Vector Problem Solving

Q1.

The plane Π has equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

where λ and μ are scalar parameters.

(a) Show that vector $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ is perpendicular to Π.

(2)

(b) Hence find a Cartesian equation of Π.

(2)

The line l has equation

$$\mathbf{r} = \begin{pmatrix} 4 \\ -5 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 6 \\ -3 \end{pmatrix}$$

where t is a scalar parameter.

The point A lies on l.

Given that the shortest distance between A and Π is $2\sqrt{29}$

(c) determine the possible coordinates of A.

(4)

(Total for question = 8 marks)

Q2.

The line l has equation

$$\frac{x+2}{1} = \frac{y-5}{-1} = \frac{z-4}{-3}$$

The plane Π has equation

$$\mathbf{r}.(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = -7$$

Determine whether the line l intersects Π at a single point, or lies in Π, or is parallel to Π without intersecting it.

(Total for question = 5 marks)

Q3.

The surface of a horizontal tennis court is modelled as part of a horizontal plane, with the origin on the ground at the centre of the court, and

- **i** and **j** are unit vectors directed across the width and length of the court respectively
- **k** is a unit vector directed vertically upwards
- units are metres

After being hit, a tennis ball, modelled as a particle, moves along the path with equation

$$\mathbf{r} = (-4.1 + 9\lambda - 2.3\lambda^2)\mathbf{i} + (-10.25 + 15\lambda)\mathbf{j} + (0.84 + 0.8\lambda - \lambda^2)\mathbf{k}$$

where λ is a scalar parameter with $\lambda \geq 0$

Assuming that the tennis ball continues on this path until it hits the ground,

(a) find the value of λ at the point where the ball hits the ground.

(2)

The direction in which the tennis ball is moving at a general point on its path is given by

$$(9 - 4.6\lambda)\mathbf{i} + 15\mathbf{j} + (0.8 - 2\lambda)\mathbf{k}$$

(b) Write down the direction in which the tennis ball is moving as it hits the ground.

(1)

(c) Hence find the acute angle at which the tennis ball hits the ground, giving your answer in degrees to one decimal place.

(4)

The net of the tennis court lies in the plane $\mathbf{r} . \mathbf{j} = 0$

(d) Find the position of the tennis ball at the point where it is in the same plane as the net.

(3)

The maximum height above the court of the top of the net is 0.9 m.

Modelling the top of the net as a horizontal straight line,

(e) state whether the tennis ball will pass over the net according to the model, giving a reason for your answer.

(1)

With reference to the model,

(f) decide whether the tennis ball will actually pass over the net, giving a reason for your answer.

(2)

(Total for question = 13 marks)

Q4.

Two birds are flying towards their nest, which is in a tree.

Relative to a fixed origin, the flight path of each bird is modelled by a straight line.

In the model, the equation for the flight path of the first bird is

$$\mathbf{r}_1 = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ a \\ 0 \end{pmatrix}$$

and the equation for the flight path of the second bird is

$$\mathbf{r}_2 = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

where λ and μ are scalar parameters and a is a constant.

In the model, the angle between the birds' flight paths is 120°

(a) Determine the value of a.

(4)

(b) Verify that, according to the model, there is a common point on the flight paths of the two birds and find the coordinates of this common point.

(5)

The position of the nest is modelled as being at this common point.

The tree containing the nest is in a park.

The ground level of the park is modelled by the plane with equation

$$2x - 3y + z = 2$$

(c) Hence determine the shortest distance from the nest to the ground level of the park.

(3)

(d) By considering the model, comment on whether your answer to part (c) is reliable, giving a reason for your answer.

(1)

(Total for question = 13 marks)

Q5.

Part of the mains water system for a housing estate consists of water pipes buried beneath the ground surface. The water pipes are modelled as straight line segments. One water pipe, W, is buried beneath a particular road. With respect to a fixed origin O, the road surface is modelled as a plane with equation $3x - 5y - 18z = 7$, and W passes through the points $A(-1, -1, -3)$ and $B(1, 2, -3)$. The units are in metres.

(a) Use the model to calculate the acute angle between W and the road surface.

(5)

A point $C(-1, -2, 0)$ lies on the road. A section of water pipe needs to be connected to W from C.

(b) Using the model, find, to the nearest cm, the shortest length of pipe needed to connect C to W.

(6)

(Total for question = 11 marks)

Q6.

An octopus is able to catch any fish that swim within a distance of 2 m from the octopus's position.

A fish F swims from a point A to a point B.

The octopus is modelled as a fixed particle at the origin O.

Fish F is modelled as a particle moving in a straight line from A to B.

Relative to O, the coordinates of A are $(-3, 1, -7)$ and the coordinates of B are $(9, 4, 11)$, where the unit of distance is metres.

(a) Use the model to determine whether or not the octopus is able to catch fish F.

(7)

(b) Criticise the model in relation to fish F.

(1)

(c) Criticise the model in relation to the octopus.

(1)

(Total for question = 9 marks)

Mark Scheme

Q1.

