Optimal Ascent Guidance

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Contents

1 Nomenclature

Symbols

a_t	thrust acceleration
f	system differential equations
F_t	thrust force
$\mathbf{g}$	gravitational force vector
$\mathbf{h}$	specific angular momentum vector
H	Hamiltonian
i	inclination
$\mathbf{I}$	identity matrix
J	performance index
$\bar{J}$	augmented performance index
$\mathbf{k}$	vertical unit vector
L	Lagrangian
m	mass or number of equations
n	number of control variables
$\mathbf{n}$	node vector
$\mathbf{p}$	unit thrust vector
q	number of constraints
$\mathbf{r}$	position vector
t	time
$\mathbf{u}$	control vector
$\mathbf{v}$	velocity vector
v_e	exhaust velocity
$\mathbf{x}$	state vector
x	x position coordinate
y	y position coordinate
z	z position coordinate
ϵ	error tolerance
θ	flight path angle or azimuth angle
$\boldsymbol{\lambda}$	variable Lagrange multiplier vector
$\boldsymbol{\Lambda}$	vector containing $\boldsymbol{\lambda}_v$ and $\dot{\boldsymbol{\lambda}}_v$
μ	gravitational parameter
$\boldsymbol{\nu}$	constant Lagrange multiplier vector
$\boldsymbol{\sigma}$	transversality condition vector
$\boldsymbol{\phi}$	terminal component of performance index
$\boldsymbol{\Phi}$	augmented terminal component of performance index
ψ	polar angle
$\boldsymbol{\psi}$	constraint vector
ω	angular rate of reference orbit

Subscripts

d	desired
f	final
m	related to mass
r	related position
v	related to velocity
x	x component of vector
y	y component of vector
z	z component of vector
0	initial
$\times$	skew matrix

2 Introduction

This paper will present an algorithm based on optimal control theory that determines the thrust vector history that maximizes the payload to an orbit with specified terminal conditions. Three different sets of terminal conditions will be discussed, unconstrained orbital plane, constrained inclination, and constrained inclination and longitude of the ascending node. All three sets will specify the terminal altitude, velocity, and the flight path angle.

The necessary conditions of optimality will be derived for a general continuous system. They will then be applied to the specific problem of maximizing the payload of a launch vehicle to orbit which is equivalent to minimizing the flight time. In general, satisfying the necessary conditions only provides an extremum, a minimum or maximum, of the cost function. Since a trajectory with a longer flight time could always be selected a maximum does not exist, therefore satisfying the necessary conditions will yield a minimum.

The trajectory will be represented by seven variables. The initial thrust vector direction and its time derivative, both three dimensional vectors, and the time of flight. The algorithm will select values for these variables that satisfy the constraints and the necessary conditions. The constraints and necessary conditions will be represented by seven non-linear guidance equations. The trajectory will be integrated numerically and a multivariate Newton's method algorithm will be used to solve the guidance equations for the unknown values. The equations defining the necessary conditions for each case are derived in appendix A and the partial derivatives necessary for integration and Newton's method are summarized in Appendix B.

3 Optimal Control Theory

In this section the necessary conditions of optimality will be derived for a general continuous system. The development closely follows that of Bryson and Ho [1] and Lewis et. al [2].

Consider a system that is governed by the following m differential equations

$$\dot{x} = f\left[x\left(t\right), u\left(t\right), t\right] \tag{3.1}$$

with q specified terminal constraints

$$\psi\left[x\left(t_f\right), t_f\right] = 0. \tag{3.2}$$

The problem is to find the n optimal control values, $u^*\left(t\right)$, that minimizes the performance index

$$J = \phi\left[x\left(t_f\right), t_f\right] + \int_{t_0}^{t_f} L\left[x\left(t\right), u\left(t\right), t\right] dt. \tag{3.3}$$

The augmented performance index can be formed by introducing a set of constant Lagrange multipliers, ν, and a set of variable Lagrange multipliers, $\lambda\left(t\right)$, and adjoining the system differential equations and constraints. The constrained minimum of the performance index is equivalent to the unconstrained minimum of the augmented performance index.

$$\bar{J} = \phi\left[x\left(t_f\right), t_f\right] + \nu^T \psi\left[x\left(t_f\right), t_f\right] + \int_{t_0}^{t_f} \left\{L\left[x\left(t\right), u\left(t\right), t\right] + \lambda^T\left(t\right)\left(f\left[x\left(t\right), u\left(t\right), t\right] - \dot{x}\right)\right\} dt \tag{3.4}$$

or

$$\bar{J} = \Phi|_{t_f} + \int_{t_0}^{t_f} \left(H - \lambda^T \dot{x}\right) dt, \tag{3.5}$$

where the Hamiltonian is defined as

$$H\left[x\left(t\right), u\left(t\right), t\right] = L\left[x\left(t\right), u\left(t\right), t\right] + \lambda^T f\left[x\left(t\right), u\left(t\right), t\right] \tag{3.6}$$

and

$$\Phi\left[x\left(t_f\right), t_f\right] = \phi\left[x\left(t_f\right), t_f\right] + \nu^T \psi\left[x\left(t_f\right), t_f\right]. \tag{3.7}$$

