Differential Equations - MATH246

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Conway - Fall 2024

Class Information

Grading

- Matlab assignments — 18% (6% each)
- Quizzes (drop two lowest) — 17%
- Two best in-class exams — 17% each
- Worst in-class exam — 8%
- Final exam — 23%

Office Hours

- Monday: 2:00 PM - 3:00 PM (in person, Kirwin 2400)
- Tuesday: 1:15 PM - 2:30 PM (in person, Kirwin 2400)
- TBA: Zoom (online)

Exams

- 3 midterms and a final exam

Lecture 1, Tuesday 8/27/2024

Course Overview: (Differential Equations)

Chapter 0:

A differential equation is an algebraic relation between functions, their derivatives, and independent variables.

Examples:

- $\left(\frac{dx}{dt}\right)^2 + x\sin(t) = \cos(x)$ *(Order = 1)*
- $y'' + ty' + y = \cos(t)$ (Note: $y' = \frac{dy}{dt}$)*(Order = 2)*
- $\frac{dy}{dt} \cdot \frac{dy}{ds} + y\frac{dz}{dt} = \sin(st)$ *(Order = 1)*

Order: The order of a differential equation is the order of the highest derivative that appears.

Notation: For $\frac{dy}{dx}$, we can write y' or $\dot{y}$ (dot notation).

An ordinary differential equation (ODE) involves no partial derivatives, as opposed to a partial differential equation (PDE).

Note: This course only deals with ODEs.

Linearity of ODEs

An ODE with function y and independent variable t is **linear** if it can be written as:

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = f(t)$$

where $y^{(n)}$ is the nth derivative of y.

Examples:

- $\left(\frac{dx}{dt}\right)^2 + x\sin(t) = \cos(x)$ *(Not linear: $\left(\frac{dx}{dt}\right)^2$ and $\cos(x)$)*
- $y'' + ty' + y = \cos(t)$ *(Linear)*
- $y^{(4)} + y^{(2)} = 2$ *(Linear)*

Systems of ODEs

A system of ODEs consists of multiple ordinary differential equations that are considered together:

$$\begin{cases} \text{ODE1} \\ \text{ODE2} \\ \vdots \\ \text{ODE}n \end{cases}$$

Chapter 1: Introduction

Section 1: First-Order ODEs

First-order ODEs can be complicated. We will focus on those that can be put into the standard form $\boxed{\dfrac{dy}{dt} = f(t, y)}$.

Example: Consider the equation $\frac{dw}{dz} = \frac{-z}{6w}$. This can be rewritten as:

$$\frac{dw}{dz} = \frac{-z}{6w}$$

A function $Y(t)$ is a solution to $y' = f(t, y)$ on the interval (a, b) if:

- $Y(t)$ and $Y'(t)$ exist on (a, b),
- $f(t, Y(t))$ exists on (a, b), and
- $Y'(t) = f(t, Y(t))$ on (a, b).

Example: Consider the equation $y'(t) = \frac{t}{y}$ with the solution $Y(t) = \sqrt{4 - t^2}$.
To check this, calculate:

$$Y'(t) = \frac{-t}{\sqrt{4 - t^2}}$$

$Y(t)$ is defined on the interval $[-2, 2]$, but $f(t, Y(t)) = \frac{t}{\sqrt{4-t^2}}$ is only defined for $(-2, 2)$, not at ± 2. Therefore, $Y(t)$ is a solution on $(-2, 2)$, not on $[-2, 2]$.

Explicit Equations

These are of the form $y' = f(t)$.
The general solution is:

$$y = \int f(t)\, dt = F(t) + C$$

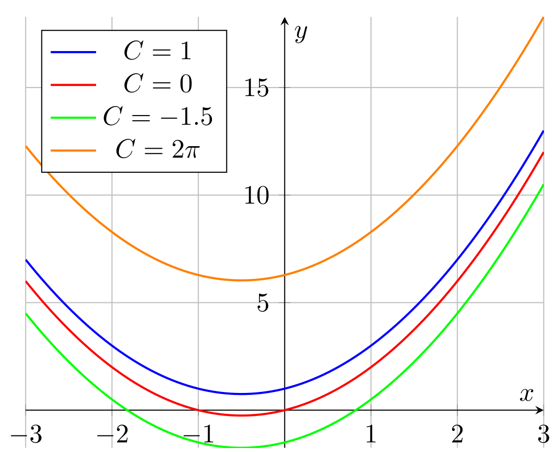
where $F(t)$ is an antiderivative of $f(t)$ (i.e., $F'(t) = f(t)$) and C is a constant.
Example: Consider the ODE

$$\frac{dy}{dx} = 2x + 1$$

The general solution is:

$$y = x^2 + x + C$$

Graph for different values of C:



To select a specific solution from the general solution, we need an initial condition: $\boxed{y(t_I) = y_I}$.
The pair $y' = f(t)$ with $y(t_I) = y_I$ is called an Initial Value Problem (IVP).
Example: Solve the IVP

$$\frac{dy}{dx} = 2x + 1 \quad \text{with} \quad y(0) = 2$$

Solution:
Start with the general solution:

$$y = x^2 + x + C$$

Using the initial condition $y(0) = 2$:

$$2 = 0^2 + 0 + C \quad \Rightarrow \quad C = 2$$

Thus, the specific solution is:

$$y = x^2 + x + 2$$

Interval of Definition/Existence

The interval of definition/existence of a solution to an IVP is the **largest** interval (a, b) where:

- $t_I \in (a, b)$
- $f(t)$ is continuous on (a, b)

Chapter 2: Linear Equations

These look like:

$$p(t)y' + q(t)y = r(t) \quad \text{where } p(t) \neq 0 \text{ for the values of } t \text{ we are considering.}$$

In standard form:

$$y' = -\frac{q(t)}{p(t)}y + \frac{r(t)}{p(t)}$$

Let:

$$a(t) = \frac{q(t)}{p(t)}, \quad f(t) = \frac{r(t)}{p(t)}$$

We write it as:

$$y' + a(t)y = f(t)$$

Here, $f(t)$ is called the forcing function.
If $f(t) = 0$, the ODE is called homogeneous; otherwise, it is non-homogeneous.

Recipe for Solving First-Order Linear ODEs

Given:

$$y' + a(t)y = f(t)$$

1. Choose an antiderivative $A(t)$ of $a(t)$. 2. Multiply both sides by $e^{A(t)}$:

$$e^{A(t)}y' + a(t)e^{A(t)}y = f(t)e^{A(t)}$$

Let:

$$f(t)e^{A(t)} = g(t)$$

This simplifies to:

$$\frac{d}{dt}\left(e^{A(t)}y\right) = g(t)$$

3. Integrate both sides:

$$e^{A(t)}y = G(t) + C \quad \Rightarrow \quad y = e^{-A(t)}G(t) + Ce^{-A(t)}$$

This is the general solution.
Example: Solve the ODE

$$\frac{dy}{dt} = -y$$

1. Rewrite as $y' + y = 0$. 2. Here, $a(t) = 1$, so choose $A(t) = t$. 3. Multiply both sides by e^t:

$$e^t y' + e^t y = 0 \quad \Rightarrow \quad \frac{d}{dt}(e^t y) = 0$$

4. Integrate:

$$e^t y = C \quad \Rightarrow \quad y = Ce^{-t}$$

This is the general solution.
Example: Consider the ODE

$$y' = -y + e^t$$

1. Rewrite as $y' + y = e^t$. 2. Here, $a(t) = 1$, so choose $A(t) = t$. 3. Multiply both sides by e^t:

$$e^t y' + e^t y = e^{2t} \quad \Rightarrow \quad \frac{d}{dt}(e^t y) = e^{2t}$$

4. Integrate:

$$e^t y = \frac{1}{2}e^{2t} + C \quad \Rightarrow \quad y = \frac{1}{2}e^t + Ce^{-t}$$

This is the general solution.
Example: Solve the IVP

$$\frac{dx}{dt} + \cos(t)x = \cos(t) \quad \text{with} \quad x\left(\frac{\pi}{2}\right) = 0$$

Solution:
1. Here, $a(t) = \cos(t)$, so choose $A(t) = \sin(t)$. 2. Multiply both sides by $e^{\sin(t)}$:

$$e^{\sin(t)}x' + \cos(t)e^{\sin(t)}x = \cos(t)e^{\sin(t)}$$

This simplifies to:

$$\frac{d}{dt}\left(e^{\sin(t)}x\right) = \cos(t)e^{\sin(t)}$$

3. Integrate:

$$e^{\sin(t)}x = \int \cos(t)e^{\sin(t)}\,dt = e^{\sin(t)} + C$$

Thus,

$$x = 1 + Ce^{-\sin(t)}$$

4. Apply the initial condition $x\left(\frac{\pi}{2}\right) = 0$:

$$0 = 1 + Ce^{-1} \quad \Rightarrow \quad C = -e$$

Thus, the specific solution is:

$$x = 1 - e^{1-\sin(t)}$$

Lecture 2, Thursady 8/29/2024

I.2 (continued)

Problem Statement

Consider the initial value problem (IVP):

$$y' + a(t)y = f(t), \quad y(t_I) = y_I$$

Theorem: If $a(t)$ and $f(t)$ are continuous over the interval (a, b) and $t_I \in (a, b)$, then there is a unique solution to the IVP that is continuous on (a, b), and it's given by our method.

Example

Consider the differential equation:

$$z' + \cot(t)z = \frac{1}{\ln(t^2)}, \quad z(4) = 3$$

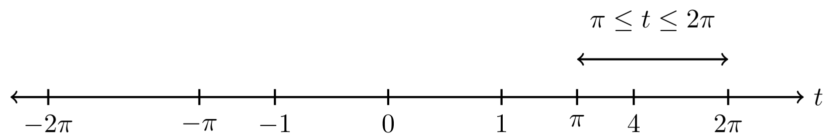
Find the largest interval on which we can guarantee a unique continuous solution to this IVP.

Solution

The function $\ln(t^2)$ is continuous on $(-\infty, 0)$ and $(0, \infty)$, but $\frac{1}{\ln(t^2)}$ is discontinuous at $t = 0$ and when $\ln(t^2) = 0$, i.e., $t = \pm 1$.

The function $\cot(t)$ has discontinuities at multiples of π.

The largest interval of continuity that includes $t = 4$ is $(\pi, 2\pi)$.



I.3: Separable Equation

A first-order ordinary differential equation (ODE) is **separable** if it can be written in the form:

$$y' = f(t)g(y)$$

Example:
Consider the differential equation:

$$y' = 2ty^2 + 3t^2y^2$$

We can factor this as:

$$y' = (2t + 3t^2)y^2$$

Here, we have:

$$f(t) = 2t + 3t^2, \quad g(y) = y^2$$

An ODE of the form $y' = g(y)$ is called **autonomous**.
A solution is called **stationary** if it is constant. If $y = C$ is a stationary solution, then:

$$y' = 0 \Rightarrow \boxed{0 = g(C)}$$

Example:
Consider the equation:

$$y' = 4y - y^3$$

To find the stationary solutions, set:

$$4y - y^3 = 0 \Rightarrow y(4 - y^2) = 0 \Rightarrow y(2 - y)(2 + y) = 0$$

Thus, the stationary solutions are:

$$y = 0, \quad y = 2, \quad y = -2$$

Non-Stationary Solutions

To find non-stationary solutions of the equation $y' = g(y)$, we proceed as follows:

$$y' = g(y) \quad \Rightarrow \quad \frac{1}{g(y)} y' = 1$$

Taking the integral on both sides:

$$\int \frac{1}{g(y)} y' \, dt = \int 1 \, dt$$

This simplifies to:

$$\int \frac{1}{g(y)} \, dy = t + C$$

The result is an implicit equation for our solution.

Why can we divide by $g(y)$? $g(y) = 0$ corresponds to stationary solutions, and we are looking for non-stationary solutions, i.e., $g(y) \neq 0$.

Example: Find All Solutions to $y' = y^2$

Stationary Solutions: Set $y^2 = 0$, which implies $y = 0$.

Non-Stationary Solutions:

Starting with the equation:

$$\frac{1}{y^2} y' = 1$$

Integrate both sides:

$$\int \frac{1}{y^2} y' \, dt = \int 1 \, dt$$

This simplifies to:

$$\int \frac{1}{y^2} \, dy = t + C$$

Evaluating the integral:

$$-\frac{1}{y} = t + C$$

We can find an explicit solution:

$$-y = \frac{1}{t + C} \quad \Rightarrow \quad y = -\frac{1}{t + C}$$

Each solution $y = -\frac{1}{t+C}$ actually represents two solutions, one defined on $(-\infty, -C)$ and the other on $(-C, \infty)$.

Note: Our solution is discontinuous even though all functions in the original equation $y' = y^2$ are continuous.

