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Introduction to Quantitative Risk Management

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Abstract

As the world enters the age of Artificial Intelligence, decentralized operations emerge, and questions regarding the legitimacy of centralized institutions are raised. People possess the skills and technologies to manage financial assets independently and thus resist relying on intermediaries such as banks and exchanges. As a potential solution, decentralized finance offers decentralized applications that allow people to conduct financial activities; for example, cryptocurrencies are traded globally without reliance on any centralized authority. Therefore, it is vital to appreciate the nature and behaviour of cryptocurrencies and explore the art of financial risk management. Under the initiative, this project will analyze the trends of six popular cryptocurrencies–BinanceCoin, Cardano, NEM, Ethereum, Dogecoin, and Bitcoin–and investigate the correlation structure between selected pairs using risk estimation models and Copula.

1 Introduction

1.1 Recent Financial Crisis and Motivation

On Nov 11, 2022, FTX, one of the largest cryptocurrency exchanges, filed for bankruptcy due to false risk estimation and liquidity crisis. The collapse of this 32-billion-worth company shook the market and sent alarm bells worldwide. With the increasing popularity of cryptocurrency, the volatility of the crypto market should not be overlooked. People need to learn proper financial risk management techniques to better understand cryptocurrencies.

1.2 Overview

Through this project, the student will learn about quantitative risk management(QRM) techniques and apply the methods to analyze the behaviours of six popular cryptocurrencies: Binance, Cardano, NEM, Ethereum, Dogecoin, and Bitcoin. The student will analyze Value at Risk(VaR) and the correlation structure between BinanceCoin and Cardano, NEM and Ethereum, and Dogecoin and BitCoin. Methods of risk estimation modelling and Copula analysis will be applied to actual data obtained from the Kaggle website [1].

2 Theoretical Background

2.1 Value-at-Risk

Value-at-Risk(VaR) is a popular statistical measure that estimates the maximum loss with a given probability in a certain period, which can be one week, one year, etc.

Let L denotes the loss distribution, α be the confidence level, then VaR is defined as:

$$VaR_\alpha = inf\{l \in \mathbb{R} : P(L > l) \leq 1 - \alpha\}$$

In other words, if an asset has a 95% confident weekly VaR of value 3, then there is 1 - 95% chance that the asset declines in value by at least 3 units in one week. Hence, VaR is a helpful metric used to approximate the risk of holding an asset(s) over a time period. In essence, VaR is the quantile of the underlying loss distribution F_L. Fix $\alpha \in$ (0,1). If F_L is normal with mean μ and variance σ^2, then

$$VaR_\alpha = \mu + \sigma\Phi^{-1}(\alpha)$$

where Φ represents the standard normal distribution function and $\Phi^{-1}(\alpha)$ is the α-quantile of Φ. This is sometimes called the delta-normal approach. If F_L is a t distribution with γ degree of freedom(dof), then

$$VaR_\alpha = \mu + \sigma t_\gamma^{-1}(\alpha)$$

where t_γ is the t distribution with γ dof and $t_\gamma^{-1}(\alpha)$ is its α-quantile. Nevertheless, the mean and variance used by the measures are invariant over time. The assumption of constant mean and variance is always impractical. Time-varying factors are common in reality, and intertwined causal effects are non-negligible. It is crucial to incorporate the information given by historical data into calculating risk measures. Hence the topic of financial time series is studied in this project.

2.2 Introduction to Financial Time Series

Time series is a sequence of successive data collected at a consistent interval over a period of time. Consider a sample of daily log returns $X_1, X_2, ..., X_n$, meaning that $X_t(t \in 1, ..., n) = log(\frac{S_t}{S_{t-1}})$ where S_t is defined as the price of a stock at day t. Experts have established empirically stylized facts of financial time series that the data usually violate the assumption of stationarity and normality. These stylized facts are as follows.

1. Return series are not independently identically distributed(i.i.d) despite showing little serial correlation.

2. Series of absolute or squared returns show profound serial correlation.

3. Conditional expected returns are close to zero.

4. Volatility appears to vary over time.

5. Return series are leptokurtic or heavy-tailed.

6. Extreme returns appear in clusters.

[2]. Volatility of $(X_t)_{t\in\{1,\ldots,n\}}$ is time-variant and conditioned on past events, and its distribution is more heavy-tailed than a normal distribution. Thus, heavy tailed models with conditional parameters are studied to help capture the hidden pattern in the data.

