Gravitational Lensing.

Thomas Woodland-Scott
University of Plymouth

Abstract

Gravitational lensing occurs when light rays, emanating from a distant source, pass by a massive object, gravity from this object causes the light rays to bend. This project presents the basics of gravitational lensing, deriving some key equations needed to understand the phenomenon. We make use of two main topics, optics and general relativity. We introduce lensing by a point mass, a spherically symmetric mass and a general mass distribution, calculating the deflection angle of a passing light ray for each. Finally, we will discuss multiple imaged objects including the formation of Einstein rings and arcs.

Analogy with Optics

The complete classical theory of electromagnetic fields is encoded into Maxwell's equations, providing us with an important relationship between physical wave optics and geometrical ray optics. This connection facilitates us to find an equation describing light rays [3]. Since the wavelengths of light are tiny with respect to the optical system, in our case gravitational lensing, it is appropriate to use *short-wave asymptotics*. By inserting the ansatz,

$$u_k(t,\mathbf{x}) = \phi_k(\mathbf{x})e^{ik[S(\mathbf{x})-ct]}, \tag{1}$$

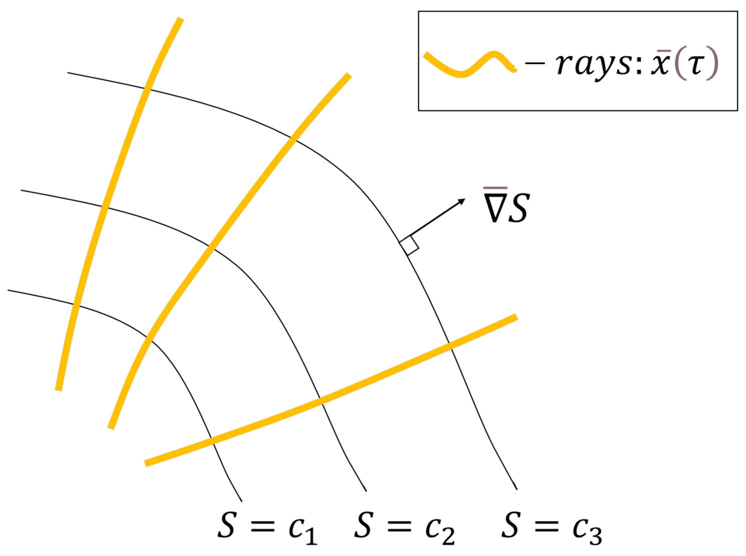
into the wave equation, we can derive, in leading order for $1/k \to 0$, the *eikonal equation* of geometrical optics:

$$(\boldsymbol{\nabla} S(\mathbf{x}))^2 = n^2(\mathbf{x}). \tag{2}$$

This is a partial differential equation for the eikonal function $S(\mathbf{x})$, where n, the *refractive index*, is a dimensionless number that describes how fast light travels through a medium.

Then, as light rays are the trajectories taken by light over time, obtained such that the curve's tangent at each point is proportional to $\boldsymbol{\nabla}(S)$, the *ray equation* is:

$$\frac{d\mathbf{x}(\tau)}{d\tau} = \boldsymbol{\nabla} S. \tag{3}$$



The simplest example of gravitational lensing is lensing due to a point mass, i.e. a massive body with zero spatial extent. A point mass has a gravitational field that acts as a medium and, as such, perturbs the light rays. We have the refractive index n as,

$$n(\mathbf{x}) = 1 - \frac{2}{c^2}\phi(\mathbf{x}), \tag{4}$$

with the *Newtonian potential*, $\phi(\mathbf{x}) = \phi(r) = -GM/r$.
The vector, $\hat{\boldsymbol{\alpha}}$, describes the change in direction from the initial unit vector (the unit vector when deflection begins) and the final unit vector (the unit vector when the light ray travels once again in a straight trajectory). Taking the modulus gives us the deflection angle,

$$|\hat{\boldsymbol{\alpha}}| \equiv \alpha = \frac{4GM}{c^2|\mathbf{x}_\perp|} \equiv \frac{4GM}{c^2 b}. \tag{5}$$

This equation is Einstein's famous result for the angle of deflection for a point mass lens.

