Differential Equations - MATH246

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Conway - Fall 2024

Class Information

Grading

- Matlab assignments — 18% (6% each)
- Quizzes (drop two lowest) — 17%
- Two best in-class exams — 17% each
- Worst in-class exam — 8%
- Final exam — 23%

Office Hours

- Monday: 2:00 PM - 3:00 PM (in person, Kirwin 2400)
- Tuesday: 1:15 PM - 2:30 PM (in person, Kirwin 2400)
- TBA: Zoom (online)

Exams

- 3 midterms and a final exam

Lecture 1, Tuesday 8/27/2024

Course Overview: (Differential Equations)

Chapter 0:

A differential equation is an algebraic relation between functions, their derivatives, and independent variables.

Examples:

- $\left(\frac{dx}{dt}\right)^2 + x\sin(t) = \cos(x)$ *(Order = 1)*
- $y'' + ty' + y = \cos(t)$ (Note: $y' = \frac{dy}{dt}$)*(Order = 2)*
- $\frac{dy}{dt} \cdot \frac{dy}{ds} + y\frac{dz}{dt} = \sin(st)$ *(Order = 1)*

Order: The order of a differential equation is the order of the highest derivative that appears.

Notation: For $\frac{dy}{dx}$, we can write y' or $\dot{y}$ (dot notation).

An ordinary differential equation (ODE) involves no partial derivatives, as opposed to a partial differential equation (PDE).

Note: This course only deals with ODEs.

Linearity of ODEs

An ODE with function y and independent variable t is **linear** if it can be written as:

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = f(t)$$

where $y^{(n)}$ is the nth derivative of y.

Examples:

- $\left(\frac{dx}{dt}\right)^2 + x\sin(t) = \cos(x)$ *(Not linear: $\left(\frac{dx}{dt}\right)^2$ and $\cos(x)$)*
- $y'' + ty' + y = \cos(t)$ *(Linear)*
- $y^{(4)} + y^{(2)} = 2$ *(Linear)*

Systems of ODEs

A system of ODEs consists of multiple ordinary differential equations that are considered together:

$$\begin{cases} \text{ODE1} \\ \text{ODE2} \\ \vdots \\ \text{ODE}n \end{cases}$$

Chapter 1: Introduction

Section 1: First-Order ODEs

First-order ODEs can be complicated. We will focus on those that can be put into the standard form $\boxed{\dfrac{dy}{dt} = f(t,y)}$.

Example: Consider the equation $\frac{dw}{dz} = \frac{-z}{6w}$. This can be rewritten as:

$$\frac{dw}{dz} = \frac{-z}{6w}$$

A function $Y(t)$ is a solution to $y' = f(t, y)$ on the interval (a, b) if:

- $Y(t)$ and $Y'(t)$ exist on (a, b),
- $f(t, Y(t))$ exists on (a, b), and
- $Y'(t) = f(t, Y(t))$ on (a, b).

Example: Consider the equation $y'(t) = \frac{t}{y}$ with the solution $Y(t) = \sqrt{4 - t^2}$.

To check this, calculate:

$$Y'(t) = \frac{-t}{\sqrt{4 - t^2}}$$

$Y(t)$ is defined on the interval $[-2, 2]$, but $f(t, Y(t)) = \frac{t}{\sqrt{4-t^2}}$ is only defined for $(-2, 2)$, not at ± 2. Therefore, $Y(t)$ is a solution on $(-2, 2)$, not on $[-2, 2]$.

Explicit Equations

These are of the form $y' = f(t)$.

The general solution is:

$$y = \int f(t)\, dt = F(t) + C$$

where $F(t)$ is an antiderivative of $f(t)$ (i.e., $F'(t) = f(t)$) and C is a constant.

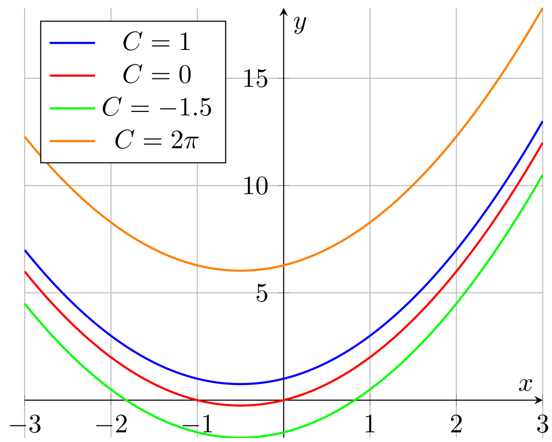
Example: Consider the ODE

$$\frac{dy}{dx} = 2x + 1$$

The general solution is:

$$y = x^2 + x + C$$

Graph for different values of C:



To select a specific solution from the general solution, we need an initial condition: $\boxed{y(t_I) = y_I}$.

The pair $y' = f(t)$ with $y(t_I) = y_I$ is called an Initial Value Problem (IVP).

Example: Solve the IVP

$$\frac{dy}{dx} = 2x + 1 \quad \text{with} \quad y(0) = 2$$

Solution:

Start with the general solution:

$$y = x^2 + x + C$$

Using the initial condition $y(0) = 2$:

$$2 = 0^2 + 0 + C \quad \Rightarrow \quad C = 2$$

Thus, the specific solution is:

$$y = x^2 + x + 2$$

Interval of Definition/Existence

The interval of definition/existence of a solution to an IVP is the **largest** interval (a, b) where:

- $t_I \in (a, b)$
- $f(t)$ is continuous on (a, b)

Chapter 2: Linear Equations

These look like:

$p(t)y' + q(t)y = r(t)$ where $p(t) \neq 0$ for the values of t we are considering.

In standard form:

$$y' = -\frac{q(t)}{p(t)}y + \frac{r(t)}{p(t)}$$

Let:

$$a(t) = \frac{q(t)}{p(t)}, \quad f(t) = \frac{r(t)}{p(t)}$$

We write it as:

$$y' + a(t)y = f(t)$$

Here, $f(t)$ is called the forcing function.
If $f(t) = 0$, the ODE is called homogeneous; otherwise, it is non-homogeneous.

Recipe for Solving First-Order Linear ODEs

Given:

$$y' + a(t)y = f(t)$$

1. Choose an antiderivative $A(t)$ of $a(t)$. 2. Multiply both sides by $e^{A(t)}$:

$$e^{A(t)}y' + a(t)e^{A(t)}y = f(t)e^{A(t)}$$

Let:

$$f(t)e^{A(t)} = g(t)$$

This simplifies to:

$$\frac{d}{dt}\left(e^{A(t)}y\right) = g(t)$$

3. Integrate both sides:

$$e^{A(t)}y = G(t) + C \quad \Rightarrow \quad y = e^{-A(t)}G(t) + Ce^{-A(t)}$$

This is the general solution.
Example: Solve the ODE

$$\frac{dy}{dt} = -y$$

1. Rewrite as $y' + y = 0$. 2. Here, $a(t) = 1$, so choose $A(t) = t$. 3. Multiply both sides by e^t:

$$e^t y' + e^t y = 0 \quad \Rightarrow \quad \frac{d}{dt}(e^t y) = 0$$

4. Integrate:

$$e^t y = C \quad \Rightarrow \quad y = Ce^{-t}$$

This is the general solution.

Example: Consider the ODE

$$y' = -y + e^t$$

1. Rewrite as $y' + y = e^t$. 2. Here, $a(t) = 1$, so choose $A(t) = t$. 3. Multiply both sides by e^t:

$$e^t y' + e^t y = e^{2t} \quad \Rightarrow \quad \frac{d}{dt}(e^t y) = e^{2t}$$

4. Integrate:

$$e^t y = \frac{1}{2}e^{2t} + C \quad \Rightarrow \quad y = \frac{1}{2}e^t + Ce^{-t}$$

This is the general solution.

Example: Solve the IVP

$$\frac{dx}{dt} + \cos(t)x = \cos(t) \quad \text{with} \quad x\left(\frac{\pi}{2}\right) = 0$$

Solution:

1. Here, $a(t) = \cos(t)$, so choose $A(t) = \sin(t)$. 2. Multiply both sides by $e^{\sin(t)}$:

$$e^{\sin(t)}x' + \cos(t)e^{\sin(t)}x = \cos(t)e^{\sin(t)}$$

This simplifies to:

$$\frac{d}{dt}\left(e^{\sin(t)}x\right) = \cos(t)e^{\sin(t)}$$

3. Integrate:

$$e^{\sin(t)}x = \int \cos(t)e^{\sin(t)}\,dt = e^{\sin(t)} + C$$

Thus,

$$x = 1 + Ce^{-\sin(t)}$$

4. Apply the initial condition $x\left(\frac{\pi}{2}\right) = 0$:

$$0 = 1 + Ce^{-1} \quad \Rightarrow \quad C = -e$$

Thus, the specific solution is:

$$x = 1 - e^{1-\sin(t)}$$

Lecture 2, 8/29/2024

I.2 (continued)

Problem Statement

Consider the initial value problem (IVP):

$$y' + a(t)y = f(t), \quad y(t_I) = y_I$$

Theorem: If $a(t)$ and $f(t)$ are continuous over the interval (a, b) and $t_I \in (a, b)$, then there is a unique solution to the IVP that is continuous on (a, b), and it's given by our method.

