}_1, \dots, \vec{v}_m$ span vector space V . Then each vector in V can be uniquely expressed as a linear combination of $\vec{v}_1, \dots, \vec{v}_m$ if and only if $\{\vec{v}_1, \dots, \vec{v}_m\}$ is linearly independent, i.e., $\{\vec{v}_1, \dots, \vec{v}_m\}$ is a basis.

Theorem: Suppose $V = \text{span}(\{\vec{v}_1, \dots, \vec{v}_m\})$. Then we can always reduce the set by deleting vectors, so that the new set of vectors is a basis of V .

Moreover, given a linearly independent set that does not span V , we can add vectors to extend to a basis of V .

Bases for $\text{col}(A)$, $\text{row}(A)$, $\text{ker}(A)$

For $\text{ker}(A)$, we simply find the solutions to $A\vec{x} = \vec{0}$ in parametric vector form, and the vectors that span the solutions form a basis for $\text{ker}(A)$.

Example: $A = \begin{bmatrix} 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$; solutions to $A\vec{x} = \vec{0}$ are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 9 \\ -2 \\ 1 \end{bmatrix}$, $x_2, x_4 \in \mathbb{R}$

\Rightarrow basis for $\text{ker}(A)$ is $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 9 \\ -2 \\ 1 \end{bmatrix} \right\}$. $\text{ker}(A) = \text{span}$ of those vectors.

Theorem: A basis for $\text{col}(A)$ consists of the pivot columns corresponding to the original matrix A , not the reduced row echelon form (RREF). A basis for $\text{row}(A)$ consists of the non-zero rows of the RREF of A .

Example: $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not a basis for $\text{col}(A)$.

\Rightarrow basis of $\text{col}(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\text{col}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

Basis of $\text{row}(A) = \{[1 \ 1 \ 0]\}$

Given the matrix $A = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 1 & 2 & -1 & 0 \\ 2 & 6 & 0 & 3 \end{bmatrix}$, after row reduction, we have $A \rightarrow \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

a) What is $\text{col}(A)$?

$$\Rightarrow \text{col}(A) = \text{span} \left(\left(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \right) \right)$$

b) Basis of $\text{col}(A)$?

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \right\}$$

is a basis.

c) Basis of $\text{row}(A)$?

$$\{[1 \ 0 \ -3 \ 0] \ [0 \ 1 \ 1 \ 0] \ [0 \ 0 \ 0 \ 1]\}$$

d) Suppose A is the matrix of a linear transformation from \mathbb{R}^4 to \mathbb{R}^3 . Find a basis for $\text{range}(A)$ and $\text{ker}(A)$.

range(A) = $A\vec{x} \rightarrow$ basis is the basis for $\text{col}(A)$.

ker(A)? Solve $A\vec{x} = \vec{0}$

$$\Rightarrow \begin{cases} x_1 = 3x_3 \\ x_2 = -x_3 \\ x_3 = x_3 \\ x_4 = 0 \end{cases}$$

$$\Rightarrow \text{Solution to } A\vec{x} = \vec{0} \text{ are } \vec{x} = x_3 \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad x_3 \in \mathbb{R}$$

$$\Rightarrow \text{ker}(A) = \text{span} \left(\left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\} \right), \text{ basis for } \text{ker}(A) \text{ is } \left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

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Given the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{bmatrix}$ with the row-reduced echelon form (RREF) $\begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

a) Determine $\text{row}(A)$.

$$\text{row}(A) = \text{span}\{[1 \ 0 \ 5 \ 0], [0 \ 1 \ -1 \ 2]\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -4 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 11 \\ 8 \end{bmatrix} \right\}$$

b) Basis of $\text{col}(A)$?

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right\}$$

Section 4.4: Coordinate Systems

Definition: Let $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis of V . Let $\vec{u} \in V$, and write $\vec{u} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$. The column vector $\vec{u}_B = [\vec{u}]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ is the coordinate vector of \vec{u} relative to basis $B = \{\vec{v}_1, \dots, \vec{v}_n\}$. The a_1, \dots, a_n are the coordinates of \vec{u} .

Example: Let $B = \{(3, 4), (7, -1)\}$. For example, $-2(3, 4) + 3(7, -1) = (15, -11)$. Therefore, $[(15, -11)]_B = [-2][3]$.

Example: Let $B = \{(1, 0), (0, 1)\}$, the standard basis. For example, $9(1, 0) + 8(0, 1) = (9, 8)$. Therefore, $[(9, 8)]_B = [9][8]$.

Example: Let $B_1 = \{(1, 0), (0, 1)\} = \{\vec{u}_1, \vec{u}_2\}$ and $B_2 = \{(2, -1), (-1, 1)\} = \{\vec{v}_1, \vec{v}_2\}$. Let $\vec{u} = (4, 5)$ and $\vec{u} = 4\vec{u}_1 + 5\vec{u}_2$. Then $(4, 5) = 3\vec{v}_1 + 2\vec{v}_2$.

Coordinates relative to the "oblique axes" formed by \vec{v}_1 and \vec{v}_2 are denoted as $[(4, 5)]_{B_2} = [3][2]$.

Definition: If $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for V , and $\vec{u} \in V$, the change-of-coordinates matrix from B to the standard basis is $P_B = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix}$. Moreover, if $\vec{u} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$, we have $\vec{u} = P_B[\vec{u}]_B$. "Multiplying P_B with the coordinate vector outputs \vec{u} in terms of the standard basis".

Moreover, with P_B having linearly independent columns, $(P_B)^{-1}$ exists, so $[\vec{u}]_B = (P_B)^{-1}\vec{u}$.

We can see $(P_B)^{-1}$ as a mapping from a vector space V to \mathbb{R}^n (vector $[\vec{u}]_B$).

Thus, we can see that any vector space with n basis vectors has a mapping $(P_B)^{-1}$, which is one-to-one and onto, will be isomorphic to \mathbb{R}^n .

Example: $\{(x, y, 0) : x, y \in \mathbb{R}\}$ is isomorphic to \mathbb{R}^2 (in \mathbb{R}^3).

Example: Consider \mathbb{P}_n and the standard basis $B = \{1, x, x^2, \dots, x^n\}$.

If $f(x) \in \mathbb{P}_n$, and $f(x) = c_0 + c_1x + c_2x^2 + \dots + c_nx^n$, then $[\vec{f}(x)]_B = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$. Therefore,

$f(x) \mapsto [f(x)]_B$ is an isomorphism from \mathbb{P}_n to \mathbb{R}^{n+1} .

Usefulness of context \Rightarrow Derivative transformation, useful as polynomials.

Ex: Is $\{3 + x^2, 1 + x + x^2, x\}$ a linearly independent set in \mathbb{P}_2 ? Using the previous info, we have:

$$\vec{v}_1 = [3 + x^2]_B = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = [1 + x + x^2]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \vec{v}_3 = [x]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Are $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly independent?

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the row-reduced matrix is an identity matrix, the set is linearly independent.

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Last time: $\mathbb{P}_n \rightarrow \mathbb{R}^{n+1}$ is an isomorphism.

$P_B = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix}$ is the change of coordinates matrix from $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ to the standard basis.

Example: Let $T : \mathbb{P}_2 \rightarrow \mathbb{P}_2$ be given by $T(a_0 + a_1x + a_2x^2) = a_2 + (a_1 - a_0)x$.

To find the matrix representation of T with respect to the standard basis $\{1, x, x^2\}$:

Map basis polynomials \rightarrow represent as vectors in \mathbb{R}^3

$$a_0 = 1, a_1 = 0, a_2 = 0: T(1) = 0 + (0 - 1)x = -x \Rightarrow [T(1)]_B = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$a_0 = 0, a_1 = 1, a_2 = 0: T(x) = 0 + (1 - 0)x = x \Rightarrow [T(x)]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$a_0 = 0, a_1 = 0, a_2 = 1: T(x^2) = 1 + (0 - 0)x = 1 \Rightarrow [T(x^2)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \text{The matrix representation is } A = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For example, if $B = \{(7, 5), (-3, -2)\}$ is a basis of \mathbb{R}^2 , to express $\vec{u} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ as a linear combination of the vectors in B , and find $[\vec{u}]_B$:

$$\text{By the earlier theorem, } P_B = \begin{bmatrix} 7 & -3 \\ 5 & -2 \end{bmatrix}. \text{ Thus, } (P_B^{-1})_B = \begin{bmatrix} -2 & 3 \\ -5 & 7 \end{bmatrix}.$$

$$\text{By the theorem, } [\vec{u}]_B = (P_B^{-1})_B \vec{u} = \begin{bmatrix} -2 & 3 \\ -5 & 7 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

So, $4(7, 5) + 7(-3, -2) = (7, 6)$.

Generalization: We use P and P^{-1} to go between ANY 2 bases! Right now we go between B and the standard basis.

Section 4.5 (5th edition \rightarrow also 4.6)

Theorem: Let vector space V contain a basis of n vectors. Then any other basis of V also has exactly n vectors. Moreover, any set of $n+1$ or more vectors is necessarily linearly independent.

Def: If vector space V has a basis of n vectors, then dimension of V is n , denoted $\dim(V) = n$. For this class we assume n is finite.

Ex:

1) $\mathbb{R}^n \Rightarrow$ Standard basis $\{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 0, 1)\} \Rightarrow \dim(\mathbb{R}^n) = n$

2) $\mathbb{P}_n \Rightarrow$ Standard basis $\{1, x, x^2, \dots, x^n\} \Rightarrow \dim(\mathbb{P}_n) = n + 1$

3) $\{0\} \Rightarrow$ Dimension is 0

Basis \rightarrow linear independent AND spans V

Now if we know $\dim(V)$, we only need to check linear independence.

Theorem: If $\dim(V) = n$, a set $\{\vec{v}_1, \dots, \vec{v}_n\}$ in V is a basis if and only if it is linearly independent or spans V .

Example: Is $\{(1, 1, 1), (0, 1, 2), (3, 0, 1)\}$ a basis for \mathbb{R}^3 ?

We check for linear independence by row reducing:

$$\begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since we have three linearly independent vectors in \mathbb{R}^3 and $\dim(\mathbb{R}^3) = 3$, the set forms a basis for \mathbb{R}^3 .

