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A generalization of the Sherman–Morrison–Woodbury formula

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ABSTRACT

In this paper, we develop conditions under which the Sherman–Morrison–Woodbury formula can be represented in the Moore–Penrose inverse and the generalized Drazin inverse forms. These results generalize the original Sherman–Morrison–Woodbury formula.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces over the same field. We denote the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ by $\mathcal{B}(\mathcal{H},\mathcal{K})$ and by $\mathcal{B}(\mathcal{H})$ when $\mathcal{H}=\mathcal{K}$. For $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$, let A^*, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ be the adjoint, the range and the null space of A, respectively. The **Moore–Penrose inverse** (for short **MP inverse**) of $T\in\mathcal{B}(\mathcal{H},\mathcal{K})$ is denoted by $T^+\in\mathcal{B}(\mathcal{K},\mathcal{H})$, and it is the unique solution to the following four operator equations:

$$TXT=T,\qquad XTX=X,\qquad TX=(TX)^*,\qquad XT=(XT)^*. \tag{1}$$

It is well known that T has the MP inverse if and only if $\mathcal{R}(T)$ is closed (see [1]). An element $T\in\mathcal{B}(\mathcal{H})$ whose spectrum $\sigma(T)$ consists of the set $\{0\}$ is said to be quasi-nilpotent. The **generalized Drazin inverse** (for short **GD inverse**) (see [1,2]) is the element $T^d\in\mathcal{B}(H)$ such that

$$TT^d=T^dT,\qquad T^dTT^d=T^d,\qquad T-T^2T^d\quad\text{is quasi-nilpotent}. \tag{2}$$

It is clear that $T^+=T^d=T^{-1}$ if $T\in\mathcal{B}(\mathcal{H})$ is invertible.

In the late 1940s and the 1950s Sherman and Morrison [3], Woodbury [4], Bartlett [5] and Bodewig [6] discovered the following result. The original **Sherman–Morrison–Woodbury** (for short **SMW**) formula has been used to consider the inverse of matrices. In this paper, we will consider the more generalized case.

Theorem 1.1 (*Sherman–Morrison–Woodbury*). *Let $A\in\mathcal{B}(\mathcal{H})$ and $G\in\mathcal{B}(\mathcal{K})$ both be invertible, and $Y,Z\in\mathcal{B}(\mathcal{K},\mathcal{H})$. Then $A+YGZ^*$ is invertible iff $G^{-1}+Z^*A^{-1}Y$ is invertible. In which case,*

$$(A+YGZ^*)^{-1}=A^{-1}-A^{-1}Y(G^{-1}+Z^*A^{-1}Y)^{-1}Z^*A^{-1}. \tag{3}$$

The operator YGZ^* in Theorem 1.1 is referred to as the update operator to the initial operator A. The SMW formula has been used in a wide variety of fields. An excellent review by Hager [7] described some of the applications to statistics, networks, structural analysis, asymptotic analysis, optimization and partial differential equations (see [8,9]). The objectives of this paper are to generalize the SMW formula to the cases when A and $A+YGZ^*$ are not invertible. The MP inverse and the GD inverse of a modified operator (see [2]) $A+YGZ^*$ can be expressed in terms of generalized SMW forms being established. As a consequence, our results generalize the results of Chen Xuzhou in [10].

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2. Main results

First, we generalize the SMW formula to the case when A and G are MP invertible. The following lemma is well known and can be found in [1] and [10, Lemma 1].

Lemma 2.1. *If $A \in \mathcal{B}(\mathcal{H})$ and $P = P^2 \in \mathcal{B}(\mathcal{H})$ then*

(i) $PA = A \Longleftrightarrow \mathcal{R}(A) \subset \mathcal{R}(P)$,
(ii) $AP = A \Longleftrightarrow \mathcal{N}(P) \subset \mathcal{N}(A)$.