A minimum of $\bar{J}$ will be at a stationary point where incremental changes in x, u, t, ν, and λ do not change the value of $\bar{J}$. This is analogous to finding the extremum of a single valued function by setting its derivative to zero. $\bar{J}$ is a functional and incremental changes in $\bar{J}$ will depend on not only the differentials dx, dt, and $d\nu$, but also on variations δx, δu, and $\delta\lambda$. The differential of $\bar{J}$ is

$$\begin{aligned} d\bar{J} = {} & \left(\Phi_x^T dx + \Phi_t dt + \psi^T d\nu\right)_{t_f} + \left(H - \lambda^T \dot{x}\right) dt|_{t_f} - \left(H - \lambda^T \dot{x}\right) dt|_{t_0} \\ & + \int_{t_0}^{t_f} \left(H_x^T \delta x + H_u^T \delta u + \left(H_\lambda - \dot{x}\right)^T \delta\lambda - \lambda^T \delta\dot{x}\right) dt. \end{aligned} \tag{3.8}$$

The last term in (3.8) can be integrated by parts to find

$$-\int_{t_0}^{t_f} \lambda^T \delta\dot{x} dt = -\lambda^T \delta x|_{t_f} + \lambda^T \delta x|_{t_0} + \int_{t_0}^{t_f} \dot{\lambda}^T \delta x dt. \tag{3.9}$$

Using the fact that $dx = \delta x + \dot{x} dt$ and substituting (3.9) into (3.8) results in

$$\begin{aligned} d\bar{J} = &\left[(\Phi_x - \lambda)^T dx + (\Phi_t + H)\, dt + \psi^T d\nu \right]_{t_f} - \left(H dt + \lambda^T dx \right)_{t_0} \\ &+ \int_{t_0}^{t_f} \left[\left(H_x + \dot{\lambda} \right)^T \delta x + H_u^T \delta u + (H_\lambda - \dot{x})^T \delta\lambda \right] dt. \end{aligned} \tag{3.10}$$

The initial values of x and t are given therefore $dt|_{t_0}$ and $dx|_{t_0}$ are both equal to zero. Setting the coefficients of dx, dt, $d\nu$, δx, δu, and $\delta\lambda$ to zero forces $d\bar{J}$ to zero and provides the necessary conditions of optimality.

In summary the necessary conditions of optimality are as follows:

$$\text{State Equation:} \qquad \dot{x} = \frac{\partial H}{\partial \lambda} \tag{3.11}$$

$$\text{Constraint Equation:} \qquad \psi|_{t_f} = 0 \tag{3.12}$$

$$\text{Costate Equation:} \qquad \dot{\lambda} = -\frac{\partial H}{\partial x} \tag{3.13}$$

$$\text{Stationary Condition:} \qquad \frac{\partial H}{\partial u} = 0 \tag{3.14}$$

$$\text{Costate Boundary Conditions:} \qquad \lambda|_{t_f} = \left. \frac{\partial \Phi}{\partial x} \right|_{t_f} \tag{3.15}$$

$$\text{Hamiltonian Boundary Conditions:} \qquad H|_{t_f} = -\left. \frac{\partial \Phi}{\partial t} \right|_{t_f} \tag{3.16}$$

4 Guidance Algorithm

4.1 Summary

In this section the guidance equations will be developed and implemented. The resulting algorithm will be capable of providing real time guidance commands corresponding to a trajectory that terminates at the desired orbit while minimizing fuel consumption. The necessary conditions of optimality derived in the previous section will all need to be satisfied. First the performance index, state equations, and control variables will be defined and the Hamiltonian can be formed from the state equation. Then the costate equation (3.13) and the stationary conditions (3.14) will be applied to the Hamiltonian. After making a simplifying assumption on the gravitational model the costate differential equations can be easily solved. The resulting solution allows for the calculation of the Lagrange multipliers associated with velocity, $\boldsymbol{\lambda}_v$, and their derivatives, $\dot{\boldsymbol{\lambda}}_v$, as a functions of their initial values and time. It will be shown that the control variables will be a direct function of $\boldsymbol{\Lambda}$ which is defined as a 6-vector having the components of $\boldsymbol{\lambda}_v$ and $\dot{\boldsymbol{\lambda}}_v$. Once the initial values of $\boldsymbol{\Lambda}$ are identified, the control variables will be known as functions of time [3][4].

The remaining necessary conditions will place constraints on the terminal values of the state and control variables. The constraint equation (3.12) represents q equations, the costate boundary conditions (3.15) represent 6 equations, and the Hamiltonian boundary condition (3.16) represents 1 equation that all need to be satisfied. In summary there are $q + 7$ equations to be satisfied by the q values of $\boldsymbol{\nu}$, 6 values of $\boldsymbol{\Lambda}$, and t_f. The combination of the constraints and boundary conditions allows for the elimination of the q unknown $\boldsymbol{\nu}$'s and q equations leaving 7 equations and unknowns. Additionally since the costate equations are homogeneous in $\boldsymbol{\Lambda}$ the boundary condition represented by (3.16) can be trivially satisfied by scaling the values of $\boldsymbol{\Lambda}$ which reduces the number of equations and unknowns to 6. Instead of arbitrarily selecting one value of $\boldsymbol{\Lambda}$ to hold constant, the final magnitude of $\boldsymbol{\lambda}_v$ will be constrained to a value of 1. This will leave a total of 7 equations and unknowns. They will be the 6 terminal constraints plus the constraint on the final magnitude of $\boldsymbol{\lambda}_v$ to be satisfied by selecting the 6 initial values of $\boldsymbol{\Lambda}$ and t_f [4][5].

In order to solve for optimal initial values of $\boldsymbol{\Lambda}$ and t_f an initial guess will be required. Then the trajectory will be integrated and Newton's method will be used to improve the initial guess. This process will be repeated until the desired degree of accuracy is obtained. Newton's method requires the derivatives of the terminal constraints with respect to the initial values of the $\boldsymbol{\Lambda}$ and t_f. Analytical expressions for these derivatives are summarized in appendix B [4].