Now let A be an $m \times n$ matrix. The row space of A , denoted $\text{row}(A)$, is a subspace of \mathbb{R}^n , and the column space of A , denoted $\text{col}(A)$, is a subspace of \mathbb{R}^m .

Theorem: If A is an $m \times n$ matrix, then $\dim(\text{row}(A)) = \dim(\text{col}(A))$.

Example: Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.

$\rightarrow \text{row}(A)$ is in \mathbb{R}^2 .

$\rightarrow \text{col}(A)$ is in \mathbb{R}^3 .

So clearly $\text{row}(A) \neq \text{col}(A)$.

Definition: The rank of a matrix A , denoted $\text{rank}(A)$, is the dimension of the column space, i.e., $\text{rank}(A) = \dim(\text{col}(A)) = \dim(\text{row}(A))$.

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Theorem (Rank-Nullity, Matrix form): Let A be an $m \times n$ matrix. Then the number of columns, n , equals the rank of A plus the dimension of the kernel of A .

Note: $\dim(\ker(A))$ is called the "nullity" or $\text{null}(A)$, denoted by number $= \dim(\text{nul}(A))$.

Theorem (Rank-Nullity, Linear Transformation form): Let $T : V \rightarrow W$ be a linear transformation. Assume $\dim(V) = n$. Then $\dim(\text{domain}) = n = \dim(\text{range}(T)) + \dim(\ker(A))$.

Example: The row-reduced echelon form (RREF) of

$$\begin{bmatrix} 1 & 2 & 0 & -2 & -3 \\ 2 & 4 & 5 & 23 & -13 \\ 1 & 2 & 1 & 3 & -5 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 2 & 0 & -2 & -3 \\ 0 & 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Verify the Rank-Nullity Theorem:

$$\Rightarrow \text{basis of } \text{col}(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} \right\} \Rightarrow \text{rank}(A) = 2$$

Kernel? Solving $A\vec{x} = \vec{0}$, observe the row-reduced echelon form (RREF) given by:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & -2 & -3 & 0 \\ 0 & 0 & 1 & 5 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

$$\Rightarrow \begin{cases} x_1 = -2x_2 + 2x_4 + 3x_5 \\ x_2 = x_2 \\ x_3 = -5x_4 + 2x_5 \\ x_4 = x_4 \\ x_5 = x_5 \end{cases}$$

$$\Rightarrow \vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -5 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_4, x_5 \in \mathbb{R}.$$

$$\Rightarrow \text{basis of the kernel is } \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \Rightarrow \dim(\ker(A)) = 3.$$

By the Rank-Nullity Theorem: $\text{rank}(A) + \dim(\ker(A)) = 2 + 3 = 5$, which equals the number of columns.

Given $\text{rank}(A) = 2$, we know the number of columns is 5, hence we expect $\dim(\ker(A)) = 3$.

Example: If A is 5×9 , how large can $\dim(\text{col}(A))$ be?

Recall, $\dim(\text{col}(A)) = \dim(\text{row}(A))$. This means $\dim(\text{col}(A))$ can be at most the minimum between the two dimensions (5 and 9). Therefore, it can be at most 5.

Thus, for example, by the Rank-Nullity Theorem, # of columns = 9 = $\text{rank}(A) + \dim(\ker(A))$. Since $\text{rank}(A)$ is at most 5, we have $9 \leq 5 + \dim(\ker(A)) \Rightarrow \dim(\ker(A)) \geq 4$. Therefore, we can conclude that $\dim(\ker(A))$ is never 3.

Ex: Verify the rank-nullity theorem for the derivative operator $T : \mathbb{P}_2 \rightarrow \mathbb{P}_1$, where $T(a_0 + a_1x + a_2x^2) = a_1 + 2a_2x$.

- $\dim(\text{domain}) = \dim(\mathbb{P}_2) = 3$.

- Kernel? We need polynomials where the output has $a_1 + 2a_2x = 0$, so we need $a_2 = 0$, $a_1 = 0$, and no restriction on a_0 . $\Rightarrow T(a_0 + 0x + 0x^2) = 0 \Rightarrow \ker(T) = \{k : k \in \mathbb{R}\}$. \Rightarrow Basis of $\ker(T)$ is $\{1\}$. $\Rightarrow \dim(\ker(T)) = 1$.

- Note: $3 = \dim(\text{domain}) = \dim(\text{range}) + \dim(\ker) = ?? + 1$.

- Observe: $\text{range}(T) = \{a_1 + 2a_2x : a_1, a_2 \in \mathbb{R}\} = \{b_1 + 2b_2x : b_1, b_2 \in \mathbb{R}\} = \mathbb{P}_1$. \Rightarrow Basis is $\{1, x\} \Rightarrow \dim(\text{range}(T)) = 2 \Rightarrow$ By rank-nullity theorem, $3 = \dim(\text{domain}) = \dim(\text{range}(T)) + \dim(\ker(T)) = 2 + 1$.

Ex: A system of linear equations has 5 variables and 3 equations. The homogeneous system has 2 free variables. If A is the coefficient matrix, does $A\vec{x} = \vec{b}$ have a solution for any \vec{b} (appropriate sized \vec{b})

Another way to phrase this problem is: If A is a linear transformation is the transformation onto/surjective.

$\Rightarrow 2$ free variables $\rightarrow \dim(\ker(A)) = 2 \Rightarrow$ By the rank-nullity theorem, number of columns = 5 = $\text{rank}(A) + \dim(\ker(A)) = \text{rank}(A) + 2 \Rightarrow \text{rank}(A) = \dim(\text{col}(A)) = 3 \Rightarrow \text{col}(A)$ is a subspace of \mathbb{R}^3 . \Rightarrow columns span $\mathbb{R}^3 \Rightarrow A\vec{x} = \vec{b}$ always has a solution for any \vec{b} .

23 Lecture 23, 3/25/2024

Section 4.6: Change of Basis

$$\text{In 4.4} \rightarrow P_B = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix}$$

Given two bases $B_1 = \{\vec{v}_1, \vec{v}_2\}$ and $B_2 = \{\vec{u}_1, \vec{u}_2\}$, to express a vector as a linear combination of both bases, we need to relate $[\vec{x}]_{B_1}$ and $[\vec{x}]_{B_2}$.

Definition: Let $B_1 = \{\vec{v}_1, \dots, \vec{v}_n\}$ and $B_2 = \{\vec{u}_1, \dots, \vec{u}_n\}$ be two bases of vector space V . The change-of-coordinates matrix, or transition matrix is $P_{B_1 \leftarrow B_2} = \begin{bmatrix} [\vec{v}_1]_{B_2} & [\vec{v}_2]_{B_2} & \dots & [\vec{v}_n]_{B_2} \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix}$.

Here, the columns are the coordinate vectors of $\vec{v}_1, \dots, \vec{v}_n$ with respect to B_2 .

Remark: If B_2 is the standard basis, then $[\vec{v}_1]_{B_2} = \vec{v}_1$, yielding the result in Section 4.4.

For example, let $B_1 = \{(7, -7), (-1, 11)\} = \{\vec{v}_1, \vec{v}_2\}$ and $B_2 = \{(1, 3), (-5, -1)\} = \{\vec{u}_1, \vec{u}_2\}$.

How to find $P_{B_1 \leftarrow B_2}$? Need to write $(7, -7)$ and $(-1, 11)$ as a linear combination of $(1, 3)$ and $(-5, -1)$:

$$\Rightarrow \text{row reduce } \begin{bmatrix} 1 & -5 & 7 \\ 3 & -1 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$$[\vec{v}_1]_{B_2} = \begin{bmatrix} -3 \\ -2 \end{bmatrix} \Rightarrow \vec{v}_1 = -3\vec{u}_1 - 2\vec{u}_2 \Rightarrow$$

$$P_{B_1 \leftarrow B_2} = \begin{bmatrix} -3 & 4 \\ -2 & 1 \end{bmatrix}$$

Theorem: Let $B_1 = \{\vec{v}_1, \dots, \vec{v}_n\}$ and $B_2 = \{\vec{u}_1, \dots, \vec{u}_n\}$ be two bases of a vector space V . Let $\vec{x} \in V$. If $P_{B_1 \leftarrow B_2}$ is the change of coordinates matrix, then $P_{B_1 \leftarrow B_2} [\vec{x}]_{B_1} = [\vec{x}]_{B_2}$.

Moreover, matrix P is invertible, and $(P_{B_1 \leftarrow B_2})^{-1} [\vec{x}]_{B_2} = P_{B_2 \leftarrow B_1} [\vec{x}]_{B_2} = [\vec{x}]_{B_1}$.

$$\text{Ex: From the last example, } B_1 = \{(7, -7), (-1, 11)\}, B_2 = \{(1, 3), (-5, -1)\}, P_{B_1 \leftarrow B_2} = \begin{bmatrix} -3 & 4 \\ -2 & 1 \end{bmatrix}$$

By inspection, what is $[(5, 15)]_{B_1}$?

$$[(5, 15)]_{B_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow P_{B_1 \leftarrow B_2} [(5, 15)]_{B_1} = \begin{bmatrix} -3 & 4 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} = [(5, 15)]_{B_2} \Rightarrow 5(1, 3) + 0(-5, -1) = (5, 15).$$

Note: With the 2x2 inverse formula:

$$(P_{B_1 \leftarrow B_2})^{-1} = P_{B_2 \leftarrow B_1} = \begin{bmatrix} \frac{1}{5} & -\frac{4}{5} \\ \frac{2}{5} & -\frac{3}{5} \end{bmatrix}$$

Ex: Let $B_1 = \{1, x, x^2\}$, $B_2 = \{4, x-1, 3x^2\}$ be the basis for \mathbb{P}_2 . Find $P_{B_1 \leftarrow B_2}$ and $P_{B_2 \leftarrow B_1}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \begin{bmatrix} 4 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Note: LHS is B_1 , RHS is B_2 , $B_1 \leftarrow B_2$

$$\Rightarrow P_{B_1 \leftarrow B_2} = \begin{bmatrix} 4 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\Rightarrow P_{B_2 \leftarrow B_1} = \begin{bmatrix} 4 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$(B_{1 \leftarrow B_2}^P)^{-1} = \begin{bmatrix} 4 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = B_{2 \leftarrow B_1}^P$$

What is $[f(x)]_{B_2}$ if $f(x) = 4 - x^2$?

