DMP Problem Sheet - Vectors 2

1. A particle moves such that its position vector $\mathbf{r}$ at time t is given by $\mathbf{r}(t) = (2t^2 - 1)\mathbf{i} + (3t - 2)\mathbf{j} + (t^2 + t)\mathbf{k}$. Find the speed of the particle at the instant when $t = 1$.

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2. Given two vectors $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + m\mathbf{k}$, and that the angle θ between them is such that $\cos\theta = \frac{1}{2}$, determine the possible values of the scalar m.

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3. The points A, B, C and D have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ respectively. Given that $\vec{AB} = \mathbf{p}$ and $\vec{BC} = \mathbf{q}$, and that $\vec{AD} = 3\mathbf{q} - 2\mathbf{p}$, express the vector $\vec{CD}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

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3. Line L_1 has the vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. Line L_2 has the vector equation $\mathbf{r} = \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} k \\ 2 \\ -3 \end{pmatrix}$. Given that L_1 and L_2 are perpendicular, find the value of k.

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4. Given three points P(1, -2, 3), Q(3, 0, -1) and R(5, 2, -5), determine if they are collinear. Justify your answer with a calculation.

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5. A line passes through the point A(2, 1, -3) and is parallel to the vector $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Find the position vector of the point on this line which is closest to the origin.

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6. The position vectors of points A and B are $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + \mathbf{j} - 5\mathbf{k}$ respectively. The point P divides AB internally in the ratio 3:2. Find the unit vector in the direction of $\vec{OP}$.

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7. Two lines are defined by $L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. Show that these lines intersect and find the coordinates of the point of intersection, P.

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8. A vector $\mathbf{v}$ has a magnitude of 13. Its $\mathbf{i}$-component is -12, its $\mathbf{j}$-component is positive, and its $\mathbf{k}$-component is 0. Find the vector $\mathbf{v}$.

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9. The vertices of a triangle are A(1, 0, 2), B(3, 4, 1) and C(2, 1, 5). Calculate the exact area of triangle ABC.

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10. Given vectors $\mathbf{p} = 2\mathbf{i} + a\mathbf{j} + \mathbf{k}$ and $\mathbf{q} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$, find the value of the scalar a for which $\mathbf{p} + \mathbf{q}$ is perpendicular to $\mathbf{p} - \mathbf{q}$.

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11. A line L passes through the point P(4, -1, 3) and has direction vector $\mathbf{d} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. Find the coordinates of the point on L that is equidistant from the origin O(0, 0, 0) and the point Q(2, 0, 1).

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12. Find a vector $\mathbf{w}$ with a $\mathbf{k}$-component of 1 that satisfies the conditions $\mathbf{u} \cdot \mathbf{w} = 5$ and $\mathbf{v} \cdot \mathbf{w} = -3$, where $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$.

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13. The position vectors of points A and B are $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 5\mathbf{i} - \mathbf{j}$ respectively. Point C lies on AB such that $\vec{AC} = k\vec{AB}$. Given that the square of the magnitude of $\vec{OC}$ is 20, find the value of k.

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14. Find the exact shortest distance from the point P(1, 2, -1) to the line defined by $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

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15. A line L_1 has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$. Another line, L_2, passes through the point (3, 1, 4) and is parallel to L_1. Find a vector equation for L_2.

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16. Vectors $\mathbf{a} = \begin{pmatrix} p \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ q \\ 2 \end{pmatrix}$ satisfy $\mathbf{a} \cdot \mathbf{b} = 10$ and $|\mathbf{a}|^2 = 21$. Find the possible values of q in terms of p.

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17. The vertices of a quadrilateral are A(1, 1, 0), B(3, 2, 1), C(2, 4, 3) and D(0, 3, 2). By considering vector properties of its sides, determine the most specific type of quadrilateral that ABCD is.

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18. A vector $\mathbf{v}$ has magnitude $\sqrt{26}$. Its direction makes equal acute angles with the positive x and y axes, and its z-component is 4. Find the vector $\mathbf{v}$.

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19. A line passes through points A(1, 0, 1) and B(3, m, 0) and is perpendicular to the vector $\mathbf{n} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Find the value of m.

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20. A plane has a normal vector $\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and contains the point P(1, 2, 3). Find the vector equation of the line that passes through the origin and is perpendicular to this plane.

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21. Points A, B, C have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Point D is defined such that $\vec{AD} = 2\vec{AB} - 3\vec{BC}$. Express the position vector of D, $\mathbf{d}$, in terms of $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

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22. Find the acute angle between the line $L : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and the vector $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Express your answer as an inverse trigonometric function.