General Separable Equations

Consider the general separable equation:

$$y' = f(t)g(y)$$

If $g(c) = 0$, then $y = c$ is a stationary solution (so set g(y) = 0).
For non-stationary solutions:

$$\frac{1}{g(y)}y' = f(t)$$

Taking the integral on both sides:

$$\int \frac{1}{g(y)}y'\,dt = \int f(t)\,dt$$

This simplifies to:

$$\int \frac{1}{g(y)}\,dy = F(t) + C$$

Example: Find All Solutions to $\frac{dz}{dx} = \frac{3x+xz^2}{z+x^2z}$

First, rewrite the equation:

$$\frac{dz}{dx} = \frac{x}{1+x^2} \cdot \frac{3+z^2}{z}$$

Thus, we identify:

$$f(x) = \frac{x}{1+x^2}, \quad g(z) = \frac{3+z^2}{z}$$

Stationary Solutions: Set $g(z) = 0$:

$$\frac{3+z^2}{z} = 0 \quad \Rightarrow \quad 3 + z^2 = 0$$

This equation has no real solution, so there are no stationary solutions.
Non-Stationary Solutions:
Start with:

$$\frac{1}{g(z)}\frac{dz}{dx} = f(x)$$

Which simplifies to:

$$\frac{z}{3+z^2} \cdot \frac{dz}{dx} = \frac{x}{1+x^2}$$

Integrate both sides:

$$\int \frac{z}{3+z^2}\frac{dz}{dx}\,dx = \int \frac{x}{1+x^2}\,dx$$

Use substitution:
- Let $u = 3+z^2$, then $du = 2z\,dz$. - Let $v = 1+x^2$, then $dv = 2x\,dx$.
The integrals become:

$$\int \frac{1}{2u}du = \int \frac{1}{2v}dv$$

This integrates to:

$$\frac{1}{2}\ln|u| = \frac{1}{2}\ln|v| + C$$

Substituting back u and v:

$$\frac{1}{2}\ln|3+z^2| = \frac{1}{2}\ln|1+x^2| + C$$

Initial Value Problems (IVPs)

Example: Solve the initial value problem:

$$y' = ty^2 - ty, \quad y(1) = 2$$

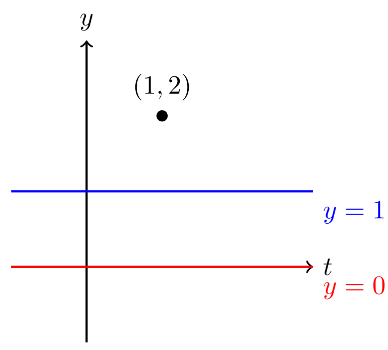
We can factor the equation as:

$$y' = t(y^2 - y)$$

Stationary Solutions: Set $y^2 - y = 0$:

$$y(y-1) = 0 \quad \Rightarrow \quad y = 0 \quad \text{or} \quad y = 1$$

Neither $y = 0$ nor $y = 1$ satisfies the initial condition $y(1) = 2$.



As shown in the graph, neither $y = 0$ nor $y = 1$ passes through the point $(1, 2)$.
Other Solutions: We solve the differential equation for non-stationary solutions:

$$\frac{1}{y^2 - y}\frac{dy}{dt} = t \quad \Rightarrow \quad \frac{1}{y^2 - y}\,dy = t\,dt$$

Integrate both sides:

$$\int \frac{1}{y^2 - y}\,dy = \int t\,dt$$

Using partial fractions:

$$\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1}$$

This leads to:

$$1 = A(y-1) + B(y)$$

By guessing $A = -1$ and $B = 1$, we get:

$$\int \left(-\frac{1}{y} + \frac{1}{y-1}\right) dy = \int t\,dt$$

Integrating both sides:

$$-\ln|y| + \ln|y-1| = \frac{t^2}{2} + C$$

Using the logarithm property $\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)$, this simplifies to:

$$\ln\left|\frac{y-1}{y}\right| = \frac{t^2}{2} + C$$

Applying the Initial Condition: Given $y(1) = 2$:

$$\ln\left|\frac{2-1}{2}\right| = \frac{1^2}{2} + C$$

$$\ln\left(\frac{1}{2}\right) = \frac{1}{2} + C \quad \Rightarrow \quad C = \ln\left(\frac{1}{2}\right) - \frac{1}{2}$$

Substituting C back into the equation:

$$\ln\left|\frac{y-1}{y}\right| = \frac{t^2}{2} + \ln\left(\frac{1}{2}\right) - \frac{1}{2}$$

Uniqueness and Existence Theorem

If $f(t)$ is continuous on (a, b) and $g(y)$ is continuous and differentiable on (c, d), then for every $t_I \in (a, b)$ and $y_I \in (c, d)$, there exists a unique continuous solution to the equation

$$y' = f(t)g(y)$$

with the initial condition $y(t_I) = y_I$, defined on some interval around t_I. The solution is determined by our method.

Example

Consider the differential equation:

$$\frac{dy}{dt} = 3y^{2/3}, \quad y(0) = 0$$

Stationary Solution: $y = 0$ is a stationary solution, and it solves our initial value problem (IVP).

However, $g(y) = 3y^{2/3}$ is not differentiable at $y = 0$, so we might have other solutions.

Finding Other Solutions:

$$\frac{1}{3y^{2/3}} \frac{dy}{dt} = 1$$

Integrating both sides:

$$\int \frac{1}{3y^{2/3}} \, dy = \int 1 \, dt$$

This simplifies to:

$$y^{1/3} = t + C$$

Raising both sides to the power of 3:

$$y = (t + C)^3$$

Applying the Initial Condition: For $y(0) = 0$, we get $C = 0$, so:

$$y = t^3$$

Thus, $y = t^3$ also solves our IVP.

Lecture 3, Tuesday 9/3/2024

Quiz tomorrow: Up to Section I.3

I.4. Theory

Consider Initial Value Problems (IVPs) of the form:

$$y' = f(t, y), \quad y(t_i) = y_i$$

We say the problem is *well-posed* if:

1. There exists a solution
2. The solution is unique
3. The solution depends continuously on the initial conditions

Existence and Uniqueness

Consider a set S of points in the (t, y) plane.

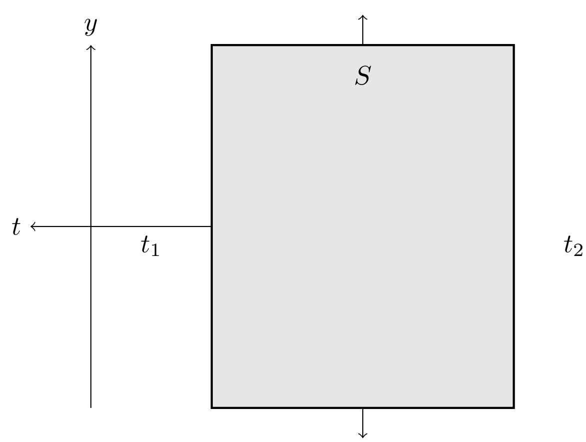


Figure 1: The set S in the (t, y) plane

If $f(t, y)$ is continuous on S and $\frac{\partial f}{\partial y}$ is continuous on S, then for any $(t_i, y_i) \in S$, there exists a unique continuous solution $Y(t)$ to the initial value problem:

$$\begin{cases} y' = f(t, y) \\ Y(t_i) = y_i \end{cases}$$

defined over some interval (a, b) containing t_i.

Moreover, the interval (a, b) can be extended as long as $(t, Y(t))$ remains inside S.

Example: Consider $y' = \frac{\sin(t+ty^2)}{1+t^2}$, $y(0) = 1$

Let's show that there is a unique solution defined on $(-1, 1)$.

Solution: For $f(t, y) = \frac{\sin(t+ty^2)}{1+t^2}$, f is continuous except at $t = \pm 1$.

$\frac{\partial f}{\partial y} = \frac{2ty\cos(t+ty^2)}{1+t^2}$, which is also continuous except at $t = \pm 1$.

By the Existence and Uniqueness Theorem, since f and $\frac{\partial f}{\partial y}$ are continuous on S, there exists a unique solution $y(t)$ to the initial value problem, defined on some interval containing $t = 0$. This interval can be extended as long as $(t, y(t))$ remains inside S, which in this case is the entire interval $(-1, 1)$.

Since $t_i = 0$, $y_i = 1 \implies (0, 1) \in S$, the theorem tells us we have a unique solution $Y(t)$ defined on a larger (a, b) such that $y(t)$ remains inside S.

Since any solution will not leave S as long as $-1 < t < 1$, we get $(a, b) = (-1, 1)$.

So, we choose S as follows:

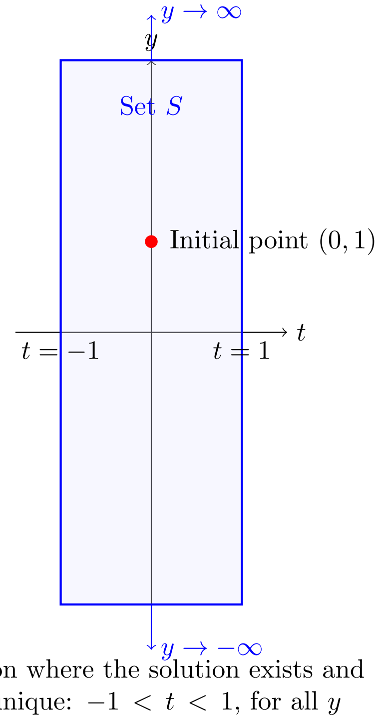


Figure 2: Set S where the solution to the differential equation is guaranteed to exist and be unique

Example: Consider the differential equation $y' = \frac{1}{t^2+y^2-1}$, with initial condition $y(0) = 0$. Find the set S where the solution is guaranteed to exist and be unique.

Solution: The function $f(t, y) = \frac{1}{t^2+y^2-1}$ is discontinuous when $t^2 + y^2 = 1$. This equation describes a circle with radius 1 centered at the origin.

The set S where the solution exists and is unique should be the interior of this circle, excluding the circle itself. We can represent this as:

$$S = \{(t, y) : t^2 + y^2 < 1\}$$

Let's visualize this set:

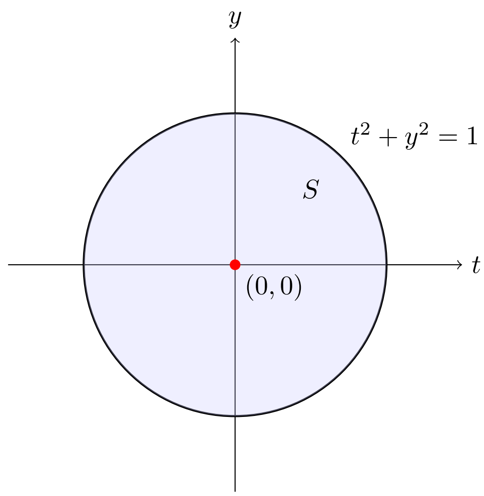


Figure 3: Set S for the given differential equation

The solution is guaranteed to exist and be unique within this circular region S, but not on or outside the boundary where $t^2 + y^2 = 1$.

Example: Consider the differential equation $y' = \frac{1}{t^2+y^2-1}$, with initial condition $y(0) = 3$. Let's visualize the set S where the solution is guaranteed to exist and be unique.

Solution: The function $f(t, y) = \frac{1}{t^2+y^2-1}$ is discontinuous when $t^2 + y^2 = 1$. This equation describes a circle with radius 1 centered at the origin.

The set S where the solution exists and is unique should be the region outside this circle, including the initial point $(0, 3)$. We can represent this as:

$$S = \{(t, y) : t^2 + y^2 > 1\}$$

Let's visualize this set:

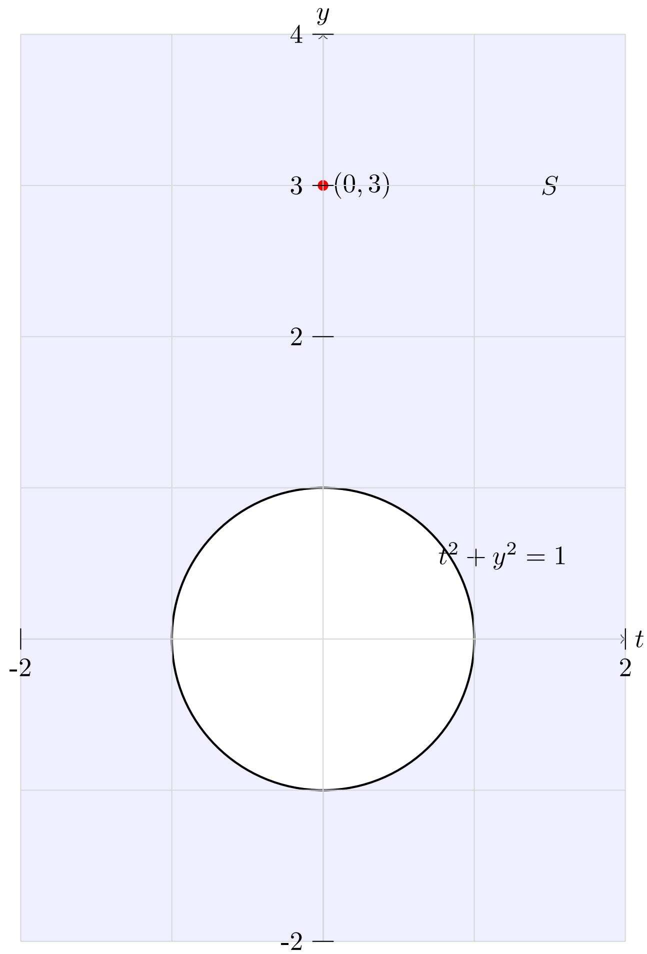


Figure 4: Set S for the given differential equation with $y(0) = 3$

The solution is guaranteed to exist and be unique within the region S, which is the area outside the circle $t^2 + y^2 = 1$, including the initial point $(0, 3)$.