Question	Scheme	Marks	AOs
(a)	$\begin{pmatrix}-1\\2\\1\end{pmatrix}.\begin{pmatrix}2\\3\\-4\end{pmatrix}=-2+6-4=0$ and $\begin{pmatrix}2\\0\\1\end{pmatrix}.\begin{pmatrix}2\\3\\-4\end{pmatrix}=4+0-4=0$ Alt: $\begin{pmatrix}-1\\2\\1\end{pmatrix}\times\begin{pmatrix}2\\0\\1\end{pmatrix}=\begin{pmatrix}2\times1-1\times0\\-(-1\times1-1\times2)\\-1\times0-2\times2\end{pmatrix}=\ldots$	M1	1.1b
	As $2\mathbf{i}+3\mathbf{j}-4\mathbf{k}$ is perpendicular to both direction vectors (two non-parallel vectors) of Π then it must be perpendicular to Π	A1	2.2a
		(2)	
(b)	$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}2\\3\\-4\end{pmatrix}=\begin{pmatrix}3\\3\\2\end{pmatrix}.\begin{pmatrix}2\\3\\-4\end{pmatrix}\Rightarrow\ldots$	M1	1.1a
	$2x+3y-4z=7$	A1	2.2a
		(2)	
(c)	$\dfrac{\lvert 2(4+t)+3(-5+6t)-4(2-3t)-7\rvert}{\sqrt{2^2+3^2+(-4)^2}}=2\sqrt{29}\Rightarrow t=\ldots$	M1	3.1a
	$t=-\dfrac{9}{8}$ and $t=\dfrac{5}{2}$	A1	1.1b
	$r=\begin{pmatrix}4\\-5\\2\end{pmatrix}-\dfrac{9}{8}\begin{pmatrix}1\\6\\-3\end{pmatrix}=\ldots$ or $r=\begin{pmatrix}4\\-5\\2\end{pmatrix}+\dfrac{5}{2}\begin{pmatrix}1\\6\\-3\end{pmatrix}=\ldots$	M1	1.1b
	$\left(\dfrac{23}{8},-\dfrac{47}{4},\dfrac{43}{8}\right)$ and $\left(\dfrac{13}{2},10,-\dfrac{11}{2}\right)$	A1	2.2a
		(4)	
		(8 marks)	

Notes:

(a)

M1: Attempts the scalar product of each direction vector and the vector $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$. Some numerical calculation is required, just "= 0" is insufficient. Alternatively, attempts the cross product (allow sign slips) with the two direction vectors.

A1: Shows that both scalar products = 0 (minimum $-2+6-4=0$ and $4-4=0$) and makes a minimal conclusion with no erroneous statements. If using cross product, the calculation must be correct, and a minimal conclusion given with no erroneous statements.

(b)

M1: Applies $\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} . \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \Rightarrow ...$

A1: $2x+3y-4z=7$

(c)

M1: A fully correct method for finding a value of t. Other methods are possible, but must be valid and lead to a value of t. Examples of other methods:

- $2\sqrt{29} = \pm\left(\dfrac{2(4+t)+3(-5+6t)-4(2-3t)}{\sqrt{2^2+3^2+(-4)^2}} - \dfrac{7}{\sqrt{29}}\right)$ using plane parallel to Π through origin and shortest distance from plane to origin.
- $2(4+t)+3(-5+6t)-4(2-3t)=7 \Rightarrow t=t_i$ (t at intersection of line and plane) and $\sin\theta = \dfrac{(2,3,-4)^T.(1,6,-3)^T}{\sqrt{29}\sqrt{46}}$ (sine of angle between line and plane) followed by $\sin\theta = \dfrac{2\sqrt{29}}{k\sqrt{46}} \Rightarrow k = ... \Rightarrow t = t_i \pm k$

A1: Correct values for t. Both are required.

M1: Uses a value of t to find a set of coordinates for A.

A1: Both correct sets of coordinates for A.

Q2.

Question	Scheme	Marks	AOs
	$(\mathbf{r}=)\begin{pmatrix}-2+\lambda\\5-\lambda\\4-3\lambda\end{pmatrix}$ or $\begin{pmatrix}-2\\5\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\-3\end{pmatrix}$ (oe)	M1	1.1b
	So meet if $\begin{pmatrix}-2+\lambda\\5-\lambda\\4-3\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix}=-7\Rightarrow(-2+\lambda)\times1+(5-\lambda)\times-2+(4-3\lambda)\times1=-7$	M1 A1	3.1a 1.1b
	$\Rightarrow 0\lambda-8=-7\Rightarrow -8=-7$ a contradiction so no intersection	A1ft	2.3
	Hence l is parallel to Π but not in it.	A1cso	3.2a
		(5)	
		(5 marks)	

Notes		
	M1	Forms a parametric form for the line. Allow one slip.
	M1	Substitutes into the equation of the plane to an equation in λ. May use Cartesian form of plane to substitute into.
	A1	Correct equation in λ
	A1ft	Simplifies and derives a contradiction and deduces line and plane do not meet. Follow through in their initial equation in λ so - contradiction so no intersection if λ disappears and constants unequal - line lies in plane if a tautology is arrived at - meet in a point if a solution for λ is found. **But do not allow for incorrect simplification from a correct initial equation in λ** Note that a miscopy/misread of 7 instead of -7 can therefore score a maximum of M1M1A0A1A0.
	A1cso	Correct deduction from correct working. This may be seen two separate statements in their working. You may see attempts at showing the line is parallel before/after deducing there is no intersection.

Alt 1	Note that some may attempt a mix of the main scheme and Alt 1. Mark under main scheme unless Alt 1 would score higher.		
	$\begin{pmatrix}1\\-1\\-3\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix}=1\times1+(-1)\times(-2)+(-3)\times1=0$	M1	3.1a
	Hence l is parallel to Π	A1	1.1b
	$(-2,5,4)$ on l, but $(1)(-2)+(-2)(5)+1(4)=-8$	M1	1.1b
	$-8\neq-7$ so $(-2,5,4)$ is not on the plane.	A1ft	2.3
	Hence l is (parallel to Π but) not in the plane.	A1cso	3.2a
		(5)	
		(5 marks)	

Alt 1 Notes		
	M1	Attempts the dot product between the two direction vectors.
	A1	Shows dot product is zero and makes the correct deduction that line is parallel to plane.
	M1	Finds a point on l and substitutes into the equation of Π (vector or Cartesian)
	A1ft	Simplifies and derives a contradiction – follow through their equation, so if arrive at a tautology, they should deduce the line is in the plane.
	A1cso	Correct deduction from correct working but may be split across working.

