Point Symmetries of Partial Differential Equations

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Overview

1. Motivation
2. Continuous Point Symmetries
3. Discrete Point Symmetries
4. Conclusion

Motivation

Known Examples

- Poincare group for QFT Lagrangians

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- Schrödinger/Galilean Group

Why do we care?

- Better understanding of symmetry groups

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- Better understanding of symmetry groups
- Finding new solutions to differential equations
- Help to explain physical phenomena

Continuous Point Symmetries

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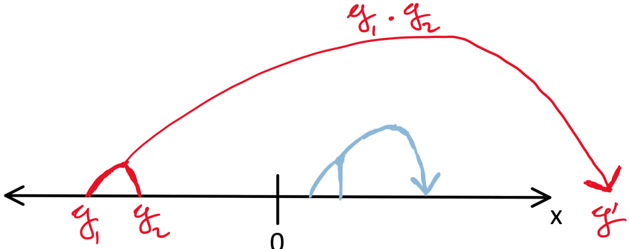
(Global) Lie group

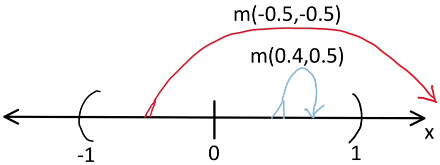
- Multiplication law
- Inverse
- Identity Element

Group of rotations $SO(3)$
Example below $(\mathbb{R}\backslash\{0\}, \times)$



Local Lie Group

- Local Multiplication
- Local Inverse



Let $V = \{x : |x| < 1\}$ with multiplication

$$m(x, y) = \frac{2xy - x - y}{xy - 1} \qquad \forall x, y \in V$$

Then $m(-\frac{1}{2}, -\frac{1}{2}) = 2 \notin V$

Lie Algebra

- Vector Space
- Generators $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_r}$

Elements can be created using the generators

$$g = \exp(\epsilon_1 \mathbf{v_1}) \exp(\epsilon_2 \mathbf{v_2}) ... \exp(\epsilon_r \mathbf{v_r})$$

Point Transformations

Consider PDE dependent on x and u

Point transformations

$$G : (x, u) \to (f(x, u), h(x, u)) = (\tilde{x}, \tilde{u})$$

Local Group of Transformations

Acting with group element g on coordinates x, u

Let $g_\epsilon = \exp(\epsilon \mathbf{v_1})$ with $\mathbf{v_1} = \partial_x$

$$(\tilde{x}, \tilde{u}) = g_\epsilon \cdot (x, u) = (x + \epsilon, u)$$

Similarly, let $h_\epsilon = \exp(\epsilon \mathbf{v_2})$ with $\mathbf{v_2} = x^2 \partial_x$

$$h_\epsilon \cdot (x, u) = \left(\frac{x}{1 - \epsilon x}, u \right) \quad \text{for} \quad |\epsilon x| < 1$$

Symmetry Group

Definition Symmetry Group
A symmetry group is a local group of transformations G, with the property that if $u = f(x)$ is a solution of the PDE, then $g \cdot f(x)$ is also a solution for all $g \in G$ (if $g \cdot f$ is defined).

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Want to find the total symmetry group

How to find these symmetry groups?

Prolongation Method

Clearly described in textbook by Olver [Olver, 1998]

Let generator have the form

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u)\frac{\partial}{\partial x^i} + \phi(x, u)\frac{\partial}{\partial u}$$

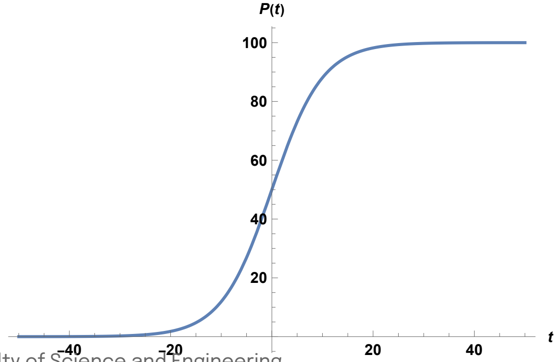
Let it act on the differential equation

Generators of the symmetry group have constraints on ξ^i and ϕ

Only finds infinitesimal transformations

Logistic Equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{k}\right), \qquad\qquad r, k \in \mathbb{R}$$



Logistic Equation

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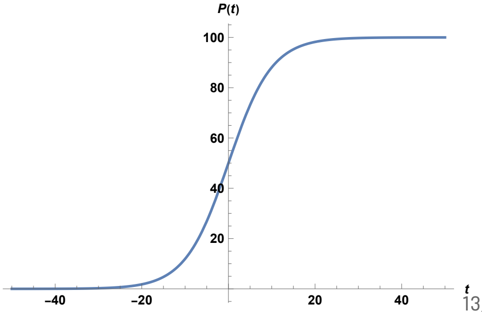
$$\mathbf{v_1} = \partial_t$$

$$G_1 : (t, P) \to (t + \epsilon, P)$$

$$\tilde{P}(t) = P(t - \epsilon)$$



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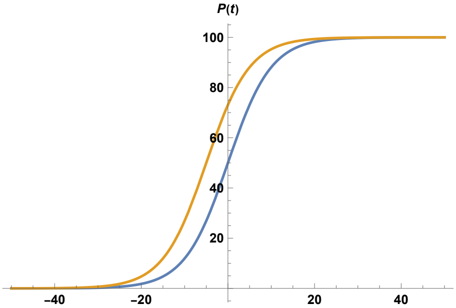

