Steps to Sketch a Curve

- you are a detective, these are the steps you **ALWAYS** follow

1) Find the x and y intercepts

2) Find any Asymptotes

- examine behavior near these asymptotes

3) Find stationary points, $f'(x) = 0$ and classify as local max or min using a sign diagram.

4) Find inflection points, $f''(x) = 0$ and classify all intervals of concavity.

5) Sketch a clear graph using the information you have found in steps 1 to 4.

$18(x-1)$

Given that $f(x)=\frac{18x-18}{x^2}$ $f'(x)=\frac{36-18x}{x^3}$ and $f''(x)=\frac{36x-108}{x^4}$

Sketch the graph of the function showing all steps.

① x-int

$0=\frac{18x-18}{x^2}$

$0=x-1$

$x=1$

$(1,0)$

② V.A

$x=0$

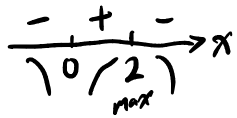
$\lim_{x\to 0^+} f(x)=-\infty$

$\lim_{x\to 0^-} f(x)=-\infty$

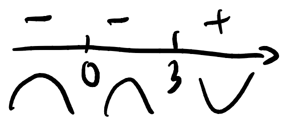
③ $f'(x)=\frac{18(2-x)}{x^3}=0$

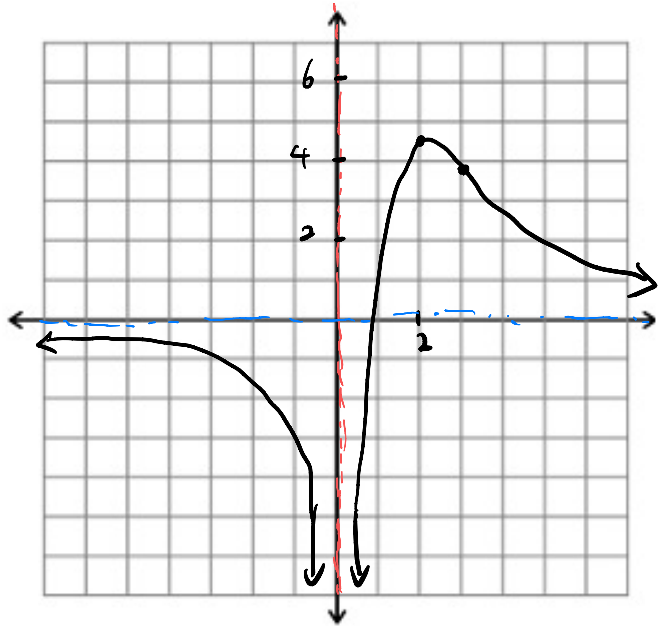
$x=2$

$y=\frac{9}{2}$



④ $f''(x)$: $0=\frac{36x-108}{x^4}=\frac{36(x-3)}{x^4}$, $x=3$ $y=4$





HL Curve Sketching

Sketch each of the following using the curve sketching model.

a) $y = x^5 - x^4$

y-int

$y = 0^5 - 0^4 = 0$

x-int

$0 = x^5 - x^4$

$x^4 = x^5$

$x = 1$

$y' = 5x^4 - 4x^3$

$0 = 5x^4 - 4x^3$

$4x^3 = 5x^4$

$x = \frac{4}{5}$ $\quad y = \left(\frac{4}{5}\right)^5 - \left(\frac{4}{5}\right)^4 = -0.08192$

