Exercise 1 - Trajectory of rotating ball

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Contents

1 Introduction

In this paper we aim at giving a comprehensive numerical analysis of a typical case study, the trajectory of a rotating ball under the influence of the Magnus effect. This analysis will be done using three related types of algorithms: Euler explicit, implicit and semi-implicit.

The algorithms used will be implemented through a C++ simulation and the obtained data analysed using classic python libraries. Aaaaaa yes no gravity aaaa yes with gravity and traine aerodynamique yes yes baguette. Some analytical results will also be proven in order to verify the soundness of our simulation by comparing them to computational results.

2 Analytical results

For the purpose of this study it is first needed to prove some analytical results that will later be achieved through numerical simulations.

We are considering a sphere (tennis ball) of mass m and radius R. It is rotating according to $\vec{\omega} = \omega \vec{e_z}$. It is moving in the gravity field $\vec{g} = -g\vec{e_y}$ and inside a fluid of density ρ which applies a force due to the Magnus effect:

$$\vec{F_p} = \mu R^3 \rho \vec{\omega} \times \vec{v} \tag{1}$$

We want to determine the movement of the ball knowing the initial velocity $\vec{v_0}$ and rotation $\vec{\omega}$. In an effort to simplify the problem we assume that the rotation is constant and we consider trajectories occuring only on the (x, y) plane.

2.1 System of differential equations

Question 1.1-(a)

Let us take the vector: $\mathbf{y} = (x, y, v_x, v_y)$

We are searching for f such that:

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \begin{pmatrix} lv_x \\ v_y \\ a_x \\ a_y \end{pmatrix} = f(\mathbf{y}) \tag{2}$$

We know that $m\vec{a} = \vec{F_p} + m\vec{g}$ and from Equation 1 we have:

$$\vec{F_p} = \mu R^3 \rho \omega \begin{pmatrix} l - v_y \\ v_x \\ 0 \end{pmatrix}$$

We get from this and $g_z = 0$ that $a_z = 0$ so the trajectory will indeed stay inside the (x, y) plane for $v_z(0) = 0$ and $z(0) = 0$. Most importantly we also get:

$$\begin{pmatrix} la_x \\ a_y \end{pmatrix} = \begin{pmatrix} l - \frac{\mu R^3 \rho \omega}{m} v_y \\ \frac{\mu R^3 \rho \omega}{m} v_x - g \end{pmatrix}$$

Which allows us to rewrite Equation 2 as:

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = f(\mathbf{y}) = \begin{pmatrix} lv_x \\ v_y \\ -\frac{\mu R^3 \rho \omega}{m} v_y \\ \frac{\mu R^3 \rho \omega}{m} v_x - g \end{pmatrix} \tag{3}$$

2.2 Mechanical energy

Question 1.1-(b)

We are searching for the total mechanical energy of the system. We know that for a rigid body such as the tennis ball considered in this problem the total mechanical energy is the sum of: the translational kinetic energy, the rotational kinetic energy and the potential so:

$$\mathrm{E}_{\mathrm{tot}} = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 + mgy$$

We get the moment of inertia of a sphere for any rotation around an axis going through the center: $I = \frac{2}{5}mR^2$ [1].

Which yields the final formula for the energy:

$$\mathrm{E}_{\mathrm{tot}} = \frac{1}{2}mv^2 + \frac{1}{5}mR^2\omega^2 + mgy \tag{4}$$

We now want to know if this energy is conserved. We know that the gravitational force from the $\vec{g}$ field is conservative. We now consider the force $\vec{F_p}$. It is strictly orthogonal to the velocity at any point from EQUATION 1 which means that $\vec{F_p}$ does not produce any work throughout the movement. We can conclude that we have no non-conservative forces doing work so the total mechanical energy must be conserved.

2.3 Zero-gravity situation

Question 1.1-(c)

All the considerations in this subsection will be done inside the (x, y) plane since we have shown that the movement always stays inside of it. The z and v_z coordinates are thus always 0.

