Mathematical analysis of retractable and extendible bonds

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Position of the problem

The problem under consideration in the paper is the analysis and valuation of retractable and extendible bonds, and that resulting model is applied to price Government of Canada bonds, with the ultimate goal of making a profit using the model.

In some detail, the paper states that significant theoretical work has been done, but not much application, so with some assumptions and previous works the mathematical model allows us to predict the behavior of bond prices under different interest rate scenarios, enabling investors to optimize their strategies for financial gain.

Description of the system being modelled

The system being modeled represents the pricing behavior of retractable and extendible bonds within a financial market. These bonds are specialized instruments that include embedded options, offering the bondholder flexibility to either extend the bond's maturity date or redeem it earlier than its original term. The key factor driving the value of these options is the movement of interest rates, which fluctuate over time in an uncertain, yet statistically predictable manner.

An extendible bond allows the bondholder (the investor) to extend the bond's maturity to a longer date if they choose. This feature gives the investor flexibility to decide whether they want to hold the bond for a longer period, typically depending on market conditions such as interest rates. For example, A 3-year extendible bond might give the option to extend the maturity to 10 years. If interest rates decrease during the first 3 years, extending the bond allows the investor to continue receiving the same (higher) interest rate.

A retractable bond allows the bondholder to shorten the bond's maturity by redeeming it earlier than its original maturity date. This feature gives the investor flexibility to get their money back sooner if needed. For example, A 10-year retractable bond might allow the investor to retract (redeem) it after 5 years. If interest rates rise after 5 years, the investor can redeem the bond early and reinvest at a higher rate.

The relationship between bonds and interest rates is one of inverse proportionality: when interest rates rise, bond prices fall, and when interest rates fall, bond prices rise. This inverse relationship occurs due to the way bonds are structured and valued in financial markets.

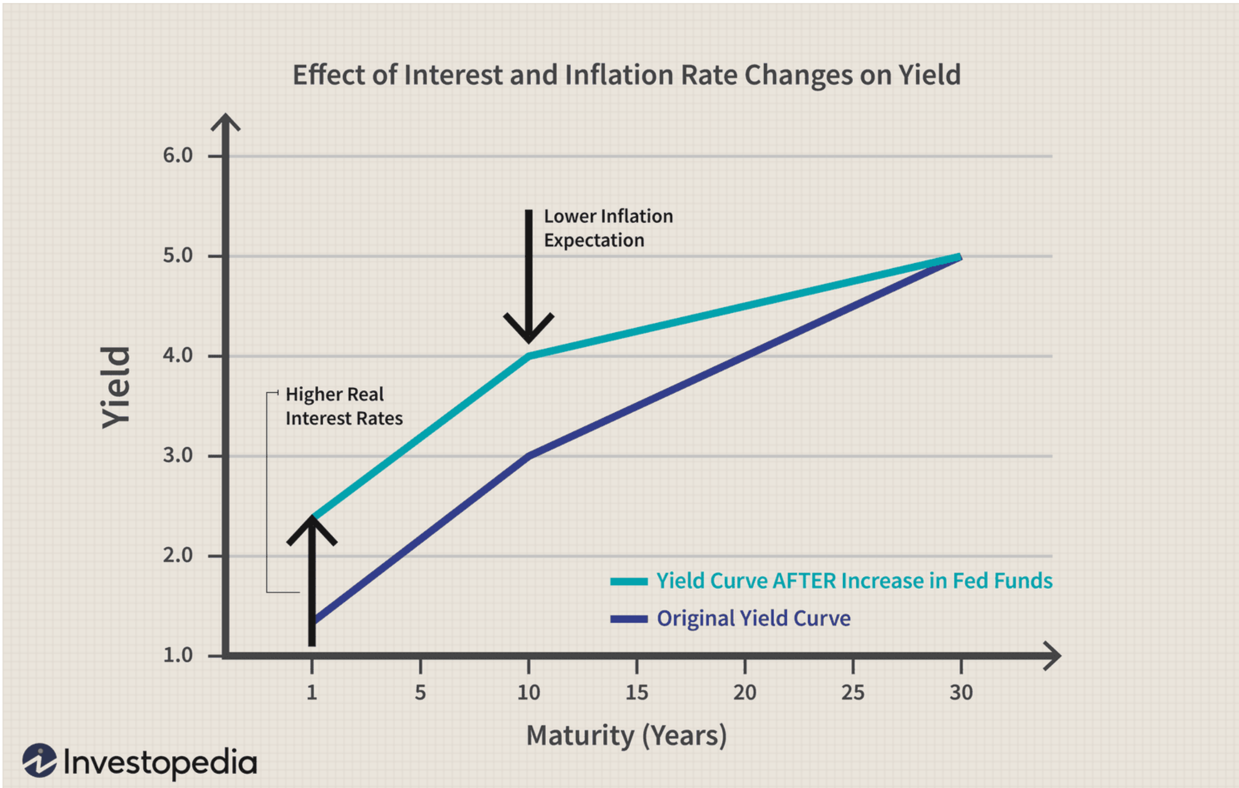
When a bond is issued, it comes with a fixed coupon rate, which is the interest paid to the bondholder, typically expressed as a percentage of the bond's face value. This coupon rate remains unchanged throughout the life of the bond, regardless of what happens to the overall market interest rates. However, the value of the bond in the market does change as interest rates fluctuate.