Example

Consider the differential equation:

$$z' + \cot(t)z = \frac{1}{\ln(t^2)}, \quad z(4) = 3$$

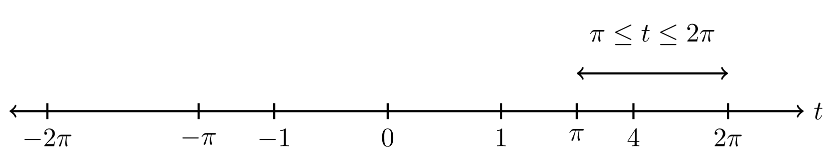
Find the largest interval on which we can guarantee a unique continuous solution to this IVP.

Solution

The function $\ln(t^2)$ is continuous on $(-\infty, 0)$ and $(0, \infty)$, but $\frac{1}{\ln(t^2)}$ is discontinuous at $t = 0$ and when $\ln(t^2) = 0$, i.e., $t = \pm 1$.

The function $\cot(t)$ has discontinuities at multiples of π.

The largest interval of continuity that includes $t = 4$ is $(\pi, 2\pi)$.



I.3: Separable Equation

A first-order ordinary differential equation (ODE) is **separable** if it can be written in the form:

$$y' = f(t)g(y)$$

Example:
Consider the differential equation:

$$y' = 2ty^2 + 3t^2y^2$$

We can factor this as:

$$y' = (2t + 3t^2)y^2$$

Here, we have:

$$f(t) = 2t + 3t^2, \quad g(y) = y^2$$

An ODE of the form $y' = g(y)$ is called **autonomous**.
A solution is called **stationary** if it is constant. If $y = C$ is a stationary solution, then:

$$y' = 0 \Rightarrow \boxed{0 = g(C)}$$

Example:
Consider the equation:

$$y' = 4y - y^3$$

To find the stationary solutions, set:

$$4y - y^3 = 0 \Rightarrow y(4 - y^2) = 0 \Rightarrow y(2 - y)(2 + y) = 0$$

Thus, the stationary solutions are:

$$y = 0, \quad y = 2, \quad y = -2$$

Non-Stationary Solutions

To find non-stationary solutions of the equation $y' = g(y)$, we proceed as follows:

$$y' = g(y) \quad \Rightarrow \quad \frac{1}{g(y)}y' = 1$$

Taking the integral on both sides:

$$\int \frac{1}{g(y)}y'\,dt = \int 1\,dt$$

This simplifies to:

$$\int \frac{1}{g(y)}\, dy = t + C$$

The result is an implicit equation for our solution.

Why can we divide by $g(y)$? $g(y) = 0$ corresponds to stationary solutions, and we are looking for non-stationary solutions, i.e., $g(y) \neq 0$.

Example: Find All Solutions to $y' = y^2$

Stationary Solutions: Set $y^2 = 0$, which implies $y = 0$.

Non-Stationary Solutions:

Starting with the equation:

$$\frac{1}{y^2} y' = 1$$

Integrate both sides:

$$\int \frac{1}{y^2} y'\, dt = \int 1\, dt$$

This simplifies to:

$$\int \frac{1}{y^2}\, dy = t + C$$

Evaluating the integral:

$$-\frac{1}{y} = t + C$$

We can find an explicit solution:

$$-y = \frac{1}{t + C} \quad \Rightarrow \quad y = -\frac{1}{t + C}$$

Each solution $y = -\frac{1}{t+C}$ actually represents two solutions, one defined on $(-\infty, -C)$ and the other on $(-C, \infty)$.

Note: Our solution is discontinuous even though all functions in the original equation $y' = y^2$ are continuous.

General Separable Equations

Consider the general separable equation:

$$y' = f(t)g(y)$$

If $g(c) = 0$, then $y = c$ is a stationary solution (so set g(y) = 0).

For non-stationary solutions:

$$\frac{1}{g(y)}y' = f(t)$$

Taking the integral on both sides:

$$\int \frac{1}{g(y)}y'\,dt = \int f(t)\,dt$$

This simplifies to:

$$\int \frac{1}{g(y)}\,dy = F(t) + C$$

Example: Find All Solutions to $\frac{dz}{dx} = \frac{3x+xz^2}{z+x^2z}$

First, rewrite the equation:

$$\frac{dz}{dx} = \frac{x}{1+x^2} \cdot \frac{3+z^2}{z}$$

Thus, we identify:

$$f(x) = \frac{x}{1+x^2}, \quad g(z) = \frac{3+z^2}{z}$$

Stationary Solutions: Set $g(z) = 0$:

$$\frac{3+z^2}{z} = 0 \quad \Rightarrow \quad 3+z^2 = 0$$

This equation has no real solution, so there are no stationary solutions.
Non-Stationary Solutions:
Start with:

$$\frac{1}{g(z)}\frac{dz}{dx} = f(x)$$

Which simplifies to:

$$\frac{z}{3+z^2} \cdot \frac{dz}{dx} = \frac{x}{1+x^2}$$

Integrate both sides:

$$\int \frac{z}{3+z^2}\frac{dz}{dx}\,dx = \int \frac{x}{1+x^2}\,dx$$

Use substitution:
- Let $u = 3+z^2$, then $du = 2z\,dz$. - Let $v = 1+x^2$, then $dv = 2x\,dx$.
The integrals become:

$$\int \frac{1}{2u} du = \int \frac{1}{2v} dv$$

This integrates to:

$$\frac{1}{2} \ln |u| = \frac{1}{2} \ln |v| + C$$

Substituting back u and v:

$$\frac{1}{2} \ln |3 + z^2| = \frac{1}{2} \ln |1 + x^2| + C$$

Initial Value Problems (IVPs)

Example: Solve the initial value problem:

$$y' = ty^2 - ty, \quad y(1) = 2$$

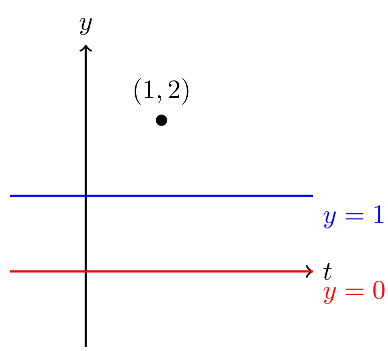
We can factor the equation as:

$$y' = t(y^2 - y)$$

Stationary Solutions: Set $y^2 - y = 0$:

$$y(y-1) = 0 \quad \Rightarrow \quad y = 0 \quad \text{or} \quad y = 1$$

Neither $y = 0$ nor $y = 1$ satisfies the initial condition $y(1) = 2$.



As shown in the graph, neither $y = 0$ nor $y = 1$ passes through the point $(1, 2)$.

Other Solutions: We solve the differential equation for non-stationary solutions:

$$\frac{1}{y^2 - y} \frac{dy}{dt} = t \quad \Rightarrow \quad \frac{1}{y^2 - y} dy = t\, dt$$

Integrate both sides:

$$\int \frac{1}{y^2 - y}\, dy = \int t\, dt$$

Using partial fractions:

$$\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1}$$

This leads to:

$$1 = A(y-1) + B(y)$$

By guessing $A = -1$ and $B = 1$, we get:

$$\int \left(-\frac{1}{y} + \frac{1}{y-1} \right) dy = \int t\, dt$$

Integrating both sides:

$$-\ln|y| + \ln|y-1| = \frac{t^2}{2} + C$$

Using the logarithm property $\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)$, this simplifies to:

$$\ln\left|\frac{y-1}{y}\right| = \frac{t^2}{2} + C$$

Applying the Initial Condition: Given $y(1) = 2$:

$$\ln\left|\frac{2-1}{2}\right| = \frac{1^2}{2} + C$$

$$\ln\left(\frac{1}{2}\right) = \frac{1}{2} + C \quad \Rightarrow \quad C = \ln\left(\frac{1}{2}\right) - \frac{1}{2}$$

Substituting C back into the equation:

$$\ln\left|\frac{y-1}{y}\right| = \frac{t^2}{2} + \ln\left(\frac{1}{2}\right) - \frac{1}{2}$$

Uniqueness and Existence Theorem

If $f(t)$ is continuous on (a, b) and $g(y)$ is continuous and differentiable on (c, d), then for every $t_I \in (a, b)$ and $y_I \in (c, d)$, there exists a unique continuous solution to the equation

$$y' = f(t)g(y)$$

with the initial condition $y(t_I) = y_I$, defined on some interval around t_I. The solution is determined by our method.

