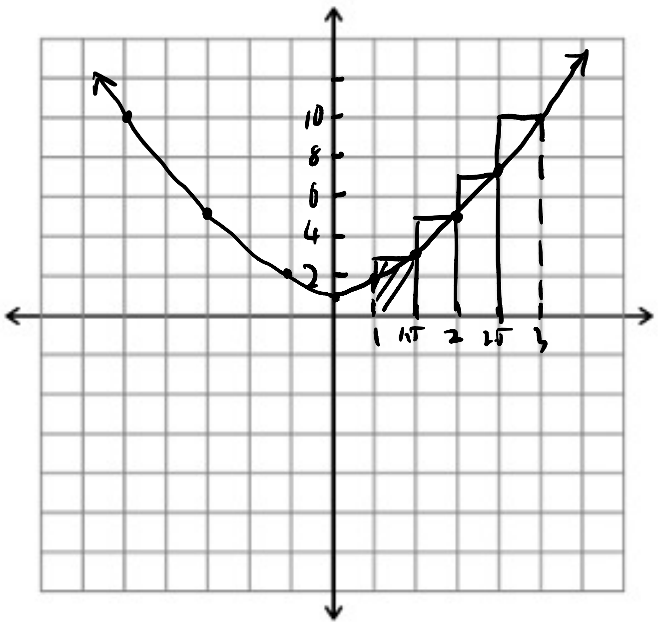
Watch:

https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-2/v/simple-riemann-approximation-using-rectangles

The video above showed a lower (left) estimation of the area under $y = x^2 + 1$ between $x = 1$ and 3. On the graph below, show a higher (right) estimation of the area. Use the same equal widths used in the video.



$f(3) \times 0.5 + f(2.5) \times 0.5 + f(2) \times 0.5 + f(1.5) \times 0.5$

$= 10 \times 0.5 + 7.25 \times 0.5 + 5 \times 0.5 + 3.25 \times 0.5$

$= 0.5 \times 25.5 = 12.75$

Given the curve y = x^2, find the area between the curve and the x-axis on the interval $0 \le x \le 1$.

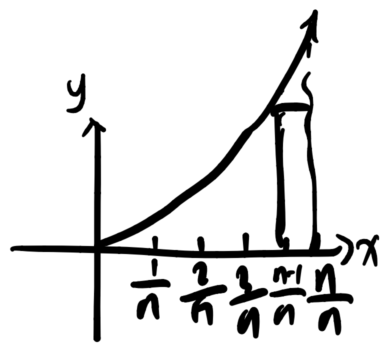
1) Suppose we divide the interval in *n* subintervals, each of width $\frac{1}{n}$.

Explain why the total area of lower rectangles can be written as:

$$A_L = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i-1}{n}\right)$$

$$A_L = \frac{1}{n}\left(f(0) + f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \dots f\left(\frac{n-1}{n}\right)\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i-1}{n}\right)$$



2) Use the following sums to show that $A_L = \dfrac{1}{3} - \dfrac{1}{2n} + \dfrac{1}{6n^2}$.

$y = x^2$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$A_L = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i-1}{n}\right)$$

$$A_L = \frac{1}{n}\sum_{i=1}^{n}\frac{(i-1)^2}{n^2} = \frac{1}{n}\sum_{i=1}^{n}\frac{i^2-2i+1}{n^2} = \frac{1}{n}\left(\sum_{i=1}^{n}\frac{i^2}{n^2} - \sum_{i=1}^{n}\frac{2i}{n^2} + \sum_{i=1}^{n}\frac{1}{n^2}\right)$$

$$= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - n(n+1) + n\right)$$

$$= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1-6)+6n}{6}\right)$$

$$= \frac{1}{n^2}\left(\frac{(n+1)(2n-5)+6}{6}\right)$$

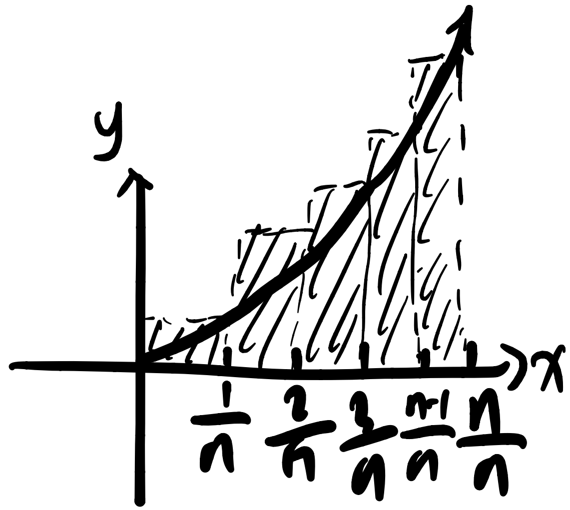
$$= \frac{2n^2-3n+1}{6n^2} = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$$