Heat Equation

(1+1) dimensional: $u(x,t)$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

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$$\begin{aligned}
\mathbf{v_1} &= \partial_x \\
\mathbf{v_2} &= \partial_t \\
\mathbf{v_3} &= u\partial_u \\
\mathbf{v_4} &= x\partial_x + 2t\partial_t \\
\mathbf{v_5} &= 2t\partial_x - xu\partial_u \\
\mathbf{v_6} &= 4tx\partial_x + 4t^2\partial_t - (x^2 + 2t)u\partial_u \\
\mathbf{v}_\alpha &= \alpha(x, t)\partial_u
\end{aligned}$$

Heat Equation

$$
\begin{aligned}
\mathbf{v_1} &= \partial_x & G_1 &: \quad (x+\epsilon, t, u) \\
\mathbf{v_2} &= \partial_t & G_2 &: \quad (x, t+\epsilon, u) \\
\mathbf{v_3} &= u\partial_u & G_3 &: \quad (x, t, e^{\epsilon}u) \\
\mathbf{v_4} &= x\partial_x + 2t\partial_t & G_4 &: \quad (e^{\epsilon}x, e^{2\epsilon}t, u) \\
\mathbf{v_5} &= 2t\partial_x - xu\partial_u & G_5 &: \quad (x+2\epsilon t, t, e^{-\epsilon x-\epsilon^2 t}u) \\
\mathbf{v_6} &= 4tx\partial_x + 4t^2\partial_t - (x^2+2t)u\partial_u & G_6 &: \quad \left(\frac{x}{1-4\epsilon t}, \frac{t}{1-4\epsilon t}, \sqrt{1-4\epsilon t}\exp\left(-\frac{\epsilon x^2}{1-4\epsilon t}\right)u\right) \\
\mathbf{v}_\alpha &= \alpha(x,t)\partial_u & G_\alpha &: \quad (x, t, u+\epsilon\alpha(x,t))
\end{aligned}
$$

Heat Equation

Heat equation cannot be described generally as a global group!

$L^2(\mathbb{R})$ (square-integrable) solutions do satisfy global group properties, transformations from $\mathbf{v_6}$ do nothing to $f \in L^2(\mathbb{R})$

Similar situation for Schrödinger equation

Discrete Point Symmetries

Motivation

Missing obvious symmetries!

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$G_4 : \quad (x, t, u) \to (x, t, e^{\epsilon} u)$$

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What about $u \to -u$?

Method described by Hydon ([Hydon, 2000])

Finding Hidden Symmetry Group

How to find these symmetries?

Prolongation gives generators $\mathbf{v_1}, ..., \mathbf{v_r}$

Each generates its own symmetry group G_i

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$$\Gamma : (x, t, u) \to (\tilde{x}, \tilde{t}, \tilde{u})$$

Then we also have the symmetry group

$$\tilde{G}_i = \Gamma G_i \Gamma^{-1} = \exp(\epsilon \Gamma \mathbf{v_i} \Gamma^{-1})$$

With generator $\tilde{\mathbf{v}}_\mathbf{i} = \Gamma \mathbf{v_i} \Gamma^{-1}$

Basis Transformation

How to find these Γ symmetries?

Prolongation gives generators $\mathbf{v_1}, ..., \mathbf{v_r}$

$$[\mathbf{v_i}, \mathbf{v_j}] = c_{ij}^k \mathbf{v_k}$$

Turns out new generators ($\tilde{\mathbf{v}}_\mathbf{i} = \Gamma \mathbf{v_i} \Gamma^{-1}$) satisfy

$$[\tilde{\mathbf{v}}_\mathbf{i}, \tilde{\mathbf{v}}_\mathbf{j}] = c_{ij}^k \tilde{\mathbf{v}}_\mathbf{k}$$

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$$[\tilde{\mathbf{v}}_\mathbf{i}, \tilde{\mathbf{v}}_\mathbf{j}] = c_{ij}^k \tilde{\mathbf{v}}_\mathbf{k}$$

New generators can be found using linear basis transformations $\mathbf{v_i} = b_i^l \tilde{\mathbf{v}}_\mathbf{l}$