Example

Consider the differential equation:

$$\frac{dy}{dt} = 3y^{2/3}, \quad y(0) = 0$$

Stationary Solution: $y = 0$ is a stationary solution, and it solves our initial value problem (IVP).

However, $g(y) = 3y^{2/3}$ is not differentiable at $y = 0$, so we might have other solutions.

Finding Other Solutions:

$$\frac{1}{3y^{2/3}} \frac{dy}{dt} = 1$$

Integrating both sides:

$$\int \frac{1}{3y^{2/3}} \, dy = \int 1 \, dt$$

This simplifies to:

$$y^{1/3} = t + C$$

Raising both sides to the power of 3:

$$y = (t + C)^3$$

Applying the Initial Condition: For $y(0) = 0$, we get $C = 0$, so:

$$y = t^3$$

Thus, $y = t^3$ also solves our IVP.

Lecture 3, Tuesday 9/3/2024

Quiz tomorrow: Up to Section I.3

I.4. Theory

Consider Initial Value Problems (IVPs) of the form:

$$y' = f(t, y), \quad y(t_i) = y_i$$

We say the problem is *well-posed* if:

1. There exists a solution
2. The solution is unique
3. The solution depends continuously on the initial conditions

Existence and Uniqueness

Consider a set S of points in the (t, y) plane.

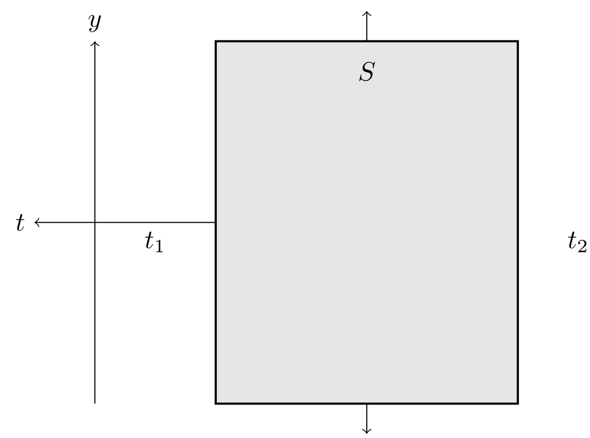


Figure 1: The set S in the (t, y) plane

If $f(t, y)$ is continuous on S and $\frac{\partial f}{\partial y}$ is continuous on S, then for any $(t_i, y_i) \in S$, there exists a unique continuous solution $Y(t)$ to the initial value problem:

$$\begin{cases} y' = f(t, y) \\ Y(t_i) = y_i \end{cases}$$

defined over some interval (a, b) containing t_i.

Moreover, the interval (a, b) can be extended as long as $(t, Y(t))$ remains inside S.

Example: Consider $y' = \frac{\sin(t+ty^2)}{1+t^2}$, $y(0) = 1$

Let's show that there is a unique solution defined on $(-1, 1)$.

Solution: For $f(t, y) = \frac{\sin(t+ty^2)}{1+t^2}$, f is continuous except at $t = \pm 1$.

$\frac{\partial f}{\partial y} = \frac{2ty\cos(t+ty^2)}{1+t^2}$, which is also continuous except at $t = \pm 1$.

By the Existence and Uniqueness Theorem, since f and $\frac{\partial f}{\partial y}$ are continuous on S, there exists a unique solution $y(t)$ to the initial value problem, defined on some interval containing $t = 0$. This interval can be extended as long as $(t, y(t))$ remains inside S, which in this case is the entire interval $(-1, 1)$.

Since $t_i = 0$, $y_i = 1 \implies (0, 1) \in S$, the theorem tells us we have a unique solution $Y(t)$ defined on a larger (a, b) such that $y(t)$ remains inside S.

Since any solution will not leave S as long as $-1 < t < 1$, we get $(a, b) = (-1, 1)$.

So, we choose S as follows:

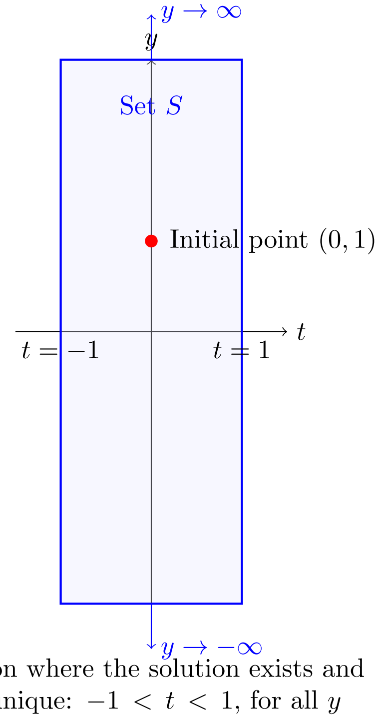


Figure 2: Set S where the solution to the differential equation is guaranteed to exist and be unique

Example: Consider the differential equation $y' = \frac{1}{t^2+y^2-1}$, with initial condition $y(0) = 0$. Find the set S where the solution is guaranteed to exist and be unique.

Solution: The function $f(t,y) = \frac{1}{t^2+y^2-1}$ is discontinuous when $t^2+y^2 = 1$. This equation describes a circle with radius 1 centered at the origin.

The set S where the solution exists and is unique should be the interior of this circle, excluding the circle itself. We can represent this as:

$$S = \{(t,y) : t^2 + y^2 < 1\}$$

Let's visualize this set:

The solution is guaranteed to exist and be unique within this circular region S, but not on or outside the boundary where $t^2 + y^2 = 1$.

Example: Consider the differential equation $y' = \frac{1}{t^2+y^2-1}$, with initial condition $y(0) = 3$. Let's visualize the set S where the solution is guaranteed to exist and be unique.

Solution: The function $f(t,y) = \frac{1}{t^2+y^2-1}$ is discontinuous when $t^2+y^2 = 1$. This equation describes a circle with radius 1 centered at the origin.

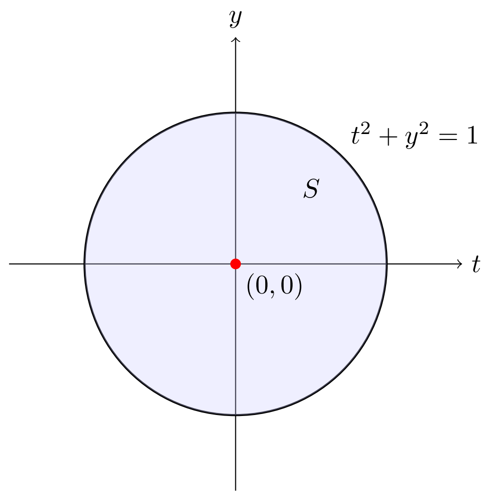


Figure 3: Set S for the given differential equation

The set S where the solution exists and is unique should be the region outside this circle, including the initial point $(0, 3)$. We can represent this as:

$$S = \{(t, y) : t^2 + y^2 > 1\}$$

Let's visualize this set:

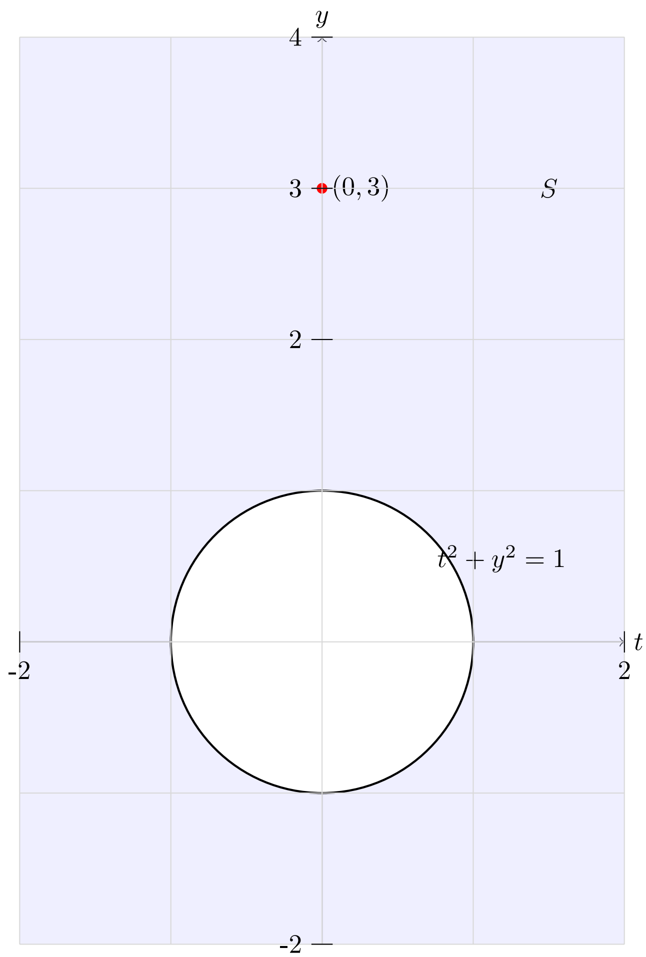


Figure 4: Set S for the given differential equation with $y(0) = 3$

The solution is guaranteed to exist and be unique within the region S, which is the area outside the circle $t^2 + y^2 = 1$, including the initial point $(0, 3)$.