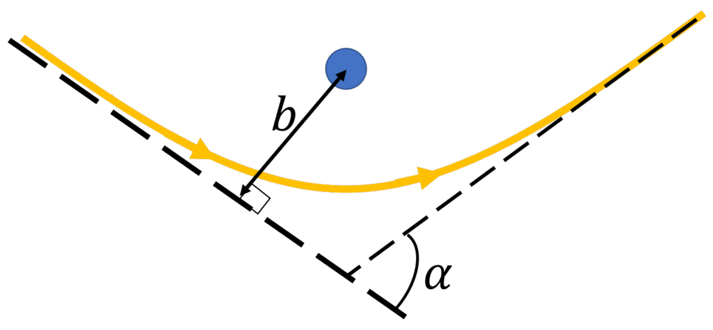


Fig. 2: A light ray being deflected by a point mass, b is the *impact parameter*.

Lensing Effects

One application of the mathematics discussed in the Optics and the General Relativity sections is to look at the images an observer would see as light is bent by gravitational fields.
Imposing conditions on the geometry of a system, the positioning of the source, lens and observer relative to each other, determines the image. A point mass or spherically symmetric source, positioned directly behind the lens, produces a ring shaped image, commonly known as an Einstein ring. Determining the radius of this ring is straight froward, assuming we know the positioning of the source, lens and observer.

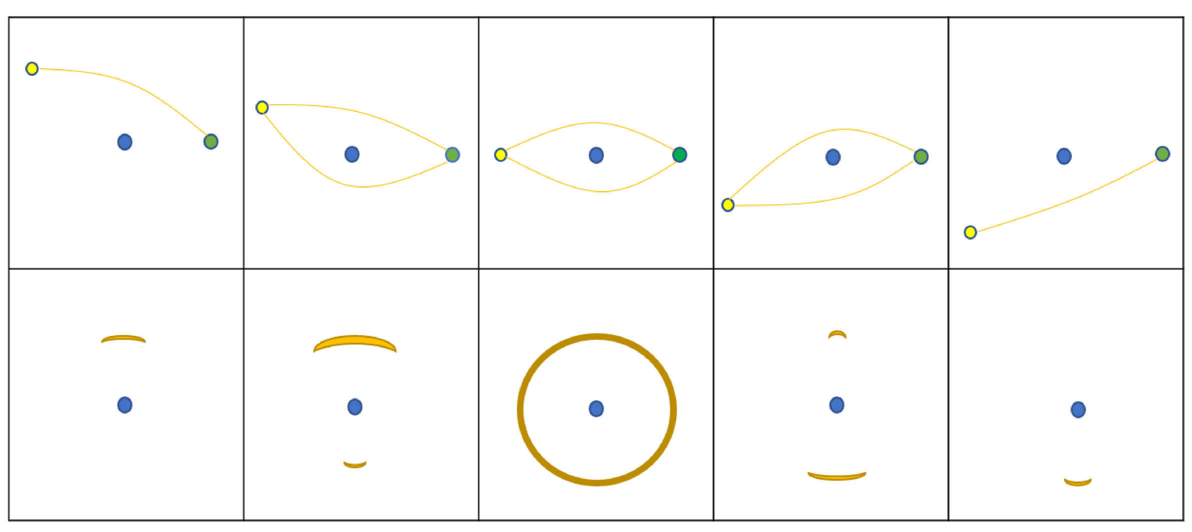


Fig. 3: The top row shows the geometry of the system and the bottom row shows the image the observer will see. The yellow, blue and green circles represent the source, lens and observer respectively.

General Relativity: Schwarzschild's Solution

In general relativity, Einstein replaced the flat Minkowski spacetime with a curved spacetime, this curvature is a result of energy and momentum produced by masses. This curvature is encoded in the metric, g, which obeys Einstein's field equations:

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu}, \tag{6}$$

$R_{\mu\nu}$ is the *Ricci tensor*, R is the *Ricci scalar*, Λ is the *cosmological constant*, $T_{\mu\nu}$ is the *Energy-Momentum tensor* and κ is a constant. The solution to the field equations that describes the gravitational field produced by a spherically symmetric, non-rotating body, was originally solved by K. Schwarzschild in 1916, hence it is known as the *Schwarzschild Solution*:

$$g = -\left(1 - \frac{r_s}{r}\right)c^2dt^2 + \frac{dr^2}{(1-\frac{r_s}{r})} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{7}$$

