ASTR 531 - Stellar Interiors and Evolution

Exam 1 | Tom Wagg

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Thriving under high pressure

Dearest Grad Student,

My name is Professor Daniel Eyemund and I'm reaching out to you with an exciting business idea - diamonds! I've realised that I could get around applying for grants by generating a large number of diamonds and discretely selling them! They might be rare on Earth but with the right conditions perhaps we could create a ***bunch*** *of them out in space. The plan is as follows:*

1. *Collect a big cloud of carbon*
2. *Squish it down and inject it into a white dwarf*
3. *Create diamonds*
4. *...*
5. *Profit?*

Your mission, should you choose to accept it (... which you have to because this is an exam), is to find out whether a white dwarf could produce diamonds, if so, what kind of white dwarf we need! I feel your experiences with stellar interiors, not to mention high pressure environments, would make you ideally suited to this task. Please note your timely response would be appreciated as I will be meeting with Elon to pitch the idea in 30 minutes.

Yours sincerely,
Prof. D. Eyemund

Part a - Complete degeneracy pressure for electrons
Given that, for complete degeneracy, the $n_e(p)$ distribution is rectangular for $p < p_\mathrm{F}$ such that $n_e(p) = 2/h^3$, show that the pressure scales as $P_e \propto n_e^{5/3}$.

Part b - Polytrope central pressure
By showing that part a is true, you've proven that we can use an $n = 3/2$ polytrope! It'll now be useful to know an expression for the central pressure of a polytrope star.

Use the fact that $P_c = K\rho_c^{1+1/n}$ to show that $P_c = A\frac{GM^2}{R^4}$ (and find the constant A in terms of D_n, π, n, M_n). Hint: You can find K by comparing an expression for α and an expression for M_n. Additionally note that $\rho_c = \bar{\rho}D_n$.

To start, we need to find an expression for K. We have the definition of α from the Lane-Emden equation (see Eq. 11.5).

$$\alpha = \left[\frac{(n+1)K}{4\pi G\rho_c^{1-1/n}}\right]^{1/2} \tag{1}$$

Next we can use Equation 11.10a to find another expression for α from the definition of M_n

$$M = 4\pi a^3 \rho_c M_n \tag{2}$$

$$\alpha = \left(\frac{M}{M_n 4\pi\rho_c}\right)^{1/3} \tag{3}$$

Combining these yields an expression for K

$$\left[\frac{(n+1)K}{4\pi G\rho_c^{1-1/n}}\right]^{1/2} = \left(\frac{M}{M_n 4\pi\rho_c}\right)^{1/3} \tag{4}$$

$$K = \left(\frac{M}{M_n 4\pi\rho_c}\right)^{2/3} \frac{4\pi G\rho_c^{1-1/n}}{(n+1)} \tag{5}$$

$$K = \left(\frac{M}{M_n}\right)^{2/3} \frac{(4\pi)^{1/3} G\rho_c^{1/3-1/n}}{(n+1)} \tag{6}$$

Now, given that $P_c = K\rho_c^{1+1/n}$, this means that we can write the central pressure as

$$P_c = \left(\frac{M}{M_n}\right)^{2/3} \frac{(4\pi)^{1/3} G\rho_c^{1/3-1/n}}{(n+1)} \rho_c^{1+1/n} \tag{7}$$

$$= \left(\frac{M}{M_n}\right)^{2/3} \frac{(4\pi)^{1/3} G\rho_c^{4/3}}{(n+1)} \tag{8}$$

$$= \left[\frac{(4\pi)^{1/3} G}{(n+1)M_n^{2/3}}\right] M^{2/3} \rho_c^{4/3} \tag{9}$$

We also know that the average density is given by $\bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3}$ and it is related to the central density as $\rho_c = \bar{\rho} D_n$. Therefore we can express the central pressure

$$P_c = \left[\frac{(4\pi)^{1/3} G}{(n+1)M_n^{2/3}}\right] M^{2/3} \left(\frac{M D_n}{\frac{4}{3}\pi R^3}\right)^{4/3} \tag{10}$$

$$= \left[\frac{(4\pi)^{1/3} G}{(n+1)M_n^{2/3}}\right] \left(\frac{D_n}{\frac{4}{3}\pi}\right)^{4/3} \frac{M^2}{R^4} \tag{11}$$

$$P_c = \underbrace{\left[\frac{(3D_n)^{4/3}}{4\pi(n+1)M_n^{2/3}}\right]}_{=A} \frac{GM^2}{R^4} \tag{12}$$

Part c - White dwarf* M-P$_\mathbf{c}$ *relation

Now let's take that and create a relation for the central pressure of a white dwarf as only a function of mass. Apply the M-R relation for WDs (assuming $\mu_e = 2$) and that $n = 3/2$ to show that $P_c = \frac{AG}{B^4} M^C$ where B and C are constants you should find.

The WD M-R relation with $\mu_e = 2$ gives that

$$R = 0.012\,\mathrm{R}_\odot \left(\frac{M}{\mathrm{M}_\odot} \right)^{-1/3} \tag{13}$$

This gives the final result of the central pressure as a function of WD mass

$$\boxed{P_{c,\mathrm{WD}} = \frac{AG}{B^4} M^C} \tag{14}$$

where $B = 0.012\,\mathrm{R}_\odot\,\mathrm{M}_\odot{}^{1/3}$ and $C = 10/3$.