1 I.5 - Graphical Methods

1.1 Phase Portraits for Autonomous Equations

Consider the autonomous differential equation:

$$\frac{dy}{dt} = g(y)$$

Our goal is to describe the qualitative behavior of solutions without explicitly solving the equation.

- When $g(y) = 0$:
  - $y' = 0$
  - We have a stationary solution
- When $g(y) > 0$:
  - $y' > 0$
  - The solution is increasing
- When $g(y) < 0$:
  - $y' < 0$
  - The solution is decreasing

Example: $y' = 4y - y^3$

Let's analyze the differential equation $y' = 4y - y^3$.

1. First, we find the stationary solutions:

$$4y - y^3 = y(4 - y^2) = y(2 - y)(2 + y) = 0$$

   Thus, the stationary solutions are $y = 0, \pm 2$.

2. Next, we determine the sign of $g(y) = 4y - y^3$ between these zeros:

$$\begin{aligned} g(1) &= 4(1) - (1)^3 = 3 > 0 \\ g(3) &= 4(3) - (3)^3 = 12 - 27 = -15 < 0 \\ g(-3) &= 4(-3) - (-3)^3 = -12 + 27 = 15 > 0 \\ g(-1) &= 4(-1) - (-1)^3 = -4 + 1 = -3 < 0 \end{aligned}$$

Based on this analysis, we can create a phase portrait:
The phase portrait shows:

- Solutions increase when $y \in (-\infty, -2) \cup (0, 2)$

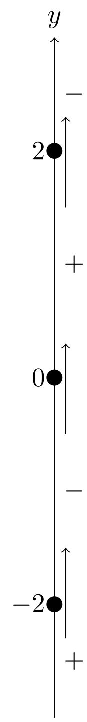


Figure 5: Phase portrait for $y' = 4y - y^3$

- Solutions decrease when $y \in (-2, 0) \cup (2, \infty)$
- Stationary solutions at $y = 0, \pm 2$

This diagram is called a phase portrait or phase line. It provides valuable information about the behavior of solutions $y(t)$ for different initial conditions:

- For $y(t)$ starting in $(-\infty, -2)$:
  - $y(t)$ increases as t increases
  - $y(t) \to -2$ as $t \to \infty$ (asymptotically approaching -2)
- For $y(t)$ starting in $(-2, 0)$:
  - $y(t)$ is decreasing
  - $y(t) \to -2$ as $t \to \infty$
- For $y(t)$ starting in $(0, 2)$:
  - $y(t)$ is increasing
  - $y(t) \to 2$ as $t \to \infty$ (asymptotically approaching 2)

- For $y(t)$ starting in $(2, \infty)$:
  - $y(t)$ is decreasing
  - $y(t) \to 2$ as $t \to \infty$ (asymptotically approaching 2)

Sketch solutions:

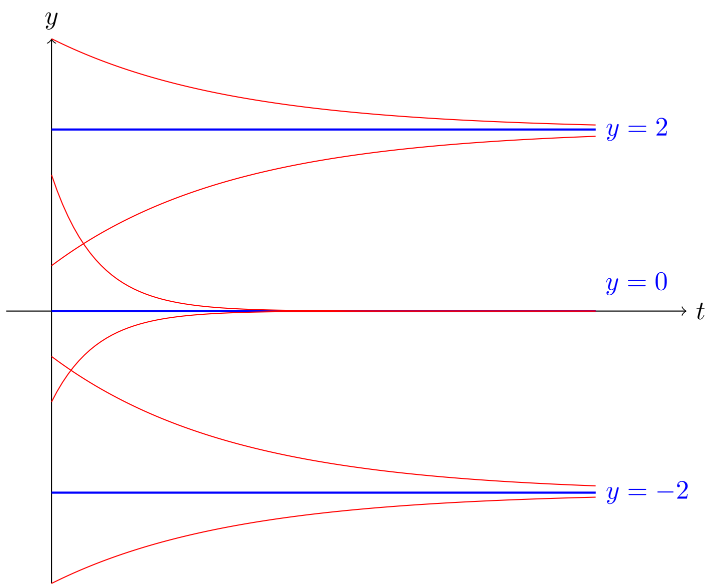


Figure 6: Sketch of solutions for $y' = 4y - y^3$

This figure illustrates:

- Constant solutions at $y = 2$, $y = 0$, and $y = -2$ (blue lines)
- Solutions approaching $y = 2$ from above and below
- Solutions approaching $y = -2$ from above and below
- Solutions approaching $y = 0$ from above and below

The red curves represent various solutions to the differential equation, showing how they behave over time depending on their initial conditions.

We classify stationary solutions as follows:

- **Stable or attracting:** All nearby solutions move towards it as $t \to \infty$.
  (In this case: $y = \pm 2$)
- **Unstable or repelling:** All nearby solutions move away from it as $t \to \infty$.
  (In this case: $y = 0$)

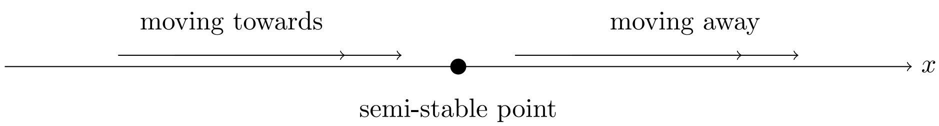


Figure 7: Behavior near a semi-stable point

- **Semi-stable:** Some solutions move towards it and some move away from it as $t \to \infty$.

Example: Consider the differential equation $y' = y^2$

- Stationary solution: $y = 0$
- For $y \neq 0$: $y^2 > 0$, implying solutions move away from $y = 0$

This example demonstrates an unstable stationary solution at $y = 0$.

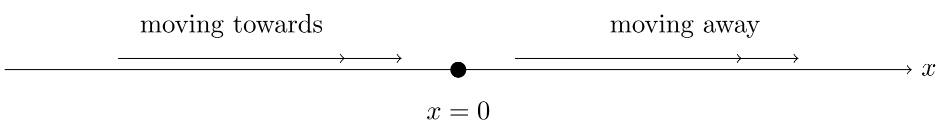


Figure 8: Behavior on x-axis for $y' = y^2$

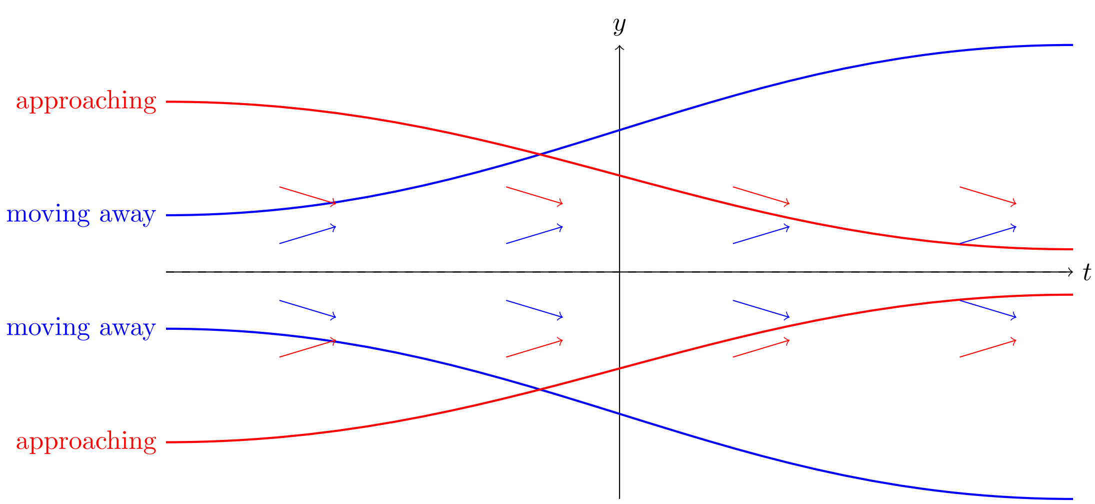


Figure 9: Behavior of solutions in t-y plane for $y' = y^2$

This figure illustrates the behavior of solutions to $y' = y^2$ near the stationary solution $y = 0$. The blue curves represent solutions moving away from $y = 0$, while the red curves represent solutions approaching $y = 0$. This demonstrates that $y = 0$ is an unstable or repelling stationary solution.