Basis Transformation

New generators are found using basis transformations $\mathbf{v_i} = b_i^l \tilde{\mathbf{v}}_\mathbf{l}$

$$\begin{pmatrix} \tilde{\mathbf{v}}_\mathbf{1} \\ \tilde{\mathbf{v}}_\mathbf{2} \\ \vdots \\ \tilde{\mathbf{v}}_\mathbf{r} \end{pmatrix} = \underbrace{\begin{pmatrix} b_1^1 & b_1^2 & \dots & b_1^r \\ b_2^1 & b_2^2 & \dots & b_2^r \\ \vdots & \vdots & \ddots & \vdots \\ b_r^1 & b_r^2 & \dots & b_r^r \end{pmatrix}}_{B} \begin{pmatrix} \mathbf{v_1} \\ \mathbf{v_2} \\ \vdots \\ \mathbf{v_r} \end{pmatrix}$$

Basis Transformation

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Because we require the same structure constants, B satisfies

$$c_{lm}^n b_i^l b_j^m = c_{ij}^k b_k^n$$

Spherical Burgers Equation

$$\frac{\partial u}{\partial t} + \frac{u}{t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

Lie algebra spanned by three generators

$$\begin{aligned}
\mathbf{v_1} &= -x\partial_x - 2t\partial_t + u\partial_u \\
\mathbf{v_2} &= \ln(t)\partial_x + \frac{1}{t}\partial_u \\
\mathbf{v_3} &= \partial_x
\end{aligned}$$

Spherical Burgers Equation

$$\mathbf{v_1} = -x\partial_x - 2t\partial_t + u\partial_u, \quad \mathbf{v_2} = \ln(t)\partial_x + \frac{1}{t}\partial_u, \quad \mathbf{v_3} = \partial_x$$

$$[\mathbf{v_1}, \mathbf{v_2}] = \mathbf{v_2} - 2\mathbf{v_3}, \qquad [\mathbf{v_1}, \mathbf{v_3}] = \mathbf{v_3}$$

Most general transformation matrix

$$B = \begin{pmatrix} b_1^1 & b_1^2 & b_1^3 \\ b_2^1 & b_2^2 & b_2^3 \\ b_3^1 & b_3^2 & b_3^3 \end{pmatrix}$$

Spherical Burgers Equation (*)

$$c^n_{lm} b^l_i b^m_j = c^k_{ij} b^n_k$$

All nonlinear conditions (n, i, j)

$$
\begin{aligned}
(1,1,2):& \qquad 0 = b^1_2 - 2b^1_3 \\
(1,1,3):& \qquad 0 = b^1_3 \\
(1,2,3):& \qquad 0 = 0 \\
(2,1,2):& \qquad b^1_1 b^2_2 = b^2_2 - 2b^2_3 \\
(2,1,3):& \qquad b^1_1 b^2_3 = b^2_3 \\
(3,1,2):& \qquad -2b^1_1 b^2_2 + b^1_1 b^3_2 = b^3_2 - 2b^3_3 \\
(3,1,3):& \qquad -2b^1_1 b^2_3 + b^1_1 b^3_3 = b^3_3
\end{aligned}
$$

Spherical Burgers Equation (*)

$$c^n_{lm} b^l_i b^m_j = c^k_{ij} b^n_k$$

All nonlinear conditions (n, i, j)

$$\begin{array}{lll}
(1,1,2): & 0 = b^1_2 - 2b^1_3 \quad \rightarrow & b^1_2 = 0 \\
(1,1,3): & 0 = b^1_3 \quad \rightarrow & b^1_3 = 0 \\
(1,2,3): & 0 = 0 & \\
(2,1,2): & b^1_1 b^2_2 = b^2_2 - 2b^2_3 \quad \rightarrow & b^2_3 = 0 \\
(2,1,3): & b^1_1 b^2_3 = b^2_3 \quad \rightarrow & b^1_1 = 1 \\
(3,1,2): & -2b^1_1 b^2_2 + b^1_1 b^3_2 = b^3_2 - 2b^3_3 & \\
(3,1,3): & -2b^1_1 b^2_3 + b^1_1 b^3_3 = b^3_3 &
\end{array}$$

Spherical Burgers Equation

Solving the nonlinear conditions we find

$$b_2^1 = 0, \qquad b_3^1 = 0, \qquad b_3^2 = 0, \qquad b_1^1 = 1$$

Most general transformation matrix

$$B = \begin{pmatrix} 1 & b_1^2 & b_1^3 \\ 0 & b_2^2 & b_2^3 \\ 0 & 0 & b_3^3 \end{pmatrix}$$

Removing Continuous Symmetries

What about the symmetries generated by $\mathbf{v_1}, ..., \mathbf{v_r}$?

