The sum of two positive numbers is 20. Find the two numbers such that the product of one and the square of the other is a maximum.

$0 \le x \le 20$, $0 \le y \le 20$ > extreme values

xy^2 maximum

$x + y = 20$

$xy^2 = z$

$\frac{dz}{dy} = (20-y)y^2$

$\frac{dz}{dy} = 20y^2 - y^3$

$\frac{dz}{dy} = 40y - 3y^2$

$y(40-3y)$

$y = 0,\ y = \frac{40}{3}$

$0 \; \overset{+}{\nearrow} \; \frac{40}{3} \; \overset{-}{\searrow} \; 20$

max

$x = 20 - \frac{40}{3} = \frac{20}{3}$

$z = x(20-x)^2$

$z = x(400 - 40x + x^2)$

$z = x^3 - 40x^2 + 400x$

$\frac{dz}{dx} = 3x^2 - 80x + 400$

$0 = 3x^2 - 80x + 400$

$-1 \quad 20$

$-3 \quad 20$

$0 = (-x+20)(-3x+20)$

$x = 20,\ x = \frac{20}{3}$

$0 \; \overset{+}{\nearrow} \; \frac{20}{3} \; \overset{-}{\searrow} \; 20$

max

$\therefore \frac{20}{3}$ and $\frac{40}{3}$.

Maximum/Minimum (Optimization) Word Problems

Optimization Steps

1) Introduce variables

2) Find a relationship between the variables

3) Find an expression to maximize or minimize

4) Simplify max/min to have only 1 variable

5) Find the derivative.

6) Set the derivative equal to 0 to find stationary points.

7) Use a sign diagram to determine if stationary points are max or min

8) Check extreme values for global max/mins.

9) Answer the question.

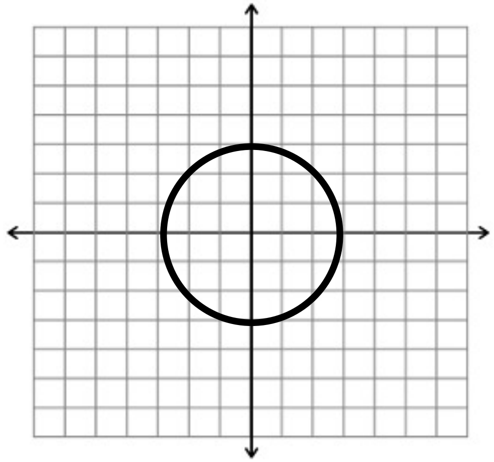
Example #1

A rectangle is inscribed in a circle $x^2 + y^2 = 9$. Find the dimensions of the rectangle with maximum area.

$x^2 + y^2 = 9$

$y^2 = 9 - x^2$

$y = \sqrt{9 - x^2}$



$\boxed{\therefore \text{ dimension are } 3\sqrt{2} \times 3\sqrt{2}}$

$2x(2y) = z$

$4xy = z$

$4x\sqrt{9 - x^2} = z$

$\frac{dz}{dx} = 4\sqrt{9 - x^2} + 2x(9 - x^2)^{-\frac{1}{2}}(-2x)$

$\frac{dz}{dx} = 4\sqrt{9 - x^2} - \frac{4x^2}{\sqrt{9 - x^2}}$

$= \frac{4}{\sqrt{9 - x^2}}(9 - x^2 - x^2)$

$= \frac{-4}{\sqrt{9 - x^2}}(9 - 2x^2)$

$$\begin{array}{ccccccc} & - & & + & & - & \\ -3 & & -\frac{3\sqrt{2}}{2} & & \frac{3\sqrt{2}}{2} & & 3 \end{array}$$

max

$-3 \le x \le 3$

$-3 \le y \le 3$

$0 = 9 - 2x^2$

$2x^2 = 9$

$x^2 = \frac{9}{2}$

$x = \pm\frac{3}{\sqrt{2}}$

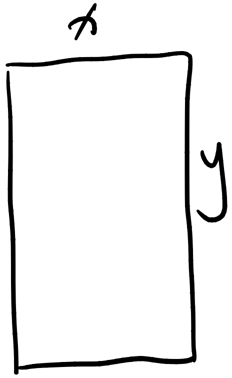
$\therefore x = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$

$y = \sqrt{9 - \frac{18}{4}}$

$y = \sqrt{\frac{18}{4}} = \frac{3\sqrt{2}}{2}$

Example #2

A rectangular sheet of metal with a perimeter of 2 m will be rolled into the curved surface of a cylindrical tank. Find the greatest possible volume of this tank.

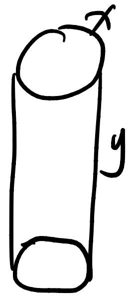


$x+y=1$

$y=1-x$

$\pi d = x$

$d=\frac{x}{\pi} \quad r=\frac{x}{2\pi}$



$Z=\pi\left(\frac{x}{2\pi}\right)^2 y$

$Z=\cancel{\pi}\left(\frac{x^2}{4\pi^{\cancel{2}}}\right)(1-x)$

$Z=\frac{x^2}{4\pi}-\frac{x^3}{4\pi}$

$Z=\frac{1}{4\pi}(x^2-x^3)$

$\frac{dZ}{dx}=\frac{1}{4\pi}(2x-3x^2)$

$2x-3x^2=0$

$x(2-3x)=0$

$x=0,\ x=\frac{2}{3}$

$+ \quad -$

$0 \quad \left(\frac{2}{3}\right) \quad 2$

$\therefore x=\frac{2}{3}, \quad y=1-\frac{2}{3}=\frac{1}{3}$

$V=\pi r^2 h$

$V=\pi\left(\frac{1}{3\pi}\right)^2\times\frac{1}{3}=\cancel{\pi}\left(\frac{1}{9\pi^{\cancel{2}}}\right)\times\frac{1}{3}=\boxed{\frac{1}{27\pi}}$

$\therefore$ Volume is $\frac{1}{27\pi}\ m^3$