Equivalence of Absolute Directional Momentum Conservation with Special Relativistic Kinematics

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Abstract

A directional decomposition of momentum is shown to be exactly equivalent to special relativistic kinematics. For each Cartesian axis $\boldsymbol{k} \in \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$ define two additive scalars $\boldsymbol{p_{k\pm}} = \boldsymbol{M} \pm \frac{1}{2}\boldsymbol{p_k}$, where $\boldsymbol{p_k}$ is the usual momentum component and $\boldsymbol{M} = \sqrt{\boldsymbol{p_f^2} + \mathbf{p^2}}$ with $\boldsymbol{p_f} = \boldsymbol{m_0 c}$. The postulate that, in any isolated system, the sums of all $\boldsymbol{p_{k+}}$ and of all $\boldsymbol{p_{k-}}$ are separately conserved for each axis is proved to be equivalent to the standard conservation of energy ($\boldsymbol{E} = \boldsymbol{cM}$) and three-momentum, i.e., to four-momentum conservation, for arbitrary isolated many-particle processes. A short uniqueness theorem shows that, under additivity, isotropy, parity, and local invertibility, the representation $\boldsymbol{p_{k\pm}} = \boldsymbol{M} \pm \frac{1}{2}\boldsymbol{p_k}$ is fixed up to a common rescaling; a natural normalization selects the canonical form used here. Immediate corollaries recover $\boldsymbol{v} = \boldsymbol{pc/M}$, $\gamma = \boldsymbol{M/p_f}$, the massless limit, and the Newtonian expansion. No modification of special relativity is implied: the directional scalars provide an equivalent, axis-wise conservation reparameterization that can be convenient in later applications.

Keywords: Special relativity; Conservation laws; Four-momentum; Kinematics; Axioms; Momentum decomposition

1 Introduction

Conservation laws in special relativity are encoded in the four-momentum P^μ. In this note a momentum-unit form is adopted, $P^\mu = (M, \mathbf{p})$, where $\mathbf{p} = (p_x, p_y, p_z)$ is the three-momentum and the time-like component M satisfies $E = c\,M$. The quantity $p_f = m_0 c$ will be called the *fermic momentum* (rest mass in momentum units). With these conventions the standard dispersion relation reads [1, 2]

$$M = \sqrt{p_f^2 + \mathbf{p}^2}, \tag{1}$$

and will be used throughout this paper. By Noether's theorem, spacetime symmetries underlie the usual conservation statements for E and $\mathbf{p}$ [3].

The central object is a directional decomposition of momentum along each Cartesian axis. For a fixed axis $k \in \{x, y, z\}$ define two additive scalars

$$p_{k^\pm} \equiv M \pm \tfrac{1}{2} p_k.$$

These quantities are scalars with respect to the Euclidean rotations of the chosen inertial frame; they are not Lorentz invariants. The term "additive" means that for an isolated composite system the values sum over constituents. The postulate studied here asserts that, in any isolated process, the sums of all p_{k^+} and of all p_{k^-} are separately conserved for each axis k. This will be referred to as *additive directional momentum conservation* (ADMC).

The main result is an exact compliance theorem: assuming the standard kinematics expressed by Eq. (1), ADMC holds if and only if the usual conservation of energy E and of three-momentum $\mathbf{p}$ holds, i.e., if and only if four-momentum is conserved. Thus the directional scalars provide an equivalent, component-wise bookkeeping of the same physics. A short uniqueness result is also established: under additivity, isotropy, parity, and local invertibility, the representation $p_{k^\pm} = M \pm \frac{1}{2} p_k$ is fixed up to a common rescaling, which a natural normalization removes. Immediate corollaries recover standard kinematic relations, including $v = p\,c/M$, $\gamma = M/p_f$, the massless limit, and the Newtonian expansion.

The scope is purely kinematic. No modification of special relativity is proposed. Section 2 states the postulate and its definitions precisely and proves the equivalence with four-momentum conservation. Section 3 provides the uniqueness argument and lists the immediate corollaries.