Each generator creates its own basis transformations

Remove these transformations using adjoint matrix

Spherical Burgers Equation

$$B = \begin{pmatrix} 1 & b_1^2 & b_1^3 \\ 0 & b_2^2 & b_2^3 \\ 0 & 0 & b_3^3 \end{pmatrix}$$

Make use of the adjoint matrices

$$A(1,\epsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\epsilon} & 2\epsilon e^{-\epsilon} \\ 0 & 0 & e^{-\epsilon} \end{pmatrix}, \quad A(2,\epsilon) = \begin{pmatrix} 1 & \epsilon & -2\epsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A(3,\epsilon) = \begin{pmatrix} 1 & 0 & \epsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Spherical Burgers Equation

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$$BA(3,\epsilon) = \begin{pmatrix} 1 & b_1^2 & b_1^3 + \epsilon \\ 0 & b_2^2 & b_2^3 \\ 0 & 0 & b_3^3 \end{pmatrix}$$

Spherical Burgers Equation

Eliminating continuous symmetries gives

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b_2^2 & 0 \\ 0 & 0 & b_3^3 \end{pmatrix}$$

Apply nonlinear conditions again

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \qquad\qquad b \in \mathbb{R}\backslash\{0\}$$

Solving PDEs (*)

Transformation matrix B gives conditions on which transformations generated by Γ are allowed

$$\Gamma : (x, t, u) \to (\tilde{x}, \tilde{t}, \tilde{u})$$

Doing the basis transformations from x, t, u to $\tilde{x}, \tilde{t}, \tilde{u}$ gives conditions on $\tilde{x}, \tilde{t}, \tilde{u}$

Lastly, the transformations have to satisfy the symmetry condition

$$\Delta(x, u^{(n)}) = 0 \qquad \to \qquad \Delta(\tilde{x}, \tilde{u}^{(n)}) = 0$$

Spherical Burgers Equation (*)

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}$$

The transformations Γ generates are

$$(\tilde{x}, \quad \tilde{t}, \quad \tilde{u}) = \left(b(x + \ln(c_1)tu + c_3\sqrt{t}), \quad c_1 t, \quad \frac{b}{c_1}u + c_2\frac{1}{\sqrt{t}} \right)$$

Symmetry condition sets $c_2 = c_3 = 0$, $c_1 = 1$ and $b = \pm 1$

$$\Gamma : (x, t, u) \to (-x, t, -u)$$

Spherical Burgers Equation

$$\frac{\partial u}{\partial t} + \frac{u}{t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

One discrete symmetry generator

$$\Gamma : (x, t, u) \to (-x, t, -u)$$

Gradient Descent Method

Other ways to find the transformation matrix?

Utilizing gradient descent

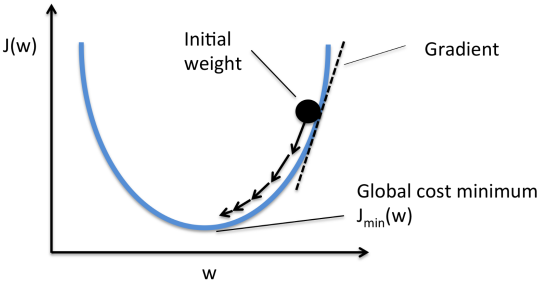


Figure: Source: https://ekamperi.github.io/machine%20learning/2019/07/28/gradient-descent.html

Gradient Descent Method

Random transformation matrix

$$B^{(0)} = \begin{pmatrix} b_1^1 & b_1^2 & \dots & b_1^r \\ b_2^1 & b_2^2 & \dots & b_2^r \\ \vdots & \vdots & \ddots & \vdots \\ b_r^1 & b_r^2 & \dots & b_r^r \end{pmatrix} \rightarrow \begin{pmatrix} 0.3142 & 1.9728 & \dots & 1.2987 \\ -0.9425 & -1.2042 & \dots & 0.8242 \\ \vdots & \vdots & \ddots & \vdots \\ 0.5528 & -1.9230 & \dots & 1.0125 \end{pmatrix}$$

Calculate nonlinear conditions

$$L(b_1^1, b_1^2, \dots, b_r^r) = \sqrt{\sum_n \sum_i \sum_j \left(c_{lm}^n b_i^l b_j^m - c_{ij}^k b_k^n \right)^2}$$

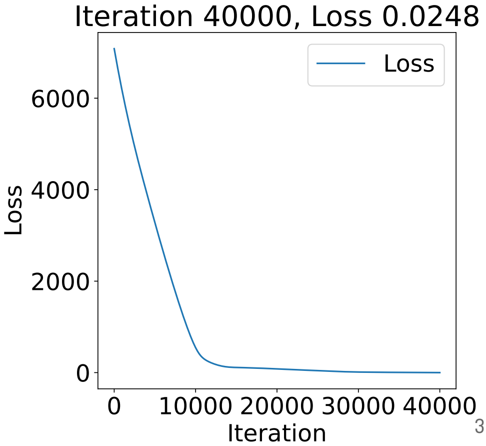
Gradient Descent Method

Update transformation matrix

$$B^{(1)} = B^{(0)} - \alpha \nabla L(b_1^1, b_1^2, ..., b_r^r)$$

Keep going until loss converges

Take average of multiple simulations

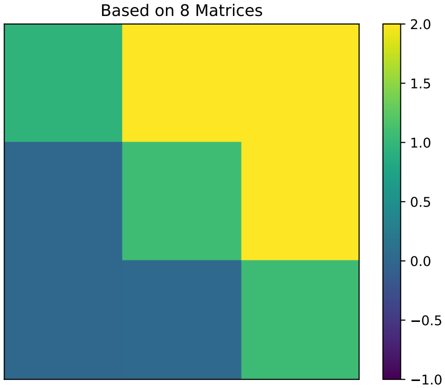


Spherical Burger’s Equation

Analytically found

$$B = \begin{pmatrix} 1 & b_1^2 & b_1^3 \\ 0 & b_2^2 & b_2^3 \\ 0 & 0 & b_3^3 \end{pmatrix}$$