$x = 0$ $\quad y = 0$

$+ \quad - \quad +$

0 max $\quad \frac{4}{5}$ min

$y'' = 20x^3 - 12x^2$

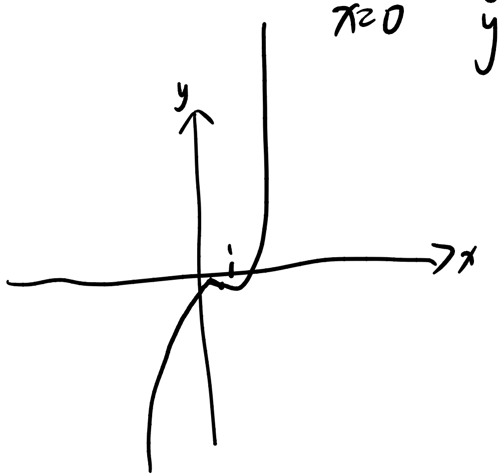
$0 = 20x^3 - 12x^2$

$12x^2 = 20x^3$

$x = 0 \quad x = \frac{3}{5}$

$- \quad - \quad +$

$\cap\ 0\ \cap\ \frac{3}{5}\ \cup$



b) $y = \dfrac{x^2 - 4}{x^2 - 1}$

c) $y = x\sqrt{1 - x^2}$

d) $y = \dfrac{x}{\sqrt{x^2 - 4}}$

e) $y = \dfrac{x^2 + 9}{x}$

f) $f(x) = e^{x\sqrt{3}} \sin x$

g) $y = \ln\left(x^2 + 5\right)$

h) $f(x) = e^{\sin^2 x}$

b) $y = \frac{x^2-4}{x^2-1}$

x-int $0 = \frac{x^2-4}{x^2-1}$

$x = 2,\ x = -2 \quad (2,0)(-2,0)$

y-int $y = \frac{-4}{-1} = 4 \quad (0,4)$

V.A. $x^2 - 1 = 0$

$x = 1,\ x = -1$

H.A.

$\lim_{x \to -\infty} f(x) = 1$

$\lim_{x \to \infty} f(x) = 1$

$y' = \frac{2x(x^2-1) - 2x(x^2-4)}{(x^2-1)^2}$

$y' = \frac{2x(\cancel{x^2} - 1 - \cancel{x^2} + 4)}{(x^2-1)^2} = \frac{6x}{(x^2-1)^2}$

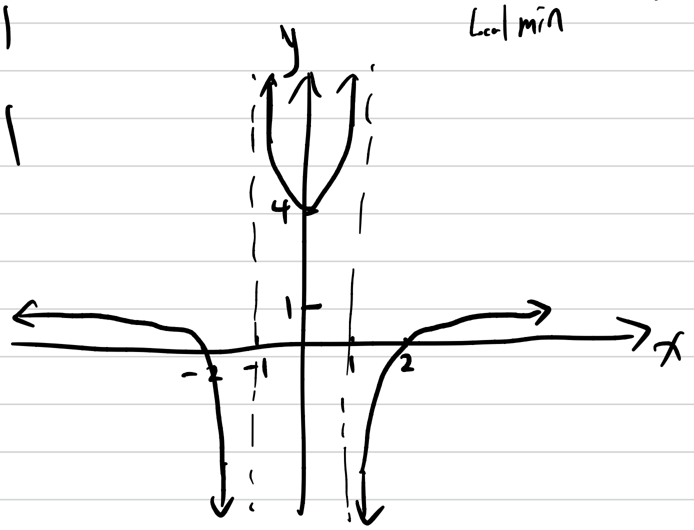
$0 = \frac{6x}{(x^2-1)^2}$

$x = 0$

$-\quad -\quad +\quad +$

$-1 \quad 0 \quad 1$

Local min



$y'' = \frac{6(x^2-1)^2 - 2(x^2-1)(2x)(6x)}{(x^2-1)^4}$

$y'' = \frac{6(\cancel{x^2-1})(x^2-1-4x^2)}{(x^2-1)^{\cancel{4}\,3}}$

$y'' = \frac{6(-3x^2-1)}{(x^2-1)^3}$

$y'' = \frac{-6(3x^2+1)}{(x^2-1)^3}$

$0 = 3x^2 + 1$

$x^2 = -\frac{1}{3}$

no inflection point

$-\quad +\quad -$ → x

$-1 \quad 1$

c) $y = x\sqrt{1-x^2}$

$1 - x^2 \geq 0$

$x^2 \leq 1$

$\boxed{-1 \leq x \leq 1}$

y-int: $(0,0)$

x-int: $(0,0), (-1,0), (1,0)$

$y' = \sqrt{1-x^2} + \frac{1}{2}x(1-x^2)^{-\frac{1}{2}}(-2x)$

$y' = \frac{1}{\sqrt{1-x^2}}(1 - x^2 - x^2)$

$y' = \frac{1-2x^2}{\sqrt{1-x^2}}$

$0 = 1 - 2x^2$

$x^2 = \frac{1}{2}$

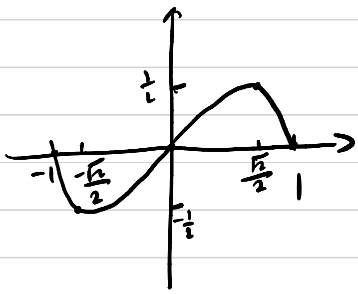
$x = \pm\frac{\sqrt{2}}{2}$

$-\quad +\quad -$ → x

$-1 \quad -\frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \quad 1$

max $\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$

min $\left(-\frac{\sqrt{2}}{2}, -\frac{1}{2}\right)$



$y'' = \frac{-4x(\sqrt{1-x^2}) - \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(1-2x^2)(-2x)}{(1-x^2)}$