B_1 is easy \Rightarrow want to use $B_1 \rightarrow B_2$

$$\Rightarrow [f(x)]_{B_1} = [4][0][-1]$$

$$\Rightarrow B_{2 \leftarrow B_1}^P [f(x)]_{B_1} = \begin{bmatrix} 1/4 & 1/4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1/3 \end{bmatrix}$$

$$\Rightarrow \text{Indeed, we have } 1(4) + 0(x-1) + (-1/3)(3x^2) = 4 - x^2$$

24 Lecture 24, 3/27/2024

Last Time: To find the change of coordinates matrix $B_{2 \leftarrow B_1}^P$, row reduce

$$\left[\begin{array}{cc|cc} \vec{u}_1 & \vec{u}_2 & \vec{v}_1 & \vec{v}_2 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array} \right]$$

Definition: Let $B_1 = \{\vec{v}_1, \vec{v}_2\}$ and $B_2 = \{\vec{u}_1, \vec{u}_2\}$.

Section 5.1, 5.2: Eigenvectors and Eigenvalues and Characteristic equation.

Definition: Let A be an $n \times n$ matrix. A scalar λ is an eigenvalue of A if there exists a vector \vec{x} that is not the zero vector such that $A\vec{x} = \lambda\vec{x}$. Vector \vec{x} is an eigenvector of A . Note if $\lambda = 1 \Rightarrow A\vec{x} = \vec{x}$ ("stability").

How to find λ and \vec{x}

To solve $A\vec{x} = \lambda\vec{x}$, we have $A\vec{x} - \lambda\vec{x} = 0 \Rightarrow (A - \lambda I)\vec{x} = 0$.

polynomial of $\lambda \rightarrow \text{find } \lambda$

Definition: If A is an $n \times n$ matrix, the characteristic polynomial of A is $\det(A - \lambda I)$, which is a polynomial of degree n in terms of λ . The characteristic equation is $\det(A - \lambda I) = 0$.

The roots of the polynomial are the eigenvalues λ .

The algebraic multiplicity of λ is the number of times λ occurs in the characteristic polynomial as a root. Once you obtain the eigenvalues λ , substitute in $(A - \lambda I)\vec{x} = 0$ to get eigenvectors \vec{x} .

Example: Find the eigenvalues and eigenvectors of matrix A :

$$A = \begin{bmatrix} 5 & 2 \\ -8 & -3 \end{bmatrix}$$

$$\Rightarrow 0 = \det(A - \lambda I) = \det \left(\begin{bmatrix} 5 - \lambda & 2 \\ -8 & -3 - \lambda \end{bmatrix} \right)$$

$$= (5 - \lambda)(-3 - \lambda) - (-16)$$

$$= \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$$

$$\Rightarrow \lambda = 1 \text{ is an eigenvalue with algebraic multiplicity } 2.$$

$$\Rightarrow \text{solve } (A - \lambda I)\vec{x} = \vec{0} \Rightarrow (A - I)\vec{x} = \vec{0} \Rightarrow \text{Row reduce } \begin{bmatrix} 4 & 2 \\ -8 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ (Always infinite solutions)}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \Rightarrow x_1 = -\frac{1}{2}x_2, \quad x_2 = x_2 \Rightarrow \vec{x} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} x_2$$

\Rightarrow For eigenvalue $\lambda = 1$, the eigenvectors are $\vec{x} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} x_2$, $x_2 \in \mathbb{R} \setminus \{0\}$ ($\vec{0}$ is not an eigenvector)

Theorem: If A is $n \times n$, the set of all eigenvectors corresponding to λ , along with $\vec{0}$, form a subspace of \mathbb{R}^n , called the eigenspace of λ .

Example: From above, the eigenspace of $\lambda = 1$ is $\text{span} \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$.

Note: $\vec{x} \neq \vec{0}$, but $\lambda = 0$ is okay. If $\lambda = 0$, then $A\vec{x} = 0$, so an eigenvector for $\lambda = 0$ is now a non-trivial solution to $A\vec{x} = 0$. This implies that matrix A cannot be invertible. Equivalently, $\det(A) = 0$.

Theorem: An $n \times n$ matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue, i.e., $\det(A) \neq 0$ if and only if $\lambda = 0$ is not an eigenvalue.

25 Lecture 25, 3/29/2024

Last time: $A\vec{v} = \lambda\vec{v}$

λ = eigenvalue \vec{v} = eigenvector

Solve $\det(A - \lambda I) = 0$ to get eigenvalues.

Algebraic multiplicity \rightarrow number of times λ is a root in $\det(A - \lambda I)$.

Eigenspace of $\lambda \Rightarrow$ All eigenvectors corresponding to λ and $\vec{0}$.

Ex: $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$. Is $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ an eigenvector?

Check $A\vec{v} = \lambda\vec{v}$. Compute $A\vec{v} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 4 \end{bmatrix}$. But $\begin{bmatrix} 4 \\ 5 \\ 4 \end{bmatrix}$ is not a multiple of

$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, i.e., there is no λ where $A\vec{v} = \lambda\vec{v} \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not an eigenvector.

What are eigenvalues and eigenvectors of A ?

Solve $0 = \det(A - \lambda I) = \det \left(\begin{bmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{bmatrix} \right) = (3-\lambda)^2(5-\lambda)$.

$\Rightarrow \lambda = 3$ (multiplicity 2) and $\lambda = 5$ (multiplicity 1) are the eigenvalues.

Eigenvectors?

$\lambda = 3$: $(A - 3I)\vec{v} = \vec{0}$; Row reduce

$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow x_1 = -x_2 + x_3, \quad x_2 = x_2, \quad x_3 = x_3$$

$$\vec{v} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{for } \lambda = 3, \text{ the eigenspace is } \text{span}\left(\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}\right).$$

For $\lambda = 3$, the set of eigenvectors is $\{x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} : x_2, x_3 \in \mathbb{R}, \text{ with } x_2 \neq 0, x_3 \neq 0 \text{ simultaneously}\}.$

$$x_3 = 0, x_2 = 1 \Rightarrow \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ is an eigenvector for } \lambda = 3.$$

$\lambda = 5$: Solve $(A - 5I)\vec{v} = \vec{0}$; row reduce

$$\begin{bmatrix} -1 & 1 & -1 \\ 2 & 0 & -2 \\ 1 & 1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow x_1 = x_3, x_2 = 2x_3, x_3 = x_3$$

$$\vec{v} = x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{for } \lambda = 5, \text{span}\left(\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}\right) \text{ is the eigenspace. The set of eigenvectors is } \{x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} : x_3 \in$$

$\mathbb{R}, \text{ and } x_3 \neq 0\}.$

Remark: In this problem, $\lambda = 3$ had algebraic multiplicity 2 and 2 linearly independent eigenvectors.

In Wednesday's example, $\lambda = 1$ had algebraic multiplicity 2, but only 1 linearly independent eigenvector.

In the last example, is A invertible? Yes, we know A is invertible if and only if $\lambda = 0$ is not an eigenvalue. So with $\lambda = 3$ and 5 as the only eigenvalues, it is invertible.

Section 5.3: Diagonalization

Definition: Let A and B be $n \times n$ matrices. We say A is similar to B if there exists an invertible matrix P such that $A = PBP^{-1}$.

Note that if $Q = P^{-1}$, we have $A = Q^{-1}B$, which implies $B = QAQ^{-1}$. Thus, B is similar to A .

Example: One can check

$$\begin{bmatrix} 7 & -10 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1}$$

In the form $A = PBP^{-1}$.

Eigenvalues of A : Solve $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 7 - \lambda & -10 \\ 3 & -4 - \lambda \end{vmatrix} = (2 - \lambda)(1 - \lambda) \Rightarrow \lambda = 1, 2 \text{ are the eigenvalues.}$$

Eigenvalues of B :

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) \Rightarrow \text{eigenvalues are } \lambda = 1, 2 \text{ also.}$$

Theorem: If A and B are similar, then they have the same eigenvalues, including multiplicities.

Definition: A square matrix A is diagonalizable if there exists an invertible matrix P such that $D = P^{-1}AP$, where D is a diagonal matrix.

That is, A is similar to a diagonal matrix.

Now write $A = PDP^{-1}$. Then,

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = (PD)(I)(DP^{-1}) = PD^2P^{-1}$$

So, in general, $A^k = PD^kP^{-1}$.

26 Lecture 26, 4/1/2024

Last time: Matrix A is diagonalizable if there exists P such that $D = P^{-1}AP$, then we get $A^k = PD^kP^{-1}$.

Let

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Then $D^{50} = \begin{bmatrix} 1^{50} & 0 \\ 0 & 2^{50} \end{bmatrix}$, which is easier to compute than A^{50} .

What about A^{200} ?

Find A^2 then $(A^2)^2$ which is $(A^4)^2 = A^8$.

Theorem: Let A be an $n \times n$ square matrix. Then, A is diagonalizable if and only if A has n linearly independent eigenvectors. Moreover, if $D = P^{-1}AP$, then the diagonal entries of D are the n eigenvalues of A , and the columns of P are the n linearly independent eigenvectors of A .

For example, let $A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$. Is A diagonalizable? We need to find the eigenvalues.

$$\det(A - \lambda I) = \det \left(\begin{bmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{bmatrix} \right) = (7-\lambda)(-1-\lambda) - (3 \cdot 3) = (8-\lambda)(2+\lambda) = 0$$

This gives us $\lambda = 8, -2$ as the eigenvalues.

For $\lambda = 8$, solve $(A - 8I)\vec{v} = \vec{0}$ which leads to row reduction:

$$\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

This gives $x_1 = 3x_2$ and $x_2 = x_2$.

So, eigenvectors are of the form $\vec{v} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, where $x_2 \in \mathbb{R} \setminus \{0\}$.

Thus, $\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is a linearly independent eigenvector for $\lambda = 8$. A basis for the eigenspace of $\lambda = 8$ is $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$.

Make sure you get infinite solutions. The $\vec{0}$ is not an Eigenvector. Recognize when the answer is unreasonable.