We are in 4-dimensional spacetime, with parameters t, r, ϕ and θ. r_s is the *Schwarzschild radius*, [2]. Using the Newtonian limit we find the integration constant,

$$r_s = \frac{2GM}{c^2}, \tag{8}$$

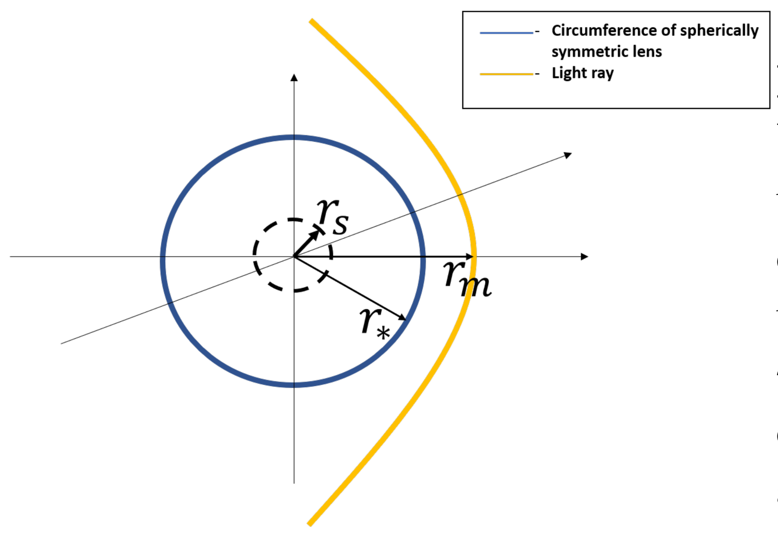
where M is the mass of the lens, G is *Newton's gravitational constant* and c is the speed of light in vacuum. Using the metric, seen in (7), we use the Lagrangian formulation for the *light-like geodesic equation*:

$$L(x,\dot{x}) = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0, \tag{9}$$

Using (9) we solve Euler-Lagrange equations of motion, resulting in the formation of an elliptic integral describing the exact deflection angle, α, of light in the Schwarzschild spacetime:

$$\alpha + \pi = 2\int_{r_m}^{\infty}\left(\left(\frac{1}{r_m^2} - \frac{r_s}{r_m^3}\right)r^4 - r^2 + r_s r\right)^{-1/2} dr, \tag{10}$$

where r_m is the minimum distance between the light ray and center of mass. This formula assumes that the Schwarzschild radius, r_s, is small relative to the physical radius from the centre of mass, r_*, (i.e. the assumption $r_s \ll r_* \leq r_m$).



Limitation: This breaks down with black holes. If a star is massive enough, the gravitational force on itself could cause it to collapse, resulting in the matter becoming so dense that the distance r_s approaches r_m, and maybe even exceeds it. This directly contradicts our assumption.

Solving the elliptical integral, seen in (10), yields the deflection angle,

$$\alpha = \frac{4GM}{c^2 r_m}. \tag{11}$$

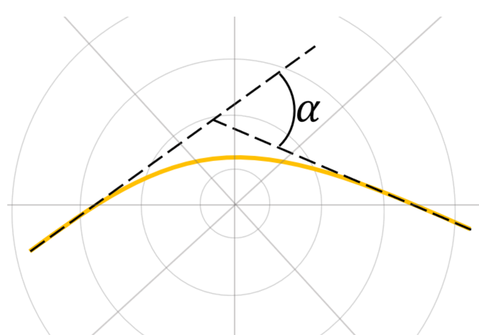


Fig. 5: deflection of a light ray in a plane of spherically symmetric spacetime

Historical Importance of Gravitational Lensing

In May 1919, an expedition headed by Sir A. Eddington was sent to the island of Príncipe, situated just off the west cost of Africa, to observe the position of stars in the Haydes cluster. At the time, this cluster was located near to the Sun during a solar eclipse. This expedition showed the image of these stars was off its expected position by an angle of 1.9 arcseconds. This was sufficiently accurate to help verify Einstein's general theory of relativity, [1] [2].

Conclusion

Gravitational lensing is a consequence of general relativity and is a useful tool that we can use to learn more about the universe we live in. This project has explored the basic mathematics, through optical short-wave asymptotics and general relativity, that form the foundation needed in order to delve deeper into the topic. We have discussed its historical relevance as well as some of its manifestations including the formation of Einstein rings.

References

[1] S. Dodelson. *Gravitational Lensing.* Cambridge University Press, 2017.
[2] V. Perlik. General relativity. *Universität Bremen*, 2012.
[3] H. Römer. *Theoretical Optics.* John Wiley & Sons, Ltd, 2004. Weinheim, Germany.