1 I.5 - Graphical Methods

1.1 Phase Portraits for Autonomous Equations

Consider the autonomous differential equation:

$$\frac{dy}{dt} = g(y)$$

Our goal is to describe the qualitative behavior of solutions without explicitly solving the equation.

- When $g(y) = 0$:
  - $y' = 0$
  - We have a stationary solution
- When $g(y) > 0$:
  - $y' > 0$
  - The solution is increasing
- When $g(y) < 0$:
  - $y' < 0$
  - The solution is decreasing

Example: $y' = 4y - y^3$

Let's analyze the differential equation $y' = 4y - y^3$.

1. First, we find the stationary solutions:

$$4y - y^3 = y(4 - y^2) = y(2 - y)(2 + y) = 0$$

   Thus, the stationary solutions are $y = 0, \pm 2$.

2. Next, we determine the sign of $g(y) = 4y - y^3$ between these zeros:

$$\begin{aligned}
g(1) &= 4(1) - (1)^3 = 3 > 0 \\
g(3) &= 4(3) - (3)^3 = 12 - 27 = -15 < 0 \\
g(-3) &= 4(-3) - (-3)^3 = -12 + 27 = 15 > 0 \\
g(-1) &= 4(-1) - (-1)^3 = -4 + 1 = -3 < 0
\end{aligned}$$

Based on this analysis, we can create a phase portrait:
The phase portrait shows:

- Solutions increase when $y \in (-\infty, -2) \cup (0, 2)$
- Solutions decrease when $y \in (-2, 0) \cup (2, \infty)$

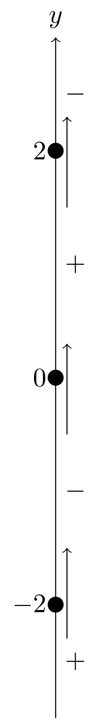


Figure 5: Phase portrait for $y' = 4y - y^3$

- Stationary solutions at $y = 0, \pm 2$

This diagram is called a phase portrait or phase line. It provides valuable information about the behavior of solutions $y(t)$ for different initial conditions:

- For $y(t)$ starting in $(-\infty, -2)$:
  - $y(t)$ increases as t increases
  - $y(t) \to -2$ as $t \to \infty$ (asymptotically approaching -2)
- For $y(t)$ starting in $(-2, 0)$:
  - $y(t)$ is decreasing
  - $y(t) \to -2$ as $t \to \infty$
- For $y(t)$ starting in $(0, 2)$:
  - $y(t)$ is increasing
  - $y(t) \to 2$ as $t \to \infty$ (asymptotically approaching 2)
- For $y(t)$ starting in $(2, \infty)$:
  - $y(t)$ is decreasing

– $y(t) \to 2$ as $t \to \infty$ (asymptotically approaching 2)

Sketch solutions:

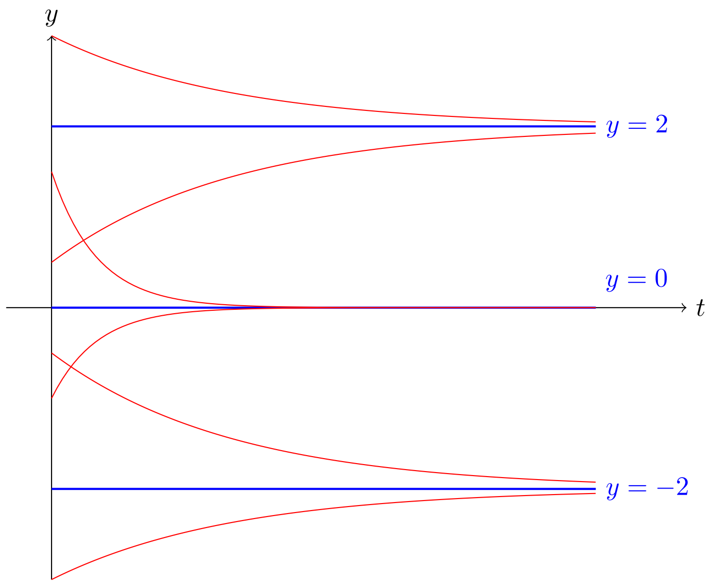


Figure 6: Sketch of solutions for $y' = 4y - y^3$

This figure illustrates:

- Constant solutions at $y = 2$, $y = 0$, and $y = -2$ (blue lines)
- Solutions approaching $y = 2$ from above and below
- Solutions approaching $y = -2$ from above and below
- Solutions approaching $y = 0$ from above and below

The red curves represent various solutions to the differential equation, showing how they behave over time depending on their initial conditions.

We classify stationary solutions as follows:

- **Stable or attracting:** All nearby solutions move towards it as $t \to \infty$.
  (In this case: $y = \pm 2$)
- **Unstable or repelling:** All nearby solutions move away from it as $t \to \infty$.
  (In this case: $y = 0$)
- **Semi-stable:** Some solutions move towards it and some move away from it as $t \to \infty$.

Example: Consider the differential equation $y' = y^2$

- Stationary solution: $y = 0$

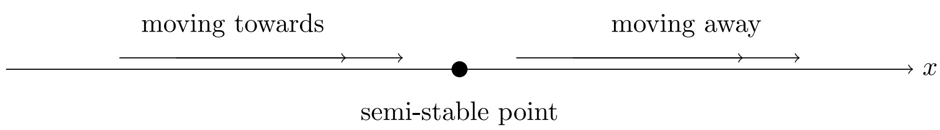


Figure 7: Behavior near a semi-stable point

- For $y \neq 0$: $y^2 > 0$, implying solutions move away from $y = 0$

This example demonstrates an unstable stationary solution at $y = 0$.

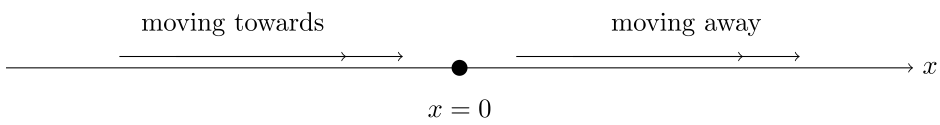


Figure 8: Behavior on x-axis for $y' = y^2$

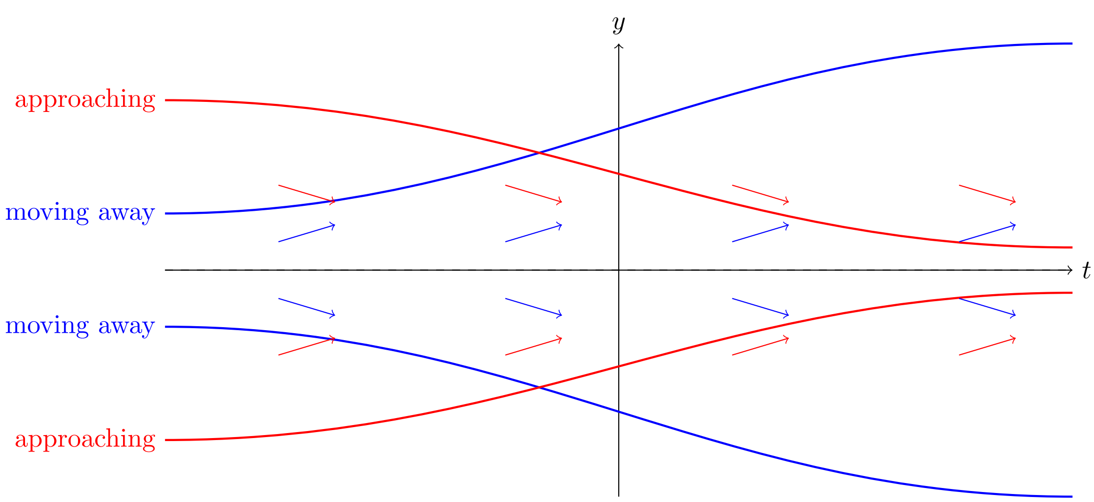


Figure 9: Behavior of solutions in t-y plane for $y' = y^2$

This figure illustrates the behavior of solutions to $y' = y^2$ near the stationary solution $y = 0$. The blue curves represent solutions moving away from $y = 0$, while the red curves represent solutions approaching $y = 0$. This demonstrates that $y = 0$ is an unstable or repelling stationary solution.

For stability think of a pendulum

This figure illustrates a pendulum moving from right to left. The top position (vertically upright) represents an unstable equilibrium, while the bottom position represents a stable equilibrium. These correspond to the stationary solutions of the pendulum's differential equation.

unstable solution is pendulum at the top and stable is pendulum at the bottom

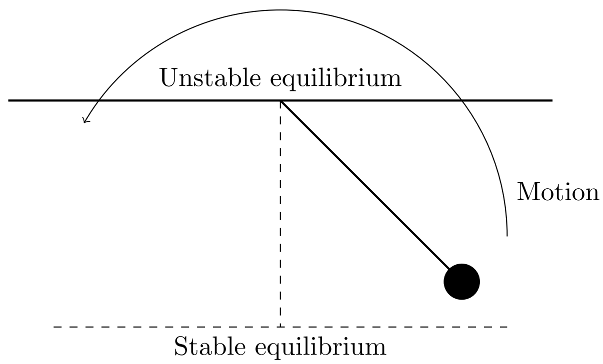


Figure 10: Pendulum illustrating stable and unstable equilibrium points

Example: Phase Line Analysis

Consider the differential equation:

$$y' = \frac{(y^2-1)(y-3)^2}{(y+3)^2}$$

Stationary Solutions:

- $y = -1$
- $y = 1$
- $y = 3$

Note: $y = -3$ is undefined in the equation.
We will now draw the phase line for this differential equation to analyze its behavior.

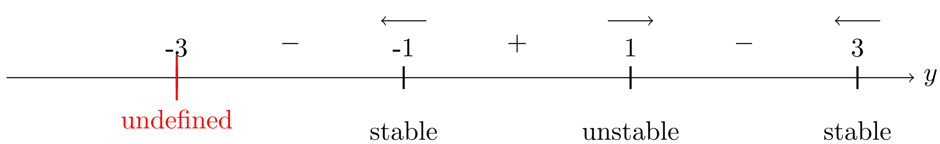


Figure 11: Phase line for $y' = (y^2-1)(y-3)^2/(y+3)^2$

Note: stability only applies to stationary solutions, not to undefined points.
Sketch
Example: Consider the differential equation $y' = t - y^2$
Procedure: To sketch the slope field:

1. Choose a representative selection of (t, y) points.
2. At each point (t, y), draw an arrow with slope $t - y^2$.
3. Connect the arrows to visualize solution curves.

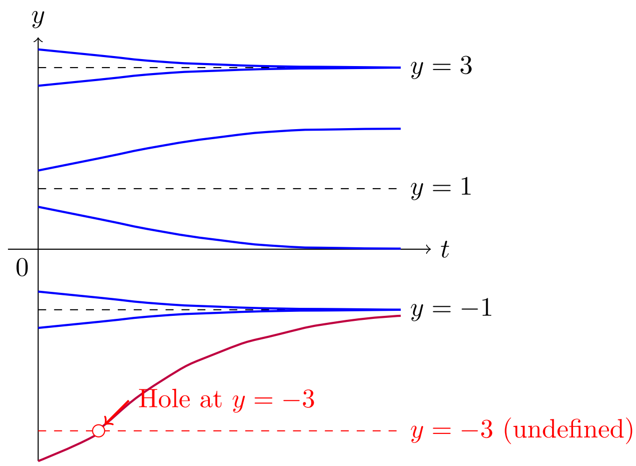


Figure 12: Sketch of solutions for $y' = (y^2-1)(y-3)^2/(y+3)^2$, including a solution crossing $y = -3$ with a clear hole

Sample calculations:

$$\begin{aligned}
\text{At } (0,0): &\quad t - y^2 = 0 \\
\text{At } (1,0): &\quad t - y^2 = 1 \\
\text{At } (-1,0): &\quad t - y^2 = -1 \\
\text{At } (0,\pm 1): &\quad t - y^2 = -1
\end{aligned}$$

Continue this process for other points to build a comprehensive slope field.