Gradient descent algorithm gives



Heat Equation

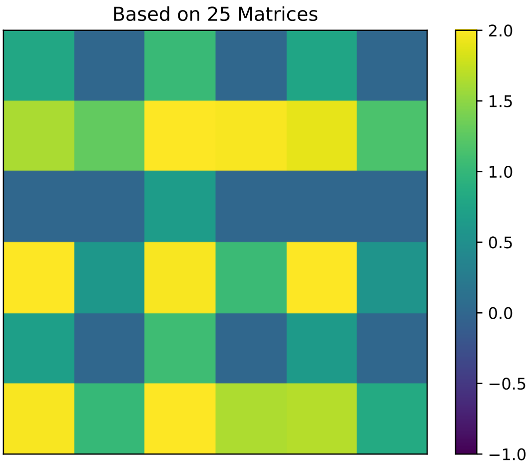
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Lie algebra consists of 6 generators

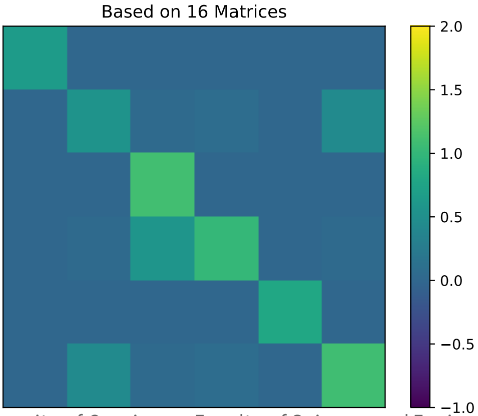
$$B = \begin{pmatrix} b_1^1 & b_1^2 & b_1^3 & b_1^4 & b_1^5 & b_1^6 \\ b_2^1 & b_2^2 & b_2^3 & b_2^4 & b_2^5 & b_2^6 \\ b_3^1 & b_3^2 & b_3^3 & b_3^4 & b_3^5 & b_3^6 \\ b_4^1 & b_4^2 & b_4^3 & b_4^4 & b_4^5 & b_4^6 \\ b_5^1 & b_5^2 & b_5^3 & b_5^4 & b_5^5 & b_5^6 \\ b_6^1 & b_6^2 & b_6^3 & b_6^4 & b_6^5 & b_6^6 \end{pmatrix}$$

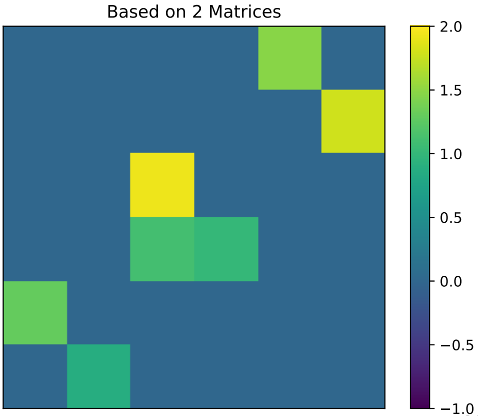
Difficult to solve the nonlinear conditions

Heat Equation



Heat Equation





Heat Equation (*)

$$B_1 = \begin{pmatrix} a e^{\epsilon_4} & 0 & 0 & 0 & 0 & 0 \\ 0 & a^2 e^{2\epsilon_4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{a} e^{-\epsilon_4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{a^2} e^{-2\epsilon_4} \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & a e^{\epsilon_4} & 0 \\ 0 & 0 & 0 & 0 & 0 & a^2 e^{2\epsilon_4} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ -\frac{1}{a} e^{-\epsilon_4} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{a^2} e^{-2\epsilon_4} & 0 & 0 & 0 & 0 \end{pmatrix}$$

Heat Equation

2 types of discrete symmetries depending on B_1, B_2. For B_1:

Real: $\Gamma_1 : (x, t, u) \to (-x, t, u), \qquad \Gamma_2 : (x, t, u) \to (x, t, -u)$

Complex: $\Gamma_3 : (x, t, u) \to (ix, -t, u), \qquad \Gamma_4 : (x, t, u) \to (x, t, iu)$

From B_2:

$$\Gamma_5 : (x, t, u) \to \left(\frac{x}{2t}, \quad -\frac{1}{4t}, \quad \sqrt{t} \exp\left(\frac{x^2}{2t}\right) u \right)$$

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From B_2:

$$\Gamma_5 : (x, t, u) \to \left(\frac{x}{2t}, \quad -\frac{1}{4t}, \quad \sqrt{t} \exp\left(\frac{x^2}{2t}\right) u \right)$$