Theorem 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$, $G \in \mathcal{B}(\mathcal{K})$ and $Y, Z \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\mathcal{R}(A)$ and $\mathcal{R}(G)$ are closed, also let $B = A + YGZ^*$ and $S = G^+ + Z^*A^+Y$ such that $\mathcal{R}(S)$ and $\mathcal{R}(B)$ are closed. If*

$$\mathcal{R}(A^*) \subset \mathcal{R}(B^*), \quad \mathcal{N}(A^*) \subset \mathcal{N}(B^*), \quad \mathcal{N}(G^*) \subset \mathcal{N}(Y) \quad \textit{and} \quad \mathcal{N}(S^*) \subset \mathcal{N}(G), \tag{4}$$

then $(A + YGZ^*)^+ = A^+ - A^+Y(G^+ + Z^*A^+Y)^+Z^*A^+$.

Proof. Since $\mathcal{R}(A^+) = \mathcal{R}(A^*) \subset \mathcal{R}(B^*) = \mathcal{R}(B^+B)$ and $\mathcal{N}(AA^+) = \mathcal{N}(A^*) \subset \mathcal{N}(B^*) = \mathcal{N}(B^+)$, by Lemma 2.1, we get $B^+BA^+ = A^+$ and $B^+AA^+ = B^+$. Hence,

$$B^+Y + B^+(B - A)A^+Y = A^+Y.$$

From $B - A = YGZ^*$ and $\mathcal{N}(G^*) \subset \mathcal{N}(Y)$, by Lemma 2.1 again, we get $B^+YGG^+ + B^+YGZ^*A^+Y = A^+Y$, i.e., $B^+YGS = A^+Y$. The condition $\mathcal{N}(S^*) \subset \mathcal{N}(G)$ implies that $B^+YG = B^+YGSS^+ = A^+YS^+$. Since $B = A + YGZ^*$, we have $B^+BA^+ = B^+AA^+ + B^+YGZ^*A^+$. Hence, by (4),

$$B^+ = A^+ - B^+YGZ^*A^+ = A^+ - A^+YS^+Z^*A^+ = A^+ - A^+Y(G^+ + Z^*A^+Y)^+Z^*A^+. \quad \Box$$

In Theorem 2.1, if S and G are invertible, we can get the following simple result.

Corollary 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$, $G \in \mathcal{B}(\mathcal{K})$ and $Y, Z \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\mathcal{R}(A)$ is closed and G is invertible, also let $B = A + YGZ^*$ and $S = G^{-1} + Z^*A^+Y$ such that $\mathcal{R}(B)$ is closed and S is invertible. If*

$$\mathcal{R}(A^*) \subset \mathcal{R}(B^*) \quad \textit{and} \quad \mathcal{N}(A^*) \subset \mathcal{N}(B^*),$$

then $(A + YGZ^*)^+ = A^+ - A^+Y(G^{-1} + Z^*A^+Y)^{-1}Z^*A^+$.

As in [11], $M_{m,n}$ denotes the space of complex-valued $m \times n$ matrices and, when $m = n$, this is shortened to M_n. When $G = I$ and $\mathcal{H}, \mathcal{K}$ are finite dimensional complex spaces, the condition that S is invertible in Corollary 2.1 can be dropped and Corollary 2.1 reduces to the following result, which is the generalization of Theorem 1 in [10].

Corollary 2.2. *Let $A, B \in M_{m,n}$, $Y \in M_{m,s}$ and $Z \in M_{n,s}$. If $B = A + YZ^*$ and*

$$\mathcal{R}(A^*) \subset \mathcal{R}(B^*) \quad \textit{and} \quad \mathcal{N}(A^*) \subset \mathcal{N}(B^*),$$

then $I + Z^*A^+Y$ is invertible, in which case $(A + YZ^*)^+ = A^+ - A^+Y(I + Z^*A^+Y)^{-1}Z^*A^+$.