Slope field for $y' = t - y^2$

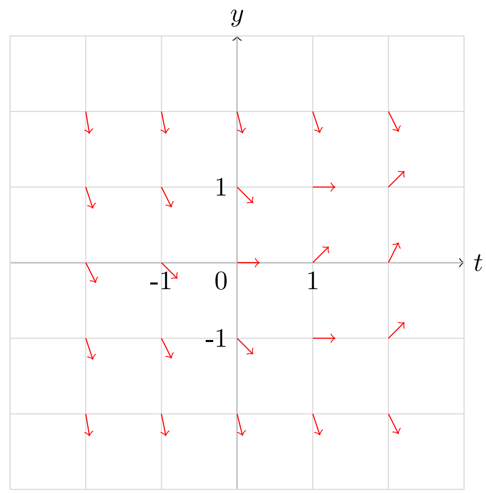


Figure 13: Slope field for the differential equation $y' = t - y^2$

If $y(0) = 1$, then the point should follow the slope field:

Slope field for $y' = t - y^2$

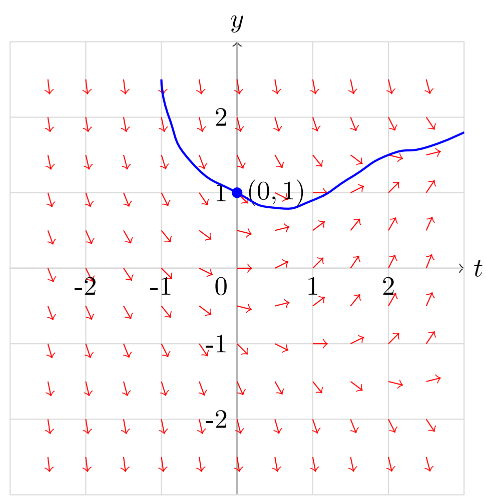


Figure 14: Slope field for $y' = t - y^2$ with initial point $y(0) = 1$ and solution curve

Lecture 4, Thursday 9/5/2024

2 I.6 Applications of Differential Equations

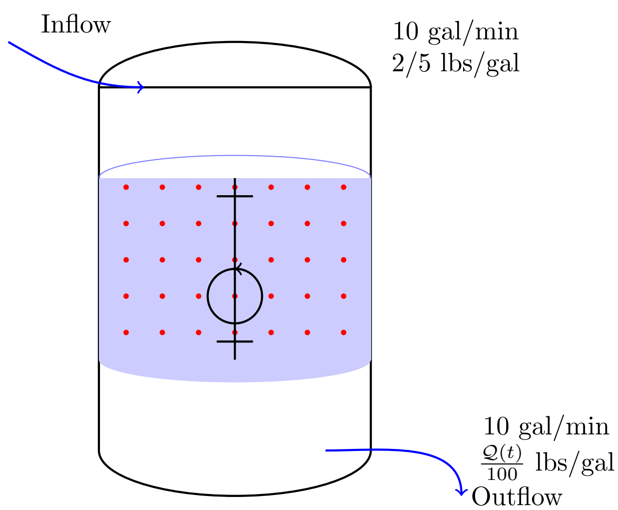
2.1 I.6.1 Tanks and Mixtures

2.1.1 I.6.1.1 IRS Method: Identify, Reduce, Solve

In tank and mixture problems, we often encounter scenarios where:

- Water flows into and out of a tank
- Salt is mixed into the water
- The concentration of salt varies over time
- The mixture in the tank is continuously leaving

Our primary interest is in understanding how the salt content changes over time.



Tank with well-mixed salt solution

Figure 15: Diagram of a tank with water inflow, outflow, and well-mixed salt solution

Example: Consider a tank with the following properties:

- Initial contents: 100 gallons of brine
- Initial salt content: 20 lbs dissolved in the brine
- Inflow: Brine containing 2/5 lbs/gal of salt at 10 gal/min
- Outflow: Well-mixed solution at 10 gal/min

Problem: Find a formula for the salt content after t minutes.
Solution: Let $\mathcal{Q}(t)$ be the quantity of salt at time t.

$$\text{Rate of salt inflow} = 10 \text{ gal/min} \cdot \frac{2}{5} \text{ lbs/gal} = 4 \text{ lbs/min}$$
$$\text{Rate of salt outflow} = 10 \text{ gal/min} \cdot \frac{\mathcal{Q}(t)}{100} \text{ lbs/gal} = \frac{\mathcal{Q}(t)}{10} \text{ lbs/min}$$

The rate of change of salt content is the difference between inflow and outflow:

$$\frac{d\mathcal{Q}}{dt} = \text{salt in} - \text{salt out} = 4 - \frac{\mathcal{Q}(t)}{10}$$

This is our differential equation. We can solve it using the integrating factor method or separation of variables.

The initial condition is:

$$\mathcal{Q}(0) = 20 \text{ lbs}$$

Therefore, our initial value problem (IVP) is:

$$\begin{cases} \mathcal{Q}' = 4 - \frac{\mathcal{Q}}{10} \\ \mathcal{Q}(0) = 20 \end{cases}$$

Solve the IVP: The equation is linear: $\mathcal{Q}' + \frac{1}{10}\mathcal{Q} = 4$

$$a(t) = \frac{1}{10} \Rightarrow A(t) = \frac{t}{10}$$
$$\text{Multiply by } e^{t/10} : e^{t/10}\mathcal{Q}' + \frac{1}{10}e^{t/10}\mathcal{Q} = 4e^{t/10}$$
$$\text{Integrate both sides} : e^{t/10}\mathcal{Q} = 40e^{t/10} + C$$
$$\Rightarrow \mathcal{Q} = 40 + Ce^{-t/10}$$

Use the initial condition to solve for C:

$$20 = 40 + C \Rightarrow C = -20$$

Therefore, the solution is:

$$\mathcal{Q}(t) = 40 - 20e^{-t/10}$$

Further Questions

Asymptotic Behavior

As $t \to \infty$, what happens to $\mathcal{Q}(t)$?

Answer:

$$\begin{aligned}\lim_{t\to\infty} \mathcal{Q}(t) &= \lim_{t\to\infty}(40 - 20e^{-t/10}) \\ &= 40 - 20\lim_{t\to\infty} e^{-t/10} \\ &= 40 - 20(0) = 40 \text{ lbs}\end{aligned}$$

Time to Reach a Specific Amount

After what time will there be more than 30 lbs of salt in the tank?

Answer: Set $\mathcal{Q}(t) = 30$ lbs and solve for t:

$$\begin{aligned}30 &= 40 - 20e^{-t/10} \\ -10 &= -20e^{-t/10} \\ \frac{1}{2} &= e^{-t/10} \\ \ln(\frac{1}{2}) &= -\frac{t}{10} \\ t &= 10\ln(2) \approx 6.93 \text{ minutes}\end{aligned}$$

General Case

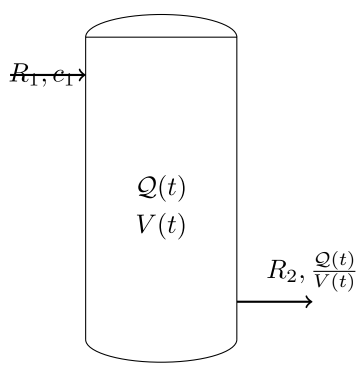


Figure 16: Cylinder with inflow and outflow

Parameters:

- R_1: Rate of inflow
- c_1: Concentration of inflow
- R_2: Rate of outflow
- $\frac{\mathcal{Q}(t)}{V(t)}$: Concentration of outflow

Initial Conditions:

$$\mathcal{Q}(0) = \mathcal{Q}_0$$
$$V(0) = V_0$$

Differential Equations:

$$V'(t) = r_1 - r_2 = v(t) = (r_1 - r_2)t + v_0$$
$$\mathcal{Q}'(t) = r_1 c_1 - r_2 \frac{\mathcal{Q}(t)}{V(t)} = r_1 c_1 - \frac{r_2 q(t)}{(r_1 - r_2)t + v0}$$

Salt Analysis:

- Salt in: $r_1 c_1$
- Salt out: $\frac{\mathcal{Q}(t)}{V(t)}$

Population Dynamics:
Let $P(t)$ be the population at time t. Consider the differential equation:

$$\frac{dP}{dt} = R(P)P - h(t)$$

where:

- $R(P)$: Growth rate
- $h(t)$: Harvest rate

Exponential Model: For the exponential model, we take $h(t) = 0$ and $R(P) = r$ (constant). The differential equation becomes:

$$\frac{dP}{dt} = rP$$

The general solution is:

$$P(t) = Ce^{rt}$$

With the initial condition $P(0) = P_0$, we get:

$$P(t) = P_0 e^{rt}$$

Example 1: Monkey Population Growth
Consider a population of monkeys with the following characteristics:

- Initial population: 100 monkeys
- Growth rate: 4% per year
- Immigration: 8 new monkeys join from surrounding tribes every year

The Initial Value Problem (IVP) describing this system is:

$$\frac{dM}{dt} = 0.04M + 8$$
$$M(0) = 100$$

where $M(t)$ represents the number of monkeys at time t.
Solution: $M(t) = 300e^{0.04t} - 200$

Example 2: Rabbit Population Growth
A population of rabbits doubles in size every year.
Question: What is the growth rate assuming exponential growth?
Solution: Let's assume the population follows the exponential growth model:

$$P(t) = P_0 e^{rt}$$

where:

- $P(t)$ is the population at time t
- P_0 is the initial population
- r is the growth rate
- t is time in years

Given:

$$P(0) = P_0$$
$$P(1) = 2P_0$$

Substituting into our model:

$$2P_0 = P_0 e^r$$

Simplifying:

$$2 = e^r$$

Taking the natural logarithm of both sides:

$$r = \ln(2) \approx 0.693$$

Therefore, the growth rate is approximately 69.3% per year.
Logistic Model: This model accounts for finite resources and competition.

$$\frac{dp}{dt} = (r - ap)p \tag{1}$$

where:

- $R(p) = r - ap$ is the per capita growth rate
- When p is small, $R(p) \approx r$
- As p grows, $R(p)$ decreases
- At $p = \frac{r}{a}$, $R(p) = 0$ (this is called the carrying capacity)

Stationary solutions: Set $(r - ap) = 0$

$$\Rightarrow p = 0 \text{ or } p = \frac{r}{a}$$

Phase-line portrait:

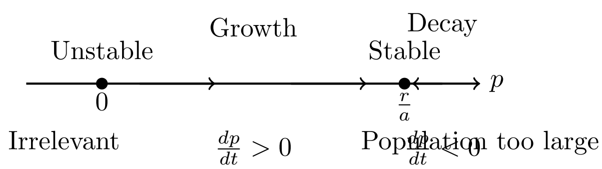
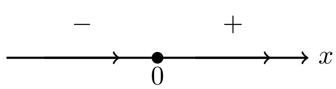
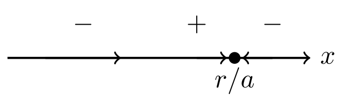


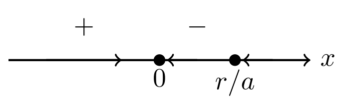
Figure 17: Phase-line portrait for the logistic model

Additional Phase Line Portraits

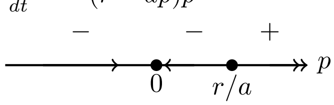
Here are three more examples of phase line portraits:

1. Simple linear model: 

2. Logistic model: 

3. Reversed logistic model: 

Variant logistic model:
$\frac{dp}{dt} = -(r - ap)p$



This phase line portrait for $\frac{dp}{dt} = -(r - ap)p$ shows:

- Two equilibrium points: at $p = 0$ and $p = r/a$
- For $p < 0$, $\frac{dp}{dt} < 0$, so p decreases (moves left)

- For $0 < p < r/a$, $\frac{dp}{dt} < 0$, so p decreases (moves left)
- For $p > r/a$, $\frac{dp}{dt} > 0$, so p increases (moves right)

This describes a context where a population needs a certtin critical size r/a to be able to grow otherwise it will die out.

Lecture 5, Tuesday 9/10/2024

I.6 Motion: Falling Objects (continued)

Consider an object with mass m and force F acting on it. Newton's Second Law of Motion states:

$$F = ma$$

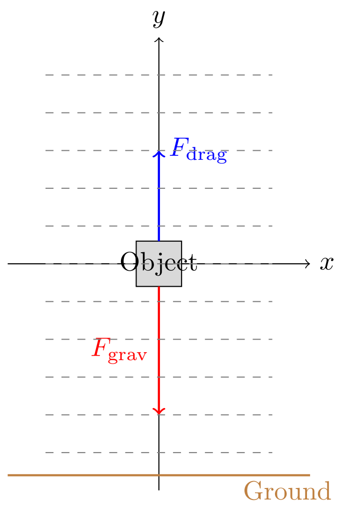
where a is the acceleration, defined as:

$$a = \frac{dv(t)}{dt}$$

and $v(t)$ is the velocity.