New solutions of the form

$$\tilde{f}(x, t) = \frac{i}{2\sqrt{t}} \exp\left(-\frac{x^2}{2t}\right) f\left(-\frac{x}{2t}, -\frac{1}{4t}\right)$$

Heat Equation

Real solutions and real transformations:
2 discrete generators, total symmetry group has 4 components

Complex solutions and complex transformations:
At least 32 distinct discrete symmetries $\rightarrow$ 32 components

Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$\begin{aligned}
\mathbf{v_1} &= \partial_x & \mathbf{v_8} &= (x^2 - y^2 - z^2)\partial_x + 2xy\partial_y + 2xz\partial_z - xu\partial_u \\
\mathbf{v_2} &= \partial_y & \mathbf{v_9} &= 2yx\partial_x + (-x^2 + y^2 - z^2)\partial_y + 2yz\partial_z - yu\partial_u \\
\mathbf{v_3} &= \partial_z & \mathbf{v_{10}} &= 2zx\partial_x + 2zy\partial_y + (-x^2 - y^2 + z^2)\partial_z - zu\partial_u \\
\mathbf{v_4} &= y\partial_z - z\partial_y & \mathbf{v_{11}} &= x\partial_x + y\partial_y + z\partial_z \\
\mathbf{v_5} &= z\partial_x - x\partial_z \\
\mathbf{v_6} &= x\partial_y - y\partial_x \\
\mathbf{v_7} &= u\partial_u
\end{aligned}$$

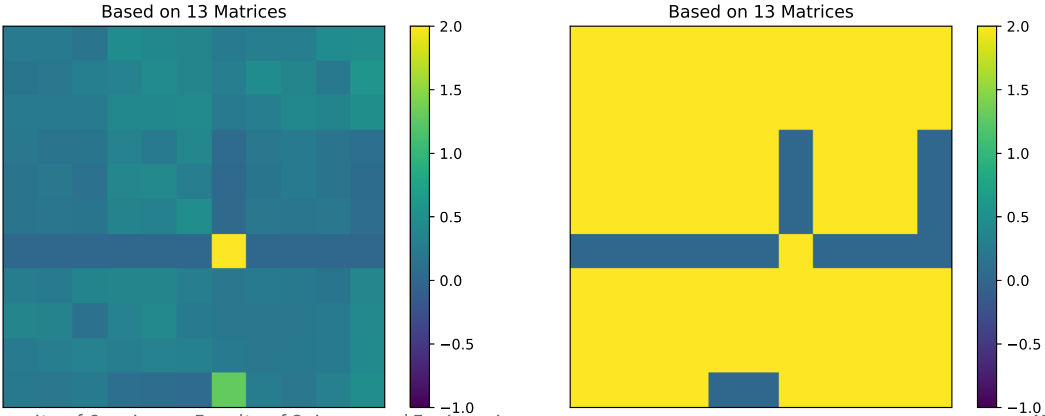
Laplace Equation

B contains 121 parameters

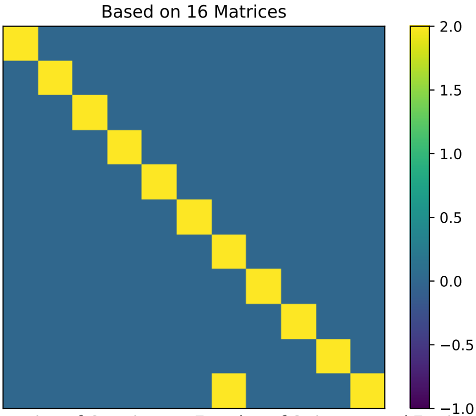
605 coupled nonlinear equations, looking like

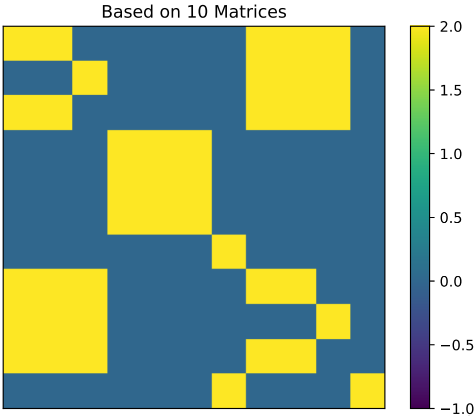
$$-b_1^{11}b_2^1 + b_1^6 b_2^2 - b_1^5 b_2^3 + b_1^3 b_2^5 - b_1^2 b_2^6 + b_1^1 b_2^{11} = 0$$
$$2b_7^{10}b_8^2 - 2b_7^9 b_8^3 + b_7^6 b_8^5 - b_7^5 b_8^6 + 2b_7^3 b_8^9 - 2b_7^2 b_8^{10} = 0$$
$$2b_4^9 - b_2^{10}b_{10}^4 + b_2^8 b_{10}^6 - b_2^6 b_{10}^8 + b_2^{11}b_{10}^9 + b_2^4 b_{10}^{10} - b_2^9 b_{10}^{11} = 0$$