Once the algorithm has converged, the solution will be used as an initial guess for the next guidance pass. The state of the vehicle will be updated and the equations will be solved again. This process will happen continuously to eliminate any accumulated error. As t approaches t_f even small errors will become impossible to eliminate within the capability of the vehicle. Therefore during the final seconds of powered flight the algorithm will transition to a terminal guidance mode where the solution of the last guidance pass will be used until t_f has been reached.

4.2 Development of the Guidance Equations

The vehicle will operate under constant thrust or constant acceleration, in either case the performance index corresponds to a minimum time problem.

$$J = \boldsymbol{\phi}\left[x\left(t_f\right), t_f\right] = t_f \tag{4.1}$$

The state variables position, velocity, and mass are represented by the vector $\mathbf{x}$ and the state differential equations by $\dot{\mathbf{x}}$.

$$\mathbf{x} = \begin{bmatrix} \mathbf{r} \\ \mathbf{v} \\ m \end{bmatrix} \qquad \dot{\mathbf{x}} = \begin{bmatrix} \mathbf{v} \\ \mathbf{g} + a_T \mathbf{p} \\ -F_T/v_e \end{bmatrix} \tag{4.2}$$

The unit thrust direction, $\mathbf{p}$, is written in terms of the control vector $\mathbf{u} = \left[\theta, \psi\right]^T$. The angles are never solved for but will be useful in showing that $\mathbf{p}$ is a function of $\boldsymbol{\lambda}_v$.

$$\mathbf{p} = \begin{bmatrix} \sin\theta\cos\psi \\ \sin\psi \\ \cos\theta\cos\psi \end{bmatrix} \tag{4.3}$$

The Hamiltonian is

$$H = \boldsymbol{\lambda}_r^T \mathbf{v} + \boldsymbol{\lambda}_v^T \left(\mathbf{g} + a_T \mathbf{p}\right) - \boldsymbol{\lambda}_m \left(F_t/v_e\right) \tag{4.4}$$

and application of the costate equation (3.13) yields an equation for each state variable.

$$\dot{\boldsymbol{\lambda}}_r = -\boldsymbol{\lambda}_v^T \frac{\partial \mathbf{g}}{\partial \mathbf{r}} \tag{4.5}$$

$$\dot{\boldsymbol{\lambda}}_v = -\boldsymbol{\lambda}_r \tag{4.6}$$

$$\dot{\boldsymbol{\lambda}}_m = -\boldsymbol{\lambda}_v^T \mathbf{p} \frac{\partial a_T}{\partial m} + \boldsymbol{\lambda}_m \frac{\partial F_t/v_e}{\partial m} \tag{4.7}$$

The equation related to the mass state variable is shown, but because $\boldsymbol{\lambda}_m$ does not impact the control variables it can be ignored. Under the assumption that the flight will take place over a small range in altitude, gravity can be taken as a linear function of position.

$$\mathbf{g} \approx -\frac{\mu}{\bar{r}^3}\mathbf{r} = -\omega^2 \mathbf{r} \tag{4.8}$$

Differentiating (4.6) and combining with (4.5) and (4.8) yields the differential equation

$$\ddot{\boldsymbol{\lambda}}_v + \omega^2 \boldsymbol{\lambda}_v = 0. \tag{4.9}$$

with the solution

$$\boldsymbol{\lambda}_v\left(t\right) = \boldsymbol{\lambda}_v\left(t_0\right)\cos\omega t + \frac{\dot{\boldsymbol{\lambda}}_v\left(t_0\right)}{\omega}\sin\omega t \tag{4.10}$$

$$\dot{\boldsymbol{\lambda}}_v\left(t\right) = -\omega\boldsymbol{\lambda}_v\left(t_0\right)\sin\omega t + \dot{\boldsymbol{\lambda}}_v\left(t_0\right)\cos\omega t. \tag{4.11}$$

Application of the stationary conditions (3.14) yields

$$a_T \boldsymbol{\lambda}_v^T \frac{\partial \mathbf{p}}{\partial \mathbf{u}} = 0. \tag{4.12}$$

After differentiating $\mathbf{p}$ with respect to θ and ψ and setting $\boldsymbol{\lambda}_v = \mathbf{p}$ it is easily verifiable that a solution to the stationary equation is given by

$$\mathbf{p}(t) = \frac{\boldsymbol{\lambda}_v(t)}{|\boldsymbol{\lambda}_v(t)|}. \tag{4.13}$$

The unit thrust direction is a direct function of $\boldsymbol{\lambda}_v$ and $\dot{\boldsymbol{\lambda}}_v$. With the following definition,

$$\boldsymbol{\Lambda} \equiv \begin{bmatrix} \boldsymbol{\lambda}_v \\ \dot{\boldsymbol{\lambda}}_v \end{bmatrix} \tag{4.14}$$

the unknown variables become the 6 values of $\boldsymbol{\Lambda}(t_0)$ and t_f.

