Exercise 1 - Trajectory of rotating ball

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Contents

1 Introduction

In this paper we aim at giving a comprehensive numerical analysis of a typical case study, the trajectory of a rotating ball under the influence of the Magnus effect. This analysis will be done using three related types of algorithms: Euler explicit, implicit and semi-implicit.

The algorithms used will be implemented through a C++ simulation and the obtained data analysed using classic python libraries. Aaaaaa yes no gravity aaaa yes with gravity and traine aerodynamique yes yes baguette. Some analytical results will also be proven in order to verify the soundness of our simulation by comparing them to computational results.

2 Analytical results

For the purpose of this study it is first needed to prove some analytical results that will later be achieved through numerical simulations.

We are considering a sphere (tennis ball) of mass m and radius R. It is rotating according to $\vec{\omega} = \omega \vec{e_z}$. It is moving in the gravity field $\vec{g} = -g\vec{e_y}$ and inside a fluid of density ρ which applies a force due to the Magnus effect:

$$\vec{F_p} = \mu R^3 \rho \vec{\omega} \times \vec{v} \tag{1}$$

We want to determine the movement of the ball knowing the initial velocity $\vec{v_0}$ and rotation $\vec{\omega}$. In an effort to simplify the problem we assume that the rotation is constant and we consider trajectories occuring only on the (x, y) plane.

2.1 System of differential equations

Question 1.1-(a)

Let us take the vector: $\mathbf{y} = (x, y, v_x, v_y)$

We are searching for f such that:

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \begin{pmatrix} v_x \\ v_y \\ a_x \\ a_y \end{pmatrix} = f(\mathbf{y}) \tag{2}$$

We know that $m\vec{a} = \vec{F_p} + m\vec{g}$ and from EQUATION 1 we have:

$$\vec{F_p} = \mu R^3 \rho\omega \begin{pmatrix} -v_y \\ v_x \\ 0 \end{pmatrix}$$

We get from this and $g_z = 0$ that $a_z = 0$ so the trajectory will indeed stay inside the (x, y) plane for $v_z(0) = 0$ and $z(0) = 0$. Most importantly we also get:

$$\begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} -\frac{\mu R^3 \rho\omega}{m} v_y \\ \frac{\mu R^3 \rho\omega}{m} v_x - g \end{pmatrix}$$

Which allows us to rewrite EQUATION 2 as:

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = f(\mathbf{y}) = \begin{pmatrix} v_x \\ v_y \\ -\frac{\mu R^3 \rho\omega}{m} v_y \\ \frac{\mu R^3 \rho\omega}{m} v_x - g \end{pmatrix} \tag{3}$$

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2.2 Mechanical energy

Question 1.1-(b)

We are searching for the total mechanical energy of the system. We know that for a rigid body such as the tennis ball considered in this problem the total mechanical energy is the sum of: the translational kinetic energy, the rotational kinetic energy and the potential so:

$$\mathrm{E_{tot}} = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 + m g y$$

We get the moment of inertia of a sphere for any rotation around an axis going through the center: $I = \frac{2}{5} m R^2$ [1].

Which yields the final formula for the energy:

$$\mathrm{E_{tot}} = \frac{1}{2}mv^2 + \frac{1}{5}mR^2\omega^2 + mgy \tag{4}$$

We now want to know if this energy is conserved. We know that the gravitational force from the $\vec{g}$ field is conservative. We now consider the force $\vec{F_p}$. It is strictly orthogonal to the velocity at any point from EQUATION 1 which means that $\vec{F_p}$ does not produce any work throughout the movement. We can conclude that we have no non-conservative forces doing work so the total mechanical energy must be conserved. #

2.3 Zero-gravity situation

Question 1.1-(c)

All the considerations in this subsection will be done inside the (x, y) plane since we have shown that the movement always stays inside of it. The z and v_z coordinates are thus always 0.

We now want to solve EQUATION 3 for $\vec{g} = 0$ and the initial conditions $\vec{x}(0) = 0$ and $\vec{v}(0) = v_0\vec{e_x}$. Let us write $\alpha = \frac{\mu R^3\rho\omega}{m}$. We have:

$$a_x = \dot{v_x} = -\alpha v_y$$
$$a_y = \dot{v_y} = \alpha v_x$$

Which yields:

$$\ddot{v_x} = -\alpha\dot{v_y} = -\alpha^2 v_x$$
$$\Rightarrow \ddot{v_x} + \alpha^2 v_x = 0$$

which we recognise as the harmonic oscillator equation. Its general solution is:

$$v_x(t) = A\cos(\alpha t + \varphi) \tag{5}$$

We can deduce all the general solutions from this:

$$x(t) = \frac{A}{\alpha}\sin(\alpha t + \varphi) + c_1 \tag{6}$$

$$\dot{v_y}(t) = \alpha v_x = A\alpha\cos(\alpha t + \varphi)$$

$$\Rightarrow v_y(t) = A\sin(\alpha t + \varphi) + c_2 \tag{7}$$

$$\Rightarrow y(t) = -\frac{A}{\alpha}\cos(\alpha t + \varphi) + c_2 t + c_3 \tag{8}$$