For stability think of a pendulum

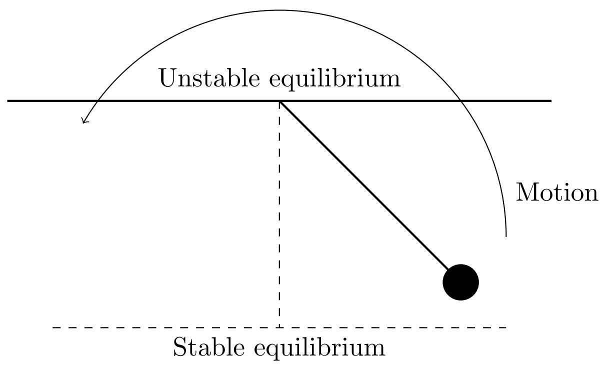


Figure 10: Pendulum illustrating stable and unstable equilibrium points

This figure illustrates a pendulum moving from right to left. The top position (vertically upright) represents an unstable equilibrium, while the bottom position represents a stable equilibrium. These correspond to the stationary solutions of the pendulum's differential equation.

unstable solution is pendulum at the top and stable is pendulum at the bottom

Example: Phase Line Analysis

Consider the differential equation:

$$y' = \frac{(y^2 - 1)(y - 3)^2}{(y + 3)^2}$$

Stationary Solutions:

- $y = -1$
- $y = 1$
- $y = 3$

Note: $y = -3$ is undefined in the equation.

We will now draw the phase line for this differential equation to analyze its behavior.

Note: stability only applies to stationary solutions, not to undefined points.

Sketch

Example: Consider the differential equation $y' = t - y^2$

Procedure: To sketch the slope field:

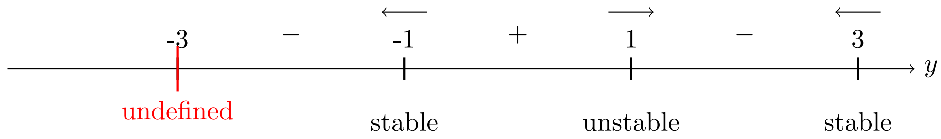


Figure 11: Phase line for $y' = (y^2 - 1)(y-3)^2/(y+3)^2$

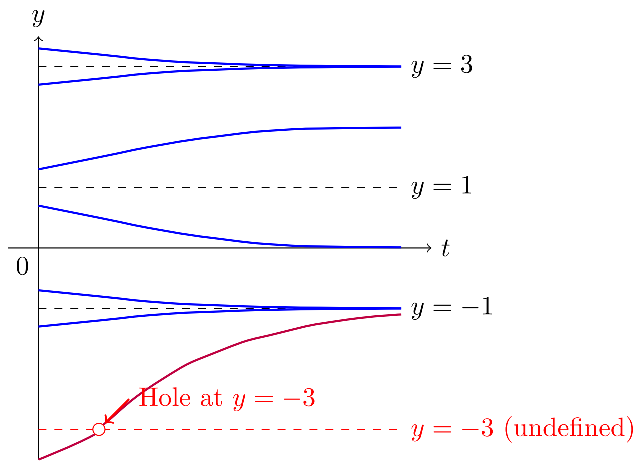


Figure 12: Sketch of solutions for $y' = (y^2 - 1)(y-3)^2/(y+3)^2$, including a solution crossing $y = -3$ with a clear hole

1. Choose a representative selection of (t, y) points.
2. At each point (t, y), draw an arrow with slope $t - y^2$.
3. Connect the arrows to visualize solution curves.

Sample calculations:

$$\begin{aligned}
\text{At } (0,0) &: \quad t - y^2 = 0 \\
\text{At } (1,0) &: \quad t - y^2 = 1 \\
\text{At } (-1,0) &: \quad t - y^2 = -1 \\
\text{At } (0,\pm 1) &: \quad t - y^2 = -1
\end{aligned}$$

Continue this process for other points to build a comprehensive slope field.

Slope field for $y' = t - y^2$

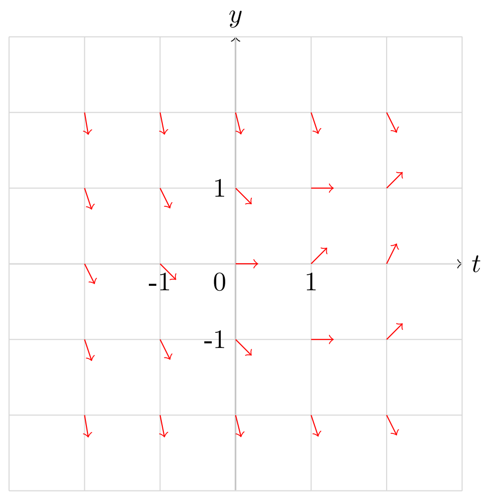


Figure 13: Slope field for the differential equation $y' = t - y^2$

If $y(0) = 1$, then the point should follow the slope field:

Slope field for $y' = t - y^2$

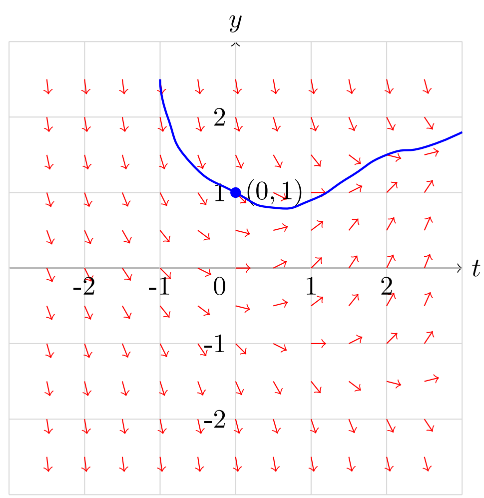


Figure 14: Slope field for $y' = t - y^2$ with initial point $y(0) = 1$ and solution curve

Lecture 4, Thursday 9/5/2024

2 I.6 Applications of Differential Equations

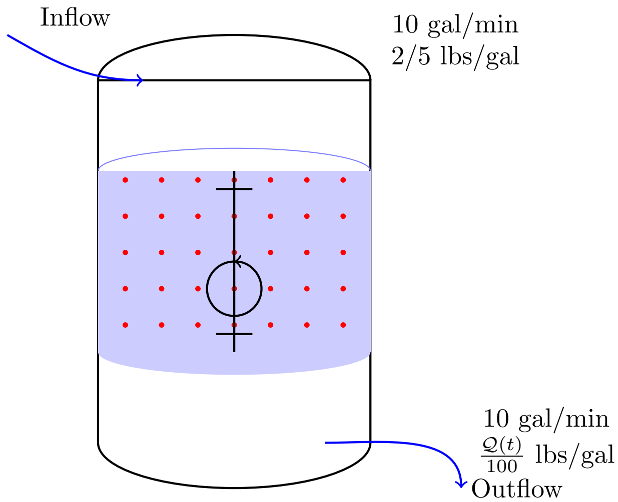
2.1 I.6.1 Tanks and Mixtures

2.1.1 I.6.1.1 IRS Method: Identify, Reduce, Solve

In tank and mixture problems, we often encounter scenarios where:

- Water flows into and out of a tank
- Salt is mixed into the water
- The concentration of salt varies over time
- The mixture in the tank is continuously leaving

Our primary interest is in understanding how the salt content changes over time.



Tank with well-mixed salt solution

Figure 15: Diagram of a tank with water inflow, outflow, and well-mixed salt solution

Example: Consider a tank with the following properties:

- Initial contents: 100 gallons of brine
- Initial salt content: 20 lbs dissolved in the brine
- Inflow: Brine containing 2/5 lbs/gal of salt at 10 gal/min

- Outflow: Well-mixed solution at 10 gal/min

Problem: Find a formula for the salt content after t minutes.
Solution: Let $\mathcal{Q}(t)$ be the quantity of salt at time t.

$$\text{Rate of salt inflow} = 10 \text{ gal/min} \cdot \frac{2}{5} \text{ lbs/gal} = 4 \text{ lbs/min}$$
$$\text{Rate of salt outflow} = 10 \text{ gal/min} \cdot \frac{\mathcal{Q}(t)}{100} \text{ lbs/gal} = \frac{\mathcal{Q}(t)}{10} \text{ lbs/min}$$

The rate of change of salt content is the difference between inflow and outflow:

$$\frac{d\mathcal{Q}}{dt} = \text{salt in} - \text{salt out} = 4 - \frac{\mathcal{Q}(t)}{10}$$

This is our differential equation. We can solve it using the integrating factor method or separation of variables.

