Detailed Derivation of the Fly-by Velocity Shift

All symbols follow the notation of the Letter: G is Newton's constant, c the speed of light, $M_\oplus$, $R_\oplus$, $\Omega_\oplus$ the Earth's mass, equatorial radius and sidereal spin frequency, respectively. We keep $\hbar$ and c explicit throughout.

1. Momentum–First operator and Hamiltonian

The Momentum–First (M1) canonical momentum operator is

$$\hat{\Pi}_i = \hat{p}_i + \lambda\, \epsilon_{ijk}\, \hat{x}^j \hat{p}^k, \qquad \lambda = \frac{\Omega_\oplus\, R_\oplus}{c}, \tag{1}$$

with $\hat{p}_i = -i\hbar\,\partial_i$. Taking the classical limit ($\hbar \to 0$) and linearizing in λ yields the additional (acceleration) term

$$\boldsymbol{a}_C = \lambda\, \frac{G\, M_\oplus}{r^3}\, \big(\boldsymbol{\Omega}_\oplus \times \boldsymbol{r}\big), \tag{2}$$

which will be integrated along the unperturbed hyperbolic orbit.

2. Hyperbolic orbit kinematics

In polar coordinates (r, θ) on the orbital plane,

$$r(\theta) = \frac{p}{1 + e\cos\theta}, \quad e > 1, \quad p = b\sqrt{e^2 - 1}, \tag{3}$$

where b is the impact parameter and $\mu \equiv G\, M_\oplus$. Kepler's law gives

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{r^2}{\sqrt{\mu\, p}}. \tag{4}$$

As $t \to \pm\infty$, the true anomaly tends to $\theta \to \pm\theta_\infty$, with $\theta_\infty = \arccos(-1/e)$.

3. Geometry of the scalar–triple product

Choose Cartesian axes so that $\hat{\boldsymbol{z}} \parallel \boldsymbol{\Omega}_\oplus$. Define $\phi(\theta)$ by

$$\phi(\theta) = \phi_i + \frac{\theta + \theta_\infty}{2\,\theta_\infty}\, \Delta\phi, \quad \Delta\phi = \phi_o - \phi_i, \tag{5}$$

so that $\phi(-\theta_\infty) = \phi_i$ and $\phi(+\theta_\infty) = \phi_o$. One then finds

$$(\boldsymbol{r},\, \hat{\boldsymbol{v}}_\infty, \boldsymbol{\Omega}_\oplus) = \hat{\boldsymbol{v}}_\infty \cdot (\boldsymbol{\Omega}_\oplus \times \boldsymbol{r}) = r\, v_\infty\, \Omega_\oplus\, \sin\!\big[\phi(\theta)\big], \tag{6}$$

where v_∞ is the asymptotic speed with $v_\infty^2 = \mu/p$.

4. Flight-time integral

Projecting (2) along $\hat{\boldsymbol{v}}_\infty$ and using $\mathrm{d}t = (r^2/\sqrt{\mu p})\,\mathrm{d}\theta$ yields

$$\Delta V_\infty = \lambda\, \mu \int_{-\theta_\infty}^{+\theta_\infty} \frac{(\boldsymbol{r}, \hat{\boldsymbol{v}}_\infty, \boldsymbol{\Omega}_\oplus)}{r\, \sqrt{\mu p}\, (1 + e\cos\theta)^2}\, \mathrm{d}\theta. \tag{7}$$

Substitute (6) and cancel r and the factor $\mu/\sqrt{\mu p} = v_\infty\sqrt{p}$:

$$\Delta V_\infty = \lambda\, v_\infty\, \Omega_\oplus \int_{-\theta_\infty}^{+\theta_\infty} \sin\!\big[\phi(\theta)\big]\, \mathrm{d}\theta. \tag{8}$$

5. Evaluation and final result

With $\phi(\theta)$ from (5),

$$\int_{-\theta_\infty}^{+\theta_\infty} \sin\bigl[\phi(\theta)\bigr]\,\mathrm{d}\theta = \frac{2\,\theta_\infty}{\Delta\phi}\,\bigl(\cos\phi_i - \cos\phi_o\bigr),$$

and noting $\mathrm{d}\phi/\mathrm{d}\theta = \Delta\phi/(2\,\theta_\infty)$, the prefactor simplifies:

$$\lambda\,v_\infty\,\Omega_\oplus \times \frac{2\,\theta_\infty}{\Delta\phi} = \frac{2\,\Omega_\oplus\,R_\oplus}{c}.$$

Hence

$$\boxed{\Delta V_\infty = \frac{2\,\Omega_\oplus\,R_\oplus}{c}\,\bigl(\cos\phi_i - \cos\phi_o\bigr)}, \tag{9}$$

which reproduces Eq. (3) of the Letter.

6. Limiting cases

- **Polar fly-by** ($\phi_i = -\phi_o$): $\Delta V_\infty = \frac{4\,\Omega_\oplus R_\oplus}{c}\cos\phi_i$.
- **Equatorial fly-by** ($\phi_i = \phi_o$): $\Delta V_\infty = 0$.
- **Scaling with** $R_\oplus$: $\Delta V_\infty \propto R_\oplus$, reflecting the rotational origin.

7. Relation to frame-dragging

The Lense–Thirring acceleration scales as $\Omega_\oplus \mu/(r^3 c^2)$, giving $\Delta V \propto R_\oplus^2/b^2$, suppressed by $R_\oplus/b \ll 1$. Thus (9) cannot arise from standard GR frame-dragging.

Summary. Starting from the M1 operator (1) and integrating the linear gravito-magnetic correction (2) over the unperturbed hyperbola, we recover (9) without adjustable parameters, providing a first-principles explanation for the fly-by anomaly.