The boundary conditions can be found from (3.15) and (3.16).

$$\boldsymbol{\lambda}|_{t_f} = \left.\frac{\partial \boldsymbol{\psi}}{\partial \mathbf{x}}\right|_{t_f}^T \boldsymbol{\nu} \tag{4.15}$$

$$H|_{t_f} = \boldsymbol{\nu}^T \left.\frac{\partial \boldsymbol{\psi}}{\partial t}\right|_{t_f} - 1 \tag{4.16}$$

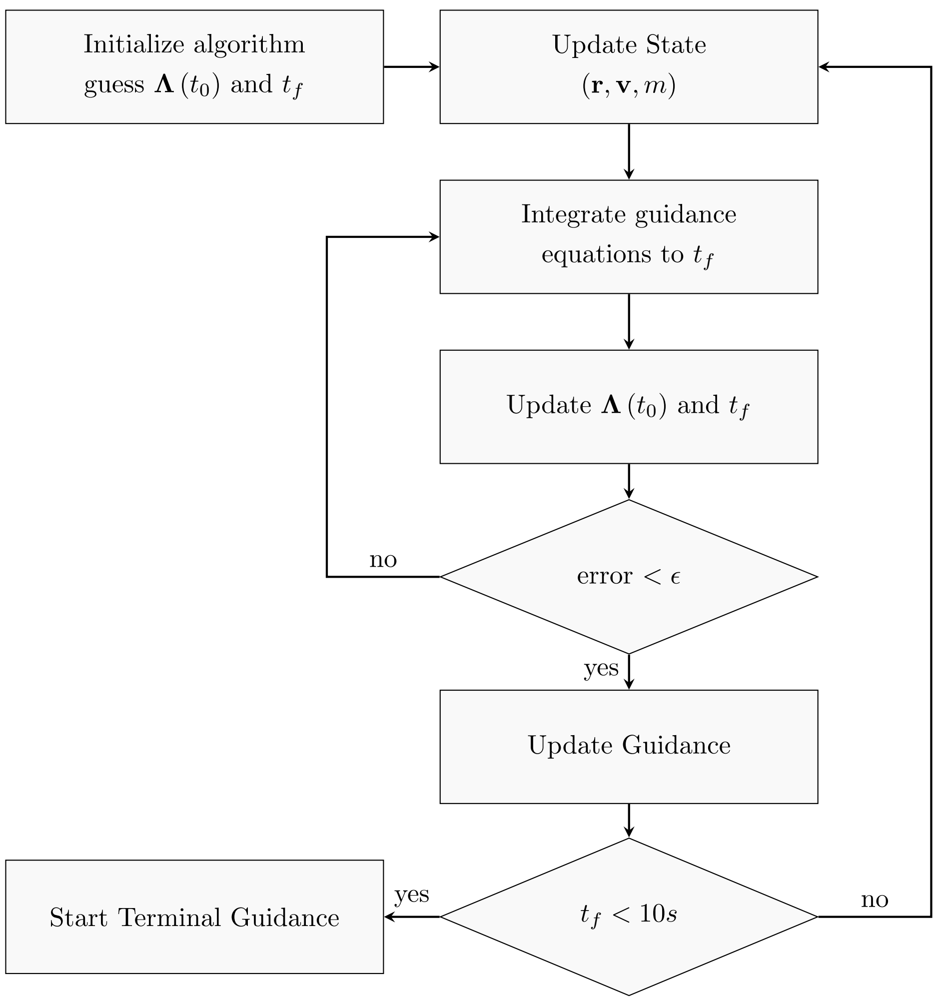
Since the costate equations are homogeneous in $\boldsymbol{\Lambda}$ (4.16) can be eliminated. Combining the constraint equation (3.12) and (4.15) will produce the 6 constraints on the terminal values of $\boldsymbol{\Lambda}$ and $\mathbf{x}$. q of these constraints will be specified and $6-q$ will be auxiliary constraints, also known as transversality conditions, determined by the boundary conditions. Terminal constraints for three different sets of specified constraints are summarized below with derivations in appendix A.

Description	Constraint	Transversality Conditions
Position magnitude	$\mathbf{r}\cdot\mathbf{r} - r_d^2 = 0$	$\mathbf{r}\times\dot{\boldsymbol{\lambda}}_v - \mathbf{v}\times\boldsymbol{\lambda}_v = 0$
Velocity magnitude	$\mathbf{v}\cdot\mathbf{v} - v_d^2 = 0$	
Flight path angle	$\mathbf{r}\cdot\mathbf{v} - r_d v_d \cos\theta_d = 0$	
Position magnitude	$\mathbf{r}\cdot\mathbf{r} - r_d^2 = 0$	$(\mathbf{r}\times\dot{\boldsymbol{\lambda}}_v - \mathbf{v}\times\boldsymbol{\lambda}_v)\cdot\mathbf{k} = 0$
Velocity magnitude	$\mathbf{v}\cdot\mathbf{v} - v_d^2 = 0$	$(\mathbf{r}\times\dot{\boldsymbol{\lambda}}_v - \mathbf{v}\times\boldsymbol{\lambda}_v)\cdot\mathbf{h} = 0$
Flight path angle	$\mathbf{r}\cdot\mathbf{v} - r_d v_d \cos\theta_d = 0$	
Inclination	$\mathbf{k}\cdot(\mathbf{r}\times\mathbf{v}) - r_d v_d \cos i_d = 0$	
Position magnitude	$\mathbf{r}\cdot\mathbf{r} - r_d^2 = 0$	$(\mathbf{r}\times\dot{\boldsymbol{\lambda}}_v - \mathbf{v}\times\boldsymbol{\lambda}_v)\cdot\mathbf{h} = 0$
Velocity magnitude	$\mathbf{v}\cdot\mathbf{v} - v_d^2 = 0$	
Flight path angle	$\mathbf{r}\cdot\mathbf{v} - r_d v_d \cos\theta_d = 0$	
Inclination	$\mathbf{k}\cdot(\mathbf{r}\times\mathbf{v}) - r_d v_d \cos i_d = 0$	
LAN	$\mathbf{n}_d\cdot(\mathbf{r}\times\mathbf{v}) = 0$	

4.3 The Algorithm

Once the vehicle has traversed the thicker portion of the atmosphere the algorithm is initialized with an estimate for the values of $\mathbf{\Lambda}(t_0)$ and t_f. The initial values of $\mathbf{\Lambda}(t_0)$ are selected to correspond to a zero angle of attack trajectory, assuming the initial portion of the flight was reasonable this should be an acceptable starting point. The initial value of t_f is selected to correspond to the amount of time required for the vehicle to change its velocity from its current value to that of the desired orbital velocity.