The initial condition is:

$$\mathcal{Q}(0) = 20 \text{ lbs}$$

Therefore, our initial value problem (IVP) is:

$$\begin{cases} \mathcal{Q}' = 4 - \frac{\mathcal{Q}}{10} \\ \mathcal{Q}(0) = 20 \end{cases}$$

Solve the IVP: The equation is linear: $\mathcal{Q}' + \frac{1}{10}\mathcal{Q} = 4$

$$a(t) = \frac{1}{10} \Rightarrow A(t) = \frac{t}{10}$$

$$\text{Multiply by } e^{t/10} : e^{t/10}\mathcal{Q}' + \frac{1}{10}e^{t/10}\mathcal{Q} = 4e^{t/10}$$

$$\text{Integrate both sides} : e^{t/10}\mathcal{Q} = 40e^{t/10} + C$$

$$\Rightarrow \mathcal{Q} = 40 + Ce^{-t/10}$$

Use the initial condition to solve for C:

$$20 = 40 + C \Rightarrow C = -20$$

Therefore, the solution is:

$$\mathcal{Q}(t) = 40 - 20e^{-t/10}$$

Further Questions

Asymptotic Behavior

As $t \to \infty$, what happens to $\mathcal{Q}(t)$?

Answer:

$$\begin{aligned}
\lim_{t\to\infty} \mathcal{Q}(t) &= \lim_{t\to\infty} (40 - 20e^{-t/10}) \\
&= 40 - 20 \lim_{t\to\infty} e^{-t/10} \\
&= 40 - 20(0) = 40 \text{ lbs}
\end{aligned}$$

Time to Reach a Specific Amount

After what time will there be more than 30 lbs of salt in the tank?

Answer: Set $\mathcal{Q}(t) = 30$ lbs and solve for t:

$$\begin{aligned}
30 &= 40 - 20e^{-t/10} \\
-10 &= -20e^{-t/10} \\
\frac{1}{2} &= e^{-t/10} \\
\ln(\frac{1}{2}) &= -\frac{t}{10} \\
t &= 10\ln(2) \approx 6.93 \text{ minutes}
\end{aligned}$$

General Case

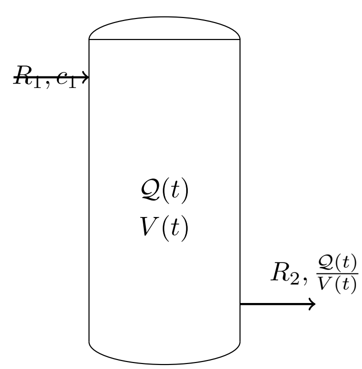


Figure 16: Cylinder with inflow and outflow

Parameters:

- R_1: Rate of inflow
- c_1: Concentration of inflow

- R_2: Rate of outflow
- $\frac{\mathcal{Q}(t)}{V(t)}$: Concentration of outflow

Initial Conditions:

$$\mathcal{Q}(0) = \mathcal{Q}_0$$
$$V(0) = V_0$$

Differential Equations:

$$V'(t) = r_1 - r_2 = v(t) = (r_1 - r_2)t + v_0$$
$$\mathcal{Q}'(t) = r_1 c_1 - r_2 \frac{\mathcal{Q}(t)}{V(t)} = r_1 c_1 - \frac{r_2 q(t)}{(r_1 - r_2)t + v0}$$

Salt Analysis:

- Salt in: $r_1 c_1$
- Salt out: $\frac{\mathcal{Q}(t)}{V(t)}$

Population Dynamics:
Let $P(t)$ be the population at time t. Consider the differential equation:

$$\frac{dP}{dt} = R(P)P - h(t)$$

where:

- $R(P)$: Growth rate
- $h(t)$: Harvest rate

Exponential Model: For the exponential model, we take $h(t) = 0$ and $R(P) = r$ (constant).

The differential equation becomes:

$$\frac{dP}{dt} = rP$$

The general solution is:

$$P(t) = Ce^{rt}$$

With the initial condition $P(0) = P_0$, we get:

$$P(t) = P_0 e^{rt}$$

Example 1: Monkey Population Growth
Consider a population of monkeys with the following characteristics:

- Initial population: 100 monkeys

- Growth rate: 4% per year
- Immigration: 8 new monkeys join from surrounding tribes every year

The Initial Value Problem (IVP) describing this system is:

$$\frac{dM}{dt} = 0.04M + 8$$
$$M(0) = 100$$

where $M(t)$ represents the number of monkeys at time t.
Solution: $M(t) = 300e^{0.04t} - 200$

Example 2: Rabbit Population Growth
A population of rabbits doubles in size every year.
Question: What is the growth rate assuming exponential growth?
Solution: Let's assume the population follows the exponential growth model:

$$P(t) = P_0 e^{rt}$$

where:

- $P(t)$ is the population at time t
- P_0 is the initial population
- r is the growth rate
- t is time in years

Given:

$$P(0) = P_0$$
$$P(1) = 2P_0$$

Substituting into our model:

$$2P_0 = P_0 e^r$$

Simplifying:

$$2 = e^r$$

Taking the natural logarithm of both sides:

$$r = \ln(2) \approx 0.693$$

Therefore, the growth rate is approximately 69.3% per year.
Logistic Model: This model accounts for finite resources and competition.

$$\frac{dp}{dt} = (r - ap)p \tag{1}$$

where:

- $R(p) = r - ap$ is the per capita growth rate
- When p is small, $R(p) \approx r$
- As p grows, $R(p)$ decreases
- At $p = \frac{r}{a}$, $R(p) = 0$ (this is called the carrying capacity)

Stationary solutions: Set $(r - ap) = 0$

$$\Rightarrow p = 0 \text{ or } p = \frac{r}{a}$$

Phase-line portrait:

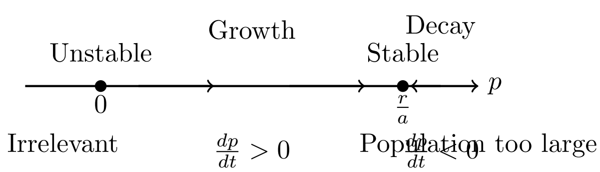
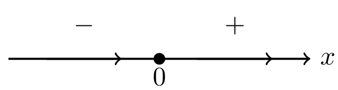
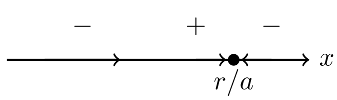


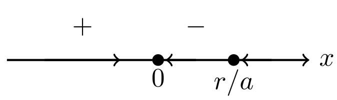
Figure 17: Phase-line portrait for the logistic model

Additional Phase Line Portraits

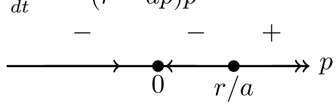
Here are three more examples of phase line portraits:

1. Simple linear model: 

2. Logistic model: 

3. Reversed logistic model: 

Variant logistic model:
$\frac{dp}{dt} = -(r - ap)p$



This phase line portrait for $\frac{dp}{dt} = -(r - ap)p$ shows:

- Two equilibrium points: at $p = 0$ and $p = r/a$
- For $p < 0$, $\frac{dp}{dt} < 0$, so p decreases (moves left)

- For $0 < p < r/a$, $\frac{dp}{dt} < 0$, so p decreases (moves left)
- For $p > r/a$, $\frac{dp}{dt} > 0$, so p increases (moves right)

This describes a context where a population needs a certtin critical size r/a to be able to grow otherwise it will die out.

Lecture 5, Tuesday 9/10/2024

I.6 Motion: Falling Objects (continued)

Consider an object with mass m and force F acting on it. Newton's Second Law of Motion states:

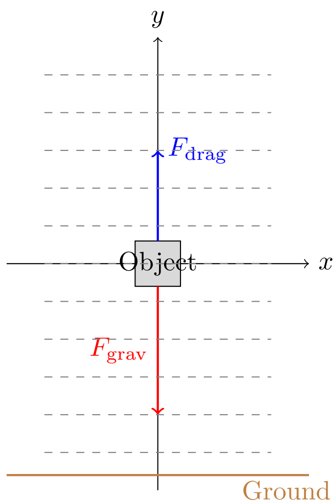
$$F = ma$$

where a is the acceleration, defined as:

$$a = \frac{dv(t)}{dt}$$

and $v(t)$ is the velocity.
For our analysis, we will use the following coordinate system:

For falling objects, the total force is composed of gravity and drag:



Where:

$$F_{\text{grav}} = \text{force from gravity} = mg$$
$$F_{\text{drag}} = \text{drag force} = -cv^2$$

Here, $g = -9.8\ \text{m/s}^2$, c is a constant, and v is velocity.
Therefore, our ODE model for a falling object is:

$$ma = m\frac{dv}{dt} = mg - cv^2$$

Let c be the drag constant.