For our analysis, we will use the following coordinate system:

For falling objects, the total force is composed of gravity and drag:



Where:

$$F_{\text{grav}} = \text{force from gravity} = mg$$
$$F_{\text{drag}} = \text{drag force} = -cv^2$$

Here, $g = -9.8$ m/s^2, c is a constant, and v is velocity.
Therefore, our ODE model for a falling object is:

$$ma = m\frac{dv}{dt} = mg - cv^2$$

Let c be the drag constant.

Cancelling out m, we get:

$$v' = g + cv^2$$

This is a non-linear equation, but it is separable.

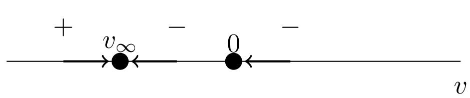
Stationary solutions are found by solving:

$$g + cv^2 = 0 \quad \Rightarrow \quad v = \pm\sqrt{-g/c}$$

For falling objects, we are interested in the negative velocity solution. Thus,

$$v_\infty = -\sqrt{-g/c}$$

is called the terminal velocity.



This horizontal phase line portrait shows:

- The equilibrium points at $v = 0$ and $v = v_\infty$
- Arrows indicating the direction of change
- Signs showing whether v' is positive or negative in each region

We can observe that:

- For $v > 0$, v' is negative, so v decreases towards 0
- For $v_\infty < v < 0$, v' is still negative, so v continues to decrease towards v_∞
- For $v < v_\infty$, v' becomes positive, so v increases back towards v_∞

This confirms that v_∞ is indeed the terminal velocity, as the system will eventually settle at this value regardless of the initial conditions.

A skydiver with mass $m = 60$ kg jumps from an airplane and assumes a position with drag coefficient $c = 0.002\ \mathrm{m}^{-1}$. Find her terminal velocity.

Solution:

$v(0) = 0, c = 0.002, g = -9.8$

$\Rightarrow v' = g + 0.002v^2$

terminal velocity $v_\infty = -\sqrt{\frac{-g}{c}}$

To find the terminal velocity, we use the formula derived earlier:

$$v_\infty = -\sqrt{\frac{-g}{c}}$$

Substituting the given values:

$$v_\infty = -\sqrt{\frac{9.8\ \mathrm{m/s}^2}{0.002\ \mathrm{m}^{-1}}} = -70.0\ \mathrm{m/s}$$

Therefore, the skydiver's terminal velocity is approximately 70.0 m/s (or 252 km/h) downward.

Now, let's solve the ODE analytically:

Rewrite the ODE:

$$\begin{aligned} v' &= g + cv^2 \\ &= c\left(\frac{g}{c} + v^2\right) \\ &= c(v^2 - v_\infty^2) \\ &= c(v - v_\infty)(v + v_\infty) \end{aligned}$$

Separate variables:

$$\frac{v'}{(v - v_\infty)(v + v_\infty)} = c$$

Integrate both sides:

$$\int \frac{v'}{(v - v_\infty)(v + v_\infty)}\, dv = \int c\, dt$$

Using partial fractions decomposition:

$$\frac{1}{2v_\infty}\left(\ln|v - v_\infty| - \ln|v + v_\infty|\right) = ct + C_0$$

Apply initial condition $v(0) = 0$:

$$C_0 = \frac{1}{2v_\infty}\left(\ln|-v_\infty| - \ln|v_\infty|\right) = 0$$

Therefore, the solution is:

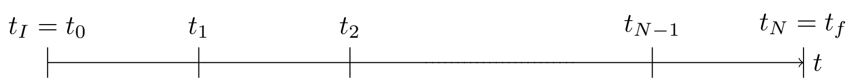
$$\ln\left|\frac{v - v_\infty}{v + v_\infty}\right| = 2cv_\infty t$$

2.2 I.7: Numerical Methods

Consider the initial value problem:

$$y' = f(t, y), \quad y(t_I) = y_I$$

Our goal is to approximate $y(t_f)$, where $t_f > t_I$.
To do this, we divide the interval $[t_I, t_f]$ into n equal steps:



We then use an iterative process to approximate the solution:

1. Use $y(t_0) = y_I$ to approximate $y(t_1)$
2. Use the approximation of $y(t_1)$ to approximate $y(t_2)$
3. $\vdots$
4. Use the approximation of $y(t_{N-1})$ to approximate $y(t_N)$

This process forms the basis of numerical methods for solving differential equations.
For good methods, the accuracy of the approximation increases as N increases.
We will use uniform step sizes:

$$\text{step size} = h = \frac{t_f - t_I}{N}$$

$$t_i = t_I + ih \quad \text{for } i = 0, 1, 2, \ldots, N$$

Euler method:
Idea: The derivative can be approximated by the difference quotient

$$f'(t) \approx \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}$$

For small h, we have the approximation:

$$f(t+h) \approx f(t) + hf'(t)$$

If $y(t)$ is the solution to $y' = f(t,y)$ with $y(t_i) = y_i$, then:

$$y(t_i + h) \approx y(t_i) + hy'(t_i) = y(t_i) + hf(t_i, y(t_i))$$

Thus,

$$y(t_1) \approx y(t_0) + hf(t_0, y(t_0))$$
$$y(t_2) \approx y(t_1) + hf(t_1, y(t_1))$$

Algorithm:

1. Set $y_0 = y_I$
2. For $i = 1, 2, \ldots, N$:
$$y_i = y_{i-1} + hf(t_{i-1}, y_{i-1})$$
3. Then $y(t_f) \approx y(t_N) = y_N$

Example: Let $y(t)$ be the solution to $y' = t^2 + y^2$, $y(0) = 1$.
Approximate $y(0.2)$ with step size $h = 0.1$ using the Euler method.
Solution:

$$t_0 = 0, \quad t_1 = 0.1, \quad t_2 = 0.2$$

$$y_0 = 1$$

$$y_1 = y_0 + hf(t_0, y_0) = 1 + 0.1(0^2 + 1^2) = 1.1$$

$$y_2 = y_1 + hf(t_1, y_1) = 1.1 + 0.1((0.1)^2 + 1.1^2) = 1.222$$

Therefore, $y(0.2) \approx 1.222$.

Note: In textbooks, this is often referred to as the explicit Euler method. We will not cover the implicit Euler method in this course.

The Euler Method approximation is derived from a Taylor series expansion:

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t) + \cdots$$

The terms $\frac{h^2}{2}y''(t) + \cdots$ are of order $O(h^2)$, which means they approach zero faster than h as $h \to 0$.

Error Analysis:

- $O(h^2)$ represents the local error, i.e., the error at each step.
- The global error (total error) is the sum of the local errors: $N \cdot O(h^2)$.
- Since $N = \frac{T_f - T_I}{h} = \frac{\text{constant}}{h}$, we have:
$$\frac{\text{constant}}{h} \cdot O(h^2) = O(h)$$

Upshot: "Error is $O(h)$" implies that if we scale h by a constant c, the error should also scale by c.

Higher-order Taylor Series Approximations

Example: Order 2

The second-order Taylor series approximation for $y(t+h)$ is:

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t) + O(h^3)$$

If $y(t)$ is the solution to $y' = f(t, y)$, then:

$$\begin{aligned}
y'(t) &= f(t, y) \\
y''(t) &= \frac{d}{dt}f(t, y(t)) = \frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t)) \cdot y'(t)
\end{aligned}$$

Substituting these into the Taylor series approximation:

$$y(t+h) \approx y(t) + hf(t, y(t)) + \frac{h^2}{2}\left(\frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t)) \cdot f(t, y(t))\right)$$

This method would have error $O(h^2)$

Lecture 6, Thursday 9/12/2024

I.7 (continued): Integral Approximations

Recall: For a function $f(t)$, we have:

$$f(t+h) - f(t) = \int_t^{t+h} f'(x)\,dx$$

Therefore,

$$f(t+h) = f(t) + \int_t^{t+h} f'(x)\,dx$$

We can approximate this integral in two ways:

1. Left Sum (Euler Method)

Approximating the integral using a rectangle:

- Area of rectangle: $h \cdot f'(t)$
- Approximation: $f(t+h) \approx f(t) + hf'(t)$

This approximation forms the basis of the Euler method.

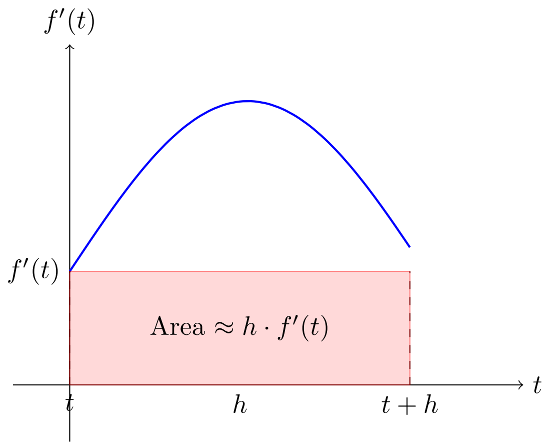


Figure 18: Left sum approximation

2. Trapezoid Method

For a trapezoid with bases h_1 and h_2 and width w:

$$\text{Area} = \frac{w(h_1 + h_2)}{2}$$

Applying this to our integral:

$$\int_t^{t+h} f'(x)\,dx \approx \frac{h(f'(t) + f'(t+h))}{2}$$

Therefore,

$$f(t+h) \approx f(t) + \frac{h(f'(t) + f'(t+h))}{2}$$

$$f(t+h) \approx f(t) + h(f'(t) + f'(t+h))/2$$

Now, let $Y(t)$ be the solution to the differential equation:

$$y' = f(t, y), \quad y(t_I) = y_i$$

We want to find $Y(t_F)$. Choose N steps, with step size $h = \frac{t_F - t_I}{N}$.
Define $t_k = t_I + kh$ for $k = 0, 1, \ldots, N$.
Initialize $y_0 = y_i$.
The trapezoid method suggests:

$$y_{k+1} = y_k + \frac{h}{2}[f(t_k, y_k) + f(t_{k+1}, y_{k+1})]$$

However, we can't use y_{k+1} to compute y_{k+1}. Instead, we replace y_{k+1} in $f(t_{k+1}, y_{k+1})$ with an Euler step approximation:

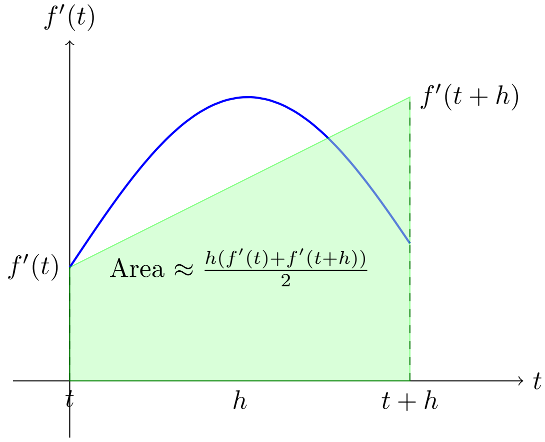


Figure 19: Trapezoid approximation

$$y_{k+1} \approx y_k + hf(t_k, y_k)$$

Therefore, we set:

$$y_{k+1} = y_k + \frac{h}{2}[f(t_k, y_k) + f(t_{k+1}, y_k + hf(t_k, y_k))]$$

It turns out that the global error of this method is $O(h^2)$.
This is called range-trapezoid method.

Example: Trapezoid Method

Consider the differential equation:

$$y' = t^2 + y^2, \quad y(0) = 1$$

Let's estimate $y(0.2)$ using the trapezoid method with $h = 0.1$.
Solution: We have $f(t, y) = t^2 + y^2$, $h = 0.1$, $t_0 = 0$, $t_1 = 0.1$, and $t_2 = 0.2$.

$$\begin{aligned}
y_1 &= y_0 + \frac{h}{2}[f(t_0, y_0) + f(t_1, y_0 + hf(t_0, y_0))] \\
&= 1 + \frac{0.1}{2}[(0^2 + 1^2) + (0.1^2 + (1 + 0.1 \cdot 1^2)^2)] \\
&= 1 + 0.1 \cdot \frac{1 + 0.01 + 1.21}{2} \\
&= 1 + 0.1 \cdot 1.11 \\
&= 1.111
\end{aligned}$$

$$
\begin{aligned}
y_2 &= y_1 + \frac{h}{2}[f(t_1, y_1) + f(t_2, y_1 + hf(t_1, y_1))] \\
&= 1.111 + \frac{0.1}{2}[(0.1^2 + 1.111^2) + (0.2^2 + (1.111 + 0.1 \cdot 1.111^2)^2)] \\
&= 1.111 + 0.1 \cdot \frac{0.010201 + 1.234543}{2} \\
&= 1.111 + 0.1 \cdot 1.122372 \\
&\approx 1.248
\end{aligned}
$$

Range-Midpoint Method

For the range-midpoint method, we use the formula:

$$
y_{k+1} = y_k + hf\left(t_k + \frac{h}{2}, y_k + \frac{h}{2}f(t_k, y_k)\right)
$$

The global error for this method is $O(h^2)$.