We now want to solve EQUATION 3 for $\vec{g} = 0$ and the initial conditions $\vec{x}(0) = 0$ and $\vec{v}(0) = v_0\vec{e_x}$. Let us write $\alpha = \frac{\mu R^3 \rho\omega}{m}$. We have:

$$a_x = \dot{v_x} = -\alpha v_y \tag{5}$$

$$a_y = \dot{v_y} = \alpha v_x \tag{6}$$

Taking the time derivative yields:

$$\ddot{v_x} = -\alpha\dot{v_y} = -\alpha^2 v_x \implies \ddot{v_x} + \alpha^2 v_x = 0 \tag{7}$$

which we recognise as the harmonic oscillator equation. Its general solution is:

$$v_x(t) = A\cos(\alpha t + \varphi) \tag{8}$$

We can deduce the general solution for $x(t)$ from this:

$$x(t) = \frac{A}{\alpha}\sin(\alpha t + \varphi) + c_1 \tag{9}$$

We can find $y(t)$ by substituing v_x into EQUATION 6

$$\dot{v_y}(t) = \alpha v_x = A\alpha\cos(\alpha t + \varphi) \tag{10}$$

$$\implies v_y(t) = A\sin(\alpha t + \varphi) + c_2 \tag{11}$$

$$\implies y(t) = -\frac{A}{\alpha}\cos(\alpha t + \varphi) + c_2 t + c_3 \tag{12}$$

Using the mentioned initial conditions inside these equations gives us 5 unknowns for 4 equations. For symmetry reasons we can set $\varphi = 0$ which gives the other constants:

$$c_1 = 0, \quad c_2 = 0, \quad c_3 = \frac{v_0}{\alpha}, \quad A = v_0$$

We now want to characterise the type of movement we are having with this ball. We show that the acceleration is orthogonal to the velocity and has a constant norm:

$$\vec{a} \cdot \vec{v} = \alpha(-v_y v_x + v_x v_y) = 0 \tag{13}$$

$$||\vec{a}||^2 = \alpha^2((-v_y)^2 + (v_x)^2) = \alpha^2 A^2 = \text{cst} \tag{14}$$

This means that the movement is circular uniform and we can write:

$$a = \frac{v^2}{r}, \quad a = \alpha v_0, \quad v = \text{cst} = v_0$$

$$\Rightarrow r = \frac{v_0}{\alpha}$$

with r the radius of the circular movement. The angular frequency is given by: $\Omega = \frac{v}{r} = \alpha$.

For clarity we rewrite EQUATION 8, EQUATION 9, EQUATION 11 and EQUATION 12 with the constants found and reminding that $\alpha = \frac{\mu R^3 \rho \omega}{m}$:

$$v_x(t) = v_0 \cos(\alpha t) \tag{15}$$

$$v_y(t) = v_0 \sin(\alpha t) \tag{16}$$

$$x(t) = \frac{v_0}{\alpha} \sin(\alpha t) \tag{17}$$

$$y(t) = \frac{v_0}{\alpha}(1 - \cos(\alpha t)) \tag{18}$$

2.4 Gravity with no initial speed situation

Question 1.1-(d)

We now want to solve the same problem with $g \neq 0$. The given initial conditions are $\vec{x}(0) = 0$ and $\vec{v}(0) = 0$.

We choose a new referential $\mathcal{R}'$ moving at a constant velocity $\vec{v}_E = \frac{m}{\omega^2 \mu R^3 \rho} \vec{\omega} \times \vec{g} = \frac{1}{\alpha} \vec{e}_z \times \vec{g}$ relative to our first referential $\mathcal{R}$.

