Differential Equations - MATH246

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Fall 2024

Class Information

Grading

- Matlab assignments — 18% (6% each)
- Quizzes (drop two lowest) — 17%
- Two best in-class exams — 17% each
- Worst in-class exam — 8%
- Final exam — 23%

Office Hours

- Monday: 2:00 PM - 3:00 PM (in person, Kirwin 2400)
- Tuesday: 1:15 PM - 2:30 PM (in person, Kirwin 2400)
- TBA: Zoom (online)

Exams

- 3 midterms and a final exam

Day 1, Tuesday 8/27/2024

Course Overview: (Differential Equations)

Chapter 0:

A differential equation is an algebraic relation between functions, their derivatives, and independent variables.

Examples:

- $\left(\frac{dx}{dt}\right)^2 + x\sin(t) = \cos(x)$ *(Order = 1)*
- $y'' + ty' + y = \cos(t)$ (Note: $y' = \frac{dy}{dt}$)*(Order = 2)*
- $\frac{dy}{dt} \cdot \frac{dy}{ds} + y\frac{dz}{dt} = \sin(st)$ *(Order = 1)*

Order: The order of a differential equation is the order of the highest derivative that appears.

Notation: For $\frac{dy}{dx}$, we can write y' or $\dot{y}$ (dot notation).

An ordinary differential equation (ODE) involves no partial derivatives, as opposed to a partial differential equation (PDE).

Note: This course only deals with ODEs.

Linearity of ODEs

An ODE with function y and independent variable t is **linear** if it can be written as:

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = f(t)$$

where $y^{(n)}$ is the nth derivative of y.

Examples:

- $\left(\frac{dx}{dt}\right)^2 + x\sin(t) = \cos(x)$ *(Not linear: $\left(\frac{dx}{dt}\right)^2$ and $\cos(x)$)*
- $y'' + ty' + y = \cos(t)$ *(Linear)*
- $y^{(4)} + y^{(2)} = 2$ *(Linear)*

Systems of ODEs

A system of ODEs consists of multiple ordinary differential equations that are considered together:

$$\begin{cases} \text{ODE1} \\ \text{ODE2} \\ \vdots \\ \text{ODE}n \end{cases}$$

Chapter 1: Introduction

Section 1: First-Order ODEs

First-order ODEs can be complicated. We will focus on those that can be put into the standard form $\boxed{\dfrac{dy}{dt} = f(t, y)}$.

Example: Consider the equation $\frac{dw}{dz} = \frac{-z}{6w}$. This can be rewritten as:

$$\frac{dw}{dz} = \frac{-z}{6w}$$

A function $Y(t)$ is a solution to $y' = f(t, y)$ on the interval (a, b) if:

- $Y(t)$ and $Y'(t)$ exist on (a, b),
- $f(t, Y(t))$ exists on (a, b), and
- $Y'(t) = f(t, Y(t))$ on (a, b).

Example: Consider the equation $y'(t) = \frac{t}{y}$ with the solution $Y(t) = \sqrt{4 - t^2}$.

To check this, calculate:

$$Y'(t) = \frac{-t}{\sqrt{4 - t^2}}$$

$Y(t)$ is defined on the interval $[-2, 2]$, but $f(t, Y(t)) = \frac{t}{\sqrt{4-t^2}}$ is only defined for $(-2, 2)$, not at ± 2. Therefore, $Y(t)$ is a solution on $(-2, 2)$, not on $[-2, 2]$.

Explicit Equations

These are of the form $y' = f(t)$.

The general solution is:

$$y = \int f(t)\, dt = F(t) + C$$

where $F(t)$ is an antiderivative of $f(t)$ (i.e., $F'(t) = f(t)$) and C is a constant.

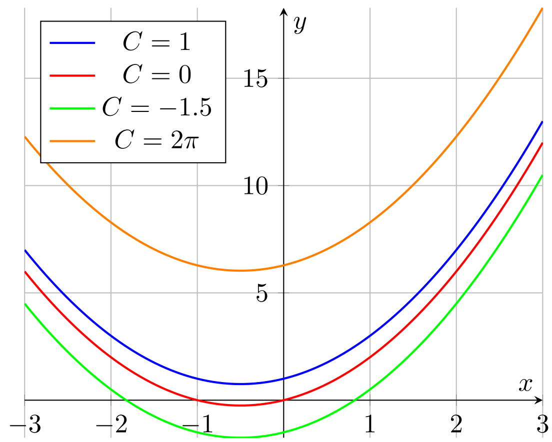
Example: Consider the ODE

$$\frac{dy}{dx} = 2x + 1$$

The general solution is:

$$y = x^2 + x + C$$

Graph for different values of C:



To select a specific solution from the general solution, we need an initial condition: $\boxed{y(t_I) = y_I}$.