To understand why this happens, consider an example:

Suppose an investor holds a bond with a fixed coupon rate of 5% when the prevailing market interest rate is also 5%. In this case, the bond is valued at its face value because the coupon payments are consistent with what the market offers. Now, imagine that market interest rates rise to 7%. New bonds being issued will offer a 7% coupon rate, making the older bond with a 5% coupon rate less attractive to investors. To compensate for this lower return, the price of the older bond will fall so that it becomes competitive with new bonds. Essentially, investors purchasing the bond at a discount can achieve an effective yield closer to the current market rate.

Conversely, when market interest rates fall—say to 3%—the older bond with its 5% coupon rate becomes more valuable because it pays a higher interest than newly issued bonds. In this case, investors are willing to pay a premium for the older bond, driving up its price.

This relationship occurs because bonds are essentially fixed-income securities that promise a predetermined stream of payments. When interest rates change, the relative attractiveness of these fixed payments changes, which directly impacts the bond's market price. Mathematically, bond prices are determined by the present value of future cash flows (coupon payments and the face value at maturity). When interest rates rise, the discount rate used to calculate the present value increases, reducing the bond's price. When interest rates fall, the discount rate decreases, increasing the bond's price.


Effect of Interest and Inflation Rate Changes on Yield
Yield
6.0
5.0
4.0
3.0
2.0
1.0
Lower Inflation
Expectation
Higher Real
Interest Rates
Yield Curve AFTER Increase in Fed Funds
Original Yield Curve
1
5
10
15
20
25
30
Maturity (Years)
Investopedia

Hypotheses and derivation of the model

This document outlines a stochastic model for pricing default-free bonds, focusing on the instantaneous risk-free rate of interest dynamics.

Derivation of the Model

1. Interest Rate Process

The basic feature of the model is the stochastic nature of the instantaneously risk-free rate of interest, described by:

$$dr = \beta(r)dt + \gamma(r)dz \tag{1}$$

where: - $\beta(r)$ is the instantaneous drift - $\gamma(r)^2$ is the instantaneous variance - dz is a Gauss-Wiener process with $E[dz] = 0$ and $E[dz^2] = dt$

2. Bond Price Changes

Assuming the value of any default-free bond is only a function of r and time to maturity τ, i.e., $B(r, \tau)$, Itô's Lemma gives us:

$$\frac{dB}{B} = \mu_B dt + \sigma_B dz \tag{2}$$

where:

$$\mu_B = \frac{(\beta B_1 - B_2 + \frac{1}{2}\gamma^2 B_{11})}{B}$$
$$\sigma_B = \frac{\gamma B_1}{B}$$