We first write down the equations of motion in the new referential using $\vec{v} = \vec{v}' + \vec{v}_E \Rightarrow \vec{a} = \vec{a}'$:

$$\begin{aligned}
\vec{a}' &= \frac{1}{m}\vec{F}_p + \vec{g} \\
&= \alpha \vec{e}_z \times \vec{v} + \vec{g} \\
&= \alpha \vec{e}_z \times \vec{v}' + \alpha \vec{e}_z \times \vec{v}_E + \vec{g} \\
&= \alpha \vec{e}_z \times \vec{v}' + \vec{e}_z \times (\vec{e}_z \times \vec{g}) + \vec{g} \\
&= \alpha \vec{e}_z \times \vec{v}' + g(-\vec{e}_z \times (\vec{e}_z \times \vec{e}_y) - \vec{e}_y) \\
&= \alpha \vec{e}_z \times \vec{v}' + g(\vec{e}_y - \vec{e}_y)
\end{aligned}$$

$$\Longrightarrow \vec{a}' = \alpha \begin{pmatrix} -v'_y \\ v'_x \\ 0 \end{pmatrix} \tag{19}$$

In the question 1.1-(c) we had an equation of motion $\vec{a} = \alpha \vec{e}_z \times \vec{v}$ which has the exact same form and thus the general solutions are the same for $\vec{a}'$ and $\vec{v}'$ as in EQUATION 8 to 12.

Assuming like before $\varphi = 0$ and using the given initial conditions we can find:

$$A = -v_E, \quad c_3 = -\frac{v_E}{\alpha}, \quad c_1 = 0, \quad c_2 = 0$$

with $\vec{v_E} = \frac{g}{\alpha}\vec{e}_x$.

This gives us our final solution by changing back to the $\mathcal{R}$ referential knowing it will have translated along $\hat{x}$ by $x = x' + v_E t$:

$$x(t) = -\frac{g}{\alpha^2}\sin(\alpha t) + \frac{g}{\alpha}t \tag{20}$$

$$y(t) = \frac{g}{\alpha^2}(\cos(\alpha t) - 1) \tag{21}$$

We want to determine if the ball will at some point get back to its starting height $y = 0$. We see easily that the movement according to y is periodic with cos being the only term depending on t. More formally, because cos is 2π-periodic, we have for $t_{\text{fin}} = 2\pi/\alpha$

$$y(t_{\text{fin}}) = \frac{g}{\alpha^2}\left(\cos\left(\frac{2\pi\alpha}{\alpha}\right) - 1\right) = \frac{g}{\alpha^2}(1-1) = 0$$

which does show that at $t_{\text{fin}} = 2\pi/\alpha = 2\pi m/\mu R^3 \rho\omega$ the ball gets back to its original height. This happens at a distance L from the origin:

$$L = x(t_{\text{fin}}) = -\frac{g}{\alpha^2}\sin\left(\frac{2\pi\alpha}{\alpha}\right) + \frac{2\pi g}{\alpha^2}$$

$$L = \frac{2\pi g}{\alpha^2} = \frac{2\pi g m^2}{(\mu R^3 \rho\omega)^2} \tag{22}$$

3 Simulations

3.1 Rotating ball in zero gravity

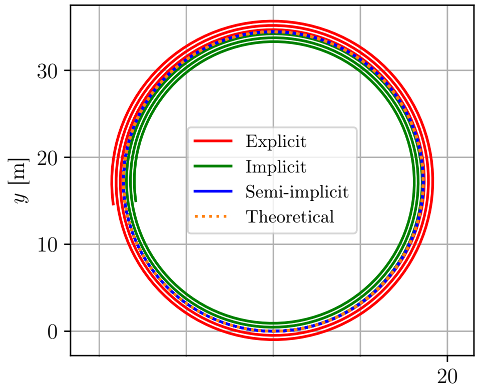
The simulations were done on a tenis ball with mass $m = 0.056$ kg and radius $R = 0.033$ m, with air density $\rho = 1.2$ kg/m^3. The coefficient $\mu = 6$ has been chosen. The ball is sent with initial conditions $\omega = 10$ rotations/s ($\omega = 20\pi$ rad/s), $\vec{x}(0) = \vec{0}$, $\vec{v}(0) = 5\vec{e_x}$ and simulated until $t_{\text{fin}} = 60$ s.