2.3 Fundamentals of Financial Analysis

The time series $(X_t)_{t\in\mathbb{Z}}$ is a stochastic process, a family of random variables indexed by integers and defined on a probability space (Ω, F, P) [2]. The process is said to be *stationary* in one or both of the two cases:

1. Strict Stationarity: The time series $(X_t)_{t\in\mathbb{Z}}$ is strictly stationary if

$$(X_{t_1}, ..., X_{t_n}) \stackrel{\text{d}}{=} (X_{t_1+k}, ..., X_{t_n+k}), \quad \forall\ t_1, ..., t_k, k \in \mathbb{Z}, n \in \mathbb{N}.$$

2. Covariance Stationarity: The time series $(X_t)_{t\in\mathbb{Z}}$(or weakly or second-order stationary) if the first two moments exist and satisfy

$$\mu(t) = \mu, \quad \gamma(t, s) = \gamma(t + k, s + k), \quad t, s, k \in \mathbb{Z}.$$

Clearly, a strict stationary time series with finite variance is covariance stationary. It may be interesting to know that some infinite-variance time series(ex. certain ARCH and GARCH processes [2]) are strictly stationary but not covariance stationary. It can be shown that the covariance of two rvs of a covariance stationary process is determined by their temporal separation, which is referred to as *lag*. Therefore, the *autocovariance function* for a covariance stationary process can be written as

$$\gamma(h) := \gamma(h, 0), \quad \forall h \in \mathbb{Z}.$$

The ratio between $\gamma(h)$ and $\gamma(0)$ is then a principal indication of serial correlation. Formally, it is called the *autocorrelation function* $\rho(h)$ of a covariance stationary process $(X_t)_{t\in\mathbb{Z}}$, and defined as

$$\rho(h) = \rho(h, 0) = \frac{\gamma(h)}{\gamma(0)}, \quad \forall h \in \mathbb{Z}.$$

2.3.1 White Noise

A time series $(X_t)_{t\in\mathbb{Z}}$ is a **white noise process** $(\epsilon_t)_{t\in\mathbb{Z}}$ if it is **covariance stationary** satisfying

$$\rho(h) = \begin{cases} 1, & \text{if } h = 0 \\ 0, & \text{otherwise} \end{cases}$$

A white noise(WN) process with mean zero and variance σ^2 is denoted as WN(0,σ^2). Using the definition of strict stationarity, we can derive the definition of a *strict white noise* processs.

2.3.2 Strict White Noise

A time series $(X_t)_{t\in\mathbb{Z}}$ is a **strict white noise process** $(Z_t)_{t\in\mathbb{Z}}$ if it is a series of **i.i.d, finite variance** rvs. A strict white noise(SWN) process with mean zero and variance σ^2 is denoted as SWN(0, σ^2).

WN and SWN are fundamental building blocks of time series analysis. They help to construct multiple time series processes used in data modelling.

2.3.3 Martingale Difference

The time series $(X_t)_{t\in\mathbb{Z}}$ is called a **marntigale-difference sequence** w.r.t the filtration $(\mathcal{F}_t)_{t\in\mathbb{Z}}$(accural information over time) if $E|X_t| < \infty$, X_t is $\mathcal{F}_t$-measurable and

$$E(X_t|\mathcal{F}_{t-1}) = 0, \quad \forall t \in \mathbb{Z}.$$

Clearly, $E(X_t) = E(E(X_t|\mathcal{F}_{t-1})) = 0,\ \forall t \in \mathbb{Z}$. In addition, if the second moment of X_t exists for all t, then autocovariances are zero.

2.3.4 ARMA Process

Let $(\epsilon_t)_{t\in\mathbb{Z}} \sim WN(0, \sigma_\epsilon^2)$. The process $(X_t)_{t\in\mathbb{Z}}$ is a **zero-mean Auto Regressive Moving Average(p,q)**(ARMA(p,q)) process if it is a **covariance stationary process** such that

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_\mathbf{p} X_{t-\mathbf{p}} = \epsilon_t + \theta_1\epsilon_{t-1} + \cdots + \theta_\mathbf{q}\epsilon_{t-\mathbf{q}} \quad \forall t \in \mathbb{Z}.$$

Note that (X_t) is an ARMA(p,q) process with **mean** μ if $(X_t - \mu)_{t\in\mathbb{Z}}$ is a zero-mean ARMA(p,q) process. A simple rearrangement of the equation yields

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_\mathbf{p} X_{t-\mathbf{p}} + \epsilon_t + \theta_1\epsilon_{t-1} + \cdots + \theta_\mathbf{q}\epsilon_{t-\mathbf{q}} \quad \forall t \in \mathbb{Z}.$$

The innovations (ϵ_t) determine the stationarity of (X_t). For instance, if they form a strictly stationary process, then clearly the ARMA process is strictly stationary. ARMA process takes into account historical information during the modelling of the presence. Each rv in the process is regressed on preceding rvs and innovations. Therefore, the process is able to reproduce a conditional relationship between variables in chronological order. A popular ARMA process used in financial modelling is **ARMA(1,1)**

$$X_t = \phi X_{t-1} + \epsilon_t + \theta\epsilon_{t-1}, \quad \forall t \in \mathbb{Z}$$

.