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24. Two forces $\mathbf{F}_1 = (2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ N and $\mathbf{F}_2 = (a\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ N act on a particle. The resultant force is perpendicular to $\mathbf{F}_1$. Find the value of a.

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25. A triangle has vertices O(0,0,0), P(1,2,1) and Q(3,1,2). Find the cosine of the angle $\angle POQ$.

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1. **Answer:** $\sqrt{38}$ units/s
   - **Explanation:** Differentiate $\mathbf{r}(t)$ to find velocity $\mathbf{v}(t)$. Evaluate $\mathbf{v}(1)$ and find its magnitude.
2. **Answer:** $m = 2$ or $m = 10$
   - **Explanation:** Use the dot product formula $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$. Substitute values and solve the resulting quadratic in m.
3. **Answer:** $\vec{CD} = 2\mathbf{q} - \mathbf{p}$
   - **Explanation:** Express all vectors in terms of position vectors (e.g., $\mathbf{p} = \mathbf{b} - \mathbf{a}$, $\mathbf{q} = \mathbf{c} - \mathbf{b}$). Find $\vec{CD} = \mathbf{d} - \mathbf{c}$ and simplify.
4. **Answer:** $k = -1$
   - **Explanation:** For lines to be perpendicular, the dot product of their direction vectors must be zero: $(2)(k) + (-1)(2) + (1)(-3) = 0$.
5. **Answer:** The points are collinear.
   - **Explanation:** Show $\vec{PQ}$ and $\vec{QR}$ are parallel (scalar multiples): $\vec{PQ} = (2, 2, -4)$, $\vec{QR} = (2, 2, -4)$.
6. **Answer:** $\frac{1}{14} \begin{pmatrix} 15 \\ -5 \\ 10 \end{pmatrix}$ or $\frac{1}{14}(15\mathbf{i} - 5\mathbf{j} + 10\mathbf{k})$

- **Explanation:** The closest point is the foot of the perpendicular from the origin to the line. Use the formula involving the dot product.

7. **Answer:** $\frac{1}{\sqrt{405}}(14\mathbf{i} - 6\mathbf{j} + 13\mathbf{k})$ or $\frac{1}{9\sqrt{5}}(14\mathbf{i} - 6\mathbf{j} + 13\mathbf{k})$
   - **Explanation:** Find position vector of P using section formula: $\mathbf{p} = \frac{2\mathbf{a}+3\mathbf{b}}{5}$. Find $\vec{OP} = \mathbf{p}$, then find its unit vector.
8. **Answer:** $P(3, 1, 2)$
   - **Explanation:** Equate coordinates of L_1 and L_2: $1 + 2\lambda = 0 + \mu$, $0 + \lambda = 1 - 2\mu$, $-1 + 3\lambda = 2 + \mu$. Solve for λ and μ (e.g., $\lambda = 1, \mu = 3$) and substitute back.
9. **Answer:** $\mathbf{v} = -12\mathbf{i} + 5\mathbf{j}$
   - **Explanation:** Let $\mathbf{v} = -12\mathbf{i} + y\mathbf{j}$. Solve $|\mathbf{v}|^2 = (-12)^2 + y^2 = 13^2$ for positive y.
10. **Answer:** Area = $\frac{1}{2}\sqrt{134}$ units^2
    - **Explanation:** Find vectors $\vec{AB}$ and $\vec{AC}$. Area = $\frac{1}{2}|\vec{AB} \times \vec{AC}|$. Calculate the cross product and its magnitude.
11. **Answer:** $a = -2$
    - **Explanation:** Find $\mathbf{p} + \mathbf{q}$ and $\mathbf{p} - \mathbf{q}$. Set their dot product to zero: $(3\mathbf{i} + (a-1)\mathbf{j} + 4\mathbf{k}) \cdot (\mathbf{i} + (a+1)\mathbf{j} - 2\mathbf{k}) = 0$.
12. **Answer:** $(5, 1, 1)$
    - **Explanation:** Parametrize line L. Find parameter t where distance to O equals distance to Q by setting the squares of the distances equal. Solve for t and find the point.
13. **Answer:** $\mathbf{w} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$
    - **Explanation:** Let $\mathbf{w} = (x, y, 1)$. Set up and solve the system: $2x - y + 3(1) = 5$ and $-x + 2y + 1(1) = -3$.
14. **Answer:** $k = \frac{2}{3}$
    - **Explanation:** Find $\vec{OC} = \mathbf{a} + k(\mathbf{b} - \mathbf{a}) = (2 + 3k, 3 - 4k, 0)$. Calculate $|\vec{OC}|^2$, set equal to 20, and solve the quadratic: $(2 + 3k)^2 + (3 - 4k)^2 = 20$.
15. **Answer:** $\frac{3\sqrt{2}}{2}$ units
    - **Explanation:** Use the shortest distance formula from a point to a line: $D = \frac{|\vec{PQ} \times \mathbf{d}|}{|\mathbf{d}|}$, where Q is a point on the line and $\mathbf{d}$ is the direction vector.
16. **Answer:** $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$
    - **Explanation:** Parallel lines share the same direction vector. Use the given point as the new position vector.
17. **Answer:** $q = \frac{11 \pm \sqrt{105}}{10}$ (This answer is in terms of an unknown p which must be found first. A full solution requires finding p from $|\mathbf{a}|^2 = p^2 + 4 + 1 = 21$, so $p = \pm 4$, then substituting to find q).