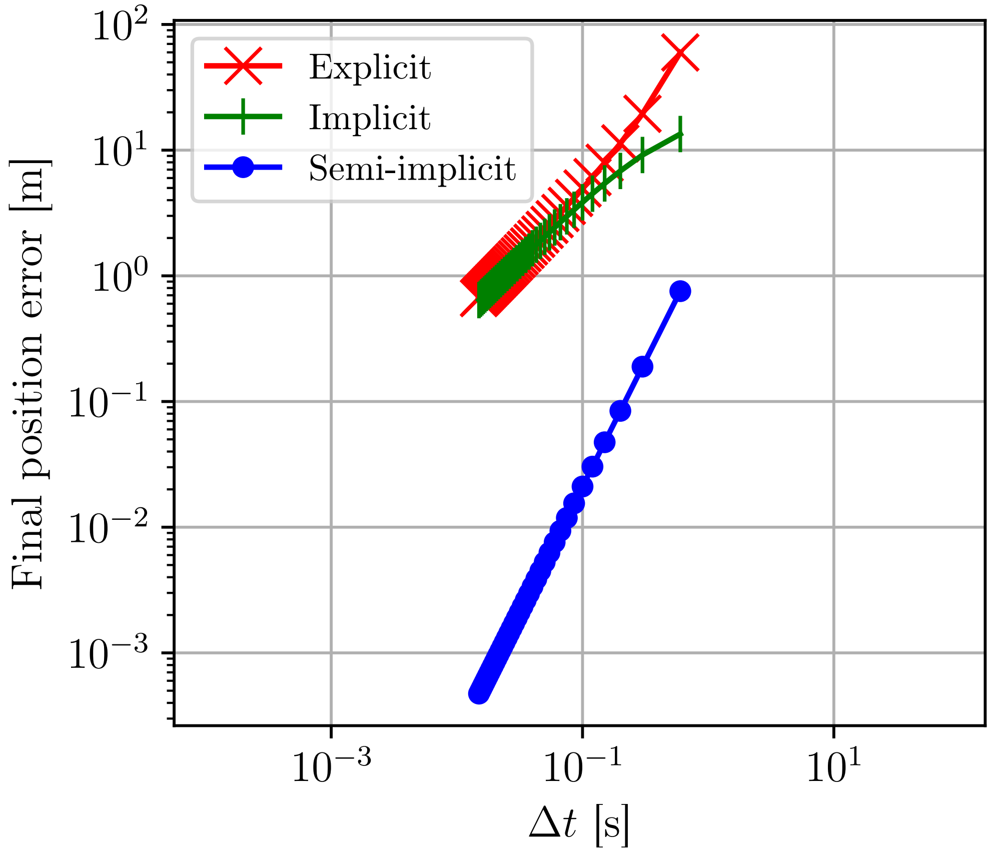
To lighten the notation, we will use EE for the Explicit Euler method, IE for Implicit Euler and SE for Semi-implicit Euler.

The position after t_{fin} is shown in FIGURE 1. We can see that the EE method overshoots the expected trajectory, while the IE method undershoots it and SE remains almost exactly on the analytical result.

3.1.1 Numeric convergence

The numeric convergence of the final position of each method is illustrated in FIGURE 2. We observe that the error on the final position tends to 0 as $\Delta t \to 0$ for every method, meaning that they all converge numerically. Furthermore, the convergence order is given by the slope of the line passing through the points. We can thus deduce that the convergence order is 1 for EE and IE, and 2 for SE.





er t_{fin} for

Figure 2: Error on final position w.r.t. analytical solution. n_{steps} was varied between 100 and 4000.

The numeric convergence of the energy for each method is also given in FIGURE 3. Again, we observe that the error clearly tends to 0, as $t \to 0$ for EE and IE. The error for SE remains quite constant and is of the order of 10^{-14}, close to the limit for double-precision floating point numbers. We thus conclude that every method converges numerically. The convergence order for EE and IE is 1, but SE does not have a convergence order as it's energy remains pretty much constant for a wide range of Δt.