The initial guess and the current vehicle state are passed to a Newton's method algorithm. The algorithm integrates the trajectory until t_f is reached. Then it formulates an error vector and the Jacobian matrix of partial derivatives relating the values of $\mathbf{\Lambda}(t_0)$ and t_f to the constraints. The error and derivative information is then used to improve the current solution. If an acceptable error tolerance has been reached the algorithm will exit and update the current guidance solution. The state will be updated and passed back to the guidance algorithm. The previous solution will be used to initiate the next guidance pass. This process will repeat continuously until a few seconds before t_f where the last solution will be used until t_f has been reached.



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[3] Jaggers, R. F. “Asymmetrical Booster Ascent Guidance and Control System Design Study Volume V.” 28 June 1974.

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A Transversality Conditions

The transversality conditions are derived through the application of the costate boundary condition (4.15) repeated here as (A.1). They must be derived separately for each set of constraints. One of the goals of this derivation will be to eliminate the constant Lagrange multipliers, $\boldsymbol{\nu}$, from the boundary conditions as this reduces the number of equations and unknowns.

$$\boldsymbol{\lambda}|_{t_f} = \left.\frac{\partial \boldsymbol{\psi}}{\partial x}\right|_{t_f}^{T} \boldsymbol{\nu} \tag{A.1}$$

A.1 Case 1: Unconstrained Orbital Plane

The constraints for case 1 are

$$\mathbf{r} \cdot \mathbf{r} - r_d^2 = 0, \tag{A.2}$$

$$\mathbf{v} \cdot \mathbf{v} - v_d^2 = 0, \tag{A.3}$$

and

$$\mathbf{r} \cdot \mathbf{v} - r_d v_d \cos \theta_d = 0. \tag{A.4}$$

Application of (A.1) to these constraints yields

$$\boldsymbol{\lambda}_r = 2\nu_1 \mathbf{r} + \nu_3 \mathbf{v} \tag{A.5}$$

and

$$\boldsymbol{\lambda}_v = 2\nu_2 \mathbf{v} + \nu_3 \mathbf{r}. \tag{A.6}$$

Since $\dot{\boldsymbol{\lambda}}_v = -\boldsymbol{\lambda}_r$, (A.6) can be written as

$$\dot{\boldsymbol{\lambda}}_v = -2\nu_1 \mathbf{r} - \nu_3 \mathbf{v}. \tag{A.7}$$

Taking the cross products of (A.6) with $\mathbf{v}$ and (A.7) with $\mathbf{r}$ will cancel the first term of each equation and leave an identical second term. They can then be combined to form the transversality condition

$$\mathbf{r} \times \dot{\boldsymbol{\lambda}}_v - \mathbf{v} \times \boldsymbol{\lambda}_v = 0. \tag{A.8}$$

A.2 Case 2: Constrained Inclination

The constraints are identical to that of case 1 with one additional constraint on the inclination of the orbit.

$$\mathbf{k} \cdot (\mathbf{r} \times \mathbf{v}) - r_d v_d \cos i_d = 0 \tag{A.9}$$

Application of (A.1) to these constraints and again using $\dot{\boldsymbol{\lambda}}_v = -\boldsymbol{\lambda}_r$ yields

$$\boldsymbol{\lambda}_v = 2\nu_2 \mathbf{v} + \nu_3 \mathbf{r} + \nu_4 (\mathbf{k} \times \mathbf{r}) \tag{A.10}$$

and

$$\dot{\boldsymbol{\lambda}}_v = -2\nu_1 \mathbf{r} - \nu_3 \mathbf{v} - \nu_4 \left(\mathbf{v} \times \mathbf{k}\right) \tag{A.11}$$

Taking the cross products of (A.10) with $\mathbf{v}$ and (A.11) with $\mathbf{r}$ will cancel the first term of each equation and leave an identical second term, however in this case there is a third term. The equations can still be combined to form

$$\mathbf{r} \times \dot{\boldsymbol{\lambda}}_v - \mathbf{v} \times \boldsymbol{\lambda}_v = \nu_4 \left(\mathbf{k} \times \mathbf{h}\right). \tag{A.12}$$

(A.12) can be thought of as requiring the left hand side to be a multiple of a vector perpendicular to both $\mathbf{k}$ and $\mathbf{h}$, which can be represented as

$$\left(\mathbf{r} \times \dot{\boldsymbol{\lambda}}_v - \mathbf{v} \times \boldsymbol{\lambda}_v\right) \cdot \mathbf{k} = 0, \tag{A.13}$$

and

$$\left(\mathbf{r} \times \dot{\boldsymbol{\lambda}}_v - \mathbf{v} \times \boldsymbol{\lambda}_v\right) \cdot \mathbf{h} = 0. \tag{A.14}$$

