ASTR 531 - Stellar Interiors and Evolution

Exam 1 — Tom Wagg

April 26, 2022

Thriving under high pressure

Dearest Grad Student,

My name is Professor Daniel Eyemund and I'm reaching out to you with an exciting business idea - diamonds! I've realised that I could get around applying for grants by generating a large number of diamonds and discretely selling them! They might be rare on Earth but with the right conditions perhaps we could create a ***bunch*** *of them out in space. The plan is as follows:*

1. *Collect a big cloud of carbon*
2. *Squish it down and inject it into the centre a white dwarf*
3. *Create diamonds (and worry about extracting them later)*
4. *...*
5. *Profit?*

Your mission, should you choose to accept it (...which you have to because this is an exam), is to find out whether a white dwarf has enough pressure at the centre to produce diamonds, and if so, what mass of white dwarf we need! I feel your experiences with stellar interiors, not to mention the high pressure environment known as grad school, would make you ideally suited to this task. Please note your timely response would be appreciated as I will be meeting with Elon to pitch the idea in 30 minutes.

Yours sincerely,
Prof. D. Eyemund

Part a - Complete degeneracy pressure for electrons

Given that, for complete degeneracy, the $n_e(p)$ distribution is rectangular for $p < p_{\rm F}$ such that $n_e(p) = 2/h^3$, show that the pressure scales as $P_e \propto n_e^{5/3}$.

First, we should solve for the Fermi momentum in terms of the number density from the fact that the integral should give the number density.

$$\begin{aligned} n_e &= \int_0^{p_{\rm F}} n_e \, {\rm d}^3 p && (1) \\ &= \frac{2}{h^3} \int_0^{p_{\rm F}} 4\pi p^2 \, {\rm d}p && (2) \\ n_e &= \frac{8\pi p_{\rm F}^3}{3h^3} && (3) \end{aligned}$$

And so we can turn this around to get an expression for the Fermi momentum in terms of n_e

$$p_{\mathrm{F}} = \left(\frac{3n_e h^3}{8\pi}\right)^{1/3} \tag{4}$$

Now, we can use this expression for the pressure integral (from Eq. 4.7) to find that the desired relation

$$\begin{aligned}
P_e &= \frac{1}{3}\int p \cdot v(p) \cdot n_e(p)\, \mathrm{d}^3 p &(5)\\
&= \frac{1}{3}\int_0^{p_{\mathrm{F}}} p \cdot \frac{p}{m_e} \cdot \frac{2}{h^3}\, \mathrm{d}^3 p &(6)\\
&= \frac{2}{3m_e h^3}\int_0^{p_{\mathrm{F}}} p^2 \cdot 4\pi p^2\, \mathrm{d}p &(7)\\
&= \frac{8\pi}{3m_e h^3} \cdot p_{\mathrm{F}}^5 &(8)\\
&= \frac{8\pi}{3m_e h^3} \cdot \left(\frac{3n_e h^3}{8\pi}\right)^{5/3} &(9)\\
&= \left(\frac{3h^3}{8\pi}\right)^{2/3} \cdot \frac{n_e^{5/3}}{m_e} &(10)\\
&\boxed{P_e \propto n_e^{5/3}} &(11)
\end{aligned}$$

Part b - Polytrope central pressure

By showing that part a is true, you've proven that we can use an $n = 3/2$ polytrope! It'll now be useful to know an expression for the central pressure of a polytrope star.

Use the fact that $P_c = K\rho_c^{1+1/n}$ to show that $P_c = A\frac{GM^2}{R^4}$ (and find the constant A in terms of D_n, π, n, M_n). Hint: You can find K by comparing an expression for α and an expression for M_n. Additionally note that $\rho_c = \bar{\rho} D_n$.

To start, we need to find an expression for K. We have the definition of α from the Lane-Emden equation (see Eq. 11.5).

$$\alpha = \left[\frac{(n+1)K}{4\pi G \rho_c^{1-1/n}}\right]^{1/2} \tag{12}$$

Next we can use Equation 11.10a to find another expression for α from the definition of M_n

$$\begin{aligned}
M &= 4\pi a^3 \rho_c M_n &(13)\\
\alpha &= \left(\frac{M}{M_n 4\pi \rho_c}\right)^{1/3} &(14)
\end{aligned}$$