3.1.2 Numeric stabilty

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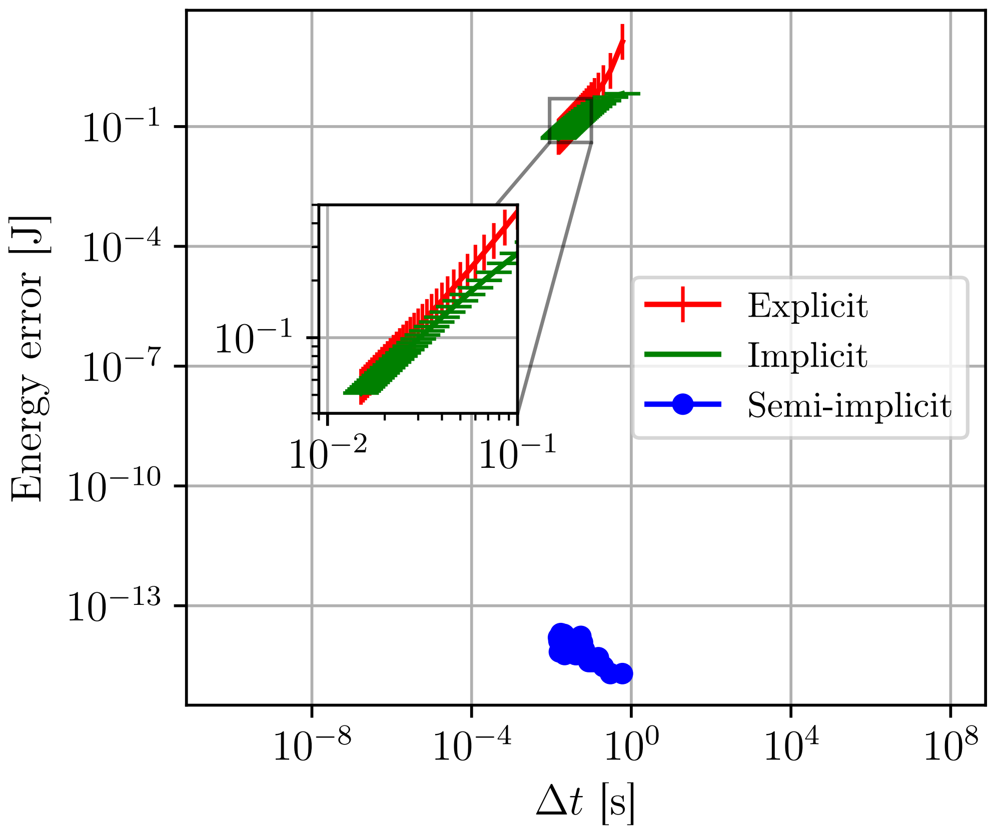


Figure 3: Error on final energy for each method where the error is defined as $\max(E_{\text{mec}}) - \min(E_{\text{mec}})$.