A.3 Case 3: Constrained Inclination and LAN

The constraints are identical to that of case 2 with one additional constraint on the longitude of the ascending node of the orbit.

$$\mathbf{n}_d \cdot \left(\mathbf{r} \times \mathbf{v}\right) = 0 \tag{A.15}$$

Again applying (A.1) to these constraints and using $\dot{\boldsymbol{\lambda}}_v = -\boldsymbol{\lambda}_r$ yields

$$\boldsymbol{\lambda}_v = 2\nu_2 \mathbf{v} + \nu_3 \mathbf{r} + \nu_4 \left(\mathbf{k} \times \mathbf{r}\right) + \nu_5 \left(\mathbf{n}_d \times \mathbf{r}\right) \tag{A.16}$$

$$\dot{\boldsymbol{\lambda}}_v = -2\nu_1 \mathbf{r} - \nu_3 \mathbf{v} - \nu_4 \left(\mathbf{v} \times \mathbf{k}\right) - \nu_5 \left(\mathbf{v} \times \mathbf{n}_d\right) \tag{A.17}$$

As in the previous two cases we take the cross product of both equations with $\mathbf{v}$ and $\mathbf{r}$ respectively and combine them to form

$$\mathbf{r} \times \dot{\boldsymbol{\lambda}}_v - \mathbf{v} \times \boldsymbol{\lambda}_v = -\nu_4 \left(\mathbf{k} \times \mathbf{h}\right) - \nu_5 \left(\mathbf{n}_d \times \mathbf{h}\right). \tag{A.18}$$

The left hand side of (A.18) is equal to a linear combination of a vector that is perpendicular to $\mathbf{k}$ and $\mathbf{h}$ and another vector that is perpendicular to $\mathbf{n}_d$ and $\mathbf{h}$. The combination of these vectors spans the plane perpendicular to $\mathbf{h}$. This requirement can be represented as

$$\left(\mathbf{r} \times \dot{\boldsymbol{\lambda}}_v - \mathbf{v} \times \boldsymbol{\lambda}_v\right) \cdot \mathbf{h} = 0. \tag{A.19}$$

B Partial Derivatives

B.1 Partial Derivatives for Integration

The partial derivatives of the constraints with respect to the control variables will be required by the Newton's method algorithm to solve the guidance equations. The constraints are formulated in terms of the final values of $\mathbf{r}$, $\mathbf{v}$, $\boldsymbol{\lambda}$, and $\dot{\boldsymbol{\lambda}}$ and the control variables $\boldsymbol{\Lambda}(t_0)$ and t_f. The derivatives of $\mathbf{r}$, $\mathbf{v}$, $\boldsymbol{\lambda}$, and $\dot{\boldsymbol{\lambda}}$ with respect to t_f and the derivatives of $\boldsymbol{\lambda}$ and $\dot{\boldsymbol{\lambda}}$ with respect to $\boldsymbol{\Lambda}(t_0)$ are easily calculated. The derivatives of $\mathbf{r}$ and $\mathbf{v}$ with respect to $\boldsymbol{\Lambda}(t_0)$ will require the integration of 36 additional equations. Including the 7 state equations a total of 43 equations will be integrated. The state equations are

$$\mathbf{x} = \begin{bmatrix} \mathbf{r} \\ \mathbf{v} \\ m \end{bmatrix} \qquad \dot{\mathbf{x}} = \begin{bmatrix} \mathbf{v} \\ \mathbf{g} + a_T \mathbf{p} \\ -F_T/v_e \end{bmatrix} \tag{B.1}$$

where the unit thrust direction will be calculated from (4.13) and (4.10), both equations are reproduced below.

$$\mathbf{p}(t) = \frac{\boldsymbol{\lambda}_v(t)}{|\boldsymbol{\lambda}_v(t)|}. \tag{B.2}$$

$$\boldsymbol{\lambda}_v(t) = \boldsymbol{\lambda}_v(t_0)\cos\omega t + \frac{\dot{\boldsymbol{\lambda}}_v(t_0)}{\omega}\sin\omega t \tag{B.3}$$

The additional 36 equations are represented by (B.4).

$$\frac{\partial \mathbf{x}(t)}{\partial \boldsymbol{\Lambda}(t_0)} = \int_{t_0}^{t} \frac{\partial \dot{\mathbf{x}}(t)}{\partial \boldsymbol{\Lambda}(t_0)} dt \tag{B.4}$$

where

$$\frac{\partial \dot{\mathbf{x}}(t)}{\partial \boldsymbol{\Lambda}(t_0)} = \frac{\partial \dot{\mathbf{x}}(t)}{\partial \mathbf{x}(t)} \frac{\partial \mathbf{x}(t)}{\partial \boldsymbol{\Lambda}(t_0)} + \frac{\partial \dot{\mathbf{x}}(t)}{\partial \boldsymbol{\Lambda}(t)} \frac{\partial \boldsymbol{\Lambda}(t)}{\partial \boldsymbol{\Lambda}(t_0)}. \tag{B.5}$$

The terms in (B.5) are listed below.