Cancelling out m, we get:

$$v' = g + cv^2$$

This is a non-linear equation, but it is separable.

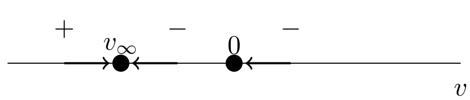
Stationary solutions are found by solving:

$$g + cv^2 = 0 \quad \Rightarrow \quad v = \pm\sqrt{-g/c}$$

For falling objects, we are interested in the negative velocity solution. Thus,

$$v_\infty = -\sqrt{-g/c}$$

is called the terminal velocity.



This horizontal phase line portrait shows:

- The equilibrium points at $v = 0$ and $v = v_\infty$
- Arrows indicating the direction of change
- Signs showing whether v' is positive or negative in each region

We can observe that:

- For $v > 0$, v' is negative, so v decreases towards 0
- For $v_\infty < v < 0$, v' is still negative, so v continues to decrease towards v_∞
- For $v < v_\infty$, v' becomes positive, so v increases back towards v_∞

This confirms that v_∞ is indeed the terminal velocity, as the system will eventually settle at this value regardless of the initial conditions.

A skydiver with mass $m = 60$ kg jumps from an airplane and assumes a position with drag coefficient $c = 0.002$ m^{-1}. Find her terminal velocity.

Solution:
$v(0) = 0, c = 0.002, g = -9.8$
$\Rightarrow v' = g + 0.002v^2$
terminal velocity $v_\infty = -\sqrt{\frac{-g}{c}}$
To find the terminal velocity, we use the formula derived earlier:

$$v_\infty = -\sqrt{\frac{-g}{c}}$$

Substituting the given values:

$$v_\infty = -\sqrt{\frac{9.8 \text{ m/s}^2}{0.002 \text{ m}^{-1}}} = -70.0 \text{ m/s}$$

Therefore, the skydiver's terminal velocity is approximately 70.0 m/s (or 252 km/h) downward.

Now, let's solve the ODE analytically:
Rewrite the ODE:

$$\begin{aligned} v' &= g + cv^2 \\ &= c\left(\frac{g}{c} + v^2\right) \\ &= c(v^2 - v_\infty^2) \\ &= c(v - v_\infty)(v + v_\infty) \end{aligned}$$

Separate variables:

$$\frac{v'}{(v - v_\infty)(v + v_\infty)} = c$$

Integrate both sides:

$$\int \frac{v'}{(v - v_\infty)(v + v_\infty)} \, dv = \int c \, dt$$

Using partial fractions decomposition:

$$\frac{1}{2v_\infty} \left(\ln |v - v_\infty| - \ln |v + v_\infty| \right) = ct + C_0$$

Apply initial condition $v(0) = 0$:

$$C_0 = \frac{1}{2v_\infty} \left(\ln |-v_\infty| - \ln |v_\infty| \right) = 0$$

Therefore, the solution is:

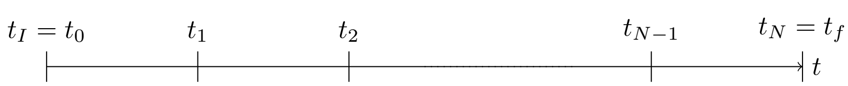
$$\ln \left| \frac{v - v_\infty}{v + v_\infty} \right| = 2cv_\infty t$$

2.2 I.7: Numerical Methods

Consider the initial value problem:

$$y' = f(t, y), \quad y(t_I) = y_I$$

Our goal is to approximate $y(t_f)$, where $t_f > t_I$.
To do this, we divide the interval $[t_I, t_f]$ into n equal steps:



We then use an iterative process to approximate the solution:

1. Use $y(t_0) = y_I$ to approximate $y(t_1)$
2. Use the approximation of $y(t_1)$ to approximate $y(t_2)$
3. $\vdots$
4. Use the approximation of $y(t_{N-1})$ to approximate $y(t_N)$

This process forms the basis of numerical methods for solving differential equations.

For good methods, the accuracy of the approximation increases as N increases.

We will use uniform step sizes:

$$\text{step size} = h = \frac{t_f - t_I}{N}$$

$$t_i = t_I + ih \quad \text{for } i = 0, 1, 2, \ldots, N$$

Euler method:
Idea: The derivative can be approximated by the difference quotient

$$f'(t) \approx \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

For small h, we have the approximation:

$$f(t+h) \approx f(t) + hf'(t)$$

If $y(t)$ is the solution to $y' = f(t, y)$ with $y(t_i) = y_i$, then:

$$y(t_i + h) \approx y(t_i) + hy'(t_i) = y(t_i) + hf(t_i, y(t_i))$$

Thus,

$$\begin{aligned} y(t_1) &\approx y(t_0) + hf(t_0, y(t_0)) \\ y(t_2) &\approx y(t_1) + hf(t_1, y(t_1)) \end{aligned}$$

Algorithm:

1. Set $y_0 = y_I$
2. For $i = 1, 2, \ldots, N$:
$$y_i = y_{i-1} + hf(t_{i-1}, y_{i-1})$$
3. Then $y(t_f) \approx y(t_N) = y_N$

Example: Let $y(t)$ be the solution to $y' = t^2 + y^2$, $y(0) = 1$.
Approximate $y(0.2)$ with step size $h = 0.1$ using the Euler method.
Solution:

$$t_0 = 0, \quad t_1 = 0.1, \quad t_2 = 0.2$$

$$y_0 = 1$$

$$y_1 = y_0 + hf(t_0, y_0) = 1 + 0.1(0^2 + 1^2) = 1.1$$

$$y_2 = y_1 + hf(t_1, y_1) = 1.1 + 0.1((0.1)^2 + 1.1^2) = 1.222$$

Therefore, $y(0.2) \approx 1.222$.
Note: In textbooks, this is often referred to as the explicit Euler method. We will not cover the implicit Euler method in this course.
The Euler Method approximation is derived from a Taylor series expansion:

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t) + \cdots$$

The terms $\frac{h^2}{2}y''(t) + \cdots$ are of order $O(h^2)$, which means they approach zero faster than h as $h \to 0$.
Error Analysis:

- $O(h^2)$ represents the local error, i.e., the error at each step.
- The global error (total error) is the sum of the local errors: $N \cdot O(h^2)$.
- Since $N = \frac{T_f - T_I}{h} = \frac{\text{constant}}{h}$, we have:

$$\frac{\text{constant}}{h} \cdot O(h^2) = O(h)$$

Upshot: "Error is $O(h)$" implies that if we scale h by a constant c, the error should also scale by c.

Higher-order Taylor Series Approximations

Example: Order 2

The second-order Taylor series approximation for $y(t+h)$ is:

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t) + O(h^3)$$

If $y(t)$ is the solution to $y' = f(t,y)$, then:

$$\begin{aligned} y'(t) &= f(t,y) \\ y''(t) &= \frac{d}{dt}f(t,y(t)) = \frac{\partial f}{\partial t}(t,y(t)) + \frac{\partial f}{\partial y}(t,y(t)) \cdot y'(t) \end{aligned}$$

Substituting these into the Taylor series approximation:

$$y(t+h) \approx y(t) + hf(t,y(t)) + \frac{h^2}{2}\left(\frac{\partial f}{\partial t}(t,y(t)) + \frac{\partial f}{\partial y}(t,y(t)) \cdot f(t,y(t))\right)$$

This method would have error $O(h^2)$

Lecture 6, Thursday 9/12/2024

I.7 (continued): Integral Approximations

Recall: For a function $f(t)$, we have:

$$f(t+h) - f(t) = \int_t^{t+h} f'(x)\,dx$$

Therefore,

$$f(t+h) = f(t) + \int_t^{t+h} f'(x)\,dx$$

We can approximate this integral in two ways:

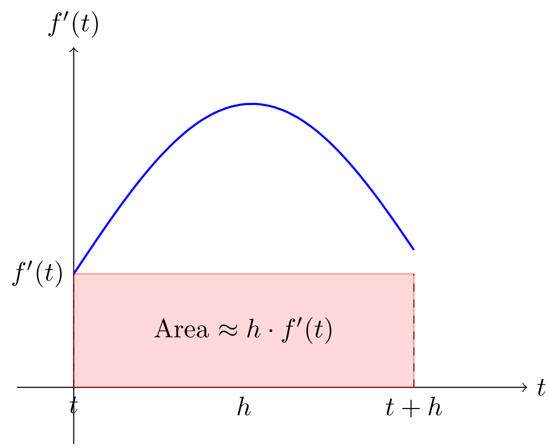


Figure 18: Left sum approximation

1. Left Sum (Euler Method)

Approximating the integral using a rectangle:

- Area of rectangle: $h \cdot f'(t)$
- Approximation: $f(t+h) \approx f(t) + hf'(t)$

This approximation forms the basis of the Euler method.

