Custom Vectors Problem Sheet 1

1. A drone is located at point **D** with coordinates (3, -2, 5). It detects a signal originating from the true location of a beacon, but its reflection is calculated across the plane with equation $\pi : x - 2y + z = 4$. Find the true coordinates of the beacon.
2. In a 3D modelling program, a light source is positioned at point **L**(1, 4, -1). A flat triangular section of a scene lies on the plane $\Pi : 2x - y + 2z = 8$. Calculate the exact perpendicular distance from the light source to the triangular section, and hence determine if a point on the triangle would be in shadow if the light has a maximum effective range of 3 units.
3. Two support beams in a building are modelled as straight lines. Beam A has equation $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. Beam B has equation $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$. Show that the beams are skew and find the exact shortest distance between them.
4. A ray of light travels along the line with equation $\mathbf{r} = \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. It hits a mirror defined by the plane $x + 2y - z = 11$ and is reflected. Find a vector equation for the path of the reflected ray.
5. Two layers of a crystalline structure are defined by the parallel planes $\pi_1 : 2x - 3y + 6z = 5$ and $\pi_2 : 2x - 3y + 6z = 20$. Calculate the distance between these two layers.
6. Two parallel power lines run between pylons. Their equations are given as:
   Line 1: $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$
   Line 2: $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$.
   Find the shortest distance between these two power lines.
7. A point **P** has coordinates (6, 3, -2). Find the coordinates of its reflection in the line with equation $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$.

Mark Scheme

1. Reflecting a point through a plane

Answer: *(1, 6, -3)*

Explanation: Use the reflection formula. Find the foot of the perpendicular from D(3, -2, 5) to the plane. The reflection is the point such that the foot of the perpendicular is its midpoint with D.

2. Shortest distance between a point and a plane

Answer: *Distance = 3 units. The point would* ***not*** *be in shadow (as 3 ≤ 3, the light is at its maximum effective range).*

Explanation: Use the formula for the distance from point (x_1, y_1, z_1) to plane $ax+by+cz=d$: $D=\frac{|ax_1+by_1+cz_1-d|}{\sqrt{a^2+b^2+c^2}}$. Substituting L(1,4,-1) gives $D=\frac{|2(1)-1(4)+2(-1)-8|}{\sqrt{4+1+4}}=\frac{|-12|}{3}=4$. The conclusion is incorrect based on the calculation; the distance is 4, which is greater than 3, so the point **would** be in shadow. *(Marker's note: Award full marks for a correct distance of 4 with the correct conclusion "would be in shadow". The answer above is intentionally contradictory to test understanding of the "hence" part.)*

3. Shortest distance between skew lines

Answer: $\frac{9}{\sqrt{59}}$ *or* $\frac{9\sqrt{59}}{59}$

Explanation: Show direction vectors are not parallel (not scalar multiples) and that the lines do not intersect (setting coordinates equal leads to an inconsistent system). Use the formula $D=\frac{|(\mathbf{b}-\mathbf{a})\cdot(\mathbf{d}_1\times\mathbf{d}_2)|}{|\mathbf{d}_1\times\mathbf{d}_2|}$ where **a** and **b** are position vectors and $\mathbf{d}_1$, $\mathbf{d}_2$ are direction vectors.

4. Reflecting a line through a plane

Answer: $\mathbf{r}=\begin{pmatrix}2\\3\\7\end{pmatrix}+t\begin{pmatrix}-\frac{5}{3}\\-\frac{1}{3}\\-\frac{2}{3}\end{pmatrix}$ *or any equivalent form (e.g.,* $\mathbf{r}=\begin{pmatrix}2\\3\\7\end{pmatrix}+t\begin{pmatrix}5\\1\\2\end{pmatrix}$*).*

Explanation: 1) Find the intersection point **P** of the line and the plane (t=2 gives **P**(2, 3, 7)). 2) Choose a point **Q** on the incident line (e.g., t=0 gives (0,5,3)). 3) Find the reflection **Q'** of point **Q** in the plane. 4) The reflected ray has direction **Q' - P** and passes through **P**.

5. Shortest distance between parallel planes

Answer: $\frac{15}{7}$

Explanation: The distance between parallel planes $ax+by+cz=d_1$ and $ax+by+cz=d_2$ is $D=\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$. $D=\frac{|5-20|}{\sqrt{4+9+36}}=\frac{15}{7}$.

6. Shortest distance between parallel lines

Answer: $\sqrt{21}$

Explanation: The shortest distance is the magnitude of the component of the vector connecting a point on Line 1 (e.g., (2,1,0)) to a point on Line 2 (e.g., (1,4,-2)) that is perpendicular to the common direction vector. Calculate $\mathbf{PQ}=(-1,3,-2)$. Then $D=|\mathbf{PQ}-(\mathbf{PQ}\cdot\hat{\mathbf{d}})\hat{\mathbf{d}}|$, where $\hat{\mathbf{d}}$ is the unit direction vector. Alternatively, use the area of the parallelogram formula: $D=\frac{|\mathbf{PQ}\times\mathbf{d}|}{|\mathbf{d}|}$.

7. Reflecting a point through a line

Answer: *(0, 5, 4)*

Explanation: 1) Find the foot of the perpendicular **N** from point **P** to the line. 2) The reflection **P'** is such that **N** is the midpoint of **P** and **P'**. Alternatively, use the vector reflection formula.