Note: See textbook page 13 for an example.

Exact ODEs & Integrating Factors

Consider a first-order ODE of the form $\frac{dy}{dx} = f(x, y)$.

Question: When do we have an implicit solution $H(x, y) = c$?

Differentiating both sides with respect to x, remembering that $y = y(x)$:

$$
\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} \cdot \frac{dy}{dx} = 0
$$

This leads to the general form of an exact ODE:

$$
M(x, y) + N(x, y)\frac{dy}{dx} = 0
$$

Alternatively written as:

$$
M(x, y)dx + N(x, y)dy = 0
$$

Question: When can we find a function $H(x, y)$ with $\frac{\partial H}{\partial x} = M$ and $\frac{\partial H}{\partial y} = N$?

If H has continuous second derivatives, then:

$$
\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial^2 H}{\partial y \partial x}
$$

Answer: If the domain of M and N has no "holes" and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then we can find a function $H(x, y)$ such that $\frac{\partial H}{\partial x} = M$ and $\frac{\partial H}{\partial y} = N$.

Such an ODE is called exact.

Example 1: Solve the IVP $(e^x y + 2x) + (2y + e^x)y' = 0$, $y(0) = 0$

Solution: We have $M = e^x y + 2x$, $N = 2y + e^x$. Let's check if the ODE is exact:

$$\frac{\partial M}{\partial y} = e^x$$
$$\frac{\partial N}{\partial x} = e^x$$

These are equal, so the ODE is exact.

We want to find $H(x, y)$ such that $\frac{\partial H}{\partial x} = M$ and $\frac{\partial H}{\partial y} = N$.

From $\frac{\partial H}{\partial x} = M$:

$$H(x, y) = \int (e^x y + 2x)\, dx = e^x y + x^2 + C(y)$$

From $\frac{\partial H}{\partial y} = N$:

$$e^x + C'(y) = 2y + e^x$$

Solving for $C(y)$:

$$C'(y) = 2y \implies C(y) = y^2$$

Therefore, $H(x, y) = e^x y + x^2 + y^2 = C$ is our general solution to the ODE.

Applying the initial condition $y(0) = 0$:

$$e^0 \cdot 0 + 0^2 + 0^2 = C \implies C = 0$$

So the solution to the IVP is $e^x y + x^2 + y^2 = 0$.

Alternative approach: Start with $\frac{\partial H}{\partial y} = 2y + e^x$

$$H(x, y) = y^2 + e^x y + C(x)$$

Then plug $\frac{\partial H}{\partial x} = e^x y + C'(x)$ into M and solve for $C(x)$.

Example 2: Solve $(3t^2 y + 8ty^2)dt + (t^3 + 8t^2 y + 12t^2 y^2)dy = 0$

Solution: We have $M = 3t^2 y + 8ty^2$, $N = t^3 + 8t^2 y + 12t^2 y^2$

Let's check if the ODE is exact:

$$\frac{\partial M}{\partial y} = 3t^2 + 16ty$$
$$\frac{\partial N}{\partial t} = 3t^2 + 16ty$$

These are equal, so the ODE is exact.

We're looking for $H(t, y)$ such that $\frac{\partial H}{\partial t} = M$ and $\frac{\partial H}{\partial y} = N$.

From $\frac{\partial H}{\partial y} = N$:

$$H(t, y) = t^3 y + 4t^2 y^2 + C(t)$$

Now, let's use $\frac{\partial H}{\partial t} = M$:

$$\frac{\partial H}{\partial t} = 3t^2y + 8ty^2 + C'(t)$$
$$3t^2y + 8ty^2 + C'(t) = 3t^2y + 8ty^2$$

Comparing the two sides, we see that $C'(t) = 0$, which means $C(t) = K$, where K is a constant. Therefore, the general solution is:

$$H(t, y) = t^3y + 4t^2y^2 + 4y^3 = C$$

This is the implicit form of the general solution to the given ODE.

Lecture 7, Tuesday 9/17/2024

Quiz 3 - I.6.3, I.7, I.9(.1-.2)

I.9 continued

Example: $\frac{dy}{dt} = -\frac{y\sin(ty)}{t\sin(ty)+y}$
Solution:

$$(t\sin(ty) + y)\frac{dy}{dt} = -y\sin(ty) \implies y\sin(ty) + (t\sin(ty) + y)\frac{dy}{dt} = 0$$

Let $M = y\sin(ty)$ and $N = t\sin(ty) + y$.

$$\frac{\partial M}{\partial y} = \sin(ty) + ty\cos(ty)$$
$$\frac{\partial N}{\partial t} = \sin(ty) + ty\cos(ty)$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, the ODE is exact.
We seek $H(t, y)$ such that $\frac{\partial H}{\partial t} = M$ and $\frac{\partial H}{\partial y} = N$.
From $\frac{\partial H}{\partial y} = N$:

$$H(t, y) = \int y\sin(ty)\, dy = -\cos(ty) + C(y)$$

Now, using $\frac{\partial H}{\partial t} = M$:

$$\frac{\partial H}{\partial t} = t\sin(ty)\cos(ty) + C'(y) = t\sin(ty) + y$$

Thus,

$$C'(y) = y \implies C(y) = \frac{y^2}{2} + K$$

Thus, the general solution is:

$$H(t, y) = -\cos(ty) + \frac{y^2}{2} = C$$

Integrating Factors

Example: $2ty + (2t^2 - e^y)y' = 0$

Find the general solution.

Let $M = 2t$ and $N = 2t^2 - e^y$.

Check:

$$\frac{\partial M}{\partial y} = 0, \quad \frac{\partial N}{\partial t} = 4t$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, the ODE is not exact.

Try multiplying the entire equation by y:

$$2ty^2 + (2yt^2 - ye^y)y' = 0$$

Now, let $M = 2ty$ and $N = 2yt^2 - ye^y$.

Check again:

$$\frac{\partial M}{\partial y} = 2t, \quad \frac{\partial N}{\partial t} = 4ty$$

Now the ODE is exact.

Solve:

$$t^2y^2 - (y-1)e^y = C$$

This y is called an integrating factor.

Compare with the standard form:

$$y' + a(t)y = 0$$

Multiplying by an integrating factor ρ:

$$\rho y' + \rho a(t)y = 0$$

Let $\rho = e^{A(t)}$, then:

$$e^{A(t)}y' + a(t)e^{A(t)}y = 0$$

In general, for $M + N\frac{dy}{dx} = 0$ which is not exact, we want to multiply by a function $\rho(x, y)$ so that $\rho M + \rho N\frac{dy}{dx} = 0$ is exact.

The function ρ is called an integrating factor.

The equation is exact if:

$$\frac{\partial(\rho M)}{\partial y} = \frac{\partial(\rho N)}{\partial x}$$

This leads to the partial differential equation (PDE):

$$\rho_y M + \rho\frac{\partial M}{\partial y} = \rho_x N + \rho\frac{\partial N}{\partial x}$$

Solving this PDE can be very challenging.

In order to solve, we make an assumption:

either (1) ρ is only a function of x

$$\rho_x = \rho', \quad \rho_y = 0$$

or (2) ρ is only a function of y

$$\rho_x = 0, \quad \rho_y = \rho'$$

For example, consider the equation:

$$2xy + (2x^2 - e^y)\frac{dy}{dx} = 0$$

Here, $M = 2x$ and $N = 2x^2 - e^y$.
We look for an integrating factor ρ such that:

$$\frac{\partial}{\partial y}(\rho M) = \frac{\partial}{\partial x}(\rho N)$$

This gives:

$$\rho_y M + \rho\frac{\partial M}{\partial y} = \rho_x N + \rho\frac{\partial N}{\partial x}$$

Substituting M and N:

$$\rho_y(2xy) + \rho(2x) = \rho_x(2x^2 - e^y) + \rho(4x)$$

Assuming ρ is only a function of y:

$$\rho_x = 0, \quad \rho_y = \rho'$$

This simplifies to:

$$\rho'(2xy) + \rho(2x) = \rho(4x)$$

Which reduces to:

$$2xy\rho' = 2x\rho$$

Solving for ρ:

$$\rho' = \frac{\rho}{y}$$

Integrating both sides:

$$\int \frac{1}{\rho}d\rho = \int \frac{1}{y}dy$$

This gives:

$$\ln|\rho| = \ln|y| + C$$

Thus:

$$|\rho| = c|y|$$

We can choose $\rho = y$.
Example: Find an integrating factor for $(1 + 3x^2 \sin y) - (x \cot y)y' = 0$
We have:

$$M = 1 + 3x^2 \sin y, \quad N = -x \cot y$$

Calculating the partial derivatives:

$$M_y = 3x^2 \cos y, \quad N_x = -\cot y$$

Since $M_y \neq N_x$, we look for an integrating factor ρ.
We start with:

$$(\rho M)_y = (\rho N)_x$$

Expanding this, we get:

$$\rho_y M + \rho M_y = \rho_x N + \rho N_x$$

Substituting M and N:

$$\rho_y(1 + 3x^2 \sin y) + \rho(3x^2 \cos y) = \rho_x(-x \cot y) + \rho(-\cot y)$$

First, try $\rho = \rho(x)$:

$$\rho_y = 0, \quad \rho_x = \rho' = \frac{d\rho}{dx}$$

This simplifies to:

$$\rho(3x^2 \cos y) = \rho' x \cot y + \rho \cot y$$

Multiplying by $\tan y$:

$$\rho(3x^2 \sin y) = x\rho' + \rho$$

Rearranging:

$$x\rho' = \rho(3x^2 \sin y - 1)$$

This gives:

$$\rho' = \frac{\rho(3x^2 \sin y - 1)}{x}$$

Since the expression for $\frac{d\rho}{dx}$ has both x and y, our assumption must not work.
Next, try $\rho = \rho(y)$:

$$\rho_x = 0, \quad \rho_y = \rho' = \frac{d\rho}{dy}$$

This simplifies to:

$$\rho'(1 + 3x^2 \sin y) + \rho(3x^2 \cos y) = \rho(-x \cot y)$$

Rearranging:

$$\rho'(1 + 3x^2 \sin y) = \rho(-\cot y - 3x^2 \cos y)$$

This gives:

$$\rho' = \frac{-\rho(\cot y + 3x^2 \cos y)}{1 + 3x^2 \sin y}$$

Simplifying further:

$$\rho' = -\rho \cot y$$

This gives:

$$\frac{1}{\rho}\frac{d\rho}{dy} = -\cot y$$

Integrating both sides:

$$\int \frac{1}{\rho} d\rho = \int -\cot y \, dy$$

This gives:

$$\ln|\rho| = -\ln|\sin y| + C$$

Thus:

$$\rho = e^{-\ln(\sin y)}$$

We can choose $\rho = e^C \sin y$. For simplicity, let $e^C = 1$, so:

$$\rho = \csc y$$

Multiplying by ρ might add or subtract solutions.

For example, consider the equation:

$$y + \left(2x + \frac{1}{y}\right) y' = 0$$

Note that $y = 0$ is not a solution.

$\rho = y$ works as an integrating factor:

$$y^2 + (2xy + 1)y' = 0$$

Now, $y = 0$ is a solution. The solution is added because $y = 0$ corresponds to $p = 0$.

Fact: Solutions can only be added when $\rho = 0$.

For example, consider the equation:

$$-y^2 + x^2 y' = 0$$

Note that $y = x$ is a solution. Check:

$$-(x)^2 + x^2(1) = 0$$

It turns out $\rho = \frac{1}{(x-y)^2}$ works as an integrating factor:

$$\Rightarrow \frac{-y^2}{(x-y)^2} + \frac{x^2}{(x-y)^2} y' = 0$$

This is exact when $y \neq x$.

Solve:

$$\frac{xy}{x-y} = C$$

This misses $y = x$ because ρ isn't defined for $y = x$.

Lecture 8, Thursday 9/19/2024

II.1, II.2: Higher-Order Linear Differential Equations

Standard Form:

$$\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = f(t)$$

where $a_{n-1}(t), \ldots, a_0(t)$ are coefficient functions and $f(t)$ is the forcing function.

If $f(t) = 0$, the ODE is homogeneous; otherwise, it is nonhomogeneous.

A function $Y(t)$ is a solution over an interval (a, b) if:

- $y(t), y'(t), \ldots, y^{(n)}(t)$ exist for all $t \in (a,b)$
- Coefficients and forcing function exist for all $t \in (a,b)$
- $y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_0(t)y(t) = f(t)$ for all $t \in (a,b)$

An IVP for an nth-order ODE has the initial conditions:

$$y(t_I) = y_0, \quad y'(t_I) = y_1, \quad \ldots, \quad y^{(n-1)}(t_I) = y_{n-1}$$

where all t_I are the same.