3) Explain why the total area of upper rectangles can be written as:

$$A_U = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)$$

$$A_u = \frac{1}{n}\left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + f\left(\frac{3}{n}\right) + f\left(\frac{4}{n}\right) \ldots f\left(\frac{n-1}{n}\right) + f\left(\frac{n}{n}\right)\right)$$

$$A_u = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)$$



4) Hence, show that $A_U = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$. $y = x^2$

$$A_U = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)$$

$$A_u = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2 = \frac{1}{n}\sum_{i=1}^{n}\frac{i^2}{n^2} = \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{1}{n^{3}}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \frac{1}{n^2}\left(\frac{2n^2+3n+1}{6}\right)$$

$$= \frac{2n^2+3n+1}{6n^2}$$

$$= \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

5) Show that as n approaches infinity, that $A_U = A_L$.

$A_L = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$ $\lim_{n\to\infty}\left(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3}$

$A_U = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$ $\lim_{n\to\infty}\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3}$

To find the area under the curve $y = f(x)$ between $a \le x \le b$:

i) If we divide the interval into n subintervals, the width of each is:

$$\frac{b-a}{n}$$

ii) The upper sum would be:

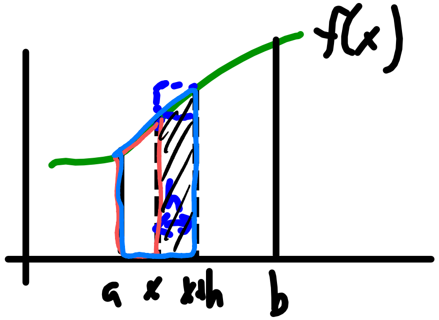
$$A_u = \frac{b-a}{n}\left(f\left(a+\frac{1}{n}\right) + f\left(a+\frac{2}{n}\right) + \dots + f(b)\right)$$

$$A_u = \frac{b-a}{n} \sum_{i=1}^{n} f\left(a+\frac{i}{n}\right) = \sum_{i=1}^{n} \underbrace{f\left(a+\frac{i}{n}\right)}_{f(x)} \overbrace{\left(\frac{b-a}{n}\right)}^{dx \text{ (difference in } x)}$$

$$\int_a^b$$

$$= \int_a^b f(x)\,dx \leftarrow \text{definite integral}$$

$= A(x)$ from a to b



if $x = a$

$A(a) = 0$

$A(a) = F(a) + c$

$0 = F(a) + c$

$\therefore c = -F(a)$

$$h f(x) \leq A(x+h) - A(x) \leq h f(x+h)$$

$$f(x) \leq \frac{A(x+h) - A(x)}{h} \leq f(x+h)$$

$$f(x) \leq A'(x) \leq f(x+h)$$

as $h \to 0$

$$f(x) \leq A'(x) \leq f(x)$$ squeeze theorem

$$\therefore f(x) = A'(x)$$

antiderivative

$$A(x) = F(x) + c$$

$$A(x) = F(x) - F(a)$$

$$A(b) = F(b) - F(a)$$

THE FUNDAMENTAL THEOREM OF CALCULUS

For a continuous function $f(x)$ with antiderivative $F(x)$, $\boldsymbol{\int_a^b f(x)\,dx = F(b) - F(a)}$.

In general, $\int_a^b f(x)\,dx$ is called a **definite integral**.

Note: F(x) is the antiderivative of f(x).

PROPERTIES OF DEFINITE INTEGRALS

The following properties of definite integrals can all be deduced from the Fundamental Theorem of Calculus:

- $\int_a^a f(x)\,dx = 0$
- $\int_a^b k\,dx = k(b-a)$ $\{k \text{ is a constant}\}$
- $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
- $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$
- $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
- $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Example #2

Find the area between the x-axis and $y = x^2 + 1$ from $x = 1$ to 3 using the Fundamental Theorem of Calculus.

$$f(x) = x^2 + 1$$

$$F(x) = \frac{x^3}{3} + x + C$$

$$\int_1^3 (x^2+1)\,dx = \left[\frac{x^3}{3} + x + C\right]_1^3$$

$$= \frac{1}{3} \times (3)^3 + 3 + C - \left(\frac{1}{3} + 1 + C\right)$$

$$= 12 + C - \frac{4}{3} - C$$

$$= \frac{32}{3}$$

Example #2

Evaluate $\int_1^3 (1+2x)\,dx$ and check using the GDC.

$$\int_1^3 (1+2x)\,dx = \left[x + x^2\right]_1^3 = 3+9-(1+1)$$

$$= 10$$