$y'' = \frac{\frac{1}{\sqrt{1-x^2}}\left(-4x(1-x^2) + x(1-2x^2)\right)}{(1-x^2)}$

$y'' = \frac{(-4x + 4x^3 + x - 2x^3)}{(1-x^2)\sqrt{1-x^2}}$

$y'' = \frac{x(2x^2-3)}{(1-x^2)^{\frac{3}{2}}}$

$0 = 2x^2 - 3$

$x = \pm\frac{\sqrt{3}}{\sqrt{2}} = \pm\frac{\sqrt{6}}{3}$ $x \neq 0$

$+\quad +\quad -\quad -$

$-1 \quad -\frac{\sqrt{6}}{3} \quad 0 \quad \frac{\sqrt{6}}{3} \quad 1$

d) $y = \dfrac{x}{\sqrt{x^2-4}}$ $y' = -\dfrac{4}{(x^2-4)^{\frac{3}{2}}}$ $y'' = \dfrac{12x}{(x^2-4)^{\frac{5}{2}}}$

V.A: $\sqrt{x^2-4} = 0$

$x^2 = 4$

$x = \pm 2$ $x \geq 2$

$x \geq -2$

$\lim_{x\to 2^-} f(x) = \text{DNE}$

$\lim_{x\to 2^+} f(x) = +\infty$

$\lim_{x\to -2^-} f(x) = -\infty$

$\lim_{x\to -2^+} f(x) = \text{DNE}$

H.A. $\lim_{x\to +\infty} y = 1$

$\lim_{x\to -\infty} y = -1$

y-int: no

x-int

$0 = \dfrac{x}{\sqrt{x^2-4}}$

$x = 0$

no

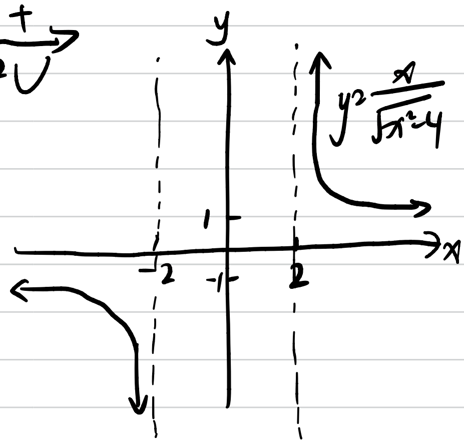
$0 = -\dfrac{4}{(x^2-4)^{\frac{3}{2}}}$

no stationary point

$-$ | $\times$ | $-$; -2, 2

$0 = \dfrac{12x}{(x^2-4)^{\frac{5}{2}}}$ no inflection

$-$ | $\times$ | $+$; -2, 2



e) $y = \dfrac{x^2+9}{x}$ $y' = \dfrac{x^2-9}{x^2}$ $y'' = \dfrac{18}{x^3}$

V.A. $x = 0$

$\lim_{x\to 0^-} y = -\infty$

$\lim_{x\to 0^+} y = +\infty$

Oblique A $y = x$

y-int: no

x-int: no

$0 = \dfrac{x^2-9}{x^2}$

$x = \pm 3$

$+$ | $-$ | $-$ | $+$; -3, 0, 3

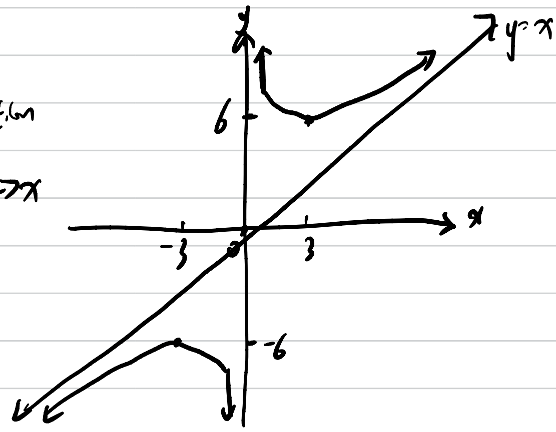