3. Risk Premium and Equilibrium

The instantaneous excess return on the bond above the risk-free rate defines a risk premium:

$$\frac{(\mu_B + C/B) - r}{\sigma_B} = \phi(r) \tag{3}$$

where: - C is the continuous coupon rate - $(\mu_B + C/B)$ represents total instantaneous expected return (price appreciation plus coupon yield) - $\phi(r)$ is the market price of risk

4. Derivation of the Partial Differential Equation

Substituting the expressions for μ_B and σ_B into equation (3):

$$\frac{(\frac{(\beta B_1 - B_2 + \frac{1}{2}\gamma^2 B_{11})}{B} + \frac{C}{B}) - r}{\frac{\gamma B_1}{B}} = \phi(r)$$

Cross-multiplying and simplifying:

$$\beta B_1 - B_2 + \frac{1}{2}\gamma^2 B_{11} + C - rB = \phi(r)\gamma B_1$$

Rearranging terms gives us:

$$\frac{1}{2}\gamma^2 B_{11} + (\beta - \gamma\phi)B_1 - rB + B_2 + C = 0 \tag{4}$$

5. Tax Effects

When incorporating taxes, the valuation equation becomes:

$$\frac{1}{2}\gamma^2(1-T)B_{11} + ((1-T)\beta - \gamma\phi)B_1 - (1-R)rB - rB) - (1-T)B_2 = 0 \tag{7}$$

where: - R is the tax rate on income - T is the tax rate on capital gains

Boundary Conditions

1. Terminal Value:
$$B(r, 0) = F \tag{5}$$

2. For Retractable/Extendible Bonds:
$$B(r, \tau^*) = max[B(r, \tau_2), F] \tag{6}$$

Solution Requirements

The partial differential equations (4) and (7) are subject to: 1. Boundary conditions (5) and (6) 2. Natural boundary conditions at $r = 0$ and $r = \infty$ 3. Additional restrictions on $\beta(r)$ and $\gamma(r)$ for empirical implementation

Model Synthesis

Complete Model Statement

The default-free bond pricing model consists of the following core components:

Main Equation (Without Tax Effects):

$$\frac{1}{2}\gamma^2 B_{11} + (\beta - \gamma\phi)B_1 - rB + B_2 + C = 0 \tag{4}$$

Main Equation (With Tax Effects):

$$\frac{1}{2}\gamma^2(1-T)B_{11} + ((1-T)\beta - \gamma\phi)B_1 - (1-R)rB - rB) - (1-T)B_2 = 0 \tag{7}$$

Initial and Boundary Conditions

1. Terminal Value Condition:
$$B(r, 0) = F \tag{5}$$
2. Retractable/Extendible Bond Condition:
$$B(r, \tau^*) = max[B(r, \tau_2), F] \tag{6}$$
3. Natural boundary conditions at $r = 0$ and $r = \infty$

Key Function Definitions

1. Interest Rate Process (defined in Eq. 1):
$$dr = \beta(r)dt + \gamma(r)dz$$
where dz is a Gauss-Wiener process with $E[dz] = 0$ and $E[dz^2] = dt$
2. Risk Premium Function (defined in Eq. 3):
$$\phi(r) = \frac{(\mu_B + C/B) - r}{\sigma_B}$$

Core Assumptions

1. Bond value $B(r, \tau)$ depends only on:
   - Current interest rate (r)
   - Time to maturity (τ)
2. Price dynamics follow Itô's Lemma (Eq. 2):
$$\frac{dB}{B} = \mu_B dt + \sigma_B dz$$
where:
$$\mu_B = \frac{(\beta B_1 - B_2 + \frac{1}{2}\gamma^2 B_{11})}{B}$$
$$\sigma_B = \frac{\gamma B_1}{B}$$