Using the mentioned initial conditions inside these equations gives us 5 unknown for 4 equations. We can set $\varphi = 0$ without any influence on the physics which gives the other constants:

$$c_1 = 0, \quad c_2 = 0, \quad c_3 = \frac{v_0}{\alpha}, \quad A = v_0$$

We now want to characterise the type of movement we are having with this ball. We show that the acceleration is orthogonal to the velocity and has a constant norm:

$$\vec{a}.\vec{v} = \alpha(-v_y v_x + v_x v_y) = 0$$
$$||\vec{a}||^2 = \alpha^2((-v_y)^2 + (v_x)^2) = \alpha^2 A^2 = \mathrm{cst}$$

This means that the movement is circular uniform and we can write:

$$a = \frac{v^2}{r}, \quad a = \alpha v_0, \quad v = \text{cst} = v_0$$
$$\Rightarrow r = \frac{v_0}{\alpha}$$

with r the radius of the circular movement. We finally want the angular frequency given by: $\Omega = \frac{v}{r} = \alpha$. #

For clarity we rewrite EQUATION 5, EQUATION 6, EQUATION 7 and EQUATION 8 with the constants found and reminding that $\alpha = \frac{\mu R^3 \rho \omega}{m}$:

$$v_x(t) = v_0 \cos(\alpha t) \tag{9}$$

$$v_y(t) = v_0 \sin(\alpha t) \tag{10}$$

$$x(t) = \frac{v_0}{\alpha} \sin(\alpha t) \tag{11}$$

$$y(t) = \frac{v_0}{\alpha}(1 - \cos(\alpha t)) \tag{12}$$

2.4 Gravity with no initial speed situation

Question 1.1-(d)

We now want to solve the same problem with $g \neq 0$. The given initial conditions are $\vec{x}(0) = 0$ and $\vec{v}(0) = 0$.

We choose a new referential $\mathcal{R}'$ moving at a constant velocity $\vec{v}_E = \frac{m}{\omega^2 \mu R^3 \rho} \vec{\omega} \times \vec{g} = \frac{1}{\alpha} \vec{e}_z \times \vec{g}$ relative to our first referential $\mathcal{R}$ of the ground.

We first write down the equations of motion in the new referential using $\vec{v} = \vec{v}' + \vec{v}_E \Rightarrow \vec{a} = \vec{a}'$:

$$\begin{aligned}
\vec{a}' &= \frac{1}{m} \vec{F}_p + \vec{g} \\
\vec{a}' &= \alpha \vec{e}_z \times \vec{v} + \vec{g} \\
\vec{a}' &= \alpha \vec{e}_z \times \vec{v}' + \alpha \vec{e}_z \times \vec{v}_E + \vec{g} \\
\vec{a}' &= \alpha \vec{e}_z \times \vec{v}' + \vec{e}_z \times (\vec{e}_z \times \vec{g}) + \vec{g} \\
\vec{a}' &= \alpha \vec{e}_z \times \vec{v}' + g(-\vec{e}_z \times (\vec{e}_z \times \vec{e}_y) - \vec{e}_y) \\
\vec{a}' &= \alpha \vec{e}_z \times \vec{v}' + g(\vec{e}_y - \vec{e}_y)
\end{aligned}$$

$$\Rightarrow \vec{a}' = \alpha \begin{pmatrix} -v'_y \\ v'_x \\ 0 \end{pmatrix} \tag{13}$$

In the question 1.1-(c) we had an equation of motion $\vec{a} = \alpha \vec{e}_z \times \vec{v}$ which has the exact same form and thus the general solutions are the same for $\vec{a}'$ and $\vec{v}'$ as in EQUATION 5 to 8.

Assuming as before $\varphi = 0$ with no influence on the physics and using the initial conditions given we can find:

$$A = -v_E, \quad c_3 = -\frac{v_E}{\alpha}, \quad c_1 = 0, \quad c_2 = 0$$

with $\vec{v_E} = g/\alpha \vec{e}_x$.

This gives us our final solution by changing back to the $\mathcal{R}$ referential knowing it will have translated along $\hat{x}$ by $x = x' + v_E t$:

$$x(t) = -\frac{g}{\alpha^2}\sin(\alpha t) + \frac{g}{\alpha}t \tag{14}$$

$$y(t) = \frac{g}{\alpha^2}(\cos(\alpha t) - 1) \tag{15}$$

We want to determine if the ball will at some point get back to its starting height $y = 0$. We see easily that the movement according to y is periodic with a cos being the only term depending on t. More formally for $t_{\text{fin}} = 2\pi/\alpha$ we have:

$$y(t_{\text{fin}}) = \frac{g}{\alpha^2}\left(\cos\left(\frac{2\pi\alpha}{\alpha}\right) - 1\right) = \frac{g}{\alpha^2}(1-1) = 0$$

which does show that at $t_{\text{fin}} = 2\pi/\alpha = 2\pi m/\mu R^3\rho\omega$ the ball gets back to its original height. This happens at a distance L from the origin:

$$L = x(t_{\text{fin}}) = -\frac{g}{\alpha^2}\sin\left(\frac{2\pi\alpha}{\alpha}\right) + \frac{2\pi g}{\alpha^2}$$

$$L = \frac{2\pi g}{\alpha^2} = \frac{2\pi g m^2}{(\mu R^3\rho\omega)^2} \tag{16}$$

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3 Simulations

4 Analysis

5 Conclusion

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A Appendix