2.3.5 ARMA Process and Conditional Mean

Stylized facts one, four, and six suggest that fluctuations are expected to be observed in financial time series. Patterns of data collected earlier are likely to vary as time progresses. Distributions conditioned on historical events should be used to depict patterns of real-life data. Let $(X_t)_{t\in\mathbb{Z}}$ with mean μ, then X_t can be expressed as

$$X_t = \mu_t + \epsilon_t, \quad \mu_t = \mu + \sum_{i=1}^{p}\phi_i(X_{t-i} - \mu) + \sum_{j=1}^{q}\theta_j\epsilon_{t-j}.$$

If assuming $(\epsilon_t)_{t\in\mathbb{Z}}$ is a martingale difference w.r.t $(\mathcal{F}_t)_{t\in\mathbb{Z}}$, then $E(X_t|\mathcal{F}_{t-1}) = \mu_t$, meaning that an ARMA process can approximate the population mean at a specific time conditioned on preceding events. Consequently, the ARMA process is excellent for estimating the present mean value based on historical data.

2.3.6 ARCH and GARCH Processes

Another important time-varying parameter to estimate is the variance of time series. The autoregressive conditionally heteroscedastic(ARCH) and genralized ARCH(GARCH) models can be used to estimate behaviours of conditional variance.

Let $(Z_t)_{t\in\mathbb{Z}} \sim$ SWN(0,1). The process $(X_t)_{t\in\mathbb{Z}}$ is calledd an **ARCH(p) process** if it is **strictly stationary** and satisfies

$$X_t = \sigma_t Z_t, \quad \sigma_t = \alpha_0 + \sum_{i=1}^{\mathbf{p}} \alpha_i X_{t-i}^2$$

$\forall t \in \mathbb{Z}$, for some strictly positive process $(\sigma_t)_{t\in\mathbb{Z}}, \alpha_0 > 0$ and $\alpha_i \geq 0 (i \in \{1, ..., p\})$. Note that expectation of X_t conditioned on filtration $\mathcal{F}_{t-1} = \sigma(\{X_s : s \leq t\})$ is zero:

$$E(X_t|\mathcal{F}_{t-1}) = E(\sigma_t Z_t|\mathcal{F}_{t-1}) = \sigma_t E(Z_t|\mathcal{F}_{t-1}) = \sigma E(Z_t) = 0$$

. Under the assumption that the second moment exists, then

$$var(X_t|\mathcal{F}_{t-1}) = E(\sigma_t^2 Z_t^2|\mathcal{F}_{t-1}) + 0 = \sigma_t^2 var(Z_t) = \sigma_t^2.$$

The volatility of a rv in $(X_t)_{t\in\mathbb{Z}}$ is then a function of the preceding values of the process. If the preceding values have large variance, then X_t will have a large variance. Such property is of interest to many experts in finance as it simulates stylized fact number six.

The generalized version of ARCH models are called GARCH. A GARCH(p,q)process $(X_t)_{t\in\mathbb{Z}}$ is defined as

$$X_t = \sigma_t Z_t, \quad \sigma_t = \alpha_0 + \sum_{i=1}^{\mathbf{p}} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$$

$\forall t \in \mathbb{Z}$, for some strictly positive process $(\sigma_t)_{t\in\mathbb{Z}}, \alpha_0 > 0$, $\alpha_i \geq 0 (i \in \{1, ..., p\})$, and $\beta_j \geq 0, (j = 1, ..., q)$. To approximate the current value, the GARCH model not only considers preceding events, but also incorporate previous volatility measures of the time series. Tt can be shown that for a GARCH(1,1) model,

$$\sigma_t^2 = \alpha_0 + (\alpha_1 Z_{t-1}^2 + \beta)\sigma_{t-1}^2.$$

2.3.7 Integration of ARMA and GARCH Model

ARMA models capture the pattern of mean values in a time series, while the GARCH models express the time series' volatility. Proper integration of the two families of models enables the modelling of a complex financial time series as it can measure both conditional centrality and conditional heteroscedasticity.