The pair $y' = f(t)$ with $y(t_I) = y_I$ is called an Initial Value Problem (IVP).

Example: Solve the IVP

$$\frac{dy}{dx} = 2x + 1 \quad \text{with} \quad y(0) = 2$$

Solution:

Start with the general solution:

$$y = x^2 + x + C$$

Using the initial condition $y(0) = 2$:

$$2 = 0^2 + 0 + C \quad \Rightarrow \quad C = 2$$

Thus, the specific solution is:

$$y = x^2 + x + 2$$

Interval of Definition/Existence

The interval of definition/existence of a solution to an IVP is the **largest** interval (a, b) where:

- $t_I \in (a, b)$
- $f(t)$ is continuous on (a, b)

Chapter 2: Linear Equations

These look like:

$p(t)y' + q(t)y = r(t)$ where $p(t) \neq 0$ for the values of t we are considering.

In standard form:

$$y' = -\frac{q(t)}{p(t)}y + \frac{r(t)}{p(t)}$$

Let:

$$a(t) = \frac{q(t)}{p(t)}, \quad f(t) = \frac{r(t)}{p(t)}$$

We write it as:

$$y' + a(t)y = f(t)$$

Here, $f(t)$ is called the forcing function.
If $f(t) = 0$, the ODE is called homogeneous; otherwise, it is non-homogeneous.

Recipe for Solving First-Order Linear ODEs

Given:

$$y' + a(t)y = f(t)$$

1. Choose an antiderivative $A(t)$ of $a(t)$. 2. Multiply both sides by $e^{A(t)}$:

$$e^{A(t)}y' + a(t)e^{A(t)}y = f(t)e^{A(t)}$$

Let:

$$f(t)e^{A(t)} = g(t)$$

This simplifies to:

$$\frac{d}{dt}\left(e^{A(t)}y\right) = g(t)$$

3. Integrate both sides:

$$e^{A(t)}y = G(t) + C \quad \Rightarrow \quad y = e^{-A(t)}G(t) + Ce^{-A(t)}$$

This is the general solution.
Example: Solve the ODE

$$\frac{dy}{dt} = -y$$

1. Rewrite as $y' + y = 0$. 2. Here, $a(t) = 1$, so choose $A(t) = t$. 3. Multiply both sides by e^t:

$$e^t y' + e^t y = 0 \quad \Rightarrow \quad \frac{d}{dt}(e^t y) = 0$$

4. Integrate:

$$e^t y = C \quad \Rightarrow \quad y = Ce^{-t}$$

This is the general solution.

Example: Consider the ODE

$$y' = -y + e^t$$

1. Rewrite as $y' + y = e^t$. 2. Here, $a(t) = 1$, so choose $A(t) = t$. 3. Multiply both sides by e^t:

$$e^t y' + e^t y = e^{2t} \quad \Rightarrow \quad \frac{d}{dt}(e^t y) = e^{2t}$$

4. Integrate:

$$e^t y = \frac{1}{2}e^{2t} + C \quad \Rightarrow \quad y = \frac{1}{2}e^t + Ce^{-t}$$

This is the general solution.

Example: Solve the IVP

$$\frac{dx}{dt} + \cos(t)x = \cos(t) \quad \text{with} \quad x\left(\frac{\pi}{2}\right) = 0$$

Solution:

1. Here, $a(t) = \cos(t)$, so choose $A(t) = \sin(t)$. 2. Multiply both sides by $e^{\sin(t)}$:

$$e^{\sin(t)}x' + \cos(t)e^{\sin(t)}x = \cos(t)e^{\sin(t)}$$

This simplifies to:

$$\frac{d}{dt}\left(e^{\sin(t)}x\right) = \cos(t)e^{\sin(t)}$$

3. Integrate:

$$e^{\sin(t)}x = \int \cos(t)e^{\sin(t)}\,dt = e^{\sin(t)} + C$$

Thus,

$$x = 1 + Ce^{-\sin(t)}$$

4. Apply the initial condition $x\left(\frac{\pi}{2}\right) = 0$:

$$0 = 1 + Ce^{-1} \quad \Rightarrow \quad C = -e$$

Thus, the specific solution is:

$$x = 1 - e^{1-\sin(t)}$$

Day 2, 8/27/2024

I.2 (continued)

Problem Statement

Consider the initial value problem (IVP):

$$y' + a(t)y = f(t), \quad y(t_I) = y_I$$

Theorem: If $a(t)$ and $f(t)$ are continuous over the interval (a, b) and $t_I \in (a, b)$, then there is a unique solution to the IVP that is continuous on (a, b), and it's given by our method.