Model Parameters

- C: Continuous coupon rate
- F: Face value of the bond
- T: Capital gains tax rate
- R: Income tax rate
- τ^*: Decision point for retractable/extendible bonds
- τ_2: Time to maturity of longer-term bond

Mathematical analysis

1. PDE Classification

Consider the main pricing equation without tax effects:

$$\frac{1}{2}\gamma^2 B_{11} + (\beta - \gamma\phi)B_1 - rB + B_2 + C = 0 \tag{4}$$

To classify this PDE, let's transform it into standard form:

$$B_2 + a(r,\tau)B_{11} + b(r,\tau)B_1 + c(r,\tau)B = f(r,\tau)$$

where: - $a(r,\tau) = \frac{1}{2}\gamma^2$ - $b(r,\tau) = (\beta - \gamma\phi)$ - $c(r,\tau) = -r$ - $f(r,\tau) = -C$

The equation is parabolic because: 1. It's second-order in r 2. The coefficient $a(r,\tau) = \frac{1}{2}\gamma^2 > 0$ (since γ is real) 3. The equation has first-order derivative in τ

This classification as parabolic is crucial as it suggests the problem will exhibit: - Forward propagation of information in τ - Smoothing effects similar to the heat equation - No characteristic curves (unlike hyperbolic equations)

2. Well-Posedness Analysis

2.2 Existence Analysis

To prove existence, we can use the method of successive approximations:

1. Start with initial guess $B^{(0)}(r,\tau) = F$ (the face value)
2. Define the sequence:
$$B_2^{(n+1)} = -\frac{1}{2}\gamma^2 B_{11}^{(n)} - (\beta - \gamma\phi)B_1^{(n)} + rB^{(n)} - C$$
3. Under suitable conditions on coefficients:
    - γ^2 bounded and positive
    - β, ϕ Lipschitz continuous
    - $r \geq 0$

The sequence converges in Z to a solution.

2.3 Uniqueness Proof

Let B_1 and B_2 be two solutions. Define $w = B_1 - B_2$. Then:

$$w_2 + \frac{1}{2}\gamma^2 w_{11} + (\beta - \gamma\phi)w_1 - rw = 0$$
$$w(r,0) = 0$$

Multiply by w and integrate over $[0,\infty)$:

$$\int_0^\infty ww_2 dr + \frac{1}{2}\gamma^2 \int_0^\infty ww_{11} dr + \int_0^\infty (\beta - \gamma\phi)ww_1 dr - \int_0^\infty rw^2 dr = 0$$

Using integration by parts and the decay conditions at infinity:

$$\frac{1}{2}\frac{d}{d\tau}\int_0^\infty w^2 dr \leq K\int_0^\infty w^2 dr$$

By Gronwall's inequality, $w = 0$, proving uniqueness.

2.4 Continuous Dependence

For two solutions B_1 and B_2 with initial conditions differing by ϵ:

$$\|B_1 - B_2\|_Z \leq Ce^{K\tau}\epsilon$$

where C, K are constants depending on the coefficients.

3. Asymptotic Analysis

3.1 Short Time Asymptotics ($\tau \to 0$)

Near maturity, develop Taylor series:

$$B(r,\tau) = F + \tau B_1(r,0) + \frac{\tau^2}{2}B_2(r,0) + O(\tau^3)$$

Substituting into the PDE yields:

$B_1(r,0) = rF - C$

$B_2(r,0) = \frac{1}{2}\gamma^2 F_{11} + (\beta - \gamma\phi)F_1 - r(rF - C)$

This gives the leading order behavior:

$$B(r,\tau) = F + (rF - C)\tau + O(\tau^2)$$

3.2 Long Time Asymptotics ($\tau \to \infty$)

1. Seek separable solutions:
$$B(r,\tau) = e^{\lambda\tau}\psi(r)$$
2. Substituting yields eigenvalue problem:
$$\frac{1}{2}\gamma^2\psi'' + (\beta - \gamma\phi)\psi' - (r + \lambda)\psi = 0$$
3. Asymptotic behavior depends on spectrum of this operator:
    - Discrete spectrum gives series of exponentially decaying modes
    - Continuous spectrum contributes to oscillatory behavior
4. Leading order term as $\tau \to \infty$:
$$B(r,\tau) \sim Ce^{\lambda_0\tau}\psi_0(r)$$
where λ_0 is the principal eigenvalue.