Theorem (Existence and Uniqueness): If $a_0(t), \ldots, a_{n-1}(t)$ and $f(t)$ are continuous on (a,b), then for any $t_I \in (a,b)$ and any initial data $y_0, \ldots, y_{n-1}$, there is a unique solution to the IVP on (a,b).

Example: Show that $\sin(t^2)$ is not a solution to any second-order homogeneous linear ODE with coefficients continuous on $(-1,1)$.

Solution: Say $y = \sin(t^2)$ is a solution to $y'' + p(t)y' + q(t)y = 0$.

$$y(0) = \sin(0) = 0$$
$$y'(t) = 2t\cos(t^2), \quad \text{so} \quad y'(0) = 0$$

Thus, $\sin(t^2)$ is the solution to the IVP:

$$\begin{cases} y'' + a_1(t)y' + a_0(t)y = 0 \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

If $a_0(t)$ and $a_1(t)$ are continuous on $(-1,1)$, then the solution to the IVP is unique.

However, $y = 0$ is also a solution to the IVP, leading to a contradiction.

Example: Determine the interval of definition for the solution to the IVP (i.e., the largest interval on which the solution exists):

$$ty'' + \frac{\tan(t)}{t-3}y' - y = e^t, \quad y(1) = 2, \quad y'(1) = 4$$

Solution: Given $t_I = 1$, we seek the largest interval containing 1 on which the coefficients and forcing function are continuous.

Rewrite in standard form:

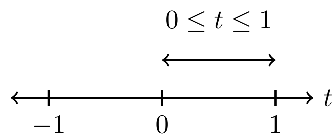
$$y'' + \frac{\tan(t)}{t(t-3)}y' - \frac{1}{t}y = \frac{e^t}{t}$$

Identify discontinuities:

$$a_1(t) = \frac{\tan(t)}{t(t-3)} \quad \text{has discontinuities at} \quad t = 0, 3, \frac{\pi}{2} + k\pi$$

$$a_0(t) = -\frac{1}{t} \quad \text{has a discontinuity at} \quad t = 0$$

$$f(t) = \frac{e^t}{t} \quad \text{has a discontinuity at} \quad t = 0$$



Thus, the largest interval of continuity including $t = 1$ is $(0, 3)$.

II.2: Homogeneous Linear Equations

A homogeneous linear differential equation has the form:

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = 0$$

If $y_1(t)$ and $y_2(t)$ are two solutions, then $c_1y_1(t) + c_2y_2(t)$ is also a solution for any constants c_1 and c_2.

Check:

$$\frac{d}{dt}(c_1y_1 + c_2y_2) = c_1y_1' + c_2y_2'$$

If y_1 and y_2 are solutions, then:

$$(c_1y_1 + c_2y_2)^{(n)} + a_{n-1}(t)(c_1y_1 + c_2y_2)^{(n-1)} + \cdots + a_1(t)(c_1y_1 + c_2y_2)' + a_0(t)(c_1y_1 + c_2y_2) = 0$$

This simplifies to:

$$c_1(y_1^{(n)} + a_{n-1}y_1^{(n-1)} + \cdots + a_1y_1' + a_0y_1) + c_2(y_2^{(n)} + a_{n-1}y_2^{(n-1)} + \cdots + a_1y_2' + a_0y_2) = 0$$

Since y_1 and y_2 are solutions:

$$c_1 \cdot 0 + c_2 \cdot 0 = 0$$

Thus, $c_1y_1 + c_2y_2$ is a solution.

Example: We know (i.e., we can check) that e^{2t} and e^{-t} are solutions to $y'' - y' - 2y = 0$. Find a solution to the IVP with $y(0) = 4$ and $y'(0) = 2$.

Solution: We can try to look for a solution of the form:

$$y = c_1e^{2t} + c_2e^{-t}$$

Then,

$$y'(t) = 2c_1e^{2t} - c_2e^{-t}$$

Using the initial conditions:

$$y(0) = c_1 + c_2 = 4$$

$$y'(0) = 2c_1 - c_2 = 2$$

Solving these equations, we get:

$$c_1 = 2, \quad c_2 = 2$$

Thus, the solution is:

$$y = 2e^{2t} + 2e^{-t}$$

Example: We can check that $\sin(2t)$ and $\cos(2t)$ are solutions to $y'' + 4y = 0$.
Find a solution to the IVP with $y(0) = 3$ and $y'(0) = -2$.
Solution: We can try to look for a solution of the form:

$$y = c_1 \sin(2t) + c_2 \cos(2t)$$

Then,

$$y'(t) = 2c_1 \cos(2t) - 2c_2 \sin(2t)$$

Using the initial conditions:

$$y(0) = c_2 = 3$$

$$y'(0) = 2c_1 = -2 \quad \Rightarrow \quad c_1 = -1$$

Thus, the solution is:

$$y = -\sin(2t) + 3\cos(2t)$$

Non-example: We know that e^{-t} and $2e^{-t}$ are solutions to $y'' - y' - 2y = 0$. Solve $y(0) = 4$ and $y'(0) = 2$.
Solution: We can try to look for a solution of the form:

$$y = c_1 e^{-t} + c_2 e^{-2t}$$

Then,

$$y'(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$$

Using the initial conditions:

$$y(0) = c_1 + c_2 = 4$$

$$y'(0) = -c_1 - 2c_2 = 2$$

Solving these equations, we get:

$$0 \neq 6$$

Thus, there is no solution.
Example: Solve the IVP $y'' - y' - 2y = 0$ with $y(0) = y_0$ and $y'(0) = y_1$.
Solution: Guess that $y = c_1 e^{-t} + c_2 e^{-2t}$ solves the ODE.
Using the initial conditions:

$$c_1 + c_2 = y_0$$

$$2c_1 - c_2 = y_1$$

Solving these equations simultaneously, we get:

$$c_1 = \frac{y_1 + y_0}{3}$$

$$c_2 = \frac{2y_0 - y_1}{3}$$

Thus, the solution to the IVP is:

$$y = \frac{y_1 + y_0}{3} e^{-t} + \frac{2y_0 - y_1}{3} e^{-2t}$$

Lecture 10, Thursday 9/26/2024

II.2 Continued: Higher-Order Linear Differential Equations

Example: Solve the IVP $y'' - y' - 2y = 0$ with $y(0) = y_0$ and $y'(0) = y_1$.

Solution: We have solutions $y = e^{-t}$ and $y = e^{-2t}$.

Assume the solution is of the form $y = c_1 e^{-t} + c_2 e^{-2t}$, where c_1 and c_2 are constants.

Using the initial conditions:

$$y(0) = c_1 + c_2 = y_0$$

$$y'(0) = -c_1 - 2c_2 = y_1$$

Turn this into a matrix equation:

$$\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$$

Solve it using linear algebra.

II.3: Systems of Linear Equations

A system of linear equations with variables can generally be written in square format with n equations and n variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_m + a_{m2}x_m + \cdots + a_{mn}x_n &= b_n \end{aligned}$$

where a_{ij} are real constants called the coefficients of the system.

If $b_1 = b_2 = \cdots = b_n = 0$, the system is homogeneous. If any b_i is not 0, the system is nonhomogeneous. A solution $x_1 = x_2 = \cdots = x_n = 0$ is called the trivial solution.

If there are two solutions $x_1, \ldots, x_n$ and $y_{11}, \ldots, y_{1n}$ for a particular nonhomogeneous system, then setting $z_1 = x_1 - y_{11}, z_2 = x_2 - y_{12}, \ldots, z_n = x_n - y_{1n}$ gives a non-trivial solution of the corresponding homogeneous system:

$$a_1 z_1 + \cdots + a_n z_n = a_{11}x_1 - a_{11}y_{11} + \cdots + a_{1n}x_n - a_{1n}y_{1n} = b_1 - b_1 = 0$$

Upshot: If a nonhomogeneous system has multiple solutions, the corresponding homogeneous system has a non-trivial solution.

Write the system in matrix form:

$$A\mathbf{x} = \mathbf{b}$$

where A is an $n \times n$ matrix, and $\mathbf{x}$ and $\mathbf{b}$ are $n \times 1$ vectors:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \quad A = (a_{ij})$$

Idea: Solve the system by inverting matrix A. Multiply both sides by A^{-1}:

$$A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b} \implies I\mathbf{x} = A^{-1}\mathbf{b}$$

Problem: A^{-1} doesn't always exist. A^{-1} exists if and only if $\det(A) \neq 0$.
Why? $\det(I) = 1$, $\det(A^{-1}) = \frac{1}{\det(A)}$.
Determinants:

$$\det(A) = |A|$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

If $\det(A) \neq 0$, then there is a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

If $\det(A) = 0$, then $A\mathbf{x} = \mathbf{b}$ might not have a solution, but if it does, it has infinitely many solutions.

For a 2×2 matrix A, the formula for A^{-1} is:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

where $\det(A) = ad - bc$.

Given the system:

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$$

We have $A\mathbf{x} = \mathbf{b}$.

Calculate the determinant:

$$\det(A) = 1 \cdot (-1) - 1 \cdot 2 = -1 - 2 = -3$$

Then, the inverse of A is:

$$A^{-1} = \frac{1}{-3} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

So,

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = A^{-1}\mathbf{b} = \frac{1}{-3} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}(y_1 + y_0) \\ \frac{1}{3}(2y_0 - y_1) \end{pmatrix}$$

Wronskians

Say we have an n-th order linear ODE with solutions $Y_1(t), \ldots, Y_n(t)$.

The Wronskian is defined as:

$$W(t) = \begin{vmatrix} Y_1(t) & Y_2(t) & \cdots & Y_n(t) \\ Y_1'(t) & Y_2'(t) & \cdots & Y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ Y_1^{(n-1)}(t) & Y_2^{(n-1)}(t) & \cdots & Y_n^{(n-1)}(t) \end{vmatrix}$$

Question: When is there a solution to the IVP with $y(t_i) = y_i$, $y'(t_i) = y_i'$, ..., $y^{(n-1)}(t_i) = y_i^{(n-1)}$?

Using initial conditions:

$$\begin{aligned}
y(t_i) = y_0 &\to c_1 Y_1(t_i) + c_2 Y_2(t_i) + \cdots + c_n Y_n(t_i) = y_0 \\
y'(t_i) = y_1 &\to c_1 Y_1'(t_i) + c_2 Y_2'(t_i) + \cdots + c_n Y_n'(t_i) = y_1 \\
&\vdots \\
y^{(n-1)}(t_i) = y_{n-1} &\to c_1 Y_1^{(n-1)}(t_i) + c_2 Y_2^{(n-1)}(t_i) + \cdots + c_n Y_n^{(n-1)}(t_i) = y_{n-1}
\end{aligned}$$

This is a system of linear equations for $c_1, \ldots, c_n$.
Write this system using matrices:

$$\begin{pmatrix} Y_1(t_i) & Y_2(t_i) & \cdots & Y_n(t_i) \\ Y_1'(t_i) & Y_2'(t_i) & \cdots & Y_n'(t_i) \\ \vdots & \vdots & \ddots & \vdots \\ Y_1^{(n-1)}(t_i) & Y_2^{(n-1)}(t_i) & \cdots & Y_n^{(n-1)}(t_i) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

There is a solution for every choice of $y_0, \ldots, y_{n-1}$ exactly when $\det(A) \neq 0$.
This determinant is called the Wronskian of $Y_1, \ldots, Y_n$ at t_i:

$$\mathrm{Wr}(Y_1, \ldots, Y_n)(t_i)$$

Example: $Y_1(t) = e^{2t}, Y_2(t) = e^{-t}$

$$W(Y_1, Y_2)(0) = \begin{vmatrix} Y_1(0) & Y_2(0) \\ Y_1'(0) & Y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3$$

More generally,

$$\mathrm{Wr}(Y_1, \ldots, Y_n)(t) = \begin{vmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{vmatrix} = -3e^t$$

Since $\mathrm{Wr}(e^{2t}, e^{-t})(t)$ is never 0, we have:

$$y = c_1 e^{2t} + c_2 e^{-t}$$

Solve IVPs:

$$\begin{cases} y'' - y' - 2y = 0 \\ y(t_i) = y_0 \\ y'(t_i) = y_1 \end{cases}$$

For any t_i and any y_0, y_1, we say that $y = c_1 e^{2t} + c_2 e^{-t}$ is the general solution to $y'' - y' - 2y = 0$.
Example: $Y_1 = e^{-t}, Y_2 = 2e^{-t}$

$$\mathrm{Wr}(e^{-t}, 2e^{-t})(t) = \begin{vmatrix} e^{-t} & 2e^{-t} \\ -e^{-t} & -2e^{-t} \end{vmatrix} = -2e^{-2t} + 2e^{-2t} = 0$$

$y = c_1 e^{-t} + c_2 e^{-t}$ cannot solve all IVPs.