2 Postulate and linear structure

This section states the postulate precisely and records the immediate linear relations that will be used in the proofs that follow.

2.1 Definitions

Fix an inertial frame and a Cartesian axis $k \in \{x, y, z\}$. Let $\mathbf{p} = (p_x, p_y, p_z)$ denote the three-momentum and let $p_f = m_0 c$ be the fermic momentum. The time-like component M is defined by the standard dispersion relation in Eq. (1) and satisfies $E = c\,M$. For the chosen axis k, define two directional scalars by

$$p_{k^\pm} \equiv M \pm \tfrac{1}{2} p_k. \tag{2}$$

These $p_{k\pm}$ are scalars with respect to Euclidean rotations of the chosen frame; they are not Lorentz invariants. Under a parity inversion, $p_k \mapsto -p_k$ and $M \mapsto M$, so that $p_{k^+} \leftrightarrow p_{k^-}$.

The term *additive* will mean that for any isolated composite system the scalar in Eq. (2) sums over constituents, e.g., $\sum_i p_{k\pm}^{(i)} = p_{k\pm}^{(\text{total})}$, with the sum taken over particles i participating in the process.

2.2 Postulate (additive directional momentum conservation)

In any isolated process the two sums $\sum_i p_{k+}^{(i)}$ and $\sum_i p_{k-}^{(i)}$ are separately conserved for each axis k:

$$\sum_i p_{k\pm}^{(i)} = \text{constant} \quad \text{for each } k \in \{x, y, z\} \text{ and each choice of } \pm\,. \tag{3}$$

This principle will be referred to as additive directional momentum conservation (ADMC).

2.3 Immediate linear relations

Equation (2) defines an invertible linear change of variables between (M, p_k) and (p_{k^+}, p_{k^-}). Explicitly,

$$\begin{pmatrix} p_{k^+} \\ p_{k^-} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} M \\ p_k \end{pmatrix}, \qquad \begin{pmatrix} M \\ p_k \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{pmatrix} \begin{pmatrix} p_{k^+} \\ p_{k^-} \end{pmatrix}. \tag{4}$$

Consequently, for any collection of particles,

$$\sum_i (p_{k^+}^{(i)} + p_{k^-}^{(i)}) = 2 \sum_i M^{(i)}, \qquad \sum_i (p_{k^+}^{(i)} - p_{k^-}^{(i)}) = \sum_i p_k^{(i)}. \tag{5}$$

It follows immediately from Eq. (5) that the two separate conservations in Eq. (3) are equivalent to the simultaneous conservation of $\sum_i M^{(i)}$ and $\sum_i p_k^{(i)}$ for the chosen axis k, and conversely. The connection to the full four-momentum conservation will be established in Section 3 using Eq. (1).

3 Equivalence with four-momentum conservation

This section establishes that the additive directional momentum conservation (ADMC) postulate is exactly equivalent to the usual conservation of energy and three-momentum, and hence to the conservation of four-momentum, for arbitrary isolated processes.

Theorem 3.1 (Equivalence) *Assume the dispersion relation of Eq. (1) for each particle and the definitions in Eq. (2). For any isolated process involving finitely many particles, the following are equivalent:*

1. *ADMC holds, i.e., for each axis $k \in \{x, y, z\}$ the two sums $\sum_i p_{k+}^{(i)}$ and $\sum_i p_{k-}^{(i)}$ are separately conserved as in Eq. (3).*
2. *The total M and the total three-momentum $\mathbf{p}$ are conserved, i.e., $\sum_i M^{(i)}$ and $\sum_i \mathbf{p}^{(i)}$ are constants during the process.*

Equivalently, with $E = c\,M$, item (2) is conservation of energy and three-momentum, i.e., conservation of four-momentum.