2. Trapezoid Method

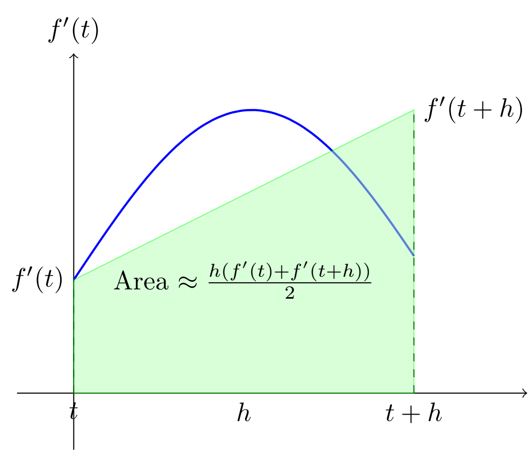


Figure 19: Trapezoid approximation

For a trapezoid with bases h_1 and h_2 and width w:

$$\text{Area} = \frac{w(h_1 + h_2)}{2}$$

Applying this to our integral:

$$\int_t^{t+h} f'(x)\,dx \approx \frac{h(f'(t) + f'(t+h))}{2}$$

Therefore,

$$f(t+h) \approx f(t) + \frac{h(f'(t) + f'(t+h))}{2}$$

$$f(t+h) \approx f(t) + h(f'(t) + f'(t+h))/2$$

Now, let $Y(t)$ be the solution to the differential equation:

$$y' = f(t, y), \quad y(t_I) = y_i$$

We want to find $Y(t_F)$. Choose N steps, with step size $h = \frac{t_F - t_I}{N}$.

Define $t_k = t_I + kh$ for $k = 0, 1, \ldots, N$.

Initialize $y_0 = y_i$.

The trapezoid method suggests:

$$y_{k+1} = y_k + \frac{h}{2}[f(t_k, y_k) + f(t_{k+1}, y_{k+1})]$$

However, we can't use y_{k+1} to compute y_{k+1}. Instead, we replace y_{k+1} in $f(t_{k+1}, y_{k+1})$ with an Euler step approximation:

$$y_{k+1} \approx y_k + hf(t_k, y_k)$$

Therefore, we set:

$$y_{k+1} = y_k + \frac{h}{2}[f(t_k, y_k) + f(t_{k+1}, y_k + hf(t_k, y_k))]$$

It turns out that the global error of this method is $O(h^2)$.

This is called range-trapezoid method.

Example: Trapezoid Method

Consider the differential equation:

$$y' = t^2 + y^2, \quad y(0) = 1$$

Let's estimate $y(0.2)$ using the trapezoid method with $h = 0.1$.

Solution: We have $f(t, y) = t^2 + y^2$, $h = 0.1$, $t_0 = 0$, $t_1 = 0.1$, and $t_2 = 0.2$.

$$
\begin{aligned}
y_1 &= y_0 + \frac{h}{2}[f(t_0, y_0) + f(t_1, y_0 + hf(t_0, y_0))] \\
&= 1 + \frac{0.1}{2}[(0^2 + 1^2) + (0.1^2 + (1 + 0.1 \cdot 1^2)^2)] \\
&= 1 + 0.1 \cdot \frac{1 + 0.01 + 1.21}{2} \\
&= 1 + 0.1 \cdot 1.11 \\
&= 1.111
\end{aligned}
$$

$$
\begin{aligned}
y_2 &= y_1 + \frac{h}{2}[f(t_1, y_1) + f(t_2, y_1 + hf(t_1, y_1))] \\
&= 1.111 + \frac{0.1}{2}[(0.1^2 + 1.111^2) + (0.2^2 + (1.111 + 0.1 \cdot 1.111^2)^2)] \\
&= 1.111 + 0.1 \cdot \frac{0.010201 + 1.234543}{2} \\
&= 1.111 + 0.1 \cdot 1.122372 \\
&\approx 1.248
\end{aligned}
$$

Range-Midpoint Method

For the range-midpoint method, we use the formula:

$$y_{k+1} = y_k + hf\left(t_k + \frac{h}{2}, y_k + \frac{h}{2}f(t_k, y_k)\right)$$

The global error for this method is $O(h^2)$.
Note: See textbook page 13 for an example.

Exact ODEs & Integrating Factors

Consider a first-order ODE of the form $\frac{dy}{dx} = f(x, y)$.
Question: When do we have an implicit solution $H(x, y) = c$?
Differentiating both sides with respect to x, remembering that $y = y(x)$:

$$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} \cdot \frac{dy}{dx} = 0$$

This leads to the general form of an exact ODE:

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

Alternatively written as:

$$M(x, y)dx + N(x, y)dy = 0$$

Question: When can we find a function $H(x,y)$ with $\frac{\partial H}{\partial x} = M$ and $\frac{\partial H}{\partial y} = N$?

If H has continuous second derivatives, then:

$$\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial^2 H}{\partial y \partial x}$$

Answer: If the domain of M and N has no "holes" and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then we can find a function $H(x,y)$ such that $\frac{\partial H}{\partial x} = M$ and $\frac{\partial H}{\partial y} = N$.

Such an ODE is called exact.

Example 1: Solve the IVP $(e^x y + 2x) + (2y + e^x)y' = 0$, $y(0) = 0$

Solution: We have $M = e^x y + 2x$, $N = 2y + e^x$. Let's check if the ODE is exact:

$$\frac{\partial M}{\partial y} = e^x$$
$$\frac{\partial N}{\partial x} = e^x$$

These are equal, so the ODE is exact.

We want to find $H(x,y)$ such that $\frac{\partial H}{\partial x} = M$ and $\frac{\partial H}{\partial y} = N$.

From $\frac{\partial H}{\partial x} = M$:

$$H(x,y) = \int (e^x y + 2x)\, dx = e^x y + x^2 + C(y)$$

From $\frac{\partial H}{\partial y} = N$:

$$e^x + C'(y) = 2y + e^x$$

Solving for $C(y)$:

$$C'(y) = 2y \implies C(y) = y^2$$

Therefore, $H(x,y) = e^x y + x^2 + y^2 = C$ is our general solution to the ODE.

Applying the initial condition $y(0) = 0$:

$$e^0 \cdot 0 + 0^2 + 0^2 = C \implies C = 0$$

So the solution to the IVP is $e^x y + x^2 + y^2 = 0$.

Alternative approach: Start with $\frac{\partial H}{\partial y} = 2y + e^x$

$$H(x,y) = y^2 + e^x y + C(x)$$

Then plug $\frac{\partial H}{\partial x} = e^x y + C'(x)$ into M and solve for $C(x)$.

Example 2: Solve $(3t^2y + 8ty^2)dt + (t^3 + 8t^2y + 12t^2y^2)dy = 0$

Solution: We have $M = 3t^2y + 8ty^2$, $N = t^3 + 8t^2y + 12t^2y^2$

Let's check if the ODE is exact:

$$\frac{\partial M}{\partial y} = 3t^2 + 16ty$$
$$\frac{\partial N}{\partial t} = 3t^2 + 16ty$$

These are equal, so the ODE is exact.

We're looking for $H(t, y)$ such that $\frac{\partial H}{\partial t} = M$ and $\frac{\partial H}{\partial y} = N$.

From $\frac{\partial H}{\partial y} = N$:

$$H(t, y) = t^3y + 4t^2y^2 + C(t)$$

Now, let's use $\frac{\partial H}{\partial t} = M$:

$$\frac{\partial H}{\partial t} = 3t^2y + 8ty^2 + C'(t)$$
$$3t^2y + 8ty^2 + C'(t) = 3t^2y + 8ty^2$$

Comparing the two sides, we see that $C'(t) = 0$, which means $C(t) = K$, where K is a constant.

Therefore, the general solution is:

$$H(t, y) = t^3y + 4t^2y^2 + 4y^3 = C$$

This is the implicit form of the general solution to the given ODE.