Example: $y_1 = \cos(2t), y_2 = \sin(2t)$

$$\mathrm{Wr}(\cos(2t), \sin(2t))(t) = \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix} = 2\cos^2(2t) + 2\sin^2(2t) = 2$$

Since $\mathrm{Wr}(y_1, y_2)(t)$ is never 0, we have:

$$y = c_1 \cos(2t) + c_2 \sin(2t)$$

This is the general solution to $y'' + 4y = 0$.

Theorem: if the coeffcients of a homogeneous linear ODE are continuous on (a, b) and $Y_1, \ldots, Y_n$ are solutions, then $\mathrm{Wr}(Y_1, \ldots, Y_n)(t)$ is never 0 on (a, b) or always 0 on (a, b).

Lecture 11, Tuesday 10/1/2024

Quiz Coverage: II.1, II.2 (up to & including 3), II.3.

Note: Thursday's class is prerecorded.

Recall: If $y_1, \ldots, y_n$ are solutions of an nth order linear homogeneous ODE, then:

$$\mathrm{Wr}(y_1, \ldots, y_n)(t) \neq 0$$

Reminder: The Wronskian is defined as:

$$\mathrm{Wr}(y_1, \ldots, y_n)(t) = \det \begin{pmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

Then functions of the form $y = c_1 y_1 + \cdots + c_n y_n$ are solutions to every IVP with initial t-value t_0.

We call such a set of functions $\{y_1, \ldots, y_n\}$ a *fundamental set of solutions* at t_0.

And $y = c_1 y_1 + \cdots + c_n y_n$ is called the *general solution* to the ODE (at t_0).

Example: We saw $y_1 = e^{2t}$, $y_2 = e^{-t}$ are solutions to $y'' - y' - 2y = 0$
and $\mathrm{Wr}(e^{2t}, e^{-t})(t) \neq 0 \Rightarrow \{e^{2t}, e^{-t}\}$ is a fundamental set of solutions at t.
And $y = c_1 e^{2t} + c_2 e^{-t}$ is the general solution at t.

Natural Fundamental Sets of Solutions

Fix an initial t-value t_I. Consider the ODE:

$$y^{(n)} + a_{n-1}(t) y^{(n-1)} + \cdots + a_1(t) y' + a_0(t) y = 0$$

Define $N_k(t)$ for $k = 0, 1, \ldots, n-1$ as the solution to the IVP:

$$\begin{cases} y^{(j)}(t_I) = 0 & \text{for } j \neq k \\ y^{(k)}(t_I) = 1 \end{cases}$$

Explicitly:

$$\begin{aligned}
N_0(t): &\quad y(t_I) = 1, \quad y'(t_I) = 0, \quad \ldots, \quad y^{(n-1)}(t_I) = 0 \\
N_1(t): &\quad y(t_I) = 0, \quad y'(t_I) = 1, \quad \ldots, \quad y^{(n-1)}(t_I) = 0 \\
&\vdots \\
N_{n-1}(t): &\quad y(t_I) = 0, \quad y'(t_I) = 0, \quad \ldots, \quad y^{(n-1)}(t_I) = 1
\end{aligned}$$

Then $\{N_0, N_1, \ldots, N_{n-1}\}$ is a fundamental set of solutions at t_I.
Why?

$$\begin{aligned}
\mathrm{Wr}(N_0, \ldots, N_{n-1})(t_I) &= \det \begin{pmatrix} N_0(t_I) & \cdots & N_{n-1}(t_I) \\ N_0'(t_I) & \cdots & N_{n-1}'(t_I) \\ \vdots & \ddots & \vdots \\ N_0^{(n-1)}(t_I) & \cdots & N_{n-1}^{(n-1)}(t_I) \end{pmatrix} \\
&= \det(I_n) = 1 \neq 0
\end{aligned}$$

This is called the *natural* fundamental set of solutions at t_I.

Why is this important? The natural fundamental set of solutions allows us to easily solve initial value problems (IVPs). Consider the IVP:

$$y^{(n-1)}(t_I) = y_{n-1}, \quad y^{(n-2)}(t_I) = y_{n-2}, \quad \ldots, \quad y'(t_I) = y_1, \quad y(t_I) = y_0$$

The solution to this IVP is given by:

$$y(t) = y_0 N_0(t) + y_1 N_1(t) + \cdots + y_{n-1} N_{n-1}(t)$$

We can verify that this solution satisfies the initial conditions:

$$\begin{aligned}
y(t_I) &= y_0 N_0(t_I) + y_1 N_1(t_I) + \cdots + y_{n-1} N_{n-1}(t_I) = y_0 \cdot 1 + y_1 \cdot 0 + \cdots + y_{n-1} \cdot 0 = y_0 \\
y'(t_I) &= y_0 N_0'(t_I) + y_1 N_1'(t_I) + \cdots + y_{n-1} N_{n-1}'(t_I) = y_0 \cdot 0 + y_1 \cdot 1 + \cdots + y_{n-1} \cdot 0 = y_1 \\
&\vdots \\
y^{(n-1)}(t_I) &= y_0 N_0^{(n-1)}(t_I) + y_1 N_1^{(n-1)}(t_I) + \cdots + y_{n-1} N_{n-1}^{(n-1)}(t_I) = y_0 \cdot 0 + y_1 \cdot 0 + \cdots + y_{n-1} \cdot 1 = y_{n-1}
\end{aligned}$$

This demonstrates that the natural fundamental set of solutions provides a straightforward method for constructing solutions to IVPs.

Example: Natural Fundamental Set and IVP Solution

Consider the differential equation $y'' + 4y = 0$ with solutions $y_1(t) = \cos(2t)$ and $y_2(t) = \sin(2t)$.

Task 1: Find a natural fundamental set of solutions at $t = 0$.

Let $N_0(t) = c_1 \cos(2t)$ and $N_1(t) = c_2 \sin(2t)$.

For $N_0(t)$:

$$N_0(0) = 1 \implies c_1 \cos(0) = 1 \implies c_1 = 1$$

For $N_1(t)$:

$$N_1(0) = 0 \quad \text{(satisfied automatically)}$$
$$N_1'(0) = 1 \implies 2c_2 \cos(0) = 1 \implies c_2 = \frac{1}{2}$$

Thus, the natural fundamental set of solutions at $t = 0$ is:

$$N_0(t) = \cos(2t)$$
$$N_1(t) = \frac{1}{2}\sin(2t)$$

Task 2: Solve the following IVP:

$$\begin{cases} y'' + 4y = 0 \\ y(0) = 7 \\ y'(0) = -3 \end{cases}$$

The solution is given by:

$$\begin{aligned} y(t) &= 7N_0(t) - 3N_1(t) \\ &= 7\cos(2t) - 3\left(\frac{1}{2}\sin(2t)\right) \\ &= 7\cos(2t) - \frac{3}{2}\sin(2t) \end{aligned}$$

Linear Independence

A set of functions $\{y_1, \ldots, y_n\}$ is linearly independent on an interval (a, b) if:

$$c_1 y_1(t) + \cdots + c_n y_n(t) = 0 \quad \text{for all } t \in (a, b)$$

implies $c_1 = \cdots = c_n = 0$.

Non-example: Consider $y_1 = e^{-t}$ and $y_2 = 2e^{-t}$. Then:

$$2y_1 - y_2 = 2e^{-t} - 2e^{-t} = 0$$

Therefore, y_1 and y_2 are linearly dependent.

Theorem 1: If $\mathrm{Wr}(y_1, \ldots, y_n)(t) \neq 0$ on (a, b), then $y_1, \ldots, y_n$ are linearly independent on (a, b).

Theorem 2: If $\mathrm{Wr}(y_1, \ldots, y_n)(t) = 0$ on (a, b), then $y_1, \ldots, y_n$ are linearly dependent on (a, b).

II.4: Homogeneous Linear ODEs with Constant Coefficients

The general form of a homogeneous linear ODE with constant coefficients is:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

where $a_0, \ldots, a_{n-1}$ are constants.

Example: Consider the equation $y'' - y' - 2y = 0$

We have already seen that $y = e^{2t}$ and $y = e^{-t}$ are solutions.

Let's guess that our solutions are of the form $y = e^{zt}$. Then:

$$\begin{aligned} y &= e^{zt} \\ y' &= ze^{zt} \\ y'' &= z^2e^{zt} \end{aligned}$$

Substituting into the original equation:

$$z^2e^{zt} - ze^{zt} - 2e^{zt} = 0$$

Dividing by e^{zt}:

$$z^2 - z - 2 = 0$$

Solving this quadratic equation:

$$z = 2 \text{ or } z = -1$$

Therefore, we have solutions $y = e^{2t}$ and $y = e^{-t}$.

In general, for a linear ODE with constant coefficients, we can use the characteristic polynomial to find the solutions.

If we plug $y = e^{zt}$ into our ODE, we get (remembering $\frac{d^k}{dt^k}e^{zt} = z^ke^{zt}$):

$$z^ne^{zt} + a_1z^{n-1}e^{zt} + \cdots + a_ne^{zt} = 0$$

Factoring out e^{zt}:

$$(z^n + a_1z^{n-1} + \cdots + a_n)e^{zt} = 0$$

This gives us the characteristic polynomial:

$$p(z) = z^n + a_1z^{n-1} + \cdots + a_n$$

Therefore, if z is any root of $p(z) = 0$, then $y = e^{zt}$ is a solution to our ODE.

Note: An nth order ODE corresponds to an nth degree polynomial, which has the potential to have n roots.

Problem: $p(z) = 0$ may have repeated roots or complex roots.

Easier case: Simple (non-repeated) real roots

In this case, we can factor $p(z)$ as:

$$p(z) = (z - r_1)(z - r_2)\cdots(z - r_n)$$

where $r_1, \ldots, r_n$ are distinct real numbers.

Consequence: $e^{r_1t}, \ldots, e^{r_nt}$ are solutions

It turns out that $\mathrm{Wr}(e^{r_1t}, \ldots, e^{r_nt})(t) \neq 0$

We can verify this by computing the Wronskian:

$$\mathrm{Wr}(e^{r_1t}, \ldots, e^{r_nt})(t) = \begin{vmatrix} e^{r_1t} & e^{r_2t} & \cdots & e^{r_nt} \\ r_1e^{r_1t} & r_2e^{r_2t} & \cdots & r_ne^{r_nt} \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1}e^{r_1t} & r_2^{n-1}e^{r_2t} & \cdots & r_n^{n-1}e^{r_nt} \end{vmatrix}$$

For $n = 2$, we have:
$$\begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = e^{(r_1+r_2)t}(r_2 - r_1) \neq 0$$
since $r_2 \neq r_1$ and $e^{(r_1+r_2)t} > 0$ for all t.
Hence, $\{e^{r_1 t}, \ldots, e^{r_n t}\}$ is a fundamental set of solutions, and the general solution is:

$$y = c_1 e^{r_1 t} + \cdots + c_n e^{r_n t}$$

Example 1: Solve $y'' + 7y' + 12y = 0$.
Solution:

$$\begin{aligned} p(z) &= z^2 + 7z + 12 \\ &= (z+3)(z+4) \end{aligned}$$

The roots are $r_1 = -3$ and $r_2 = -4$. Therefore, e^{-3t} and e^{-4t} are solutions.
The general solution is:
$$y = c_1 e^{-4t} + c_2 e^{-3t}$$
Example 2: Solve $y'' - 2y' - y = 0$. Solution:
$$\begin{aligned} p(z) &= z^2 - 2z - 1 \\ &= (z - (1+\sqrt{2}))(z - (1-\sqrt{2})) \end{aligned}$$
The characteristic equation has two distinct real roots:
$$r_1 = 1 + \sqrt{2} \quad \text{and} \quad r_2 = 1 - \sqrt{2}$$
Therefore, the general solution is:

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} = c_1 e^{(1+\sqrt{2})t} + c_2 e^{(1-\sqrt{2})t}$$

where c_1 and c_2 are arbitrary constants.
Example 3: Solve $y''' + 2y'' - y' - 2y = 0$.
Solution: Let's find the characteristic equation and its roots:

$$\begin{aligned} p(z) &= z^3 + 2z^2 - z - 2 \\ &= (z+2)(z+1)(z-1) \end{aligned}$$

To find the roots, we can use the Rational Root Theorem:
Theorem: If $p(z) = a_n z^n + \cdots + a_1 z + a_0$ is a polynomial with integer coefficients, then any rational root of $p(z) = 0$ is of the form $\pm\frac{p}{q}$, where p is a factor of a_0 and q is a factor of a_n.
In our case, $a_0 = -2$ and $a_3 = 1$. The possible rational roots are $\pm 1, \pm 2$.
Testing these values, we find the roots:

$$r_1 = -2, \quad r_2 = -1, \quad r_3 = 1$$

Therefore, the general solution is:

$$y = c_1 e^{-2t} + c_2 e^{-t} + c_3 e^{t}$$

where c_1, c_2, and c_3 are arbitrary constants.