Question	Scheme	Marks	AOs
Alt 2	Attempts to solve $\dfrac{x+2}{1}=\dfrac{y-5}{-1}=\dfrac{z-4}{-3}$ and $x-2y+z=-7$ simultaneously – eliminates one variable for M mark.	M1	3.1a
	e.g. $y=-(x+2)+5=-x+3 \Rightarrow x-2(-x+3)+z=-7 \Rightarrow 3x+z=-1$ (oe)	A1	1.1b
	Solves reduced equations, e.g. $-3(x+2)=z-4 \Rightarrow 3x+z=-2$ and $3x+z=-1 \Rightarrow (3x+z)-(3x+z)=-2-(-1)$	M1	1.1b
	$\Rightarrow 0=-1$ a contradiction so no intersection	A1ft	2.3
	Hence l is parallel to Π but not in it.	A1cso	3.2a
		(5)	
	(5 marks)		

Alt 2 notes		
	M1	Attempts to solve the Cartesian equation of the line and plane, using the plane equation to eliminate one variable for the M.
	A1	Correct elimination of their chosen variable. (E.g may see $3-3y+z=-7$ or $-2x-2y-2=-7$ etc)
	M1	Solves the reduced equations in two variables…
	A1ft	… and derives a contradiction/line and plane do not meet. Follow through their result, so may reach a tautology and deduce lies in plane, or find single solution and deduce meet in a point.
	A1cso	Correct deduction from correct working.

Q3.

Question	Scheme	Marks	AOs
(a)	Need **k** component to be zero at ground, so $0.84+0.8\lambda-\lambda^2=0 \Rightarrow \lambda=...$	M1	1.1b
	$\lambda=-\frac{3}{5},\frac{7}{5}$, but $\lambda \geqslant 0$ so $\lambda=\frac{7}{5}$	A1	1.1b
		(2)	
(b)	Direction is $(9-4.6\times1.4)\mathbf{i}+15\mathbf{j}+(0.8-2\times1.4)$ $=2.56\mathbf{i}+15\mathbf{j}-2\mathbf{k}$ or $\frac{64}{25}\mathbf{i}+15\mathbf{j}-2\mathbf{k}$	B1ft	2.2a
		(1)	

(c)	Direction perpendicular to ground is $a\mathbf{k}$, so angle to perpendicular is given by $(\cos\theta) = \dfrac{a\mathbf{k}.(2.56\mathbf{i}+15\mathbf{j}-2\mathbf{k})}{a\times\lvert 2.56\mathbf{i}+15\mathbf{j}-2\mathbf{k}\rvert}$ or $\dfrac{\begin{pmatrix}2.56\\15\\-2\end{pmatrix}\bullet\begin{pmatrix}0\\0\\a\end{pmatrix}}{\begin{vmatrix}2.56\\15\\-2\end{vmatrix}\begin{vmatrix}0\\0\\a\end{vmatrix}}$ or angle between $\begin{pmatrix}2.56\\15\\-2\end{pmatrix}$ and $\begin{pmatrix}2.56\\15\\0\end{pmatrix}$ is given by $(\cos\theta) = \dfrac{\begin{pmatrix}2.56\\15\\-2\end{pmatrix}\bullet\begin{pmatrix}2.56\\15\\0\end{pmatrix}}{\begin{vmatrix}2.56\\15\\-2\end{vmatrix}\begin{vmatrix}2.56\\15\\0\end{vmatrix}}$	M1	1.1b
	$= \dfrac{-2}{\sqrt{2.56^2+15^2+(-2)^2}}\ (= -0.130\ldots)$ Or $= \dfrac{231.5536}{\sqrt{2.56^2+15^2+(-2)^2}\sqrt{2.56^2+15^2+(0)^2}} = 0.991\ldots$	M1	1.1b
	$90° - \arccos('-0.130...') = -7.48...$ or $\arccos(0.991...)$	ddM1	3.1b
	So the tennis ball hits ground at angle of $7.5°$ (1d.p.) cao	A1	3.2a
	Alternative Finds the length of the vector in the **ij** plane $= \sqrt{2.56^2+15^2}$	M1	1.1b
	$\tan\theta = \dfrac{2}{\sqrt{2.56^2+15^2}}$	M1	1.1b
	$\theta = \arctan\left(\dfrac{2}{\sqrt{2.56^2+15^2}}\right)$ or $\theta = 90 - \arctan\left(\dfrac{\sqrt{2.56^2+15^2}}{2}\right)$	ddM1	3.1b