Solving $(A + 2I)\vec{v} = \vec{0}$, we find the eigenvectors of $\lambda = -2$ are of the form $\vec{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} x_1$, $x_1 \in \mathbb{R} \setminus \{0\}$. Therefore, the basis of the eigenspace for $\lambda = -2$ is $\left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\}$. This means we have 2 linearly independent eigenvectors in a 2×2 matrix, hence A is diagonalizable.

$$\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

This gives us $3x_1 = -x_2$ and $x_2 = x_2$.

So, $\vec{v} = x_2 \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$.

$$\begin{aligned} \text{Then we have } D &= \begin{bmatrix} 8 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} \\ &= -\frac{1}{10} \begin{bmatrix} -3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} \end{aligned}$$

(Note that this is of the form $D = P^{-1}AP$).

Matrix P has its columns being the linearly independent eigenvectors. They must be in the order of the eigenvalues from D .

What is A^5 ? $D = P^{-1}AP \Rightarrow A = PDP^{-1}$

$$A^5 = PD^5P^{-1} = -\frac{1}{10} \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 8^5 & 0 \\ 0 & (-2)^5 \end{bmatrix} \begin{bmatrix} -3 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 29488 & 9846 \\ 9840 & 3248 \end{bmatrix}$$

Def: The geometric multiplicity of eigenvalue λ is the dimension of the eigenspace of λ , i.e., the number of linearly independent eigenvectors corresponding to λ .

$$\text{Ex: Let } A = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}.$$

$\det(A - \lambda I) = (2 - \lambda)^2 \cdot (4 + \lambda) \Rightarrow \lambda = 2$ has algebraic multiplicity 2 and $\lambda = -4$ has algebraic multiplicity 1.

Geometric multiplicity? For $\lambda = -4$, we find the eigenvectors (solve $(A + 4I)\vec{v} = \vec{0}$) are of the form $\vec{v} = k \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $k \in \mathbb{R} \setminus \{0\}$. Therefore, the geometric multiplicity of $\lambda = -4$ is 1.

Inconclusive on diagonalization. Need to check $\lambda = 2$.

27 Lecture 27, 4/3/2024

Last time:

Geometric multiplicity: Dimension of eigenspace for λ .

Diagonalizable if and only if there are n linearly independent eigenvectors.

$$D = P^{-1}AP$$

For $A = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$:

$$\det(A - \lambda I) = (2 - \lambda)^2(4 + \lambda) \Rightarrow \lambda = -4$$

We find an eigenvector: $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ - only one linearly independent eigenvector.

One linearly independent eigenvector and algebraic multiplicity 1. What about $\lambda = 2$?

$$(A - 2I)\vec{v}_2 = \vec{0}$$

We row reduce:

$$\begin{bmatrix} 1 & -1 & 1 \\ 7 & -7 & 1 \\ 6 & -6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $x_1 = x_2$, $x_2 = x_2$, $x_3 = 0$.

$$\vec{x} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad x_2 \in \mathbb{R} \setminus \{0\}$$

Thus, there is only 1 linearly independent eigenvector.

Geometric multiplicity: 1. Observe algebraic multiplicity is 2.

For this 3×3 matrix, we only have 2 linearly independent eigenvectors.

Not Diagonalizable.

Theorem:

1. The sum of all algebraic multiplicities is n . 2. The algebraic multiplicity of any eigenvalue is at least the geometric multiplicity. So, if the algebraic multiplicity is 1, then the geometric multiplicity must be 1. 3. Matrix A is diagonalizable if and only if the algebraic multiplicities equal the geometric multiplicities for all eigenvalues. 4. If A has n distinct eigenvalues, then A is diagonalizable.

Ex: True or False: If A is diagonalizable, then A has n distinct eigenvalues. False!

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$(A - \lambda I) = \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix} \Rightarrow \lambda = 0$$

with algebraic multiplicity 2.

Eigenvectors?

$$(A - \lambda I)\vec{v} = \vec{0} \Rightarrow \text{Row reduce}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow x_1 = x_1, x_2 = x_2$$

So, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are two linearly independent eigenvectors.

A 2×2 matrix with 2 linearly independent eigenvectors, so A is diagonalizable, but A does not have distinct eigenvalues.

Ex: Let $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$. Is A diagonalizable?

$$\det(A - \lambda I) = (2 + \lambda)^2(-4 + \lambda)$$

(algebraic multiplicity 2).

For $\lambda = -2$:

$$(A + 2I)\vec{v}_2 = \vec{0} \Rightarrow \begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = x_2 - x_3, \quad x_2 = x_2, \quad x_3 = x_3$$

So, $\vec{v}_2 = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $x_2, x_3 \in \mathbb{R}$, not both 0. Basis of eigenspace: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$$\lambda = -2$$

has geometric multiplicity 2.

For $\lambda = 4$:

$$(A - 4I)\vec{v}_1 = \vec{0}$$

Row reduce:

$$\begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = \frac{1}{2}x_3, \quad x_2 = \frac{1}{2}x_3, \quad x_3 = x_3$$

So, $\vec{v}_1 = x_3 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$, $x_3 \in \mathbb{R} \setminus \{0\}$ are the eigenvectors for $\lambda = 4$. Eigenspace for $\lambda = 4$ is $\text{span} \left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}$.

We found A , which is 3×3 , has 3 linearly independent eigenvectors. A is diagonalizable or for each eigenvalue, algebraic multiplicity equaled the geometric multiplicity. A is diagonalizable.

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Given $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$:

For $\lambda = 4$: $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is an eigenvector.

For $\lambda = -2$: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ forms a basis for the eigenspace.

The matrix A is diagonalizable. We write:

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \left(\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \right)$$

$$D = P^{-1}AP$$

Ex: What can be said about diagonalizable matrices and being invertible?

For $\lambda = 0$, it means A is not invertible.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 - \lambda & 0 \\ 0 & -\lambda \end{bmatrix} \Rightarrow \det(A - \lambda I) = -\lambda(1 - \lambda) = 0 \Rightarrow \lambda = 1, 0$$

Two distinct eigenvalues, not invertible, yet diagonalizable.

Can matrix A ever be diagonalizable and invertible?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \det(A) = 2$$

Two distinct eigenvalues, hence invertible and diagonalizable.

Not invertible, not diagonalizable? Need $\lambda = 2$ as eigenvalue for multiplicity 2, otherwise, two distinct eigenvalues.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \det(A - \lambda I) = \lambda^2 = 0 \Rightarrow \lambda = 2$$

Only one linearly independent eigenvector, hence not diagonalizable.

Invertible but not diagonalizable?

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow \lambda = 1$$

is the only eigenvalue.

Only one linearly independent eigenvector, hence not diagonalizable.

There is no direct relation between diagonalization and being invertible.

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Algebraic multiplicity: Number of times λ is a root of $\det(A - \lambda I)$

Geometric multiplicity: Number of linearly independent corresponding eigenvectors to λ = Dimension of the eigenspace of λ

Ex: Matrix A has characteristic polynomial $\det(A - \lambda I) = (2 - \lambda)(\lambda)(5 - \lambda)^7$

Can we determine the following?

a) Size of A ? Yes, 9×9 , sum of exponents (algebraic multiplicity).
 b) Diagonalizable (need n linearly independent eigenvectors), so no, cannot determine; don't know geometric multiplicity of $\lambda = 5$.

c) $\text{rank}(A)$: At most 9. $\Rightarrow \lambda = 0$ is an eigenvalue. $\Rightarrow A$ is not invertible. \Rightarrow Not row equivalent to I . $\Rightarrow \text{rank}(A)$ is less than 9, but we don't know the exact value.

Geometric multiplicity helps with diagonalizability.

Section: 5.4 Eigenvectors and linear transformations. Consider a linear transformation $T : V \rightarrow W$. Let B_1 be a basis of V , B_2 a basis of W .

How do we express $T(\vec{v})$ in terms of B_2 ?

Theorem: Let $T : V \rightarrow W$ be a linear transformation, $\dim(V) = n$, $\dim(W) = m$.

Let $B_1 = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis of V , $B_2 = \{\vec{w}_1, \dots, \vec{w}_m\}$ be a basis of W .

Then if $\vec{v} \in V$, $m[\vec{v}]_{B_1} = [T(\vec{v})]_{B_2}$, where $m = \begin{bmatrix} [T(\vec{v}_1)]_{B_2} & [T(\vec{v}_2)]_{B_2} & \dots & [T(\vec{v}_n)]_{B_2} \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix}$

called the matrix of T relative to basis B_1 and B_2 .

$$\begin{array}{ccc} \vec{v} & \rightarrow & T(\vec{v}) \\ \downarrow & & \downarrow \\ \vec{v}_{B_1} & \rightarrow & [T(\vec{v})]_{B_2} \end{array}$$

(Note \vec{v}_{B_1} is supposed to be $[\vec{v}]_{B_1}$ but it had formatting issues)

Ex: Let $B_1 = \{(-3, 1), (1, 4)\} = \{\vec{v}_1, \vec{v}_2\}$ and $B_2 = \{(1, 2), (-2, 3)\} = \{\vec{w}_1, \vec{w}_2\}$.

Suppose $T(\vec{v}_1) = (-4, 13)$, $T(\vec{v}_2) = (6, 5)$. Find the matrix of T relative to B_1 and B_2 .

$$\text{Row reduce } \begin{bmatrix} 1 & -2 & -4 & 6 \\ 2 & 3 & 13 & 5 \end{bmatrix} (\mathbf{A}\vec{x} = \vec{b}) \rightarrow \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & -1 \end{bmatrix}$$

$$[T(\vec{v}_1)]_{B_2} = [2][3] = 2(1, 2) + 3(-2, 3) = (-4, 13)$$

So, $M = \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}$ is the matrix of T relative to B_1 and B_2 .

Ex: Let $T : \mathbb{P}_2 \rightarrow \mathbb{P}_2$ be the standard derivative operator. Find the B-matrix of T with respect to $B = \{1, x, x^2\}$.

$$\begin{aligned} T(1) &= 0 = 0(1) + 0(x) + 0(x^2) \\ T(x) &= 1 = 1(1) + 0(x) + 0(x^2) \\ T(x^2) &= 2x = 0(1) + 2(x) + 0(x^2) \end{aligned}$$

$$\text{So, } [T]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

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Ex: Let $T : \mathbb{P}_2 \rightarrow \mathbb{P}_3$ be given by $T(f(x)) = f(x) + xf(x)$. Find the matrix of T relative to basis $B_1 = \{1, x, x^2\}$ and $B_2 = \{1, x, x^2, x^3\}$.