Laplace Equation



Laplace Equation





Laplace Equation

Real transformations from B_1, map (x, y, z, u) to

$$\Gamma_1 : (-x, y, z, u), \quad \Gamma_2 : (x, -y, z, u), \quad \Gamma_3 : (x, y, -z, u), \quad \Gamma_4 : (x, y, z, -u)$$

Complex transformations from B_1

$$\Gamma_1 : (ix, y, z, u), \quad \Gamma_2 : (x, iy, z, u), \quad \Gamma_3 : (x, y, iz, u), \quad \Gamma_4 : (x, y, z, iu)$$

From B_2

$$\Gamma_5 : \left(\frac{x}{x^2 + y^2 + z^2}, \frac{y}{x^2 + y^2 + z^2}, \frac{z}{x^2 + y^2 + z^2}, (x^2 + y^2 + z^2)^{1/2} u \right)$$

Laplace Equation

From B_2

$$\Gamma_5 : \left(\frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2}, (x^2+y^2+z^2)^{1/2} u \right)$$

Hence if $u = f(x, y, z)$ is a solution, so is

$$\tilde{f}(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}} f\left(\frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2} \right)$$

Laplace Equation

From B_2

$$\Gamma_5 : \left(\frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2}, (x^2+y^2+z^2)^{1/2} u \right)$$

Hence if $u = f(x, y, z)$ is a solution, so is

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Additionally, all permutations between x, y, z can be found as discrete symmetries

Conclusion

Conclusion

- Heat and Schrödinger equation do not have global symmetry groups
- Discrete point symmetries can also be found using an optimization algorithm
- Discrete point symmetries provide additional information about the total symmetry group structure

Outlook

Future Research

- Alter algorithm to increase accuracy of finding transformation matrices
- Automate steps between solving nonl. conds and adjoint matrices
- Combine the results of continuous and discrete point transformations

References

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European Journal of Differential Equations 11(5), 515 – 527.

Thank you for your attention

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Local Transformation Group

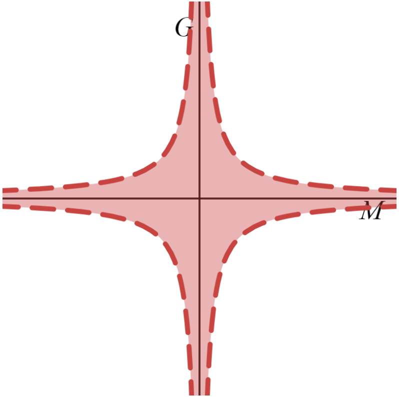
Acting with group element g on coordinates x, u

Let $g_\epsilon = \exp(\epsilon \mathbf{v_1})$ with $\mathbf{v_1} = \partial_x$

$$g_\epsilon \cdot (x, u) = (x + \epsilon, u)$$

Similarly, let $h_\epsilon = \exp(\epsilon \mathbf{v_2})$ with $\mathbf{v_2} = x^2 \partial_x$

$$h_\epsilon \cdot (x, u) = \left(\frac{x}{1 - \epsilon x}, u \right) \quad \text{for} \quad |\epsilon x| < 1$$



Prolongation Method

Infinitesimal Criterion Theorem
Suppose $\Delta(x, u^{(n)}) = 0$ is a differential equation defined over $M \subseteq X \times U$. If G is a local group of transformations acting on M and

$$\mathrm{pr}^{(n)}\mathbf{v}[\Delta(x, u^{(n)})] = 0 \tag{1}$$

whenever $\Delta(x, u^{(n)}) = 0$ for every infinitesimal generator $\mathbf{v}$ of G, then G is a symmetry group of the system.

Prolongation Method

Generator Prolongation

Let $\mathbf{v}$ be a vector field, defined on subset $M \subseteq X \times U$. The n-th order prolongation of $\mathbf{v}$ is the vector field

$$\mathrm{pr}^{(n)}\mathbf{v} = \mathbf{v} + \sum_{\mathbf{J}} \phi^{\mathbf{J}}(x, u^{(n)}) \frac{\partial}{\partial u_{\mathbf{J}}}$$

defined on $M^{(n)} \subseteq X \times U^{(n)}$, where $\mathbf{J} = (j_1, j_2, ..., j_k)$ with $1 \leq j_k \leq p$ and $1 \leq k \leq n$. The coefficient functions $\phi^{\mathbf{J}}$ are given by

$$\phi^{\mathbf{J}}(x, u^{(n)}) = D_{\mathbf{J}}\left(\phi - \sum_{i=1}^{p} \xi^i \frac{\partial u}{\partial x^i}\right) + \sum_{i=1}^{p} \xi^i \frac{\partial u_{\mathbf{J}}}{\partial x^i}$$

where $D_{\mathbf{J}}$ is the total derivative w.r.t. $\mathbf{J}$ and $u_{\mathbf{J}} = \frac{\partial u}{\partial x^{\mathbf{J}}}$