Proof. Since $\mathcal{R}(A^*) \subset \mathcal{R}(B^*)$ and $\mathcal{N}(A^*) \subset \mathcal{N}(B^*)$, by the proof of Theorem 2.1, we have

$$B^+ = A^+ - B^+YZ^*A^+ \quad \text{and} \quad B^+Y = A^+Y - B^+YZ^*A^+Y.$$

Hence $B^+Y(I + Z^*A^+Y) = A^+Y$. If $x \in \mathcal{N}(I + Z^*A^+Y)$, then $x = -Z^*A^+Yx = -Z^*B^+Y(I + Z^*A^+Y)x = 0$, which implies that $I + Z^*A^+Y$ is nonsingular and $B^+Y = A^+Y(I + Z^*A^+Y)^{-1}$. So

$$B^+ = A^+ - B^+YZ^*A^+ = B^+ = A^+ - A^+Y(I + Z^*A^+Y)^{-1}Z^*A^+. \quad \Box$$

Theorem 2.2. *Let $A \in \mathcal{B}(\mathcal{H})$, $G \in \mathcal{B}(\mathcal{K})$ and $Y, Z \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\mathcal{R}(A)$ and $\mathcal{R}(G)$ are closed, also let $B = A + YGZ^*$ and $S = G^+ + Z^*A^+Y$. If the range of $\mathcal{R}(S)$ is closed and*

$$\begin{aligned} &\mathcal{N}(A) \subset \mathcal{N}(Z^*), \quad &&\mathcal{N}(G^*) \subset \mathcal{N}(Y), \quad &&\mathcal{N}(S) \subset \mathcal{N}(Y), \\ &\mathcal{R}(Z^*) \subset \mathcal{R}(G^*), \quad &&\mathcal{R}(Z^*) \subset \mathcal{R}(S), \quad &&\mathcal{R}(Y) \subset \mathcal{R}(A), \end{aligned}$$

then $(A + YGZ^*)^+ = A^+ - A^+Y(G^+ + Z^*A^+Y)^+Z^*A^+$.

Proof. Define $X = A^+ - A^+YS^+Z^*A^+$. We shall prove that X satisfies definition (1). Since $\mathcal{N}(A) \subset \mathcal{N}(Z^*)$, $\mathcal{N}(S) \subset \mathcal{N}(Y)$ and $\mathcal{R}(Z^*) \subset \mathcal{R}(G^*)$, by Lemma 2.1, we have $Z^*(I - A^+A) = 0$, $Y(I - S^+S) = 0$ and $(I - G^+G)Z^* = 0$. Hence

$$\begin{aligned} XB &= (A^+ - A^+YS^+Z^*A^+)(A + YGZ^*) \\ &= A^+A + A^+YGZ^* - A^+YS^+Z^*A^+A - A^+YS^+Z^*A^+YGZ^* \\ &= A^+A + A^+YGZ^* - A^+YS^+Z^* - A^+YS^+(S - G^+)GZ^* \\ &= A^+A + A^+Y(I - S^+S)GZ^* - A^+YS^+(I - G^+G)Z^* \\ &= A^+A. \end{aligned}$$

If $\mathcal{N}(G^*) \subset \mathcal{N}(Y)$, $\mathcal{R}(Z^*) \subset \mathcal{R}(S)$ and $\mathcal{R}(Y) \subset \mathcal{R}(A)$, then $Y(I - GG^+) = 0$, $(I - SS^+)Z^* = 0$ and $(I - AA^+)Y = 0$. We have

$$\begin{aligned} BX &= (A + YGZ^*)(A^+ - A^+YS^+Z^*A^+) \\ &= AA^+ + YGZ^*A^+ - AA^+YS^+Z^*A^+ - YGZ^*A^+YS^+Z^*A^+ \\ &= AA^+ + YGZ^*A^+ - YS^+Z^*A^+ - YG(S - G^+)S^+Z^*A^+ \\ &= AA^+ + YG(I - SS^+)Z^*A^+ - Y(I - GG^+)S^+Z^*A^+ \\ &= AA^+. \end{aligned}$$

Thus $(XB)^* = XB$ and $(BX)^* = BX$. Moreover, we have $BXB = AA^+(A + YGZ^*) = A + AA^+YGZ^* = B$ and $XBX = A^+A(A^+ - A^+YS^+Z^*A^+) = X$. $\square$

In Theorem 2.2, if $G = I$ and $I + Z^*A^+Y$ is invertible, then we can get the following result, which is a generalization of Theorem 2 in [10].