Map basis vectors:

$$T(1) = 1 + x = 1(1) + 1(x) + 0(x^2) + 0(x^3)$$

$$T(x) = x + x^2 = 0(1) + 1(x) + 1(x^2) + 0(x^3)$$

$$T(x^2) = x^2 + x^3 = 0(1) + 0(x) + 1(x^2) + 1(x^3)$$

So, writing M in terms of B_2 :

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

What is this telling us?

Find $[T(7 - x^2)]_{B_2}$:

$$[7 - x^2]_{B_1} = \begin{bmatrix} 7 \\ 0 \\ -1 \end{bmatrix}$$

Thus,

$$M \times \begin{bmatrix} 7 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ -1 \\ -1 \end{bmatrix}$$

Therefore,

$$T(7 - x^2) = 7(1) + 7(x) - 1(x^2) - 1(x^3) = 7 + 7x - x^2 - x^3$$

(one can check and find $T(7 - x^2)$ directly).

Theorem: Suppose $T : V \rightarrow V$, $\dim(V) = n$ is a linear transformation. Let A be the (standard) matrix representation of T .

Assume A is similar to C ($A = PCP^{-1}$). If B is a basis of V , say $B = \{\vec{v}_1, \dots, \vec{v}_n\}$, and $\vec{v}_1, \dots, \vec{v}_n$ are the columns of P , then matrix C is the B -matrix of the transformation. In particular, if B is a basis of eigenvectors ("n" of them - diagonalizable), then the B -matrix is $D =$

$$\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}.$$

That is $C = P^{-1}AP = [T]_B$. We have this information.

Ex: What is the B -matrix for mapping $\vec{x} \mapsto Ax$, where $A = \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}$, and $B = \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \end{bmatrix} \right\}$?

$$C = P^{-1}AP = \left(\begin{bmatrix} 1 & -5 \\ 4 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & -5 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} \frac{13}{7} & -\frac{16}{7} \\ \frac{18}{7} & \frac{8}{7} \end{bmatrix}$$

Mapped vector:

$$A \begin{bmatrix} 77 \\ 14 \end{bmatrix} = \begin{bmatrix} 35 \\ 182 \end{bmatrix} = 45 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$

Section 5.5: Complex Eigenvalues

Definition: The complex numbers \mathbb{C} are of the form $a + bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. We define \mathbb{C}^n to be the n -tuples (z_1, z_2, \dots, z_n) , where $z_1, z_2, \dots, z_n \in \mathbb{C}$.

Definition: Given $z = c + bi$, the complex conjugate of z is $\bar{z} = c - bi$. The absolute value (modulus) is $|z| = \sqrt{a^2 + b^2}$.

The real and imaginary parts of $\vec{x} \in \mathbb{C}^n$, denoted $\text{Re}(\vec{x})$, $\text{Im}(\vec{x})$ are vectors in \mathbb{R}^n consisting of the real and imaginary values in the entries of \vec{x} .

Example:

$$\vec{x} = \begin{bmatrix} 1 - i \\ 3 \end{bmatrix} \Rightarrow \text{Re}(\vec{x}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \text{Im}(\vec{x}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

How to find complex eigenvalues, eigenvectors?

Example: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow \det(A - \lambda I) = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$. (Note: if complex eigenvalues, they come in pairs, λ and its complex conjugate.)

Eigenvectors:

For $\lambda = i$: $A - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$. Solve $(A - iI)\vec{v} = \vec{0}$:

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & -i & 0 \\ 1 & -i & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$\Rightarrow x_1 = ix_2, x_2 = x_2 \Rightarrow \vec{v} = x_2 \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$\lambda = i$ has eigenvector $\begin{bmatrix} i \\ 1 \end{bmatrix}$.

According to the book, eigenvector $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ is a multiple of the first one by a scalar i .

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Last time: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow$ eigenvalues $\lambda = \pm i$.

Shortcut for the 2×2 case to find the eigenvectors (once you have the eigenvalues): You know row 1 must be a multiple of row 2 \Rightarrow You do not need to row reduce. We solve $(A - iI)\vec{v} = \vec{0}$, we have:

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array} \right]$$

We can just work with equation 1 (row 1) to find the eigenvectors for $\lambda = i$. $ix_1 - x_2 = 0 \Rightarrow -ix_1 = x_2 \Rightarrow x_1 = ix_2 \Rightarrow x_1 = ix_2, x_2 = x_2$. Letting $x_2 = 1$, an eigenvector is $\vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

Section 6.1: Inner Products, Length, and Orthogonality

Definition: An inner product on a (real) vector space V is a function that assigns a pair of vectors $\vec{u}, \vec{v} \in V$ to a real number, denoted $\langle \vec{u}, \vec{v} \rangle$. This function satisfies the following for any $\vec{u}, \vec{v}, \vec{w} \in V$ and $c \in \mathbb{R}$,

- a) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$
- b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$
- c) $\langle c\vec{u}, \vec{v} \rangle = c\langle \vec{u}, \vec{v} \rangle$
- d) $\langle \vec{u}, \vec{u} \rangle \geq 0$ and $\langle \vec{u}, \vec{u} \rangle = 0$ if and only if $\vec{u} = \vec{0}$

Example: Over \mathbb{R}^n , the "standard" inner product is the dot product: If $\vec{u} = (u_1, u_2, \dots, u_n)$, $\vec{v} = (v_1, v_2, \dots, v_n)$ in \mathbb{R}^n , $\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + \dots + u_nv_n = [u_1 u_2 \dots u_n] \cdot [v_1 v_2 \dots v_n] = \vec{u}^T \cdot \vec{v}$ (this means dot product)

Example: $\langle f(x), g(x) \rangle = \int_0^1 f(x)g(x) dx$

From here, we focus specifically on the dot product.

Definition: Let V be a vector space with an inner product. The length or norm of $\vec{v} \in V$ is $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$

In \mathbb{R}^n , if $\vec{v} = (v_1, v_2, \dots, v_n)$, $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$

Definition: The distance between \vec{u} and \vec{v} in \mathbb{R}^n is $\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$

Definition: If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis, then $\left\{ \frac{1}{\|\vec{v}_1\|}, \frac{1}{\|\vec{v}_2\|}, \frac{1}{\|\vec{v}_3\|} \right\}$ is a normalized basis.

Definition: Two vectors \vec{u} and \vec{v} in \mathbb{R}^n are perpendicular or orthogonal if and only if $\vec{u} \cdot \vec{v} = 0$ (more generally, $\langle \vec{u}, \vec{v} \rangle = 0$).

Definition: Let W be a subspace of \mathbb{R}^n . If vector \vec{v} is orthogonal to all vectors in W , we say \vec{v} is orthogonal to W . The orthogonal complement of W is denoted as W^\perp , defined as $W^\perp = \{\vec{v} \in \mathbb{R}^n : \vec{v} \cdot \vec{w} = 0 \text{ for all } \vec{w} \in W\}$.

Example: Let W be a plane through the origin. Then $W^\perp = \text{span}(\{\vec{n}\})$ where \vec{n} is a vector orthogonal to the plane.

Theorem: For any subspace W of \mathbb{R}^n , W^\perp is a subspace of \mathbb{R}^n .

Property: $W \cap W^\perp = \{\vec{0}\}$

Example: Let A be an $m \times n$ matrix. What is $(\text{row}(A))^\perp$?

Given $A = \begin{bmatrix} 1 & 3 & 7 & -2 \\ 5 & 5 & 3 & 1 \\ 0 & 4 & 0 & -2 \end{bmatrix}$, we have:

$$[\vec{v}_1 \rightarrow \vec{v}_2 \rightarrow \vec{v}_3 \rightarrow] \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

What \vec{z} satisfies this? It's the kernel of A !

Theorem: Let A be an $m \times n$ matrix. Then $(\text{row}(A))^\perp = \ker(A)$ and $(\text{col}(A))^\perp = \ker(A^T)$.

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Given that W is a subspace, $W^\perp = \{\vec{v} \in \mathbb{R}^n : \vec{v} \cdot \vec{w} = 0 \text{ for all } \vec{w} \in W\}$.

For a matrix A of size $m \times n$, we have $(\text{row}(A))^\perp = \ker(A)$.

Given matrix $A = \begin{bmatrix} 1 & 3 & 7 & -2 \\ 5 & 5 & 3 & 1 \\ 0 & 4 & 0 & -2 \end{bmatrix}$,

we find $(\text{row}(A))^\perp$ as follows:

$$\begin{bmatrix} 1 & 0 & 0 & \frac{13}{16} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{16} \end{bmatrix}$$

$$\Rightarrow x_4 = 16, x_3 = 3, x_2 = 8, x_1 = -13$$

$$\Rightarrow (\text{row}(A))^\perp = \text{span} \left\{ \begin{bmatrix} -13 \\ 8 \\ 3 \\ 16 \end{bmatrix} \right\}$$

This represents the subspace orthogonal to the rows of matrix A .

Section 6.2: Orthogonal Sets

Definition: A set of vectors $\{\vec{v}_1, \dots, \vec{v}_m\}$ in \mathbb{R}^n is an orthogonal set if $\vec{v}_i \cdot \vec{v}_j = 0$ for any $1 \leq i \neq j \leq m$. An orthogonal set that forms a basis of a subspace is called an orthogonal basis.

Theorem: An orthogonal set that does not contain $\vec{0}$ is linearly independent.

Proof: Let $\{\vec{v}_1, \dots, \vec{v}_m\}$ be an orthogonal set.

Suppose $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m = \vec{0}$.

Take the dot product of \vec{v}_1 on both sides:

$$\vec{v}_1 \cdot (c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m) = \vec{0} \cdot \vec{v}_1 = 0$$

$$c_1(\vec{v}_1 \cdot \vec{v}_1) + c_2(\vec{v}_1 \cdot \vec{v}_2) + \dots + c_m(\vec{v}_1 \cdot \vec{v}_m) = 0$$

Since $\vec{v}_1 \cdot \vec{v}_1 > 0$, we can conclude that $c_1 = 0$.

Repeat this process for $\vec{v}_2, \vec{v}_3, \dots, \vec{v}_m$ to show that c_2, c_3, \dots, c_m must also be 0. Therefore, $\{\vec{v}_1, \dots, \vec{v}_m\}$ is linearly independent.