	So the tennis ball hits ground at angle of $7.5°$ (1d.p.)	A1	3.2a
		(4)	
(d)	In same plane as net when $\mathbf{r}.\mathbf{j}=0$, $\begin{pmatrix}-4.1+9\lambda-2.3\lambda^2\\-10.25+15\lambda\\0.84+0.8\lambda-\lambda^2\end{pmatrix}\bullet\begin{pmatrix}0\\1\\0\end{pmatrix}$ leading to $-10.25+15\lambda=0\Rightarrow\lambda=...$ $\left(=\frac{41}{60}=0.683333...\right)$	M1	3.1b
	So is at position $\left(-4.1+9\times\frac{41}{60}-2.3\left(\frac{41}{60}\right)^2\right)\mathbf{i}+0\mathbf{j}+\left(0.84+0.8\times\frac{41}{60}-\left(\frac{41}{60}\right)^2\right)\mathbf{k}$	M1	1.1b
	$=$ awrt $0.976\mathbf{i}+$ awrt $0.920\mathbf{k}$ or $=$ awrt $0.976\mathbf{i}+0.92\mathbf{k}$ (to 3 s.f.) or $=$ awrt $0.976\mathbf{i}+\frac{3311}{3600}\mathbf{k}$	A1	1.1b
		(3)	
(e)	Modelling as a line, height of net is 0.9m along its length so as **0.92 > 0.9** the ball **will** pass over the net according to the model.	B1ft	3.2a
		(1)	
(f)	Identifies a suitable feature of the model that affects the outcome And uses it to draw a compatible conclusion. For example • The ball is **not a particle** and will have **diameter/radius**, therefore it will hit the net and not pass over. • As above, but so the **ball will clip the net** but it's momentum will take it over as it is mostly above the net. • The **model** says that the ball will clear the net by 2cm which may be smaller than the balls **diameter** • The **net will not be a straight line/taut** so will not be 0.9m high, so the ball **will** have enough clearance to pass over the net.	M1 A1	3.2b 2.2b
		(2)	
			(13 marks)

Notes:

Accept any alternative vector notations throughout.

(a)

M1: Attempts to solve the quadratic from equating the **k** component to zero.

A1: Correct value, must select positive root, so accept 1.4 oe.

Correct answer only M1 A1

(b)

B1ft: For $(2.56, 15, -2)$ o.e or follow through $(9-4.6\times'\lambda', 15, 0.8-2\times'\lambda')$ for their λ.

(c)

M1: Recognises the angle between the perpendicular and direction vector is needed, and identifies the perpendicular as a**k** for any non-zero a (including 1), and attempts dot product

Alternatively recognises the dot product of $(2.56, 15, -2)$ and $(2.56, 15, 0)$

M1: Applies the dot product formula $\frac{a \cdot b}{|a||b|}$ correctly between *any* two vectors, but must have dot product and modulus evaluated.

ddM1: Dependent on both previous marks. A correct method to proceed to the required angle, usually $90° - \arccos('-0.130...')$ as shown in scheme but may e.g. use $\sin\theta$ instead of $\cos\theta$ in formula.

Alternatively is using dot product of $(2.56, 15, -2)$ and $(2.56, 15, 0)$ finds arccos(0.991…)

A1: For 7.5° cao

Alternative

M1: Finds the length of the vector in the **ij** plane.

M1: Finds the tan of any angle the

ddM1: Dependent on both previous marks. Finds the required angle

A1: For 7.5° cao

(d)

M1: Attempts to find value of λ that gives zero **j** component.

M1: Uses their value of λ in the equation of the path to find position.

A1: Correct position.

(e)

B1ft: States that $0.920 > 0.9$ so according to the model the ball will pass over the net. Follow through on their **k** component and draws an appropriate conclusion. May stay the value of k > 0.92

(f)

M1: There must be some reference to the model to score this mark. See scheme for examples. It is likely to be either the ball is not a particle, or the top of the net is not a straight line. Accept references to the ball crossing a long way from the middle.

Do not accept reasons such as "there may be wind/air resistance" as these are not referencing the given model.

A1: For a reasonable conclusion based on their reference to the model.

For example

The ball is not a particle; therefore, it will not go over the net is M1A0 as not explained why – needs reference to radius/diameter

Q4.

Question	Scheme	Marks	AOs
(a)	A complete method to use the scalar product of the direction vectors and the angle 120° to form an equation in a $$\frac{\begin{pmatrix}2\\a\\0\end{pmatrix}\bullet\begin{pmatrix}0\\1\\-1\end{pmatrix}}{\sqrt{2^2+a^2}\sqrt{1^2+(-1)^2}}=\cos 1\,20$$	M1	3.1b
	$$\frac{a}{\sqrt{4+a^2}\sqrt{2}}=-\frac{1}{2}$$	A1	1.1b
	$2a=-\sqrt{4+a^2}\sqrt{2}\Rightarrow 4a^2=8+2a^2\Rightarrow a^2=4\Rightarrow a=\ldots$	M1	1.1b
	$a=-2$	A1	2.2a
		(4)	
(b)	Any two of **i**: $-1+2\lambda=4$ (1) **j**: $5+\text{'their}-2\text{'}\lambda=-1+\mu$ (2) **k**: $2=3-\mu$ (3)	M1	3.4
	Solves the equations to find a value of $\lambda\left\{=\frac{5}{2}\right\}$ **and** $\mu\{=1\}$	M1	1.1b
	$r_1=\begin{pmatrix}-1\\5\\2\end{pmatrix}+\frac{5}{2}\begin{pmatrix}2\\\text{'their}-2\text{'}\\0\end{pmatrix}$ or $r_2=\begin{pmatrix}4\\-1\\3\end{pmatrix}+1\begin{pmatrix}0\\1\\-1\end{pmatrix}$	dM1	1.1b
	$(4,0,2)$ or $\begin{pmatrix}4\\0\\2\end{pmatrix}$	A1	1.1b
	Checks the third equation e.g. $\lambda=\frac{5}{2}$: **L HS** $=5-2\lambda=5-5=0$ $\mu=1$: **R HS** $=-1+\mu=-1+1=0$ therefore **common point/intersect/consistent/tick** or substitutes the values of λ and μ into the relevant lines and achieves the same coordinate	B1	2.1
		(5)	