Proof (1) $\Rightarrow$ (2): For a fixed axis k, add and subtract the two conserved sums in Eq. (3). Using Eq. (5),

$$\sum_i (p_{k+}^{(i)} + p_{k-}^{(i)}) = 2 \sum_i M^{(i)}, \qquad \sum_i (p_{k+}^{(i)} - p_{k-}^{(i)}) = \sum_i p_k^{(i)}.$$

Since the left-hand sides are constants by ADMC, both $\sum_i M^{(i)}$ and $\sum_i p_k^{(i)}$ are constant for the chosen k. Doing this for $k = x, y, z$ yields conservation of $\sum_i \mathbf{p}^{(i)}$ as a vector and of $\sum_i M^{(i)}$. With $E = c\,M$, energy is conserved as well.

(2) $\Rightarrow$ (1): If $\sum_i M^{(i)}$ and $\sum_i p_k^{(i)}$ are conserved, then by the linear change of variables in Eq. (4),

$$\sum_i p_{k\pm}^{(i)} = \sum_i M^{(i)} \pm \tfrac{1}{2} \sum_i p_k^{(i)}$$

is constant for each choice of sign and for each axis k. This is exactly Eq. (3). □

Remark.

Writing $P^\mu = (M, \mathbf{p})$ in momentum units with $E = c\,M$, the second item of the theorem is the standard statement that the total four-momentum is conserved. Thus ADMC provides an equivalent, component-wise conservation bookkeeping within any inertial frame. The derivation relied only on linearity and additivity, together with the definitions in Eq. (2); no dynamics beyond the dispersion relation of Eq. (1) was used.

4 Uniqueness of the directional scalars

This section shows that the directional scalars are fixed in functional form by basic structural assumptions. The result is stated first in a scale-ambiguous form and then calibrated by a natural normalization.

4.1 Assumptions

Throughout this section the following are imposed for each fixed axis $k \in \{x, y, z\}$.

- **Additivity.** For any isolated composite system, $p_{k\pm}$ sums over constituents.
- **Isotropy.** The functional form is the same for all axes, and for a given axis k the dependence on momentum components other than p_k enters only through scalars of the frame, here taken as M from Eq. (1).
- **Parity.** Under $p_k \mapsto -p_k$ with $M \mapsto M$, one has $p_{k+} \leftrightarrow p_{k-}$.
- **Local invertibility.** The map $(M, p_k) \mapsto (p_{k+}, p_{k-})$ is locally invertible on an open set of interest (in particular at timelike states with $M > 0$).

4.2 Affine form from additivity

Define the sum and difference variables

$$S_k \equiv p_{k^+} + p_{k^-}, \qquad D_k \equiv p_{k^+} - p_{k^-}. \tag{6}$$

By parity, S_k is even in p_k and D_k is odd in p_k. By isotropy, S_k and D_k depend on the state through M and p_k only. Additivity over constituents then yields Cauchy-type functional equations for the maps $(M, p_k) \mapsto S_k$ and $(M, p_k) \mapsto D_k$. Standard regularity implied by local invertibility rules out pathological solutions, so both maps are linear:

$$S_k = u\, M, \qquad D_k = v\, p_k, \tag{7}$$

with axis-independent constants $(u, v \in \mathrm{R})$. The constants are nonzero because of local invertibility.

4.3 Uniqueness up to a common rescaling

Combining Eq. (6) with Eq. (7) gives

$$p_{k^\pm} = \tfrac{u}{2}\, M \pm \tfrac{v}{2}\, p_k. \tag{8}$$

Thus, under additivity, isotropy, parity, and local invertibility, the pair (p_{k^+}, p_{k^-}) is necessarily of the affine form $\alpha M \pm \beta p_k$ with constants $\alpha = u/2$ and $\beta = v/2$. Up to an overall rescaling of both $p_{k^\pm}$, the remaining freedom is a single dimensionless ratio v/u. Isotropy ensures α, β are the same for $k = x, y, z$.