Given an orthogonal set, we can find the coefficients to form a linear combination of a vector.

Suppose $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m$.

The coefficients can be found using the formula $c_i = \frac{\vec{v}_i \cdot \vec{w}}{\vec{v}_i \cdot \vec{v}_i}$.

Example: Let $\{(1, 2, 1), (4, -2, 0), (2, 4, -10)\}$ be an orthogonal set. Write $(5, 1, 1)$ as a linear combination of this set.

$$(5, 1, 1) = \vec{w} = \dots + \frac{8}{6}(1, 2, 1) + \frac{18}{20}(4, -2, 0) + \frac{4}{120}(2, 4, -10)$$

Projection onto a Vector:

Theorem: For vectors \vec{u} and \vec{y} , the projection of \vec{y} onto \vec{u} , denoted \hat{y} , is given by:

$$\hat{y} = \text{proj}_{\vec{u}}(\vec{y}) = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \cdot \vec{u}$$

Example: Write $\vec{y} = (1, 2)$ as a sum of a vector in $\text{span}(\{(-1, 5)\})$ and a vector orthogonal to $(-1, 5)$. Here, $\vec{u} = (-1, 5)$.

First, find $\text{proj}_{\vec{u}}(\vec{y})$:

$$\text{proj}_{\vec{u}}(\vec{y}) = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \cdot \vec{u} = \frac{9}{26} \cdot (-1, 5)$$

Then,

$$\begin{aligned}\vec{y} &= (1, 2) = \hat{y} + \hat{z} \\ \vec{z} &= (1, 2) - \frac{9}{26}(-1, 5) = \left(\frac{35}{26}, \frac{7}{26}\right)\end{aligned}$$

So,

$$(1, 2) = \frac{9}{26}(1, 5) + \left(\frac{35}{26}, \frac{7}{26}\right)$$

Definition: A set $\{\vec{v}_1, \dots, \vec{v}_m\}$ is an orthonormal set if it is an orthogonal set such that each vector is a unit vector ("normalized"). Such a set spanning subspace W is called an orthonormal basis.

Example: $\{(1, 2), (-2, 1)\}$ is an orthogonal set but not orthonormal. $\left\{\left(\frac{1}{\sqrt{5}}, 2\sqrt{5}\right), \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)\right\}$ is an orthonormal set.

Suppose $\{\vec{v}_1, \dots, \vec{v}_m\}$ is an orthonormal set.

$$A^T \cdot A = I$$

since the diagonal is all 1's (unit vectors) and the rest will be 0.

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Last time, we discussed an orthogonal set $\{\vec{v}_1, \dots, \vec{v}_n\}$, which satisfies the following properties: every vector is a unit vector ($\|\vec{v}_i\| = 1$), and any two distinct vectors are orthogonal, i.e., $\vec{v}_i \cdot \vec{v}_j = 0$.

Theorem: Let A be an $m \times n$ matrix whose columns form an orthogonal set. This holds if and only if $A^T A = I$. Moreover, for any $\vec{x}, \vec{y} \in \mathbb{R}^n$,

- 1) $\|A\vec{x}\| = \|\vec{x}\|$ (mapping preserves magnitudes)
- 2) $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$ (Also, $\vec{x} \cdot \vec{y} = 0$ if and only if $(A\vec{x}) \cdot (A\vec{y}) = 0$)

Example: Let $A = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. If $\vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, find $\|A\vec{x}\|$.

Orthonormal columns

Finding the magnitude directly will be unnecessary.

By the theorem:

$$\|A\vec{x}\| = \|\vec{x}\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5.$$

Definition: An orthogonal matrix is an invertible matrix (thus square) such that $A^{-1} = A^T$ (so if you know A , just take A^T to get the inverse).

Theorem: Let A be a square matrix. The following are equivalent:

- 1) A is an orthogonal matrix.
- 2) The columns of A form an orthonormal set.
- 3) The rows of A form an orthonormal set.

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \text{Not an orthogonal matrix.}$$

Section 6.3: Orthogonal Projections

$$\begin{aligned}\vec{y} &= \text{proj } \vec{u}\vec{y} + \vec{z} \\ \vec{z} &\text{ is orthogonal to } \text{proj } \vec{u}\vec{y} \text{ and is in } \text{span}(\vec{u})\end{aligned}$$

Over \mathbb{R}^n , we express \vec{y} as a sum of two vectors, one in subspace W , and the other in the space orthogonal to the vectors in W (i.e., in W^\perp). Then the projection vector \vec{y} will be orthogonal to W . The projection vector is denoted $\text{proj}_W \vec{y}$.

Theorem: Let $\vec{y} \in \mathbb{R}^n$, W be a subspace of \mathbb{R}^n . Then \vec{y} can be uniquely written as $\vec{y} = \vec{w} + \vec{z}$, where $\vec{w} \in W$, $\vec{z} \in W^\perp$. Moreover,

$$\vec{w} = \text{proj}_W \vec{y} = \left(\frac{\vec{y} \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1} \right) \vec{w}_1 + \dots + \left(\frac{\vec{y} \cdot \vec{w}_m}{\vec{w}_m \cdot \vec{w}_m} \right) \vec{w}_m$$

where $\{\vec{w}_1, \dots, \vec{w}_m\}$ is an orthogonal basis of W .

Remark: In the case where $\{\vec{u}_1, \dots, \vec{u}_m\}$ is an orthonormal basis of W , we have $\|\vec{u}_1\|^2 = 1 = \vec{u}_1 \cdot \vec{u}_1$. Then,

$$\text{proj}_W \vec{y} = (\vec{y} \cdot \vec{u}_1)\vec{u}_1 + \dots + (\vec{y} \cdot \vec{u}_m)\vec{u}_m = UU^T \vec{y}$$

Example: Express $\vec{y} = (-1, 4, 3)$ as a sum $\vec{w} + \vec{z}$, where $\vec{w} \in \text{span}\{(1, 1, 1), (-1, 3, -2)\} = W$, and $\vec{z} \in W^\perp$. Let $\vec{v}_1 = (1, 1, 1)$, $\vec{v}_2 = (-1, 3, -2)$.

Check $\vec{v}_1 \cdot \vec{v}_2 = 0 \Rightarrow \{\vec{v}_1, \vec{v}_2\}$ is an orthogonal basis for W .

$$\begin{aligned}\Rightarrow \vec{y} &= (-1, 4, 3) \\ &= \text{proj}_W \vec{y} + \vec{z} \\ &= \left(\frac{\vec{y} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \right) \vec{v}_1 + \left(\frac{\vec{y} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \right) \vec{v}_2 + \vec{z} \\ &= \frac{6}{3}(1, 1, 1) + \frac{7}{14}(-1, 3, -2) + \vec{z} \\ &= \left(\frac{3}{2}, \frac{7}{2}, 1 \right) + \vec{z} \quad (\text{we can get } \vec{z}) \\ &= (-1, 4, 3) - \left(\frac{3}{2}, \frac{7}{2}, 1 \right) \\ &= \left(-\frac{5}{2}, \frac{1}{2}, 2 \right) \\ \Rightarrow \vec{y} &= (-1, 4, 3) = \left(\frac{3}{2}, \frac{7}{2}, 1 \right) + \left(-\frac{5}{2}, \frac{1}{2}, 2 \right).\end{aligned}$$

Check that the vectors are orthogonal.

Theorem (Best Approximation): Let W be a subspace of \mathbb{R}^n , and $\vec{y} \in \mathbb{R}^n$. The point $\text{proj}_W \vec{y}$ is the point in W closest to \vec{y} . Moreover, the distance is $\|\vec{y} - \text{proj}_W \vec{y}\|$.

Example: What is the closest point in W to $(-1, 4, 3)$? We want $\text{proj}_W \vec{y} = \left(\frac{3}{2}, \frac{7}{2}, 1 \right)$.

34 Lecture 34, 4/22/2024

Last time:

Given vectors in vector space V , we want to apply the Gram-Schmidt process:

$$\vec{y} = \vec{w} + \vec{z}, \quad \vec{w} \in W, \quad \vec{z} \in W^\perp$$

where

$$\vec{w} = \text{proj}_W \vec{y} = \sum_{i=1}^m \left(\frac{\vec{y} \cdot \vec{w}_i}{\vec{w}_i \cdot \vec{w}_i} \right) \vec{w}_i$$

with $\{\vec{w}_1, \dots, \vec{w}_m\}$ as an orthogonal basis of W .

If the basis is not orthogonal, we use the Gram-Schmidt process to construct an orthogonal basis:

$$\begin{aligned} \vec{v}_1 &= \vec{z}_1, \\ \vec{v}_2 &= \vec{z}_2 - \text{proj}_{\vec{v}_1} \vec{z}_2, \\ \vec{v}_3 &= \vec{z}_3 - \text{proj}_{\vec{v}_1} \vec{z}_3 - \text{proj}_{\vec{v}_2} \vec{z}_3, \\ &\vdots \\ \vec{v}_i &= \vec{z}_i - \sum_{j=1}^{i-1} \text{proj}_{\vec{v}_j} \vec{z}_i, \end{aligned}$$

where

$$\text{proj}_{\vec{v}_j} \vec{z}_i = \left(\frac{\vec{z}_i \cdot \vec{v}_j}{\vec{v}_j \cdot \vec{v}_j} \right) \vec{v}_j.$$

This results in an orthogonal basis $\{\vec{v}_1, \dots, \vec{v}_m\}$ for V .