(c)	Full attempt to find the minimum distance from the point of intersection (nest) to the plane (ground) E.g. Minimum distance $=\frac{\lvert 2\times\text{'4'}+(-3)\times\text{'0'}+1\times\text{'2'}-2\rvert}{\sqrt{2^2+(-3)^2+1)^2}}=\ldots$ **Alternatively** $\mathbf{r}=\begin{pmatrix}\text{'4'}\\\text{'0'}\\\text{'2'}\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\\1\end{pmatrix}$ $2(\text{'4'}+2\lambda)-3(\text{'0'}-3\lambda)+(\text{'2'}+\lambda)=2\Rightarrow$ $\lambda=\ldots\left\{-\frac{4}{7}\right\}$	M1 A1ft	3.1b 3.4

	$$\mathbf{r} = \begin{pmatrix} '4' \\ '0' \\ '2' \end{pmatrix} + '-\frac{4}{7}'\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{20}{7} \\ \frac{12}{7} \\ \frac{10}{7} \end{pmatrix}$$ Minimum distance $= \sqrt{\left(2 \times -\frac{4}{7}'\right)^2 + \left(-3 \times -\frac{4}{7}'\right)^2 + \left(1 \times -\frac{4}{7}\right)^2} = \ldots$ $$= \sqrt{\left('4' - '\frac{20}{7}'\right)^2 + \left('0' - '\frac{12}{7}'\right)^2 + \left('2' - '\frac{10}{7}'\right)^2} = \ldots$$		
	$\frac{8}{\sqrt{14}}$ or $\frac{4\sqrt{14}}{7}$or awrt 2.1	A1	2.2b
		(3)	
	Alternative Find perpendicular distance from plane to the origin $2x - 3y + z = 2$ $\lvert n \rvert = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}$ shortest distance $= \frac{2}{\sqrt{14}}$ Find perpendicular distance from the plane containing the point of intersection to the origin $2x - 3y + z = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 10$ shortest distance $= \frac{10}{\sqrt{14}}$ Minimum distance $= \frac{10}{\sqrt{14}} - \frac{2}{\sqrt{14}}$	M1 A1ft	3.1b 3.4
	$\frac{8}{\sqrt{14}}$ or $\frac{4\sqrt{14}}{7}$or awrt 2.1	A1	2.2b
		(3)	
(d)	For example Not reliable as the birds will not fly in a straight line Not reliable as angle between flights paths will not always be 120° Not reliable/reliable as the ground will not be flat/smooth Not reliable as bird's nest is not a point	B1	3.2b
		(1)	
			(13 marks)

Notes:

(a)

M1: See scheme, allow a sign slip and cos 60

A1: Correct simplified equation in a, cos 120 must be evaluated to – ½ and dot product calculated

Note: If the candidate states either $\left|\frac{a \square b}{|a||b|}\right| = \cos\theta$ or $\left|\frac{a}{\sqrt{4+a^2}\sqrt{2}}\right| = \cos 6\,0$then has the equation $\frac{a}{\sqrt{4+a^2}\sqrt{2}} = \frac{1}{2}$ award this mark. If the module of the dot product is not seen then award A0 for this equation.

dM1: Solve a quadratic equation for a, by squaring and solving an equation of the form $a^2 = K$ where $K > 0$

A1: Deduces the correct value of a from a correct equation. Must be seen in part (a) using the angle between the lines.

Alternative cross product method

M1: $\begin{vmatrix} 2 & a & 0 \\ 0 & 1 & -1 \end{vmatrix} = \sqrt{2^2 + a^2}\sqrt{1^2 + (-1)^2}\,sin\,1\,20$

A1: $\sqrt{a^2 + 8} = \sqrt{4 + a^2}\sqrt{2}\frac{\sqrt{3}}{2}$

Then as above

Note If they use the point of intersection to find a value for *a* this scores no marks

(b)

M1: Uses the model to write down any two correct equations

M1: Solve two equations simultaneously to find a value for μ and λ

dM1: Dependent on previous method mark. Substitutes μ and λ into a relevant equation. If no method shown two correct ordinates implies this mark.

A1: Correct coordinates. May be seen in part (c)

B1: Shows that the values of μ and λ give the same third coordinate or point of intersection and draws the conclusion that the **lines intersect/common point/consistent** or tick.

Note: If an incorrect value for a is found in part (a) but in part (b) they find that $a = -2$ this scores **B0** but all other marks are available

(c) This is M1M1A1 on ePen marking as M1 A1ft A1

M1: Full attempt to find the minimum distance from a point to a plane. Condone a sign slip with the value of d.

A1ft: Following through on their point of intersection. Uses the model to find a correct expression for minimum distance from the nest to the ground

A1: Correct distance

Alternative

M1: Find the shortest distance from a point to plane by finding the perpendicular distance from the given plane to the origin and the perpendicular distance from the plane contacting their point of intersection to the origin and subtracts

A1ft: Following through on their point of intersection. Uses the model to find a correct expression for minimum distance from the nest to the ground

A1: Correct distance

(d)

B1: Comments on one of the models

- Flight path of the birds modelled as a straight line
- Angle between flight paths modelled as 120°
- The bird's nest is modelled as a point
- Ground modelled as a plane

Then states unreliabl

Any correct answer seen, ignore any other incorrect answers

Q5.