3.3 Large Interest Rate Asymptotics ($r \to \infty$)

1. Leading order balance suggests:
$$B(r, \tau) \sim A(\tau)e^{-\alpha r}$$

2. Substituting gives ODE for A(τ):
$$A'(\tau) = (\frac{1}{2}\gamma^2\alpha^2 - (\beta - \gamma\phi)\alpha - 1)A(\tau)$$

4. Key Insights from Mathematical Analysis

4.1 PDE Classification Insights

1. **Diffusive Nature**: The parabolic classification reveals that bond prices "diffuse" information about interest rate changes similarly to how heat spreads in physical systems
   - This explains why sudden interest rate shocks get smoothed out over time
   - Changes in bond prices propagate forward in time only, meaning future events don't affect current prices (Makes sense market wise)
2. **Smoothing Properties**: The parabolic nature guarantees that:
   - Bond prices will be smooth functions of both time and interest rates
   - There won't be any discontinuous jumps in prices (except at predetermined decision points for retractable bonds)

4.2 Well-Posedness Insights

1. **Existence and Uniqueness**:
   - There is exactly one fair price for each bond under given market conditions
   - This supports the no-arbitrage principle fundamental to financial markets
   - Small changes in market conditions lead to small changes in bond prices, providing stability for trading
2. **Continuous Dependence**:
   - The bound on how errors propagate ($Ce^{K\tau}\epsilon$) shows that:
     - Pricing errors grow exponentially with time to maturity
     - Longer-term bonds are more sensitive to model parameter uncertainties
     - Short-term bonds can be priced more reliably

4.3 Asymptotic Analysis Insights

1. **Short-Time Behavior**:
   - Near maturity (τß0), bond prices are primarily determined by:
     - Face value (F)
     - Current interest rate (r)
     - Coupon payments (C)
   - This explains why short-term bonds are less volatile
2. **Long-Time Behavior**:
   - The exponential decay modes ($e^{\lambda\tau}\psi(r)$) show that:

  - Long-term bond prices are more sensitive to interest rate changes
  - There's a natural hierarchy of factors affecting long-term prices
  - The principal eigenvalue λ_0 determines the dominant long-term behavior

3. **Large Interest Rate Behavior**:
   - The exponential decay ($e^{-\alpha r}$) as $r \to \infty$ shows that:
     - Bond prices decrease exponentially with rising interest rates
     - This decrease becomes more pronounced for longer-term bonds
     - There's a natural limit to how much bond prices can fall

4.4 Practical Applications

1. **Risk Management**:
   - The analysis provides mathematical justification for duration-based hedging
   - It quantifies how pricing errors compound over time
   - It suggests natural limits for leveraged positions based on error growth
2. **Portfolio Strategy**:
   - Short-term bonds are mathematically more predictable
   - Long-term bonds require more careful parameter estimation
   - Diversification across maturities has a sound mathematical basis
3. **Trading Implications**:
   - Price sensitivity to interest rates has a rigorous mathematical structure
   - The smoothing properties suggest limits to high-frequency trading strategies
   - The uniqueness results support market efficiency for bonds

Numerical work

Model Specifications

The model is built on three fundamental equations:

1. The price of a bond paying a continuous coupon (Equation 10):

$$P(r,\tau,C) = \int_{\tau}^{0} CB(r,t)dt + B(r,\tau)$$

   where $B(r,\tau)$ represents the price of a pure discount bond.