Let $(Z_t)_{t\in\mathbb{Z}} \sim SWN(0,1)$. The process $(X_t)_{t\in\mathbb{Z}}$ is an ARMA(p_1, q_1) process with GARCH(p_2, q_2) errors if it is covariance stationary and satisfies:

$$X_t = \mu_t + \sigma_t Z_t,$$

$$\mu_t = \mu + \sum_{i=1}^{p_1} \phi_i (X_{t-i} - \mu) + \sum_{j=1}^{q_1} \theta_j (X_{t-j} - \mu_{t-j}),$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p_2} \alpha_i (X_{t-i} - \mu_{t-i})^2 + \sum_{j=1}^{q_2} \beta_j \sigma_{t-j}^2,$$

where $\alpha_0 > 0,\ \alpha_i \geq 0 (i = 1, ..., p_2),\ \beta_j \geq 0 (j = 1, .., q_2)$, and $\sum_{i=1}^{p_2} \alpha_i + \sum_{j=1}^{q_2} \beta_j < 1$.

By combining an ARMA process with a GARCH process, the mean and variance of the process at a given time are expressed as a function of preceding values. Formally, the dependence structure can be shown by conditional expectations.

Let $(\mathcal{F}_t)_{t\in\mathbb{Z}}$ denote the natural filtration of $(X_t)_{t\in\mathbb{Z}}$. It can be shown that both σ_t and μ_t are $\mathcal{F}_{t-1}$-measurable, and that

$$\mu_t = E(X_t|\mathcal{F}_{t-1}), \quad \sigma_t = var(X_t|\mathcal{F}_{t-1}).$$

Therefore, the mean and variance of the new ARMA + GARCH process are conditioned on historical information.

2.3.8 Model Fitting

The approach to fitting ARMA + GARCH models is maximum likelihood. It can be shown that the likelihood function is for an ARMA model with GARCH errors is

$$L(\theta; \mathbf{X}) = \prod_{t=1}^{n} \frac{1}{\sigma_t} g(\frac{X_t - \mu_t}{\sigma_t})$$

where g(·) is the density function of $(Z_t)_{t\in\mathbb{Z}}$. The parameter estimates are calculated by solving

$$\frac{\delta}{\delta\theta} \ln\ L(\theta; \mathbf{X}) = \sum_{t=1}^{n} \frac{\delta l_t(\theta)}{\delta\theta} = 0$$

. The appropriate order(hyper-parameters) of GARCH and ARMA models can be found by computing the Akaike information criterion(AIC) and the Bayesian information criterion(BIC) for each model and choosing the one with the smallest values. The formula for AIC and BIC are as follows:

$$AIC = 2k - ln(\hat{L}),$$

$$BIC = kln(n) - 2ln(\hat{L})$$

where k is the number of estimated parameters, n is the number of observations, an $\hat{L}$ is the maximized value of the likelihood function.

2.4 Copula Dependence Structure Modelling

A copula is a multivariate cumulative distribution function on $[0, 1]^d$. It provides an approach to isolate and analyze the correlation structure of the factors of interest. Copulas express dependence on a quantile scale, which is useful for describing the dependence of extreme outcomes and is natural in a risk-management context [2]. Let $C(\cdot)$ be a d-dimesional copula and F_i be the marginal cumulative distribution function for the i^{th} dimension, then the copula of the d factors can be written as

$$C(u_1, ..., u_d) = \Pr\{U_1 \leq F_1^{-1}(u_1), ..., U_d \leq F_d^{-1}(u_d)\}, \quad u_i \in [0,\ 1]\ \forall i \in \{1, ..., d\}$$

2.4.1 Definition of Copulas

A d-dimensional copula is a distribution function on $[0,1]^d$ with standard uniform marginal distributions, satisfying the following properties [2]:

1. $C(u_1, ..., u_d)$ is increasing in each component u_i
2. $C(1, ..., 1, u_i, 1, ..., 1) = u_i, \quad \forall i \in \{1, ..., d\},\ u_i \in [0,\ 1]$
3. For all $(a_1, ..., a_d),\ (b_1, ..., b_d) \in [0,\ 1]^d$ with $a_i \leq b_i$,

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\cdots+i_d} C(u_{1_{i_1}}, ..., u_{d_{i_d}}) \geq 0$$

where $u_{j1} = a_j,\ u_{j2} = b_j,\ \forall j \in \{1, ..., d\}$. [2]

The first property is a requirement of a distribution function, the second property is an attribute of uniform marginal distributions, and the third uses the rectangle inequality to ensure that $P(a_1 \leq U_1 \leq b_1, ..., a_d \leq X_d \leq b_d)$ is non-negative, where $(U_1, ..., U_d)$ has a distribution function C.