4.4 Natural normalization

A convenient and frame-transparent calibration removes the free constants. First, fix the momentum scale by identifying the odd combination with the coordinate momentum,

$$D_k \equiv p_{k^+} - p_{k^-} = p_k, \tag{9}$$

which sets $v = 1$ in Eq. (7). Second, fix the scalar scale by identifying the even combination with twice the time-like component,

$$S_k \equiv p_{k^+} + p_{k^-} = 2M, \tag{10}$$

which sets $u = 2$. With Eqs. (9)–(10), Eq. (8) reduces to

$$p_{k^\pm} = M \pm \tfrac{1}{2} p_k, \tag{11}$$

i.e., Eq. (2). This completes the uniqueness claim: the directional scalars are fixed by the structural assumptions up to a common rescaling, and the normalization in Eqs. (9)–(10) selects the canonical representation used throughout.

5 Immediate corollaries

With four-momentum conservation established in Section 3 and the momentum-unit identity $E = c\,M$ with the dispersion relation in Eq. (1), standard kinematic relations follow directly.

Corollary (Speed–momentum relation).
For a free particle the kinematic identity $\mathbf{v} = \partial E/\partial \mathbf{p}$ yields

$$\mathbf{v} = \frac{\partial}{\partial \mathbf{p}}\Big(c\sqrt{p_f^2 + \mathbf{p}^2}\Big) = \frac{c^2\,\mathbf{p}}{E} = \frac{c\,\mathbf{p}}{M}. \tag{12}$$

In particular, $v = p\,c/M$ for $p = \|\mathbf{p}\|$.

Corollary (Lorentz factor).
Using $E = \gamma m_0 c^2$ and $E = c\,M$,

$$\gamma = \frac{E}{m_0 c^2} = \frac{c\,M}{m_0 c^2} = \frac{M}{p_f}. \tag{13}$$

Corollary (Massless limit).
Let $p_f \to 0$. Then from Eq. (1),

$$M \to \|\mathbf{p}\|, \qquad E \to c\,\|\mathbf{p}\|, \qquad \mathbf{v} \to c\,\frac{\mathbf{p}}{\|\mathbf{p}\|}, \tag{14}$$

so that $v \to c$ for nonzero $\mathbf{p}$.

Corollary (Newtonian expansion).
For $p/p_f \ll 1$,

$$M = p_f\sqrt{1 + \frac{\mathbf{p}^2}{p_f^2}} = p_f\left(1 + \frac{\mathbf{p}^2}{2p_f^2} - \frac{\mathbf{p}^4}{8p_f^4} + \cdots\right), \tag{15}$$

and therefore

$$E = c\,M = m_0 c^2 + \frac{\mathbf{p}^2}{2m_0} - \frac{\mathbf{p}^4}{8m_0^3 c^2} + \cdots, \tag{16}$$

which recovers the rest energy plus the Newtonian kinetic term at leading order, with relativistic corrections following.

6 Conclusion

A directional decomposition of momentum into the axis-labeled additive scalars $p_{k^\pm} = M \pm \frac{1}{2}p_k$ was shown to be exactly equivalent to the conservation of energy and three-momentum, and hence to four-momentum conservation, assuming the standard dispersion relation $M^2 = p_f^2 + \mathbf{p}^2$. The directional postulate (ADMC) therefore does not modify special relativity; it re-expresses the same conserved quantities in a component-wise bookkeeping form within any inertial frame. A short structural analysis established that, under additivity, isotropy, parity, and local invertibility, the representation is unique up to a common rescaling, which a natural calibration removes.

The formulation may be useful wherever conservation checks dominate the kinematic analysis (e.g., multi-particle reactions, threshold problems, and numerical event bookkeeping), while leaving dynamical assumptions untouched. Extensions, applications, and origins of this representation are pursued elsewhere.

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Use of large language models. A large language model was used to assist with editing, reference formatting, and consistency checks. All technical content, derivations, and conclusions were devised, validated, and approved by the author.

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