- **Explanation:** Solve $|\mathbf{a}|^2 = p^2 + 2^2 + (-1)^2 = 21$ for p ($p = \pm 4$). Then solve $\mathbf{a} \cdot \mathbf{b} = 3p + 2q - 2 = 10$ for q using each value of p.

18. **Answer:** Rectangle
    - **Explanation:** Show opposite sides are parallel (e.g., $\vec{AB} = \vec{DC}$ and $\vec{AD} = \vec{BC}$) and adjacent sides are perpendicular (e.g., $\vec{AB} \cdot \vec{AD} = 0$). Sides are not all equal, so it's a rectangle, not a square.
19. **Answer:** $\mathbf{v} = \mathbf{i} + \mathbf{j} + 4\mathbf{k}$
    - **Explanation:** Equal angles with x and y axes implies components are equal: $\mathbf{v} = (a, a, 4)$. Solve $|\mathbf{v}|^2 = a^2 + a^2 + 16 = 26$ for positive a.
20. **Answer:** $m = -2$
    - **Explanation:** The vector $\vec{AB} = (2, m, -1)$ must be parallel to the normal vector $\mathbf{n}$ since the line is perpendicular to $\mathbf{n}$. So $(2, m, -1)$ must be a scalar multiple of $(2, -1, 3)$. This is only possible if $m = -1$ and the scalar is 1, but then the third component would be 3, not -1. The correct approach is that $\vec{AB}$ must be perpendicular to $\mathbf{n}$ since it lies in a plane normal to $\mathbf{n}$. So $\vec{AB} \cdot \mathbf{n} = 0$: $(2)(2) + (m)(-1) + (-1)(3) = 0$.
21. **Answer:** $\mathbf{r} = \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$
    - **Explanation:** The line perpendicular to the plane is parallel to the normal vector. It passes through the origin.
22. **Answer:** $\mathbf{d} = \mathbf{a} - 2\mathbf{b} + 3\mathbf{c}$
    - **Explanation:** Express in terms of position vectors: $\vec{AD} = \mathbf{d} - \mathbf{a}$, $\vec{AB} = \mathbf{b} - \mathbf{a}$, $\vec{BC} = \mathbf{c} - \mathbf{b}$. Substitute and solve for $\mathbf{d}$.
23. **Answer:** $\theta = \arccos\left(\frac{1}{3\sqrt{2}}\right)$
    - **Explanation:** Find the angle ϕ between the direction vector of the line $\mathbf{d} = (2, -2, 1)$ and $\mathbf{v}$. Use $\cos\phi = \frac{\mathbf{d}\cdot\mathbf{v}}{|\mathbf{d}||\mathbf{v}|} = \frac{(0-2+1)}{\sqrt{4+4+1}\sqrt{0+1+1}}$.
24. **Answer:** $a = -3$
    - **Explanation:** Resultant $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = ((2+a)\mathbf{i} + (1)\mathbf{j} + (2)\mathbf{k})$. For $\mathbf{R} \perp \mathbf{F}_1$, their dot product is zero: $(2)(2+a) + (-1)(1) + (3)(2) = 0$.
25. **Answer:** $\cos(\angle POQ) = \frac{7}{\sqrt{6}\sqrt{14}} = \frac{7}{\sqrt{84}} = \frac{7}{2\sqrt{21}}$
    - **Explanation:** $\cos(\angle POQ) = \frac{\vec{OP}\cdot\vec{OQ}}{|\vec{OP}||\vec{OQ}|} = \frac{(1)(3)+(2)(1)+(1)(2)}{\sqrt{1+4+1}\sqrt{9+1+4}}$.