Example: Apply Gram-Schmidt to $\{(-1, 1, -1, 1), (-1, 3, -1, 3), (1, 3, 3, 7)\}$:

$$\begin{aligned} \vec{v}_1 &= (-1, 1, -1, 1), \\ \vec{v}_2 &= (-1, 3, -1, 3) - \left(\frac{(-1, 3, -1, 3) \cdot (-1, 1, -1, 1)}{(-1, 1, -1, 1) \cdot (-1, 1, -1, 1)} \right) (-1, 1, -1, 1) \\ &= (1, 1, 1, 1), \\ \vec{v}_3 &= (1, 3, 3, 7) - \text{proj}_{\vec{v}_1} (1, 3, 3, 7) - \text{proj}_{\vec{v}_2} (1, 3, 3, 7) \\ &= (-2, -2, 2, 2). \end{aligned}$$

Check orthogonality and normalize:

$$\text{Check: } \vec{v}_1 \cdot \vec{v}_2 = 0, \quad \vec{v}_2 \cdot \vec{v}_3 = 0, \quad \vec{v}_1 \cdot \vec{v}_3 = 0,$$

$$\text{Orthonormal basis: } \left\{ \frac{\vec{v}_1}{\|\vec{v}_1\|}, \frac{\vec{v}_2}{\|\vec{v}_2\|}, \frac{\vec{v}_3}{\|\vec{v}_3\|} \right\}.$$

Theorem: (QR Factorization)

Let A be an $n \times m$ matrix. Assume A has m linearly independent vectors. Then A can be factored as $A = QR$, where Q is an $n \times m$ matrix with m orthonormal column vectors (by Gram-Schmidt, normalized). Note that if $m = n$, Q is an orthogonal matrix and R is an $m \times m$ upper triangular matrix.

If $A = QR$, then:

$$\begin{aligned} Q^T A &= Q^T QR \\ &= I \cdot R \\ &= R \end{aligned}$$

Therefore, $R = Q^T A$.

Example: Find the QR-factorization of

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 4 \\ 1 & 1 & 5 \end{bmatrix}$$

First, find the orthogonal vectors:

$$\begin{aligned} \vec{z}_1 &= (1, 1, 1) \\ \vec{z}_2 &= (2, 0, 1) \\ \vec{z}_3 &= (2, 4, 5) \end{aligned}$$

Apply Gram-Schmidt, then normalize to get:

$$\begin{aligned} \vec{v}_1 &= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\ \vec{v}_2 &= \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right) \\ \vec{v}_3 &= \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right) \end{aligned}$$

Thus,

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

Now, calculate $R = Q^T A$:

$$\begin{aligned} R &= Q^T A \\ &= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 4 \\ 1 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{3}{\sqrt{3}} & \frac{11}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{2}} & -\frac{2}{\sqrt{2}} \\ 0 & 0 & \frac{4}{\sqrt{6}} \end{bmatrix} \end{aligned}$$

This decomposition is **not unique**.

35 Lecture 35, 4/24/2024

Computing Projections and Least-Squares Problems

To compute the projection of a vector \vec{y} onto a subspace W , an orthogonal basis for W is required.

Section 6.5: Least-Squares Problems

In practice, a system of equations may not have a solution. The goal then is to find a vector \hat{x} such that the norm $\|\vec{b} - A\hat{x}\|$ is minimized over all vectors \vec{x} in the domain, which means \hat{x} is a least-squares solution to $A\vec{x} = \vec{b}$. If \vec{b} is not in the column space of A , $\text{col}(A)$, we look for $\hat{b} = \text{proj}_{\text{range}(A)} \vec{b} = \text{proj}_{\text{col}(A)} \vec{b}$.

The least squares solution to $A\vec{x} = \vec{b}$ is such an \hat{x} that $A\hat{x} = \hat{b}$. We observe that $\vec{b} - A\hat{x}$ is orthogonal to everything in $\text{range}(A) = \text{col}(A)$.

Normal Equations

The least squares solution(s) of $A\vec{x} = \vec{b}$ are the solutions to the normal equations:

$$A^T A \vec{x} = A^T \vec{b}$$

Example Problem

Find the least squares solution to:

$$x_1 + 2x_2 = -1$$

$$3x_1 + 4x_2 = 3$$

$$5x_1 + 6x_2 = -5$$

$$\text{Given } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} -1 \\ 3 \\ -5 \end{bmatrix}.$$

Calculating $A^T A$ and $A^T \vec{b}$, we can row reduce the augmented matrix to find \hat{x} .

Another method to find \hat{x} is using the pseudoinverse of A , given by $\hat{x} = (A^T A)^{-1} A^T \vec{b}$ if $(A^T A)^{-1}$ exists.

Least Squares Error

The least squares error for $A\vec{x} = \vec{b}$ is $\|\vec{b} - A\hat{x}\|$ where \hat{x} is the least squares solution.

Orthogonal Columns in A

If the columns of A are orthogonal, finding \hat{x} is straightforward.

QR-factorization Method

If A has linearly independent columns, and $A = QR$ where Q has orthonormal columns and R is upper triangular, we can find \hat{x} by solving $R\hat{x} = Q^T \vec{b}$ and then performing backward substitution.

36 Lecture 36, 4/26/2024

Recap: Least Squares Solutions To find the least squares solution \hat{x} to the system $A\vec{x} = \vec{b}$, solve the normal equation $A^T A \hat{x} = A^T \vec{b}$. In some cases, we can use $(A^T A)^{-1}$, the projection method if the columns of A are orthogonal, or the QR-factorization.

Section 6.6: Applications to Linear Models We consider the system $A\vec{x} = \vec{b}$ as $X\vec{B} = \vec{y}$, where X is the design matrix, \vec{B} is the parameter vector, and \vec{y} is the observation vector. Given data points $(x_1, y_1), \dots, (x_n, y_n)$, we seek a line $y = B_0 + B_1 x$ closest to the data.

The residual for a point is given by the distance from the observed value to the line, i.e., $|y_1 - (B_0 + B_1x_1)|$. We aim for the data point (x_1, y_1) to lie on the line, hence $y_1 = B_0 + B_1x_1$. To find B_0 and B_1 , we consider the system:

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

This leads to $X\vec{B} = \vec{y}$, which usually is inconsistent, so we apply the least-squares method to find the closest solution.

We minimize the norm $\|X\vec{B} - \vec{y}\|$, which corresponds to minimizing the squares of the residuals.

Example: Least-Squares Line Given the points $(1, 0)$, $(2, 4)$, and $(3, 7)$, find the least-squares line $y = B_0 + B_1x$.

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad \vec{B} = \begin{bmatrix} B_0 \\ B_1 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 0 \\ 4 \\ 7 \end{bmatrix}$$

Solving $X^T X \vec{B} = X^T \vec{y}$ yields $\vec{B} = \begin{bmatrix} -10/3 \\ 7/2 \end{bmatrix}$, hence the least-squares line is $y = -\frac{10}{3} + \frac{7}{2}x$.

Generalizations We can extend this to fit other models, such as $y = B_0 \cos(x) + B_1 \sin(x) + B_2 \sin^2(x)$, and find the least-squares solution similarly.

Example: Least-Squares Parabola Find the least squares parabola $y = B_0 + B_1x + B_2x^2$ for the points $(1, 5)$, $(2, 2)$, $(3, 3)$, $(4, 8)$.

Solving $X^T X \vec{B} = X^T \vec{y}$ for $X\vec{B} = \vec{y}$, we get $\vec{B} = \begin{bmatrix} 12 \\ -9 \\ 2 \end{bmatrix}$, leading to the least-squares parabola

$$y = 12 - 9x + 2x^2.$$

In the context of least-squares problems, note that the matrix $A^T A$ is always symmetric, that is, $A = A^T$.

37 Lecture 37, 4/29/2024

Section 7.1: Diagonalization of Symmetric Matrices

Recall a matrix A is symmetric if $A^T = A$. We say A is diagonalizable if it can be written in the form $D = P^{-1}AP$, where D is a diagonal matrix.

Definition: An $n \times n$ matrix is *orthogonally diagonalizable* if there is an orthogonal matrix P (so $P^{-1} = P^T$) and a diagonal matrix D such that $D = P^{-1}AP = P^TAP$. (It can be shown that D is still the matrix of eigenvalues.)

Theorem: An $n \times n$ matrix is orthogonally diagonalizable if and only if A is symmetric.

Example: Orthogonally Diagonalizable

Consider matrices A and B :

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Choose A (since B is not symmetric).

We find the characteristic polynomial:

$$\det(A - \lambda I) = (8 - \lambda)(1 + \lambda)^2 = 0 \Rightarrow \lambda = 8, -1, -1$$

Eigenvectors:

For $\lambda_1 = 8$:

$$(A - 8I)\vec{z}_1 = \vec{0} \Rightarrow \vec{z}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

For $\lambda_2, \lambda_3 = -1, -1$:

$$(A + I)\vec{z} = \vec{0}.$$

We find two linearly independent eigenvectors:

$$\vec{z}_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \quad \vec{z}_3 = \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$$

\vec{z}_1 is orthogonal to \vec{z}_2 and \vec{z}_3 .

Applying Gram-Schmidt:

$$\vec{u}_1 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$\vec{v}_3 = \vec{z}_3 - \frac{\vec{z}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 = \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} - \frac{9}{5} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$\|\vec{v}_3\| = \sqrt{\left(\frac{-4}{5}\right)^2 + \left(\frac{-2}{5}\right)^2 + 1} = \frac{3}{\sqrt{5}}$$

$$\vec{u}_3 = \frac{1}{3/\sqrt{5}} \vec{v}_3 = \begin{bmatrix} -\frac{4}{3\sqrt{5}} \\ -\frac{2}{3\sqrt{5}} \\ \frac{\sqrt{5}}{3} \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} \\ \frac{1}{3} & -\frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{\sqrt{5}}{3} \end{bmatrix}, \quad D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$D = P^T A P \quad (P^{-1} = P^T)$$

Theorem (Spectral Decomposition): Let A be an $n \times n$ symmetric matrix. Then:

1. All eigenvalues are real.
2. The geometric multiplicities equal the algebraic multiplicities for all eigenvalues.

3. The eigenspaces are mutually orthogonal. Moreover, there is an orthonormal basis of eigenvectors that span \mathbb{R}^n .
4. Matrix A is orthogonally diagonalizable.

$$A = PDP^T = \sum_{i=1}^n \lambda_i \vec{u}_i \vec{u}_i^T$$

Each matrix $\vec{u}_i \vec{u}_i^T$ is a projection matrix.

Example: For $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$, we find $\lambda_1 = 6$ with eigenvector $\vec{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\lambda_2 = -4$ with eigenvector $\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

$$A = 6 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} - 4 \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

38 Lecture 38, 5/3/2024

Last time: $A = \sum_{i=1}^n \lambda_i \mathbf{u}_i \mathbf{u}_i^T$ is the spectral decomposition of A , where λ_i 's are the eigenvalues, and \mathbf{u}_i 's are the normalized, orthogonal eigenvectors corresponding to eigenvalues λ_i 's.