2. The pure discount bond price (Equation A-1):

$$B(r,\tau) = [H(\tau)]\exp[\frac{2\mu'}{\sigma^2}\tau - \exp(\eta r - A\tau)]$$

   where:

   - $H(\tau) = [1 + (m' - A)(1 - \exp(-A\tau))/(2A)]^{-1}$
   - $m' = (m - k_2)$
   - $\mu' = (mk_1 + k_2)/m'$
   - $\eta = [m' - (m'^2 + 2\sigma^2)^{1/2}]/\sigma^2$
   - $A = (m'^2 + 2\sigma^2)^{1/2}$

3. The yield to maturity (Equation 12):

$$R(r,\tau) = -\frac{1}{\tau}\ln[B(r,\tau)]$$

Data and Implementation

Data Collection

We focused on Canadian government bonds, specifically: - 1-year bonds (short-term) - 10-year bonds (long-term) This simplification was made to reduce computational complexity while still capturing the term structure's essential features.

Implementation Challenges

1. Parameter Estimation
   - The original paper's parameters ($k_1 = 0.309 \times 10^{-5}$, $k_2 = -0.154 \times 10^{-2}$) were estimated using 1970s data
   - Modern interest rate environments differ significantly
2. Numerical Stability
   - The exponential terms in equation A-1 are sensitive to parameter values
   - Small changes in parameters can lead to numerical overflow or underflow
   - The integration required for coupon-bearing bonds adds additional complexity