Example

Consider the differential equation:

$$z' + \cot(t)z = \frac{1}{\ln(t^2)}, \quad z(4) = 3$$

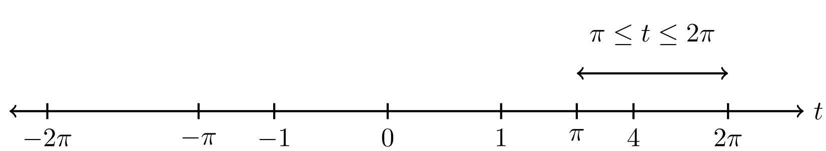
Find the largest interval on which we can guarantee a unique continuous solution to this IVP.

Solution

The function $\ln(t^2)$ is continuous on $(-\infty, 0)$ and $(0, \infty)$, but $\frac{1}{\ln(t^2)}$ is discontinuous at $t = 0$ and when $\ln(t^2) = 0$, i.e., $t = \pm 1$.

The function $\cot(t)$ has discontinuities at multiples of π.

The largest interval of continuity that includes $t = 4$ is $(\pi, 2\pi)$.



I.3: Separable Equation

A first-order ordinary differential equation (ODE) is **separable** if it can be written in the form:

$$y' = f(t)g(y)$$

Example:
Consider the differential equation:

$$y' = 2ty^2 + 3t^2y^2$$

We can factor this as:

$$y' = (2t + 3t^2)y^2$$

Here, we have:

$$f(t) = 2t + 3t^2, \quad g(y) = y^2$$

An ODE of the form $y' = g(y)$ is called **autonomous**.

A solution is called **stationary** if it is constant. If $y = C$ is a stationary solution, then:

$$y' = 0 \Rightarrow \boxed{0 = g(C)}$$

Example:
Consider the equation:

$$y' = 4y - y^3$$

To find the stationary solutions, set:

$$4y - y^3 = 0 \Rightarrow y(4 - y^2) = 0 \Rightarrow y(2 - y)(2 + y) = 0$$

Thus, the stationary solutions are:

$$y = 0, \quad y = 2, \quad y = -2$$

Non-Stationary Solutions

To find non-stationary solutions of the equation $y' = g(y)$, we proceed as follows:

$$y' = g(y) \quad \Rightarrow \quad \frac{1}{g(y)}y' = 1$$

Taking the integral on both sides:

$$\int \frac{1}{g(y)}y'\,dt = \int 1\,dt$$

This simplifies to:

$$\int \frac{1}{g(y)}\,dy = t + C$$

The result is an implicit equation for our solution.

Why can we divide by $g(y)$? $g(y) = 0$ corresponds to stationary solutions, and we are looking for non-stationary solutions, i.e., $g(y) \neq 0$.

Example: Find All Solutions to $y' = y^2$

Stationary Solutions: Set $y^2 = 0$, which implies $y = 0$.

Non-Stationary Solutions:

Starting with the equation:

$$\frac{1}{y^2}y' = 1$$

Integrate both sides:

$$\int \frac{1}{y^2}y'\,dt = \int 1\,dt$$

This simplifies to:

$$\int \frac{1}{y^2}\,dy = t + C$$

Evaluating the integral:

$$-\frac{1}{y} = t + C$$

We can find an explicit solution:

$$-y = \frac{1}{t+C} \quad \Rightarrow \quad y = -\frac{1}{t+C}$$

Each solution $y = -\frac{1}{t+C}$ actually represents two solutions, one defined on $(-\infty, -C)$ and the other on $(-C, \infty)$.

Note: Our solution is discontinuous even though all functions in the original equation $y' = y^2$ are continuous.

General Separable Equations

Consider the general separable equation:

$$y' = f(t)g(y)$$

If $g(c) = 0$, then $y = c$ is a stationary solution (so set g(y) = 0).

For non-stationary solutions:

$$\frac{1}{g(y)}y' = f(t)$$

Taking the integral on both sides:

$$\int \frac{1}{g(y)}y'\,dt = \int f(t)\,dt$$

This simplifies to:

$$\int \frac{1}{g(y)}\,dy = F(t) + C$$

Example: Find All Solutions to $\frac{dz}{dx} = \frac{3x+xz^2}{z+x^2z}$

First, rewrite the equation:

$$\frac{dz}{dx} = \frac{x}{1+x^2} \cdot \frac{3+z^2}{z}$$

Thus, we identify:

$$f(x) = \frac{x}{1+x^2}, \quad g(z) = \frac{3+z^2}{z}$$

Stationary Solutions: Set $g(z) = 0$:

$$\frac{3+z^2}{z} = 0 \quad \Rightarrow \quad 3+z^2 = 0$$

This equation has no real solution, so there are no stationary solutions.
Non-Stationary Solutions:
Start with:

$$\frac{1}{g(z)}\frac{dz}{dx} = f(x)$$

Which simplifies to:

$$\frac{z}{3+z^2} \cdot \frac{dz}{dx} = \frac{x}{1+x^2}$$

Integrate both sides:

$$\int \frac{z}{3+z^2}\frac{dz}{dx}\,dx = \int \frac{x}{1+x^2}\,dx$$

Use substitution:
- Let $u = 3+z^2$, then $du = 2z\,dz$. - Let $v = 1+x^2$, then $dv = 2x\,dx$.
The integrals become:

$$\int \frac{1}{2u} du = \int \frac{1}{2v} dv$$

This integrates to:

$$\frac{1}{2} \ln |u| = \frac{1}{2} \ln |v| + C$$

Substituting back u and v:

$$\frac{1}{2} \ln |3 + z^2| = \frac{1}{2} \ln |1 + x^2| + C$$

Initial Value Problems (IVPs)

Example: Solve the initial value problem:

$$y' = ty^2 - ty, \quad y(1) = 2$$

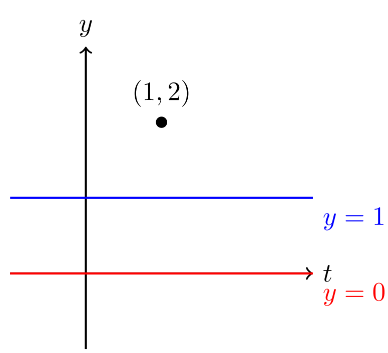
We can factor the equation as:

$$y' = t(y^2 - y)$$

Stationary Solutions: Set $y^2 - y = 0$:

$$y(y-1) = 0 \quad \Rightarrow \quad y = 0 \quad \text{or} \quad y = 1$$

Neither $y = 0$ nor $y = 1$ satisfies the initial condition $y(1) = 2$.



As shown in the graph, neither $y = 0$ nor $y = 1$ passes through the point $(1, 2)$.

Other Solutions: We solve the differential equation for non-stationary solutions:

$$\frac{1}{y^2 - y} \frac{dy}{dt} = t \quad \Rightarrow \quad \frac{1}{y^2 - y} dy = t\, dt$$

Integrate both sides:

$$\int \frac{1}{y^2 - y}\, dy = \int t\, dt$$

Using partial fractions:

$$\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1}$$

This leads to:

$$1 = A(y-1) + B(y)$$

By guessing $A = -1$ and $B = 1$, we get:

$$\int \left(-\frac{1}{y} + \frac{1}{y-1} \right) dy = \int t\, dt$$

Integrating both sides:

$$-\ln|y| + \ln|y-1| = \frac{t^2}{2} + C$$

Using the logarithm property $\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)$, this simplifies to:

$$\ln\left|\frac{y-1}{y}\right| = \frac{t^2}{2} + C$$

Applying the Initial Condition: Given $y(1) = 2$:

$$\ln\left|\frac{2-1}{2}\right| = \frac{1^2}{2} + C$$

$$\ln\left(\frac{1}{2}\right) = \frac{1}{2} + C \quad \Rightarrow \quad C = \ln\left(\frac{1}{2}\right) - \frac{1}{2}$$

Substituting C back into the equation:

$$\ln\left|\frac{y-1}{y}\right| = \frac{t^2}{2} + \ln\left(\frac{1}{2}\right) - \frac{1}{2}$$

Uniqueness and Existence Theorem

If $f(t)$ is continuous on (a, b) and $g(y)$ is continuous and differentiable on (c, d), then for every $t_I \in (a, b)$ and $y_I \in (c, d)$, there exists a unique continuous solution to the equation

$$y' = f(t)g(y)$$

with the initial condition $y(t_I) = y_I$, defined on some interval around t_I. The solution is determined by our method.

Example

Consider the differential equation:

$$\frac{dy}{dt} = 3y^{2/3}, \quad y(0) = 0$$

Stationary Solution: $y = 0$ is a stationary solution, and it solves our initial value problem (IVP).

However, $g(y) = 3y^{2/3}$ is not differentiable at $y = 0$, so we might have other solutions.

Finding Other Solutions:

$$\frac{1}{3y^{2/3}} \frac{dy}{dt} = 1$$

Integrating both sides:

$$\int \frac{1}{3y^{2/3}}\, dy = \int 1\, dt$$

This simplifies to:

$$y^{1/3} = t + C$$

Raising both sides to the power of 3:

$$y = (t + C)^3$$

Applying the Initial Condition: For $y(0) = 0$, we get $C = 0$, so:

$$y = t^3$$

Thus, $y = t^3$ also solves our IVP.