Question	Scheme	Marks	AOs
(a)	Attempts the scalar product between the direction of W and the normal to the road and uses trigonometry to find an angle.	M1	3.1a
	$\left(\begin{pmatrix}1\\2\\-3\end{pmatrix}-\begin{pmatrix}-1\\-1\\-3\end{pmatrix}\right)\bullet\begin{pmatrix}3\\-5\\-18\end{pmatrix}=-9$ or $\left(\begin{pmatrix}-1\\-1\\-3\end{pmatrix}-\begin{pmatrix}1\\2\\-3\end{pmatrix}\right)\bullet\begin{pmatrix}3\\-5\\-18\end{pmatrix}=9$	M1 A1	1.1b 1.1b
	$\sqrt{(2)^2+(3)^3+(0)^2}\sqrt{(3)^2+(-5)^3+(-18)^2}\cos\alpha="-9"$ $\theta=90-\arccos\left(\frac{9}{\sqrt{13}\sqrt{358}}\right)$ or $\theta=\arcsin\left(\frac{9}{\sqrt{13}\sqrt{358}}\right)$ Angle between pipe and road $=7.58°$ (3sf) or 0.132 radians (3sf) (Allow $-7.58°$ or -0.132 radians)	M1 A1	1.1b 3.2a
		(5)	

(b)	$W:\begin{pmatrix}-1\\-1\\-3\end{pmatrix}+t\begin{pmatrix}2\\3\\0\end{pmatrix}$ or $\begin{pmatrix}1\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\0\end{pmatrix}$	B1ft	1.1b
	$C\text{ to }W:\left\{\begin{pmatrix}-1\\-1\\-3\end{pmatrix}+t\begin{pmatrix}2\\3\\0\end{pmatrix}-\begin{pmatrix}-1\\-2\\0\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}1\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\0\end{pmatrix}-\begin{pmatrix}-1\\-2\\0\end{pmatrix}\right\}$	M1	3.4
	$\begin{pmatrix}2t\\3t+1\\-3\end{pmatrix}\bullet\begin{pmatrix}2\\3\\0\end{pmatrix}=0\Rightarrow t=\ldots$ or $\begin{pmatrix}2+2\lambda\\4+3\lambda\\-3\end{pmatrix}\bullet\begin{pmatrix}2\\3\\0\end{pmatrix}=0\Rightarrow\lambda=\ldots$ or $(2t)^2+(3t+1)^2+(-3)^2=\ldots$ or $(2+2t)^2+(4+3t)^2+(-3)^2=\ldots$	M1	3.1b
	$t=-\frac{3}{13}$ or $\lambda=-\frac{16}{13}\Rightarrow(C\text{ to }W)_{\min}$ is $-\frac{6}{13}\mathbf{i}+\frac{4}{13}\mathbf{j}-3\mathbf{k}$ or $(2t)^2+(3t+1)^2+(-3)^2=13\left(t+\frac{3}{13}\right)^2+\frac{121}{13}$ or $(2+2t)^2+(4+3t)^2+(-3)^2=13\left(\lambda+\frac{16}{13}\right)^2+\frac{121}{13}$ or $\frac{\mathrm{d}\left((2t)^2+(3t+1)^2+(-3)^2\right)}{\mathrm{d}t}=0\Rightarrow t=-\frac{3}{13}\Rightarrow C\text{ to }W\text{ is }-\frac{6}{13}\mathbf{i}+\frac{4}{13}\mathbf{j}-3\mathbf{k}$ Or $\frac{\mathrm{d}\left((2+2t)^2+(4+3t)^2+(-3)^2\right)}{\mathrm{d}t}=0\Rightarrow t=-\frac{16}{13}\Rightarrow(C\text{ to }W)_{\min}\text{ is }-\frac{6}{13}\mathbf{i}+\frac{4}{13}\mathbf{j}-3\mathbf{k}$	A1	1.1b
	$d=\sqrt{\left(-\frac{6}{13}\right)^2+\left(\frac{4}{13}\right)^2+(-3)^2}$ or $d=\sqrt{\frac{121}{13}}$	ddM1	1.1b

	Shortest length of pipe needed is 305 or 305 cm or 3.05 m	A1	3.2a
		(6)	
			(11 marks)

Notes

(a)

M1: Realises the scalar product between the direction of W and the normal to the road is needed and so applies it and uses trigonometry to find an angle

M1: Calculates the scalar product between $\pm\left(\begin{pmatrix}1\\2\\-3\end{pmatrix}-\begin{pmatrix}-1\\-1\\-3\end{pmatrix}\right)$ and $\pm\begin{pmatrix}3\\-5\\-18\end{pmatrix}$ (Allow sign slips as long as the intention is clear)

A1: $\begin{pmatrix}2\\3\\0\end{pmatrix}\bullet\begin{pmatrix}3\\-5\\-18\end{pmatrix}=-9$ or $\begin{pmatrix}-2\\-3\\0\end{pmatrix}\bullet\begin{pmatrix}3\\-5\\-18\end{pmatrix}=9$ or $\begin{pmatrix}2\\3\\0\end{pmatrix}\bullet\begin{pmatrix}-3\\5\\18\end{pmatrix}=9$ or $\begin{pmatrix}-2\\-3\\0\end{pmatrix}\bullet\begin{pmatrix}-3\\5\\18\end{pmatrix}=-9$

M1: A fully complete and correct method for obtaining the acute angle

A1: Awrt 7.58° or awrt 0.132 radians (**must see units**). Do not isw and withhold this mark if extra answers are given.

(b)

B1ft: Forms the correct parametric form for the pipe W. Follow through their direction vector for W from part (a).