Corollary 2.3. *Let $A \in \mathcal{B}(\mathcal{H})$ with $\mathcal{R}(A)$ be closed, $Y, Z \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $B = A + YZ^*$. If $I + Z^*A^+Y$ is invertible and*

$$\mathcal{N}(A) \subset \mathcal{N}(Z^*) \quad \text{and} \quad \mathcal{R}(Y) \subset \mathcal{R}(A),$$

then $(A + YZ^*)^+ = A^+ - A^+Y(I + Z^*A^+Y)^{-1}Z^*A^+$.

Next, let us generalize the SMW formula to the case of GD inverse.

Theorem 2.3. *Let $A \in \mathcal{B}(\mathcal{H})$, $G \in \mathcal{B}(\mathcal{K})$ and $Y, Z \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that A and G are GD invertible, also let $B = A + YGZ^*$ and $T = G^d + Z^*A^dY$ such that B and T are GD invertible. If*

$$\mathcal{R}(A^d) \subset \mathcal{R}(B^d), \qquad \mathcal{N}(A^d) \subset \mathcal{N}(B^d), \qquad \mathcal{N}(G^d) \subset \mathcal{N}(Y) \quad \text{and} \quad \mathcal{N}(T^d) \subset \mathcal{N}(G),$$

then $(A + YGZ^*)^d = A^d - A^dY(G^d + Z^*A^dY)^dZ^*A^d$.

Proof. Since $\mathcal{R}(A^d) \subset \mathcal{R}(B^d) = \mathcal{R}(B^dB)$ and $\mathcal{N}(AA^d) = \mathcal{N}(A^d) \subset \mathcal{N}(B^d)$, by Lemma 2.1, we get $B^dBA^d = A^d$ and $B^dAA^d = B^d$, which imply $B^dY + B^d(B - A)A^dY = A^dY$. Note that $B - A = YGZ^*$ and $\mathcal{N}(GG^d) = \mathcal{N}(G^d) \subset \mathcal{N}(Y)$. By Lemma 2.1 again, $B^dYGG^d + B^dYGZ^*A^dY = A^dY$, i.e., $B^dYGT = A^dY$. Since $\mathcal{N}(TT^d) = \mathcal{N}(T^d) \subset \mathcal{N}(G)$, we get $B^dYG = B^dYGTT^d = A^dYT^d$. From $B = A + YGZ^*$ we deduce that $B^dBA^d = B^dAA^d + B^dYGZ^*A^d$. Hence,

$$B^d = A^d - B^dYGZ^*A^d = A^d - A^dYT^dZ^*A^d = A^d - A^dY(G^d + Z^*A^dY)^dZ^*A^d. \quad \square$$

Corollary 2.4. *Let $A \in \mathcal{B}(\mathcal{H})$, $G \in \mathcal{B}(\mathcal{K})$ and $Y, Z \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that A is GD invertible and G is invertible, also let $B = A + YGZ^*$ and $T = G^{-1} + Z^*A^dY$ such that B is GD invertible and T is invertible. If*

$$\mathcal{R}(A^d) \subset \mathcal{R}(B^d) \quad \text{and} \quad \mathcal{N}(A^d) \subset \mathcal{N}(B^d),$$

then $(A + YGZ^*)^d = A^d - A^dY(G^{-1} + Z^*A^dY)^{-1}Z^*A^d$.

When $G = I$ and $\mathcal{H}$, $\mathcal{K}$ are finite dimensional complex spaces, Theorem 2.3 reduces to the following result, which is the generalization of Theorem 3 in [10].

Corollary 2.5. *Let $A, B \in M_n$ and $Y, Z \in M_{n,s}$. If $B = A + YZ^*$ and*

$$\mathcal{R}(A^d) \subset \mathcal{R}(B^d) \quad \text{and} \quad \mathcal{N}(A^d) \subset \mathcal{N}(B^d),$$

then $I + Z^*A^dY$ is invertible. In which case $(A + YZ^*)^d = A^d - A^dY(I + Z^*A^dY)^{-1}Z^*A^d$.