Finding Transformed Functions

For transformations of type

$$(\tilde{x}, \tilde{u}) = g \cdot (x, u) = (\Xi_g(x), \Phi_g(x)u)$$

Then the solutions $u = f(x)$ transform as

$$\tilde{f}(x) = \Phi_g(\Xi_g^{-1}(x))f(\Xi_g^{-1}(x))$$

For transformations of type

$$(\tilde{x}, \tilde{u}) = g \cdot (x, u) = (\Xi_g(x, u), \Phi_g(x, u))$$

Then the solutions $u = f(x)$ transform as (I is identity function, $I(x) = x$)

$$\tilde{f}(x) = [\Phi_g \circ (I \times f)] \circ [(\Xi_g \circ (I \times f))]^{-1}$$

Heat Eq Transformed Solutions

$$\tilde{f}(x,t) = f(x-\epsilon, t)$$
$$\tilde{f}(x,t) = f(x, t-\epsilon)$$
$$\tilde{f}(x,t) = e^{\epsilon} f(x,t)$$
$$\tilde{f}(x,t) = f(e^{-\epsilon}x, e^{-2\epsilon}t)$$
$$\tilde{f}(x,t) = e^{-\epsilon x+\epsilon^2 t} f(x-2\epsilon t, t)$$
$$\tilde{f}(x,t) = \frac{1}{\sqrt{1+4\epsilon t}} \exp\left(\frac{-\epsilon x^2}{1+4\epsilon t}\right) f\left(\frac{x}{1+4\epsilon t}, \frac{t}{1+4\epsilon t}\right)$$
$$\tilde{f}(x,t) = f(x,t) + \epsilon\alpha(x,t)$$

Spherical Burger's Symmetry Condition

$$\tilde{u}_{\tilde{t}} + \frac{\tilde{u}}{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} = \tilde{u}_{\tilde{x}\tilde{x}}$$

$$J = \begin{vmatrix} D_x\tilde{x} & D_x\tilde{t} \\ D_t\tilde{x} & D_t\tilde{t} \end{vmatrix} = bc_1$$

$$\tilde{u}_{\tilde{x}} = \frac{1}{J}\begin{vmatrix} D_x\tilde{u} & D_x\tilde{t} \\ D_t\tilde{u} & D_t\tilde{t} \end{vmatrix} = \frac{1}{c_1}u_x$$

$$\tilde{u}_{\tilde{x}\tilde{x}} = \frac{1}{J}\begin{vmatrix} D_x\tilde{u}_{\tilde{x}} & D_x\tilde{t} \\ D_t\tilde{u}_{\tilde{x}} & D_t\tilde{t} \end{vmatrix} = \frac{1}{bc_1}u_{xx}$$

$$\tilde{u}_{\tilde{t}} = \frac{1}{J}\begin{vmatrix} D_x\tilde{x} & D_x\tilde{u} \\ D_t\tilde{x} & D_t\tilde{u} \end{vmatrix} = \frac{b}{c_1^2}u_t - \frac{1}{2}\frac{c_2}{c_1}t^{-3/2} - \frac{b}{c_1}\ln(c_1)uu_x - \frac{1}{2}\frac{bc_3}{c_1^2}t^{-1/2}u_x$$

Spherical Burger's Symmetry Condition

Original:

$$u_t + \frac{u}{t} + uu_x = u_{xx}$$

Transformed PDE

$$\frac{b}{c_1^2}u_t + \frac{b}{c_1^2}\frac{u}{t} + \frac{b}{c_1^2}(1 - \ln(c_1))uu_x + \frac{1}{2}\frac{c_2}{c_1}t^{-3/2} + \left(\frac{c_2}{c_1} - \frac{1}{2}b\frac{c_3}{c_1^2}\right)u_x t^{-1/2} = \frac{1}{bc_1}u_{xx}$$

Only holds for $c_2 = c_3 = 0$, $c_1 = 1$ and $b = \pm 1$

Laplace Equation Transformation Matrix

Transformation matrix before removing the cont. symm. from the spatial rotations.

$$
B_1 = \begin{pmatrix}
 & & & & & & & b_1^8 & b_1^9 & b_1^{10} & \\
 & & & & & & & b_2^8 & b_2^8 & b_2^{10} & \\
 & & & & & & & b_3^8 & b_3^9 & b_3^{10} & \\
 & & & b_4^4 & b_4^5 & b_4^6 & & & & & \\
 & & & b_5^4 & b_5^5 & b_5^6 & & & & & \\
 & & & b_6^4 & b_6^5 & b_6^6 & & & & & \\
 & & & & & & b_7^7 & & & & \\
b_8^1 & b_8^2 & b_8^3 & & & & & & & & \\
b_9^1 & b_9^2 & b_9^3 & & & & & & & & \\
b_{10}^1 & b_{10}^2 & b_{10}^3 & & & & & & & & \\
 & & & & & & b_{11}^7 & & & & -1
\end{pmatrix}.
$$

Local Transformation Group

1

2