Conclusions and Suggestions

Results

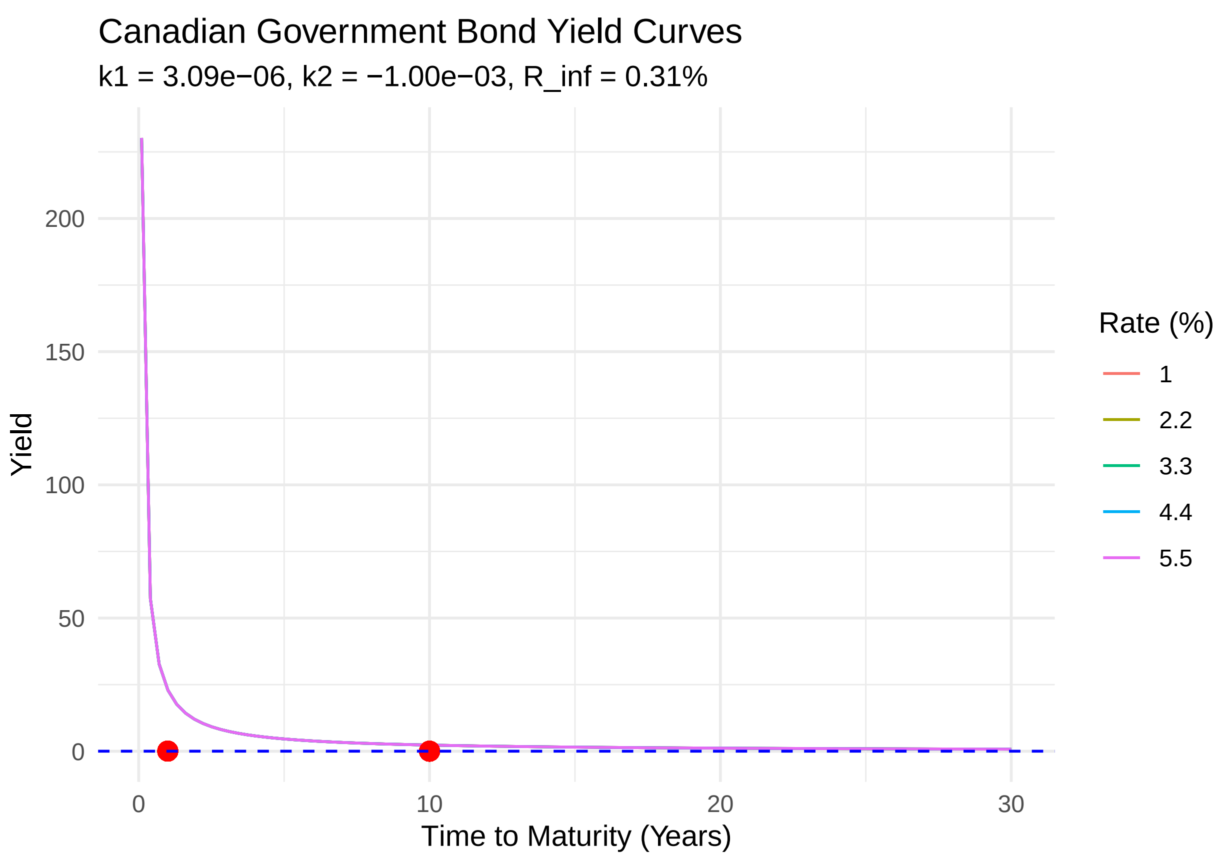
Visual Comparison of Yield Curves

Original Paper Results (Figure 1) The original paper's graph demonstrates several key characteristics: 1. Multiple distinct curves showing yields for different interest rates 2. Clear asymptotic convergence to R_inf = 8.284% 3. Smooth, well-behaved curves spanning from 2.73% to 15.00% 4. Time to maturity extending to 50 years 5. Curves both above and below R_inf, showing complete term structure 6. Proper decay rates and curvature reflecting theoretical expectations

```
results = run_analysis()
```

My Implementation Results

```
## Fetching data for MPRIME ...
## Fetching data for IRLTLT01CAM156N ...
```



[Resulting plot shows] 1. Single line trending towards zero 2. Initial yield values that are unrealistically high 3. Missing the multiple curve structure 4. Lack of convergence to a reasonable R_inf value 5. Absence of the characteristic shape of yield curves