max   min

$(-3,-6)$ $(3,6)$

$0 = \dfrac{18}{x^3}$

no inflection

$-$ | $+$; 0



f/ $f(x) = e^{x\sqrt{3}} \sin x$

V.A no

H.A: $\lim_{x\to-\infty} f(x) = 0$ $y=0$

y-int: $f(0) = e^0 \sin 0$

$= 0$

$(0,0)$

x-int: $(0,0)$

$(2\pi,0)$

$(-2\pi,0)$

$\vdots$

$f'(x) = e^{\sqrt{3}x}(\sqrt{3}\sin x + \cos x)$

$0 = e^{\sqrt{3}x}(\sqrt{3}\sin x + \cos x)$

$\sqrt{3}\sin x = -\cos x$

$-\tan x = \frac{1}{\sqrt{3}}$

$\tan x = -\frac{1}{\sqrt{3}}$

$x = \frac{5\pi}{6}, \frac{11\pi}{6}$

$-\quad +\quad +\quad -\quad +$

$-\frac{7\pi}{6} \quad -\frac{\pi}{6} \quad 0 \quad \frac{5\pi}{6} \quad \frac{11\pi}{6} \quad 2\pi \quad x$

max min

$\left(\frac{5\pi}{6}, \frac{e^{\frac{5\sqrt{3}\pi}{6}}}{2}\right)$ $\left(\frac{11\pi}{6}, -\frac{e^{\frac{11\sqrt{3}\pi}{6}}}{2}\right)$

$f''(x) = 2e^{\sqrt{3}x}(\sin x + \sqrt{3}\cos x)$

$0 = 2e^{\sqrt{3}x}(\sin x + \sqrt{3}\cos x)$

$\sqrt{3} = -\tan x$

$\tan x = -\sqrt{3}$

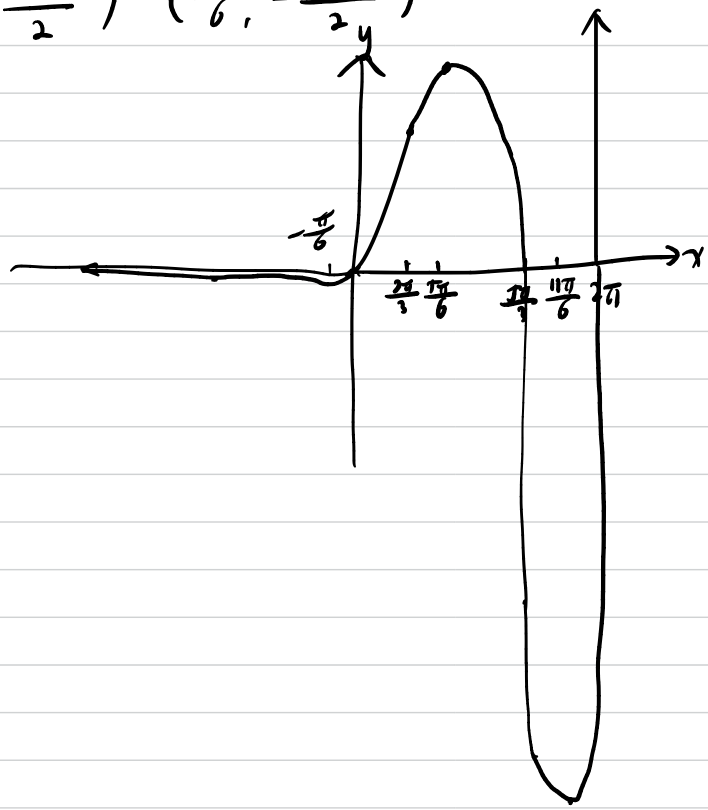
$x = \frac{2\pi}{3}, \frac{5\pi}{3}$

$+\quad -\quad +$

$0 \quad \frac{2\pi}{3} \quad \frac{5\pi}{3} \quad 2\pi \quad x$

$\left(\frac{2\pi}{3}, \frac{\sqrt{3}e^{\frac{2\sqrt{3}\pi}{3}}}{2}\right)$

$\left(\frac{5\pi}{3}, -\frac{\sqrt{3}e^{\frac{5\sqrt{3}\pi}{3}}}{2}\right)$



g) $y=\ln(x^2+5)$

y-int: $(0, \ln 5)$

x-int: $0=\ln(x^2+5)$

$x^2+5=1$

$x^2=-4$

$\therefore$ no x-int.

