Derivation of the Effective Connectivity $J_{\text{eff}}(x, a, b)$

This note derives the Jacobian of the x-dynamics with respect to x, treating the adaptation variables a and the synaptic depression variables b as constants (i.e., "frozen" at a given time t_1). The result is an effective connectivity matrix $J_{\text{eff}}(x, a, b)$ that depends on the current state. Rather that considering the entire Jacobian in which the adaptation variables are included in the state, this approach helps the reader visualize how adaptation can be viewed as modulating connectivity, which is a concept investigated in the manuscript.

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1. Model definition

We consider a network of N neurons (indices $i, j = 1, \ldots, N$) with the following equations:

1.1 x-dynamics

The membrane / rate state variable x_i evolves as

$$\dot{x}_i = \frac{-x_i + u_i + \sum_{j=1}^{N} w_{ij}\, r_j}{\tau_d},$$

where

- τ_d is the (global) decay time constant,
- u_i is an external input (ignored when computing the Jacobian),
- w_{ij} is the (structural) weight from neuron j to neuron i,
- r_j is the firing rate of neuron j.

1.2 Nonlinearity and adaptation

The firing rate r_i is given by

$$r_i = b_i\, \phi\left(x_i - a_{0_i} - c\sum_{k=1}^{K} a_{ik} \right),$$

where

- $\phi(\cdot)$ is a static nonlinearity (e.g., sigmoid),
- b_i is a synaptic depression / gain factor,
- a_{0_i} is an offset parameter (e.g., preferred input or threshold shift),
- c is an adaptation gain,
- a_{ik} are adaptation state variables for neuron i and component $k = 1, \ldots, K$.

The dynamics of a_{ik} and b_i are not needed to compute J_{eff}, since we explicitly treat a and b as constants during this derivation.

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2. Vector-matrix notation

Let

- $x = (x_1, \ldots, x_N)^T$,
- $r = (r_1, \ldots, r_N)^T$,
- $u = (u_1, \ldots, u_N)^T$,
- $b = (b_1, \ldots, b_N)^T$,
- $W = (w_{ij}) \in \mathbb{R}^{N \times N}$,
- 1 be the N-vector of ones.

The recurrent input term can be written as

$$\left(\sum_{j=1}^{N} w_{ij} r_j \right)_i = (Wr)_i.$$

To make it explicit that r_j scales the columns of W, we can also write

$$Wr = W \operatorname{diag}(r) \, 1,$$

since

$$(W \operatorname{diag}(r) \, 1)_i = \sum_j w_{ij} r_j.$$

Thus, the x-dynamics in vector form is

$$\dot{x} = \frac{-x + u + Wr}{\tau_d} = \frac{-x + u + W \operatorname{diag}(r) \, 1}{\tau_d}.$$

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3. Jacobian of r with respect to x

We treat a_{ik} and b_i as constants. For each neuron i,

$$r_i = b_i \, \phi \left(x_i - a_{0_i} - c \sum_{k=1}^{K} a_{ik} \right).$$

Define the constant

$$\operatorname{const}_i = a_{0_i} + c \sum_{k=1}^{K} a_{ik}.$$

Then

$$r_i = b_i \, \phi(x_i - \operatorname{const}_i).$$

The partial derivative of r_i with respect to x_j is

$$\frac{\partial r_i}{\partial x_j} = b_i \, \phi'(x_i - \operatorname{const}_i) \, \delta_{ij},$$

where δ_{ij} is the Kronecker delta.

Define the gain vector

$$g_i = b_i \, \phi' \left(x_i - a_{0_i} - c \sum_{k=1}^{K} a_{ik} \right),$$

and the corresponding diagonal matrix

$$G = \operatorname{diag}(g_1, \ldots, g_N).$$

In matrix form, the Jacobian of r with respect to x is

$$\frac{\partial r}{\partial x} = G.$$

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4. Jacobian of $\dot{x}$ with respect to x

We now compute

$$J_{\text{eff}}(x, a, b) = \frac{\partial \dot{x}}{\partial x}.$$

Recall

$$\dot{x} = \frac{-x + u + Wr}{\tau_d}.$$

4.1 Contribution from the leak term

For the leak term $-x/\tau_d$,

$$\frac{\partial}{\partial x_j}\left(-\frac{x_i}{\tau_d}\right) = -\frac{1}{\tau_d}\delta_{ij}.$$

Thus, in matrix form this contributes

$$-\frac{1}{\tau_d}I,$$

where I is the $N \times N$ identity matrix.

4.2 Contribution from the recurrent term

For the recurrent term $(1/\tau_d)Wr$,

$$\frac{\partial}{\partial x_j}\left(\frac{1}{\tau_d}\sum_{k=1}^{N} w_{ik}r_k\right) = \frac{1}{\tau_d}\sum_{k=1}^{N} w_{ik}\frac{\partial r_k}{\partial x_j}.$$

Using $\partial r_k/\partial x_j = g_k\delta_{kj}$, we get

$$\frac{\partial}{\partial x_j}\left(\frac{1}{\tau_d}\sum_{k=1}^{N} w_{ik}r_k\right) = \frac{1}{\tau_d}w_{ij}g_j.$$

So the entries of the Jacobian are

$$J_{\text{eff},ij} = \frac{\partial \dot{x}_i}{\partial x_j} = -\frac{1}{\tau_d}\delta_{ij} + \frac{1}{\tau_d}w_{ij}g_j.$$

In matrix form, note that $(WG)_{ij} = \sum_k w_{ik}G_{kj} = w_{ij}g_j$, since G is diagonal. Therefore

$$J_{\text{eff}}(x, a, b) = \frac{1}{\tau_d}\left(-I + WG\right).$$

Explicitly,

$$G = \text{diag}\!\left(b_i\,\phi'\!\left(x_i - a_{0_i} - c\sum_{k=1}^{K} a_{ik}\right)\right).$$

5. Summary

- The effective connectivity Jacobian for the x-dynamics, treating a and b as constants and ignoring external input u in the derivative, is

$$J_{\text{eff}}(x, a, b) = \frac{1}{\tau_d}\left(-I + WG\right),$$

where

$$G = \text{diag}\!\left(b_i\,\phi'\!\left(x_i - a_{0_i} - c\sum_{k=1}^{K} a_{ik}\right)\right).$$

- Entrywise, this is

$$J_{\text{eff},ij} = -\frac{1}{\tau_d}\delta_{ij} + \frac{1}{\tau_d}w_{ij}\,b_j\,\phi'\!\left(x_j - a_{0_j} - c\sum_{k=1}^{K} a_{jk}\right).$$