$$\frac{\partial \dot{\mathbf{x}}(t)}{\partial \mathbf{x}(t)} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{A} & \mathbf{0} \end{bmatrix} \tag{B.6}$$

$$\mathbf{A}(t) = \begin{bmatrix} \frac{\partial \ddot{x}}{\partial x} & \frac{\partial \ddot{x}}{\partial y} & \frac{\partial \ddot{x}}{\partial z} \\ \frac{\partial \ddot{y}}{\partial x} & \frac{\partial \ddot{y}}{\partial y} & \frac{\partial \ddot{y}}{\partial z} \\ \frac{\partial \ddot{z}}{\partial x} & \frac{\partial \ddot{z}}{\partial y} & \frac{\partial \ddot{z}}{\partial z} \end{bmatrix} = \mu \begin{bmatrix} \frac{3x^2}{r^5} - \frac{1}{r^3} & \frac{3xy}{r^5} & \frac{3xz}{r^5} \\ \frac{3xy}{r^5} & \frac{3y^2}{r^5} - \frac{1}{r^3} & \frac{3yz}{r^5} \\ \frac{3xz}{r^5} & \frac{3yz}{r^5} & \frac{3z^2}{r^5} - \frac{1}{r^3} \end{bmatrix} \tag{B.7}$$

$$\frac{\partial \dot{\mathbf{x}}(t)}{\partial \boldsymbol{\Lambda}(t)} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \tag{B.8}$$

$$\mathbf{B}(t) = \frac{a_t}{\lambda_v^3} \begin{bmatrix} \lambda_v^2 - \lambda_{v,x}^2 & -\lambda_{v,x}\lambda_{v,y} & -\lambda_{v,x}\lambda_{v,z} \\ -\lambda_{v,x}\lambda_{v,y} & \lambda_v^2 - \lambda_{v,y}^2 & -\lambda_{v,y}\lambda_{v,z} \\ -\lambda_{v,x}\lambda_{v,z} & -\lambda_{v,y}\lambda_{v,z} & \lambda_v^2 - \lambda_{v,z}^2 \end{bmatrix} \tag{B.9}$$

$$\frac{\partial \mathbf{\Lambda}(t)}{\partial \mathbf{\Lambda}(t_0)} = \begin{bmatrix} \cos\omega t\mathbf{I} & \frac{sin\omega t}{\omega}\mathbf{I} \\ -\omega\sin\omega t\mathbf{I} & \cos\omega t\mathbf{I} \end{bmatrix} \tag{B.10}$$

B.2 Partial Derivatives for Newton's Method

After integration has reached t_f the residuals of the constraint equation and their derivatives with respect to $\mathbf{\Lambda}(t_0)$ and t_f can be calculated. The derivatives are all functions of the final values of position, velocity, and Lagrange multipliers. The time derivatives are easily calculated since the derivatives of position and velocity with respect to time, velocity and acceleration, are known. The derivatives with respect to $\mathbf{\Lambda}(t_0)$ can be obtained using the result of the integration of (B.4).

The constraints on the final position and velocity magnitudes and the flight path angle are all simple dot product relationships. Their derivatives are easily derived from the derivative of the dot product.

$$\frac{\partial(\mathbf{r}\cdot\mathbf{r})}{\partial\mathbf{\Lambda}(t_0)} = 2\left(\mathbf{r}\cdot\frac{\partial\mathbf{r}}{\partial\mathbf{\Lambda}(t_0)}\right) \qquad \frac{\partial(\mathbf{r}\cdot\mathbf{r})}{\partial t} = 2(\mathbf{r}\cdot\mathbf{v}) \tag{B.11}$$

$$\frac{\partial(\mathbf{v}\cdot\mathbf{v})}{\partial\mathbf{\Lambda}(t_0)} = 2\left(\mathbf{v}\cdot\frac{\partial\mathbf{v}}{\partial\mathbf{\Lambda}(t_0)}\right) \qquad \frac{\partial(\mathbf{v}\cdot\mathbf{v})}{\partial t} = 2(\mathbf{v}\cdot\dot{\mathbf{v}}) \tag{B.12}$$

$$\frac{\partial(\mathbf{r}\cdot\mathbf{v})}{\partial\mathbf{\Lambda}(t_0)} = \frac{\partial\mathbf{r}}{\partial\mathbf{\Lambda}(t_0)}\cdot\mathbf{v} + \frac{\partial\mathbf{v}}{\partial\mathbf{\Lambda}(t_0)}\cdot\mathbf{r} \qquad \frac{\partial(\mathbf{r}\cdot\mathbf{v})}{\partial t} = \mathbf{v}\cdot\mathbf{v} + \mathbf{r}\cdot\dot{\mathbf{v}} \tag{B.13}$$

Recognizing that $\mathbf{k}\cdot(\mathbf{r}\times\mathbf{v}) = \mathbf{h}_z$ and applying the product rule to the derivatives related to inclination yields

$$\frac{\partial(\mathbf{k}\cdot(\mathbf{r}\times\mathbf{v}))}{\partial\mathbf{\Lambda}(t_0)} = \frac{\partial\mathbf{h}_z}{\partial\mathbf{r}}\frac{\partial\mathbf{r}}{\partial\mathbf{\Lambda}(t_0)} + \frac{\partial\mathbf{h}_z}{\partial\mathbf{v}}\frac{\partial\mathbf{v}}{\partial\mathbf{\Lambda}(t_0)} \tag{B.14}$$

and

$$\frac{\partial(\mathbf{k}\cdot(\mathbf{r}\times\mathbf{v}))}{\partial t} = \frac{\partial\mathbf{h}_z}{\partial\mathbf{r}}\cdot\mathbf{v} + \frac{\partial\mathbf{h}_z}{\partial\mathbf{v}}\cdot\dot{\mathbf{v}} \tag{B.15}$$

where

$$\frac{\partial\mathbf{h}_z}{\partial\mathbf{r}} = \begin{bmatrix} v_y \\ -v_x \\ 0 \end{bmatrix} \qquad \frac{\partial\mathbf{h}_z}{\partial\mathbf{v}} = \begin{bmatrix} -r_y \\ r_x \\ 0 \end{bmatrix}. \tag{B.16}$$