From the example last time:

$$\begin{aligned} A &= 6 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} - 4 \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= 6 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} - 4 \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix} \end{aligned}$$

Note this is the form $A = PDP^T$.

Orthogonally diagonalize A .

Ex: If $\mathbf{u} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ from above, what is $\text{proj}_{\mathbf{u}}(7, 2)$?

$$\Rightarrow \text{proj}_{\mathbf{u}}(7, 2) = \left(\frac{\mathbf{u}_1 \cdot (7, 2)}{\mathbf{u}_1 \cdot \mathbf{u}_1} \right) \mathbf{u}_1 = (\mathbf{u}_1 \cdot (7, 2)) \mathbf{u}_1$$

Note: $\mathbf{u}_1 \cdot \mathbf{u}_1 = 1$ since it is a unit vector.

OR

$$\mathbf{u}_1 \mathbf{u}_1^T \begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{9}{2} \\ \frac{9}{2} \end{bmatrix}$$

Section 7.4: Singular Value Decomposition (SVD)

We look at "diagonalizing" an arbitrary $m \times n$ matrix. From the least squares section, we found $A^T A$ is always symmetric \Rightarrow Spectral theorem and results in the last section hold.

Def: The singular values of A are the square roots of the eigenvalues of $A^T A$. The values are denoted $\sigma_1, \sigma_2, \dots, \sigma_n$ (so $\sigma = \sqrt{\lambda_1}$).

Theorem (SVD): Let A be $m \times n$ with rank r . Then we can write

$$A = U \Sigma V^T$$

U is an $m \times m$ orthogonal matrix, Σ is an $m \times n$ matrix whose upper-left submatrix is a diagonal matrix whose diagonal entries are the first r singular values, in non-increasing order, V is an $n \times n$ orthogonal matrix.

$$\text{Ex: Let } A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow A^T A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$$

Eigenvalues and eigenvectors of $A^T A$:

$$\lambda_1 = 5, \lambda_2 = 1, \lambda_3 = 0$$

$$\Rightarrow \sigma_1 = \sqrt{5}, \sigma_2 = \sqrt{1} = 1 \text{ are the singular values}$$

$$\Rightarrow \text{SVD has } \Sigma = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Image Compression:

Let A be a 256×512 matrix corresponding to an image whose entries are grayscale values of each pixel.

$$\Rightarrow \text{Store } 256 \times 512 = 131072$$

By looking at the SVD of A , we see

$$A = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} \sigma_1 & \rightarrow & \rightarrow \\ 0 & \sigma_2 & \rightarrow \\ 0 & 0 & \ddots \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

These matrices are: 256×256 , 256×512 , and 512×512 respectively.

$$\sigma_1 = 7500$$

We assume $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{256}$

We expect σ_{256} to be very small.

$$\sigma_{256} = 0.001, \sigma_{255} = 0.02$$

What if we simply change the small singular values to 0?

This alters the image by a little.

Suppose we only work with the first 60 largest singular values (so $\sigma_{61} = \dots = \sigma_{256} = 0$ now)

\Rightarrow We now only need to store $(256)(60) + (60)(60) + (60)(512) = 49680$, which is less than half the original amount needed.

How well does this preserve the image? The "metric" is determined by the singular values

$$\frac{(\sigma_1 + \sigma_2 + \dots + \sigma_k)}{(\sigma_1 + \sigma_2 + \dots + \sigma_r)}$$

k is what to only use and r is the total.

39 Lecture 39, 5/5/2024

Review:

1. Matrix Inversion Given matrices A and B^{-1} , determine $(BA)^{-1}$.

$$A = \begin{bmatrix} 1 & -2 & 8 \\ 1 & -1 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} -2 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$(BA)^{-1} = A^{-1}B^{-1} = \begin{bmatrix} 2 & 3 & 6 \\ 6 & -3 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

2. Orthogonal Matrix and Dot Product Let A be an $n \times n$ orthogonal matrix. Show for $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$(A\vec{v}) \cdot (A\vec{u}) = \vec{v} \cdot \vec{u}$$

$$(A\vec{v})^T (A\vec{u}) = (\vec{v}^T A^T)(A\vec{u}) = \vec{v}^T (A^T A)\vec{u} = \vec{v}^T I\vec{u} = \vec{v}^T \vec{u} = \vec{v} \cdot \vec{u}$$

3. Linear Dependence of Row Vector **True or False:** If A is a 7×3 matrix, the row vectors must form a linearly dependent set.

Answer: True, any set of 7 vectors in \mathbb{R}^3 is linearly dependent.

4. Linear Transformation from \mathbb{R}^4 to \mathbb{R}^5 Construct a linear transformation $T: \mathbb{R}^4 \rightarrow \mathbb{R}^5$ where the range is spanned by $(7, 9, 9, 3, 2)$ and $(-15, 1, 0, 7, 7)$, or explain why it is impossible.

$$T(\vec{x}) = \begin{bmatrix} 7 & -15 & 0 & 0 \\ 9 & 1 & 0 & 0 \\ 9 & 0 & 0 & 0 \\ 3 & 7 & 0 & 0 \\ 2 & 7 & 0 & 0 \end{bmatrix} \vec{x}$$

5. Diagonalizability of Similar Matrices Suppose A is diagonalizable. Assume B is similar to A . Show B is diagonalizable.

$$D = P^{-1}AP \quad \text{and} \quad A = QBQ^{-1}$$

$$\Rightarrow D = P^{-1}QBQ^{-1}P, \quad \text{let} \quad R = Q^{-1}P, \quad R^{-1} = P^{-1}Q$$

$$\therefore D = R^{-1}BR \quad \text{so} \quad B \text{ is diagonalizable.}$$

6. Subspace Criteria Is the set $\{(x_1, x_2, x_3) : x_1 = x_3, x_1, x_2, x_3 \in \mathbb{R}\}$ a subspace of \mathbb{R}^3 ?

$$(x_1, x_2, x_3) = (x_3, x_2, x_3) = x_3(1, 0, 1) + x_2(0, 1, 0)$$

The original set equals to $\text{span}(\{x_3(1, 0, 1) + x_2(0, 1, 0)\})$, and the span of any set is a subspace.

7. Determinant of a Matrix Product If A is a 2×2 and $\det(A) = 7$, $B = \begin{bmatrix} 3 & -4 \\ 2 & -2 \end{bmatrix}$, find $\det(3A^{-1}B^{-1})$.

$$\det(B) = -6 + 8 = 2$$

$$\det(3A^{-1}B^{-1}) = 3^2 \det(A^{-1}) \det(B^{-1}) = 9 \cdot \frac{1}{\det(A)} \cdot \frac{1}{\det(B)} = \frac{9}{14}$$

8. Linear Transformation $T : \mathbb{P}_2 \rightarrow \mathbb{P}_3$ Given by $T(a_0 + a_1x + a_2x^2) = a_1$.

Basis of kernel and range:

Range \Rightarrow basis of range is $\{1\}$

Kernel \Rightarrow Necessarily $a_1 = 0$

No restriction on a_0 or $a_2 \Rightarrow \ker(T) = \{a_0 + a_2x^2 : a_0, a_2 \in \mathbb{R}\}$

\Rightarrow Basis for $\ker(T)$ is $\{1, x^2\}$

True or False: Suppose B contains 5 polynomials in \mathbb{P}_4 , and suppose $f(x) = x^2$ is one of these polynomials. It is given that $f(x)$ is a linear combination of the other polynomials in B . Then B can never span \mathbb{P}_4 .

True, since B is given to be linearly dependent, the dimension of B can be at most 4, even though there are 5 polynomials. However, the dimension of \mathbb{P}_4 is 5, which means B cannot span \mathbb{P}_4 .

40 Lecture 40, 5/7/2024

Review:

1. Let $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^5$. Assume the set of vectors is linearly independent. Suppose T is a linear transformation. Show that $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ may be linearly independent. We can choose the matrix of T to be the zero matrix. Then $\ker(T) \Rightarrow T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3) = \vec{0}$, and any set with the zero vector is linearly dependent.

Now suppose $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent. Show $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is always linearly dependent. Hence, there exist c_1, c_2, c_3 (not all zero) where $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$. Therefore, $T(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = T(\vec{0}) = \vec{0}$. It follows that $c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + c_3T(\vec{v}_3) = \vec{0}$.

2. Let $\vec{u} \in \mathbb{R}^5$ and assume $\vec{u} \neq 0$. What is the rank of $\vec{u}\vec{u}^T$?

For example in \mathbb{R}^3 , $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then $\vec{u}\vec{u}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$. Since each new row is a multiple of the first, the rank must be 1.

3. What is the eigenspace for $\lambda = 0$ if $A = \begin{bmatrix} -2 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix}$? Solving $(A - 0I)\vec{v} = \vec{0}$, the RREF

of the augmented matrix B is $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore, the eigenspace is spanned by $\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$.

4. True or False: Let A and B be matrices of the same dimension. If $\text{row}(A) = \text{row}(B)$, then $\text{col}(A) = \text{col}(B)$. This statement is **False** as demonstrated by the matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, which have the same row space but different column spaces.

5. Let $T : \mathbb{P}_2 \rightarrow \mathbb{P}_2$ be given by $T(a_0 + a_1x + a_2x^2) = -a_0 + a_1x^2$:

(a) Find the matrix representation of T with respect to the basis $\{1, x, x^2\}$:

$$\text{Matrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(b) Basis for the kernel and range:

$$\text{Kernel: Basis is } \{x^2\}, \quad \text{Range: Basis is } \{1, x^2\}$$

6. True or False:

(a) A system of linear equations has a 10×3 augmented matrix. Then, with more equations than variables, the system is always inconsistent. **False**.

(b) Now suppose the 10×3 matrix is the matrix of a linear transformation. Then the transformation is never onto/surjective. **True**.

7. Let $B_1 = \{(1, 0), (0, 1)\}$, $B_2 = \{(3, 4), (1, 2)\}$:

- Determine the change of coordinates matrix from B_1 to B_2 :

$$\text{Matrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{bmatrix}$$

- How to express $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$ as a linear combination of vectors in B_2 , written in terms of B_1 :

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \Rightarrow 2(3, 4) - 2(1, 2) = (4, 4)$$