$y'=\frac{2x}{x^2+5}$

$y'=0$

$x=0$ $\quad(0,\ln 5)$

$-$ $\quad$ $+$ (at 0, x)

$y''=\frac{2(x^2+5)-4x^2}{(x^2+5)^2}=\frac{2(x^2+5-2x^2)}{(x^2+5)^2}$

$y''=\frac{-2(x^2-5)}{(x^2+5)^2}$

$y''=0$

$x^2-5=0$

$x^2=5$

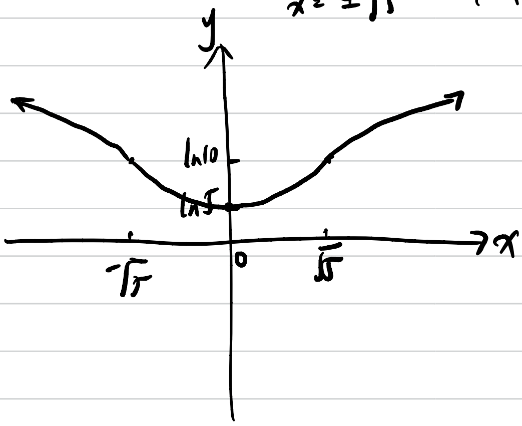
$x=\pm\sqrt{5}$

$(-\sqrt{5}, \ln 10)$

$(\sqrt{5}, \ln 10)$

$-$ $\quad$ $+$ $\quad$ $-$

$\cap$ $-\sqrt{5}$ $\cup$ $\sqrt{5}$ $\cap$



h) $y=e^{\sin^2 x}$

y-int $(0,1)$

x int $0=e^{\sin^2 x}$

no x-int

$y>0$

$y'=2\cos x\, e^{\sin^2 x}\sin x$

$0=2\cos x\, e^{\sin^2 x}\sin x$

$0=\cos x$ $\quad$ $0=\sin x$

$x=\frac{\pi}{2}, \frac{2\pi}{2}$ $\quad$ $x=0,\pi,2\pi$

$y''=2e^{\sin^2 x}((2\cos^2(x)-1)\sin^2 x+\cos^2 x)$

$0=-2e^{\sin^2 x}(2\sin^4 x-1)$

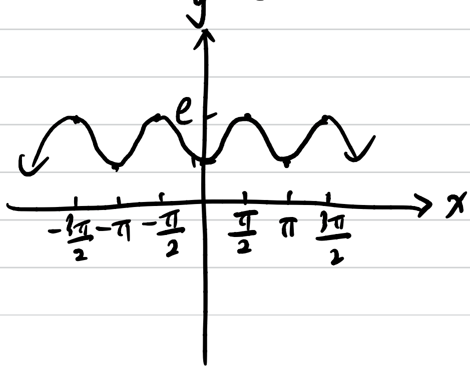
$\sin^4 x=\frac{1}{2}$

$\sin x=\frac{1}{\sqrt[4]{2}}$

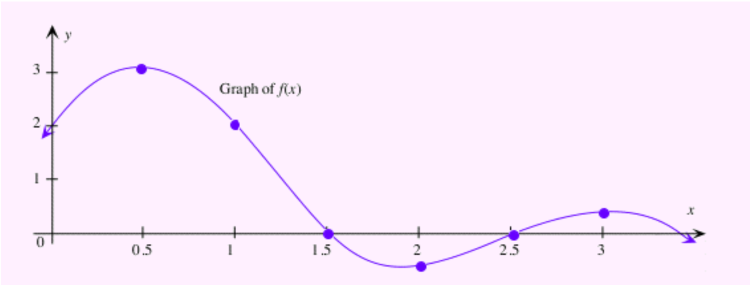
$+ \; - \; + \; - \; + \; - \; + \; -$ (x)

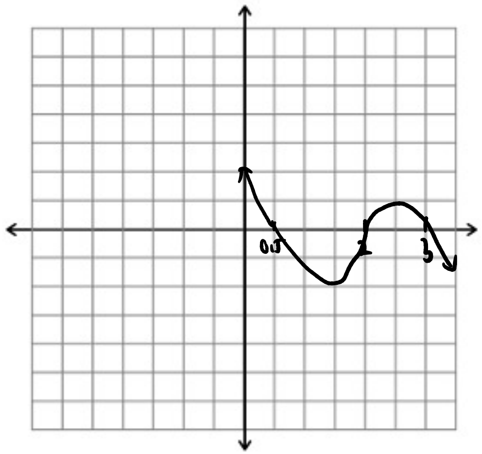
$-2\pi \quad -\frac{3\pi}{2} \quad -\pi \quad -\frac{\pi}{2} \quad 0 \quad \frac{\pi}{2} \quad \pi \quad \frac{3\pi}{2} \quad 2\pi$