M1: Identifies the need to and forms a vector connecting C to W using a parametric form for W

M1: Uses the model to form the scalar product of C to W and the direction of W to find the value of their parameter **or** finds the distance C to W or (C to W)^2 in terms of their parameter

A1: Correct vector or correct completion of the square

ddM1: Correct use of Pythagoras on their vector CW or appropriate method to find the shortest distance between the point and the pipe. **Dependent on both previous method marks.**

A1: Correct length for the required section of pipe is 305 or 305 cm or 3.05 m

Alternatives for part (b):

(b) Way 2	$\mathbf{AC} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}, \quad \mathbf{AB} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$	B1ft	1.1b
	$\mathbf{AC.AB} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} . \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = 3$	M1	3.4
	$\Rightarrow \cos CAB = \dfrac{3}{\sqrt{10}\sqrt{13}} \Rightarrow CAB = \ldots$	M1	3.1b
	$CAB = 74.74....°$	A1	1.1b
	$d = \sqrt{10}\sin 74.74....°$	ddM1	1.1b
	Shortest length of pipe needed is 305 or 305 cm or 3.05 m	A1	3.2a
		(6)	

	Notes		
	(b) B1ft: Forms the correct vectors. Follow through their direction vector for W from part (a). M1: Identifies the need to and forms the scalar product between **AC** and **AB** M1: Uses the model to form the scalar product and uses this to find the angle CAB A1: Correct angle ddM1: Correct method using their values or appropriate method to find the shortest distance between the point and the pipe. **Dependent on both previous method marks.** A1: Correct length for the required section of pipe is 305 or 305 cm or 3.05 m		

(b) Way 3	$\mathbf{AC}=\begin{pmatrix}0\\1\\-3\end{pmatrix},\quad \mathbf{AB}=\begin{pmatrix}2\\3\\0\end{pmatrix}$	B1ft	1.1b
	$\mathbf{AC}\times\mathbf{AB}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&-1&3\\2&3&0\end{vmatrix}=\begin{pmatrix}-9\\6\\2\end{pmatrix}$	M1	3.4
	$\lvert\mathbf{AC}\times\mathbf{AB}\rvert=\sqrt{9^2+6^2+2^2}=\ldots$	M1	3.1b
	$=11$	A1	1.1b
	$d=\frac{11}{\lvert\mathbf{AB}\rvert}=\frac{11}{\sqrt{2^2+3^2}}=\ldots$	ddM1	1.1b
	Shortest length of pipe needed is 305 or 305 cm or 3.05 m	A1	3.2a
		(6)	
	Notes		
	(b) B1ft: Forms the correct vectors. Follow through their direction vector for W from part (a). M1: Identifies the need to and forms the vector product between **AC** and **AB** M1: Uses the model to find the magnitude of their vector product A1: Correct value ddM1: Correct method using their values or appropriate method to find the shortest distance between the point and the pipe. **Dependent on both previous method marks.** A1: Correct length for the required section of pipe is 305 or 305 cm or 3.05 m		

Q6.

Question	Scheme	Marks	AOs
(a)	$\overrightarrow{AB} = \begin{pmatrix} 9 \\ 4 \\ 11 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} \left\{= \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix}\right\}$ or $\mathbf{d} = \begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix}$	M1	3.1a
	$\left\{\overrightarrow{OF} = \mathbf{r} = \right\} \begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix}$	M1	1.1b
	$\left\{\overrightarrow{OF} \bullet \overrightarrow{AB} = 0 \Rightarrow \right\} \begin{pmatrix} -3+12\lambda \\ 1+3\lambda \\ -7+18\lambda \end{pmatrix} \bullet \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix} = 0$	M1	1.1b
	$\Rightarrow -36 + 144\lambda + 3 + 9\lambda - 126 + 324\lambda = 0 \Rightarrow 477\lambda - 159 = 0$		
	$\Rightarrow \lambda = \frac{1}{3}$	A1	1.1b
	$\left\{\overrightarrow{OF} = \right\} \begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and minimum distance $= \sqrt{(1)^2 + (2)^2 + (-1)^2}$	M1	3.1a
	$= \sqrt{6}$ or $2.449\ldots$	A1	1.1b
	> 2, so the octopus is not able to catch the fish F.	A1ft	3.2a
		(7)	
(b)	E.g. Fish F may not swim in an exact straight line from A to B. Fish F may hit an obstacle whilst swimming from A to B. Fish F may deviate his path to avoid being caught by the octopus.	B1	3.5b
		(1)	
(c)	E.g. Octopus is effectively modelled as a particle – so we may need to look at where the octopus's mass is distributed. Octopus may during the fish F's motion move away from its fixed location at O.	B1	3.5b
		(1)	
			(9 marks)