Combining these yields an expression for K

$$\left[\frac{(n+1)K}{4\pi G \rho_c^{1-1/n}}\right]^{1/2} = \left(\frac{M}{M_n 4\pi \rho_c}\right)^{1/3} \tag{15}$$

$$K = \left(\frac{M}{M_n 4\pi\rho_c}\right)^{2/3} \frac{4\pi G\rho_c^{1-1/n}}{(n+1)} \tag{16}$$

$$K = \left(\frac{M}{M_n}\right)^{2/3} \frac{(4\pi)^{1/3} G\rho_c^{1/3-1/n}}{(n+1)} \tag{17}$$

Now, given that $P_c = K\rho_c^{1+1/n}$, this means that we can write the central pressure as

$$P_c = \left(\frac{M}{M_n}\right)^{2/3} \frac{(4\pi)^{1/3} G\rho_c^{1/3-1/n}}{(n+1)} \rho_c^{1+1/n} \tag{18}$$

$$= \left(\frac{M}{M_n}\right)^{2/3} \frac{(4\pi)^{1/3} G\rho_c^{4/3}}{(n+1)} \tag{19}$$

$$= \left[\frac{(4\pi)^{1/3} G}{(n+1)M_n^{2/3}}\right] M^{2/3} \rho_c^{4/3} \tag{20}$$

We also know that the average density is given by $\bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3}$ and it is related to the central density as $\rho_c = \bar{\rho} D_n$. Therefore we can express the central pressure

$$P_c = \left[\frac{(4\pi)^{1/3} G}{(n+1)M_n^{2/3}}\right] M^{2/3} \left(\frac{M D_n}{\frac{4}{3}\pi R^3}\right)^{4/3} \tag{21}$$

$$= \left[\frac{(4\pi)^{1/3} G}{(n+1)M_n^{2/3}}\right] \left(\frac{D_n}{\frac{4}{3}\pi}\right)^{4/3} \frac{M^2}{R^4} \tag{22}$$

$$\boxed{P_c = \underbrace{\left[\frac{(3D_n)^{4/3}}{4\pi(n+1)M_n^{2/3}}\right]}_{=A} \frac{GM^2}{R^4}} \tag{23}$$

Part c - White dwarf* $\mathbf{M\text{-}P_c}$ *relation

Now let's take that and create a relation for the central pressure of a white dwarf as only a function of mass. Apply the M-R relation for WDs (assuming $\mu_e = 2$) and that $n = 3/2$ to show that $P_c = \frac{AG}{B^4} M^C$ where B and C are constants you should find.

The WD M-R relation with $\mu_e = 2$ gives that

$$R = 0.012\,\mathrm{R}_\odot \left(\frac{M}{\mathrm{M}_\odot}\right)^{-1/3} \tag{24}$$

This gives the final result of the central pressure as a function of WD mass

$$\boxed{P_{c,\mathrm{WD}} = \frac{AG}{B^4} M^C} \tag{25}$$

where $B = 0.012\,\mathrm{R}_\odot\,{\mathrm{M}_\odot}^{1/3}$ and $C = 10/3$.

Part d - Minimum white dwarf mass to produce diamonds

Carbon requires a pressure of at least $\sim 10^{10}\,\mathrm{Pa}$ *to form diamonds. Find the minimum mass a white dwarf would need to convert carbon to diamonds in its core (assuming* $A = 0.77$, $B = 1.05 \times 10^{17}\,\mathrm{m\,kg^{1/3}}$, $C = 10/3$ *and* $G = 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^2}$ *and recalling that* $\mathrm{M}_\odot = 2 \times 10^{30}\,\mathrm{kg}$*).*

What does this imply for which sorts of white dwarfs could work for Prof. Eyemund? How would you answer change qualitatively if the professor could only inject carbon near the surface rather than at the centre?

First we can solve for the minimum mass as

$$M_{\mathrm{min}} = \left(\frac{10^{10} 10^{17 \cdot 4}}{0.77 \cdot 6.67 \times 10^{-11}} \right)^{3/10} \mathrm{kg} \tag{26}$$

$$= 3.25 \times 10^{26}\,\mathrm{kg} \tag{27}$$

$$\boxed{M_{\mathrm{min}} = 1.6 \times 10^{-4}\,\mathrm{M}_\odot} \tag{28}$$

This answer implies that **all** white dwarfs could create diamonds at their cores! If the professor could only inject carbon closer to the surface then the minimum mass would probably increase since the pressure declines sharply as you near the surface.

Extra Credit - The Verdict

Craft a pithy yet informative response to Prof. Eyemund summarising the situation. Bonus points will be awarded for each dreadful pun.

The solution is left as an exercise for the reader :D