Theorem 2.4. *Let $A \in \mathcal{B}(\mathcal{H})$, $G \in \mathcal{B}(\mathcal{K})$ and $Y, Z \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that A and G are GD invertible, also let $B = A + YGZ^*$ and $T = G^d + Z^*A^dY$. If T is GD invertible and*

$$\begin{aligned} &\mathcal{N}(A^d) \subset \mathcal{N}(Z^*), \qquad \mathcal{N}(G^d) \subset \mathcal{N}(Y), \qquad \mathcal{N}(T^d) \subset \mathcal{N}(Y), \\ &\mathcal{R}(Z^*) \subset \mathcal{R}(G^d), \qquad \mathcal{R}(Z^*) \subset \mathcal{R}(T^d), \qquad \mathcal{R}(Y) \subset \mathcal{R}(A^d), \end{aligned} \tag{5}$$

then $(A + YGZ^*)^d = A^d - A^dY(G^d + Z^*A^dY)^dZ^*A^d$.

Proof. Since $P_{\mathcal{R}(A^d),\mathcal{N}(A^d)} = AA^d = A^dA$, by Lemma 2.1, the conditions in (5) imply that

$$(I - AA^d)Y = 0, \qquad (I - TT^d)Z^* = 0, \qquad (I - GG^d)Z^* = 0,$$
$$Y(I - GG^d) = 0, \qquad Z^*(I - AA^d) = 0, \qquad Y(I - TT^d) = 0.$$

Define $\Gamma = A^d - A^dYT^dZ^*A^d$. We shall prove that Γ satisfies definition (2). Firstly, from

$$\begin{aligned}
\Gamma B &= (A^d - A^dYT^dZ^*A^d)(A + YGZ^*) \\
&= A^dA + A^dYGZ^* - A^dYT^dZ^*A^dA - A^dYT^dZ^*A^dYGZ^* \\
&= A^dA + A^dYGZ^* - A^dYT^dZ^* - A^dYT^d(T - G^d)GZ^* \\
&= A^dA + A^dY(I - T^dT)GZ^* - A^dYT^d(I - G^dG)Z^* \\
&= A^dA
\end{aligned}$$

and

$$\begin{aligned}
B\Gamma &= (A + YGZ^*)(A^d - A^dYT^dZ^*A^d) \\
&= AA^d + YGZ^*A^d - AA^dYT^dZ^*A^d - YGZ^*A^dYT^dZ^*A^d \\
&= AA^d + YGZ^*A^d - YT^dZ^*A^d - YG(T - G^d)T^dZ^*A^d \\
&= AA^d + YG(I - TT^d)Z^*A^d - Y(I - GG^d)T^dZ^*A^d \\
&= AA^d,
\end{aligned}$$

we get $\Gamma B = B\Gamma$. Secondly, $\Gamma B\Gamma = A^dA\Gamma = A^dA(A^d - A^dYT^dZ^*A^d) = \Gamma$ and $B - B^2\Gamma = B(I - B\Gamma) = (A + YGZ^*)(I - AA^d) = A(I - AA^d)$ is quasi-nilpotent. So we have $\Gamma = (A + YGZ^*)^d$. □

In Theorem 2.4, if $G = I$ and $I + Z^*A^dY$ is invertible, then we can get the following result, which is a generalization of Theorem 4 in [10].

Corollary 2.6. *Let $A \in \mathcal{B}(\mathcal{H})$ with A being GD invertible, $Y, Z \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $B = A + YZ^*$. If $I + Z^*A^dY$ is invertible and*

$$\mathcal{N}(A^d) \subset \mathcal{N}(Z^*) \quad \text{and} \quad \mathcal{R}(Y) \subset \mathcal{R}(A^d),$$

then $(A + YZ^*)^d = A^d - A^dY(I + Z^*A^dY)^{-1}Z^*A^d$.

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