3.2 Rotating ball with gravity

3.2.1 Method comparaison

3.2.2 Adding drag force

The aerodynamic air drag force

$$\vec{F}_t = -\frac{1}{2} C_t \rho S v \vec{v} \tag{23}$$

was added to the simulation by adding a term to $f(\mathbf{y})$ from EQUATION 3

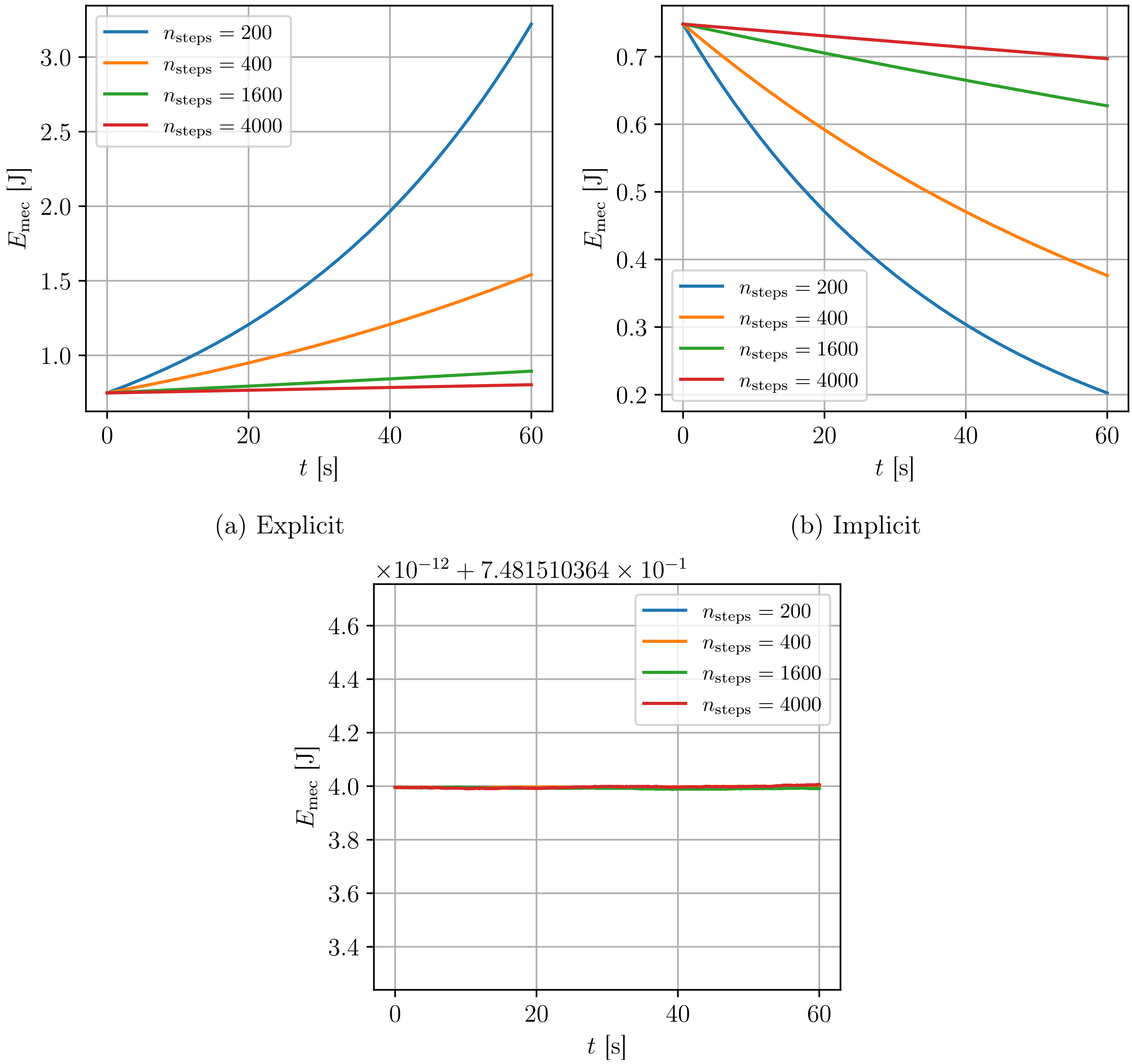
$$f(\mathbf{y}) = \begin{pmatrix} l v_x \\ v_y \\ -\frac{\mu R^3 \rho \omega}{m} v_y - \frac{1}{2m} C_t \rho S \|v\| v_x \\ \frac{\mu R^3 \rho \omega}{m} v_x - g - \frac{1}{2m} C_t \rho S \|v\| v_y \end{pmatrix} \tag{24}$$

then show some results but now its bedtime and am *zleeepy*

4 Analysis

5 Conclusion

Through this report, we learned the differences between the EE, IE, and SE methods and analysed their numeric convergence and stabilty under different conditions. While the EE and



(a) Explicit

(b) Implicit

(c) Semi-implicit

Figure 4

IE methods drifted off the analytical solution, the SE method remained very close or exactly on the analytical solution.

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