Question	Scheme	Marks	AOs
(a) ALT 1	$\overrightarrow{AB}=\begin{pmatrix}9\\4\\11\end{pmatrix}-\begin{pmatrix}-3\\1\\-7\end{pmatrix}\left\{=\begin{pmatrix}12\\3\\18\end{pmatrix}\right\}$ or $\mathbf{d}=\begin{pmatrix}4\\1\\6\end{pmatrix}$	M1	3.1a
	$\left\{\overrightarrow{OA}=\begin{pmatrix}-3\\1\\-7\end{pmatrix}\text{ and }\overrightarrow{AB}=\begin{pmatrix}12\\3\\18\end{pmatrix}\Rightarrow\right\}\begin{pmatrix}-3\\1\\-7\end{pmatrix}\bullet\begin{pmatrix}12\\3\\18\end{pmatrix}$	M1	1.1b
	$\cos\theta\left\{=\dfrac{\overrightarrow{OA}\bullet\overrightarrow{AB}}{\lvert\overrightarrow{OA}\rvert.\lvert\overrightarrow{AB}\rvert}\right\}=\dfrac{\pm\left(\begin{pmatrix}-3\\1\\-7\end{pmatrix}\bullet\begin{pmatrix}12\\3\\18\end{pmatrix}\right)}{\sqrt{(-3)^2+(1)^2+(-7)^2}.\sqrt{(12)^2+(3)^2+(18)^2}}$	dM1	1.1b
	$\left\{\cos\theta=\dfrac{-36+3-126}{\sqrt{59}.\sqrt{477}}=\dfrac{-159}{\sqrt{59}.\sqrt{477}}\right\}$		
	$\theta=161.4038029...$ or $18.59619709...$ or $\sin\theta=0.3188964021...$	A1	1.1b
	minimum distance $=\sqrt{(-3)^2+(1)^2+(-7)^2}\sin(18.59619709...)$	dM1	3.1a
	$=\sqrt{6}$ or $2.449...$	A1	1.1b
	>2, so the octopus is not able to catch the fish F.	A1ft	3.2a
		(7)	

(a) ALT 2	$\overrightarrow{AB}=\begin{pmatrix}9\\4\\11\end{pmatrix}-\begin{pmatrix}-3\\1\\-7\end{pmatrix}\left\{=\begin{pmatrix}12\\3\\18\end{pmatrix}\right\}$ or $\mathbf{d}=\begin{pmatrix}4\\1\\6\end{pmatrix}$	M1	3.1a
	$\left\{\overrightarrow{OF}=\mathbf{r}=\right\}\begin{pmatrix}-3\\1\\-7\end{pmatrix}+\lambda\begin{pmatrix}12\\3\\18\end{pmatrix}$	M1	1.1b
	$\lvert\overrightarrow{OF}\rvert^2=(-3+12\lambda)^2+(1+3\lambda)^2+(-7+18\lambda)^2$	dM1	1.1b
	$=9-72\lambda+144\lambda^2+1+6\lambda+9\lambda^2+49-252\lambda+324\lambda^2$		
	$=477\lambda^2-318\lambda+59$	A1	1.1b
	$=53(3\lambda-1)^2+6$	dM1	3.1a
	minimum distance $=\sqrt{6}$ or $2.449...$	A1	1.1b
	>2, so the octopus is not able to catch the fish F.	A1ft	3.2a
		(7)	

	Question Notes	
(a)	M1	Attempts to find $\overrightarrow{OB}-\overrightarrow{OA}$ **or** $\overrightarrow{OA}-\overrightarrow{OB}$ **or** the direction vector **d**.
	M1	Applies $\overrightarrow{OA}+\lambda$(their $\overrightarrow{AB}$ or their $\overrightarrow{BA}$ or their **d**) or equivalent.
	M1	Depends on previous M mark. Writes down (their $\overrightarrow{OF}$ which is in terms of λ)$\bullet$(their $\overrightarrow{AB}$) $=0$. Can be implied.
	A1	Lambda is correct. e.g. $\lambda=\frac{1}{3}$ for $\overrightarrow{AB}=\begin{pmatrix}12\\3\\18\end{pmatrix}$ or $\lambda=1$ for $\mathbf{d}=\begin{pmatrix}4\\1\\6\end{pmatrix}$
	M1	Depends on previous M mark. Complete method for finding $\left\lvert\overrightarrow{OF}\right\rvert$.
	A1	$\sqrt{6}$ or awrt 2.4
	A1ft	Correct follow through conclusion, which is in context with the question.
(a) ALT 1	M1	Attempts to find $\overrightarrow{OB}-\overrightarrow{OA}$ **or** $\overrightarrow{OA}-\overrightarrow{OB}$ **or** the direction vector **d**.
	M1	Realisation that the dot product is required between $\overrightarrow{OA}$ and their $\overrightarrow{AB}$. (o.e.)
	M1	Depends on previous M mark. Applies dot product formula between $\overrightarrow{OA}$ and their $\overrightarrow{AB}$. (o.e.)
	A1	$\theta=$ awrt 161.4 or awrt 18.6 or $\sin\theta=$ awrt 0.319
	M1	Depends on previous M mark. (their OA)sin(their θ)
	A1	$\sqrt{6}$ or awrt 2.4
	A1ft	Correct follow through conclusion, which is in context with the question.
(a) ALT 2	M1	Attempts to find $\overrightarrow{OB}-\overrightarrow{OA}$ **or** $\overrightarrow{OA}-\overrightarrow{OB}$ **or** the direction vector **d**.
	M1	Applies $\overrightarrow{OA}+\lambda$(their $\overrightarrow{AB}$ or their $\overrightarrow{BA}$ or their **d**) or equivalent.
	M1	Depends on previous M mark. Applies Pythagoras by finding $\left\lvert\overrightarrow{OF}\right\rvert^2$, o.e.
	A1	$\left\lvert\overrightarrow{OF}\right\rvert^2=477\lambda^2-318\lambda+59$
	M1	Depends on previous M mark. Method of completing the square or differentiating their $\left\lvert\overrightarrow{OF}\right\rvert^2$ w.r.t. λ.
	A1	$\sqrt{6}$ or awrt 2.4
	A1ft	Correct follow through conclusion, which is in context with the question.
(b)	B1	An acceptable criticism for fish F, which is in context with the question.
(c)	B1	An acceptable criticism for the octopus, which is in context with the question.