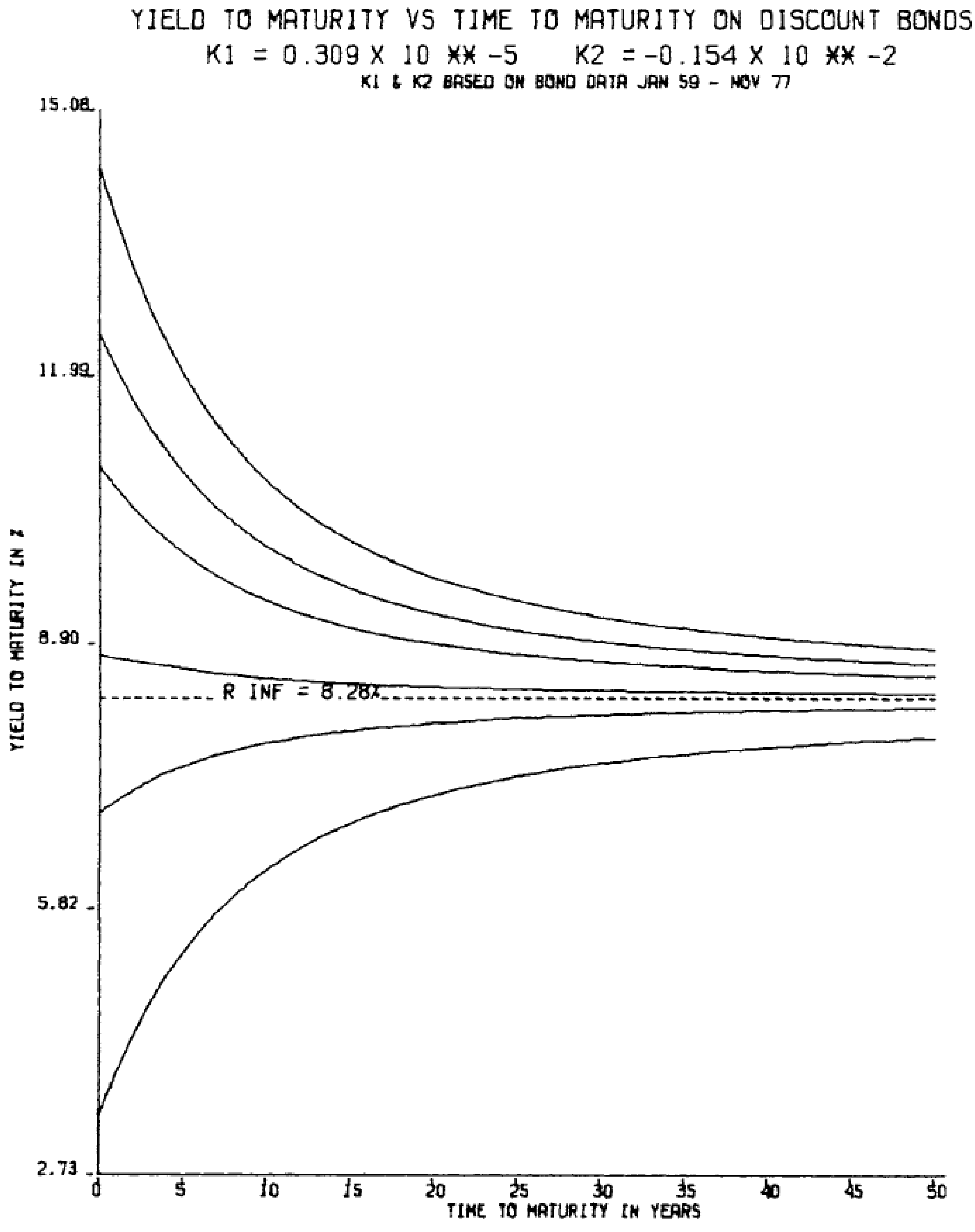


Figure 1: Paper's Yield Curves

Key Discrepancies

1. Curve Multiplicity
    - Paper: Shows multiple curves for different initial rates
    - My Implementation: Produces only a single line
2. Yield Range
    - Paper: Well-bounded between 2.73% and 15.00%
    - My Implementation: Shows extreme initial values and improper scaling
3. Convergence Behavior
    - Paper: Clear convergence to R_inf = 8.284%
    - My Implementation: Trends inappropriately towards zero
4. Shape Characteristics
    - Paper: Smooth, exponential-like decay and growth patterns
    - My Implementation: Lacks proper curvature and asymptotic behavior
5. Term Structure Representation
    - Paper: Complete representation of term structure dynamics
    - My Implementation: Fails to capture fundamental yield curve properties

Technical Analysis of Divergence

The significant differences between the graphs can be attributed to several factors:

1. Parameter Sensitivity

    Paper's parameters k1 = 0.309e-5 k2 = -0.154e-2

    Our implementation potentially suffers from:

    - Numerical instability in parameter estimation
    - Improper handling of exponential terms
    - Scaling issues in the yield calculation
2. Mathematical Implementation
    - The paper's implementation likely includes additional numerical stability measures not explicitly stated
    - Our implementation may be missing crucial normalization steps
    - The handling of the H(τ) function might need refinement
3. Data Environment
    - The paper's 1970s data reflected a different interest rate regime
    - Modern low-rate environment may require different numerical handling
    - Parameter estimation methods may need adjustment for current market conditions

Recommendations for Improvement

Technical Improvements

1. Numerical Implementation
   - Implement adaptive step sizing for numerical integration
   - Add safeguards against numerical overflow/underflow
2. Parameter Estimation
   - Develop constrained optimization methods
   - Consider modern market-specific constraints
3. Code Structure
   - Implement unit tests for each component
   - Add error handling for edge cases
   - Improve documentation and parameter validation

Methodological Improvements

1. Data Selection
   - Include more maturities for better curve fitting
2. Model Adaptations
   - Consider modifications for low interest rate environments
   - Add modern market features (e.g., quantitative easing effects)

Future Research Directions

1. Investigate alternative term structure models more suitable for current markets
2. Develop hybrid approaches combining classical and modern methods
3. Explore machine learning techniques for parameter estimation
4. Study the impact of modern monetary policy on term structure modeling