Applying the product rule to the derivatives related to the longitude of the ascending node yields

$$\frac{\partial(\mathbf{n}_d\cdot(\mathbf{r}\times\mathbf{v}))}{\partial\mathbf{\Lambda}(t_0)} = \frac{\partial(\mathbf{n}_d\cdot(\mathbf{r}\times\mathbf{v}))}{\partial\mathbf{r}}\frac{\partial\mathbf{r}}{\partial\mathbf{\Lambda}(t_0)} + \frac{\partial(\mathbf{n}_d\cdot(\mathbf{r}\times\mathbf{v}))}{\partial\mathbf{v}}\frac{\partial\mathbf{v}}{\partial\mathbf{\Lambda}(t_0)} \tag{B.17}$$

and

$$\frac{\partial\left(\mathbf{n}_d\cdot(\mathbf{r}\times\mathbf{v})\right)}{\partial t}=\frac{\partial\left(\mathbf{n}_d\cdot(\mathbf{r}\times\mathbf{v})\right)}{\partial \mathbf{r}}\cdot\mathbf{v}+\frac{\partial\left(\mathbf{n}_d\cdot(\mathbf{r}\times\mathbf{v})\right)}{\partial \mathbf{v}}\cdot\dot{\mathbf{v}} \tag{B.18}$$

where

$$\frac{\partial\left(\mathbf{n}_d\cdot(\mathbf{r}\times\mathbf{v})\right)}{\partial \mathbf{r}}=\mathbf{v}\times\mathbf{n}_d \tag{B.19}$$

$$\frac{\partial\left(\mathbf{n}_d\cdot(\mathbf{r}\times\mathbf{v})\right)}{\partial \mathbf{v}}=\mathbf{n}_d\times\mathbf{r}. \tag{B.20}$$

Defining

$$\boldsymbol{\sigma}=\mathbf{r}\times\dot{\boldsymbol{\lambda}}_v-\mathbf{v}\times\boldsymbol{\lambda}_v \tag{B.21}$$

the derivatives related to the transversality conditions for the case of unconstrained inclination and longitude of ascending node are

$$\frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{\Lambda}(t_0)}=\frac{\partial\mathbf{r}}{\partial\boldsymbol{\Lambda}(t_0)}\times\dot{\boldsymbol{\lambda}}_v+\mathbf{r}\times\frac{\partial\dot{\boldsymbol{\lambda}}_v}{\partial\boldsymbol{\Lambda}(t_0)}-\frac{\partial\mathbf{v}}{\partial\boldsymbol{\Lambda}(t_0)}\times\boldsymbol{\lambda}_v-\mathbf{v}\times\frac{\partial\boldsymbol{\lambda}_v}{\partial\boldsymbol{\Lambda}(t_0)} \tag{B.22}$$

and

$$\frac{\partial\boldsymbol{\sigma}}{\partial t}=-\mathbf{r}\times\boldsymbol{\lambda}_v-\dot{\mathbf{v}}\times\boldsymbol{\lambda}_v. \tag{B.23}$$

For the case of constrained inclination the derivative related to the transversaility conditions are

$$\frac{\partial\left(\boldsymbol{\sigma}\cdot\mathbf{k}\right)}{\partial\boldsymbol{\Lambda}(t_0)}=\frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{\Lambda}(t_0)}\cdot\mathbf{k} \tag{B.24}$$

and

$$\frac{\partial\left(\boldsymbol{\sigma}\cdot\mathbf{k}\right)}{\partial t}=\frac{\partial\boldsymbol{\sigma}}{\partial t}\cdot\mathbf{k}. \tag{B.25}$$

For the case of constrained inclination and longitude of ascending node the derivatives related to the transversaility conditions are

$$\frac{\partial\left(\boldsymbol{\sigma}\cdot\mathbf{h}\right)}{\partial\boldsymbol{\Lambda}(t_0)}=\frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{\Lambda}(t_0)}\cdot\mathbf{h}+\boldsymbol{\sigma}\cdot\frac{\partial\mathbf{h}}{\partial\boldsymbol{\Lambda}(t_0)} \tag{B.26}$$

and

$$\frac{\partial\left(\boldsymbol{\sigma}\cdot\mathbf{h}\right)}{\partial t}=\frac{\partial\boldsymbol{\sigma}}{\partial t}\cdot\mathbf{h}+\boldsymbol{\sigma}\cdot\frac{\partial\mathbf{h}}{\partial t} \tag{B.27}$$

where

$$\frac{\partial\mathbf{h}}{\partial\boldsymbol{\Lambda}(t_0)}=\frac{\partial\mathbf{h}}{\partial\mathbf{r}}\frac{\partial\mathbf{r}}{\partial\boldsymbol{\lambda}(t_0)}+\frac{\partial\mathbf{h}}{\partial\mathbf{v}}\frac{\partial\mathbf{v}}{\partial\boldsymbol{\Lambda}(t_0)} \tag{B.28}$$

where the derivatives of $\mathbf{h}$ with respect $\mathbf{r}$ and $\mathbf{v}$ are the skew matricies representing the cross product operation.

$$\frac{\partial\mathbf{h}}{\partial\mathbf{r}}=-\mathbf{v}_\times\qquad\frac{\partial\mathbf{h}}{\partial\mathbf{v}}=\mathbf{r}_\times \tag{B.29}$$