$(-\frac{\pi}{2}, e)$ $\quad(\frac{\pi}{2}, e)$ $\quad(\frac{3\pi}{2}, e)$ $\quad(-\frac{3\pi}{2}, e)$

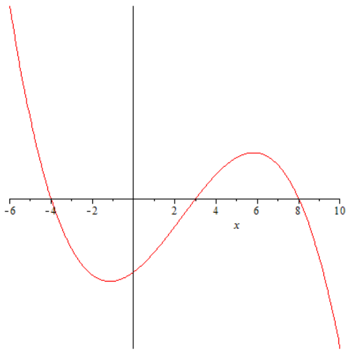


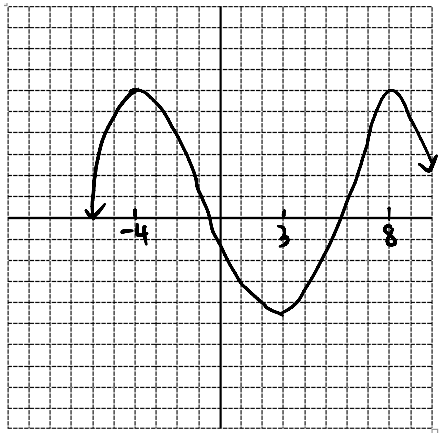
Let $f(x)$ have the graph shown below. Graph f'(x).





Given f'(x), graph f(x).





Consider the function f defined by $f(x) = 3x \arccos(x)$ where $-1 \le x \le 1$.

(a) Sketch the graph of f indicating clearly any intercepts with the axes and the coordinates of any local maximum or minimum points. [3]

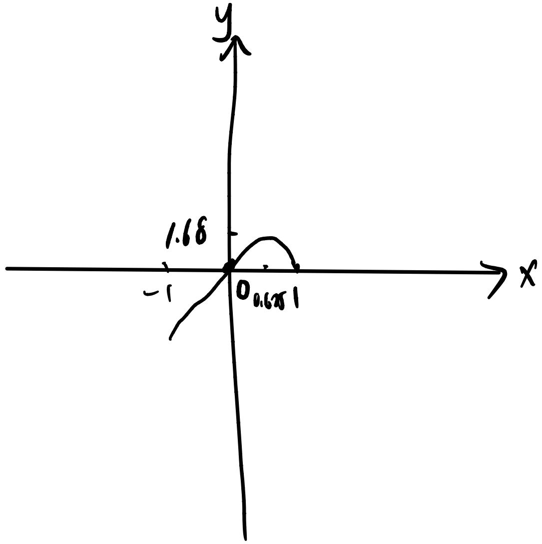
y-int: $f(x) = 0$

$(0,0)$

x-int: $0 = 3x\arccos(x)$

$x = 0$ or $x = 1$

$-1 \le x \le 1$



$f'(x) = 3\arccos x - 3x\frac{1}{\sqrt{1-x^2}}$

$f'(x) = 3\left(\arccos x - \frac{x}{\sqrt{1-x^2}}\right)$

$0 = \arccos x - \frac{x}{\sqrt{1-x^2}}$

$\frac{x}{\sqrt{1-x^2}} = \arccos x$

$x = 0.652$

$+ \quad -$

$-1 \quad 0.625 \quad 1 \quad x$

$f''(x) = \frac{3}{\sqrt{1-x^2}} - \left(\frac{3\sqrt{1-x^2} - \frac{3}{2}x(1-x^2)^{-\frac{1}{2}}(-2x)}{1-x^2}\right)$

$= \frac{3}{\sqrt{1-x^2}} - \frac{\frac{3}{\sqrt{1-x^2}}(1-x^2+x^2)}{1-x^2}$

$= \frac{3}{\sqrt{1-x^2}} - \frac{3}{(1-x^2)^{\frac{3}{2}}}$

$f''(x) = 0$

$x = 0$

$- \quad -$

$-1 \cap 0 \cap 1 \quad x$