2.4.2 Quantile Transformation

Let G be a distribution function an $G^{\leftarrow}$ be its quantile function. The for $U \sim U(0,1)$, $P(G^{\leftarrow}(U) \leq x) = G(x)$. Furthermore, if Y has a distribution function G, then $G(y) \sim U(0,1)$. The former technique helps to simulate a distribution function by sampling from the distribution. At the same time, combining the two enables the risks with a specific distribution function to be transformed into having a different distribution function.

2.4.3 Sklar's Theorem

The Sklar's theorem draws the connection between a multivariate distribution function, a copula, and the marginal distribution. It highlights the importance of copulas and suggests a means to study correlation structure through marginal behaviours.

Let F be a joint distribution function with margins $F_1, ..., F_d$. Then, $\exists$ a copula C such that $\forall\ x_1, ..., x_d \in \bar{\mathbb{R}}$,

$$F(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d)).$$

For each joint distribution, a copula can be constructed and can be used in conjunction with univariate distribution functions to construct multivariate distribution functions [2].

Construct $u_1, ..., u_d \ \in [0,1]$ such that $x_i = F^{\leftarrow}(u_1)\, i \in \{1, ..., d\}$, then the Sklar's theorem states that

$$C(u_1, ..., u_d) = F(F_1^{\leftarrow}(u_1), ..., F_d^{\leftarrow}(u_d)),$$

which shows explicitly how to express express C through F and its margins. It allows a correlation analysis on a quantile scale as $C(u_1, ..., u_d)$ calculates the joint probability that X_1 lies below its u_1-quantile, ..., and X_d lies below its u_d-quantile.

2.4.4 Stationarity Condition

Copulas help model the dependence structure between two random variables; however, they do not work well with heterogeneous data since it is difficult to find a competent multivariate model that describes both marginal behaviour and correlations in this case [2]. One of the methods to obtain stationary time series from non-stationary observed data is the GARCH-Copula approach. The first step of the process is to fit a GARCH model for each margin, then the residuals of the fitted models are treated as pseudo-observations. In practice, a good time series model fitting will produce residuals that behave similarly to a white noise process, which is covariance stationary if the models are correctly fitted or strictly stationary if the models are fitted perfectly. Hence, these residuals are used to fit a copula to model the dependence relationship between the two variables.The GARCH-Copula approach can be easily extended. Pseudo-observations of margins can be generated differently using, for example, ARMA+GARCH models without disrupting the residual's property of stationarity.

3 Risk Analysis of Cryptocurrencies

Since 2020, government institutions and financial organizations have undergone several adverse events: the COVID-19 recession, the halls of Congress invasion in America, and the bankruptcy of FTX, one of the largest cryptocurrency companies. Implementing a risk metric that measures the likelihood of hazards is vital for businesses, corporations, and social security. Moreover, a financial crisis is always highly correlated with other issues, such as the COVID-19 outburst. It is crucial to analyze the correlation between different risk factors and implement effective defensive measures accordingly. In this project, the student studied the risk and correlation structure of six cryptocurrencies using statistical measures taught in [2]. Further details about the study are discussed in the following sections.

3.1 Data

The dataset used in this project is a collection of cryptocurrency historical prices downloaded from Kaggle, an online platform that provides authoritative datasets from different fields. The historical prices of each cryptocurrency are structured as seven variables:

1. **Date**: date of observation(from 2013-2021)
2. **Open**: the opening price on the given day
3. **High**: the highest price on the given day
4. **Low**: the lowest price on the given day
5. **Close**: the closing price on the given day
6. **Volume**: the volume of transactions on the given day
7. **Market Cap**: market capitalization in USD.

Only the **Date, Open,** and **High** values are considered for this project.

3.2 Evaluation

3.2.1 Data Preprocessing

To perform a thorough risk analysis of cryptocurrencies, the student first calculated the weekly log returns for each cryptocurrency according to the following formula:

$$log(\ Close_{\text{ end of the week}}\ /\ Open_{\text{ beginning of the week}}\ ).$$

Then, three pairs of cryptocurrencies with the same time interval are selected: BinanceCoin vs Cardano, NEM vs Ethereum, and Bitcoin and Dogecoin. Specifically, BinanceCoin and Cardano weekly log returns begin on 10-08-2017; NEM and Ethereum weekly log returns begin on 08-14-2015; Bitcoin and Dogecoin weekly log returns start on 12-22-2013. All three pairs of weekly log returns end on 07-06-2021.

3.2.2 Weekly Log Returns

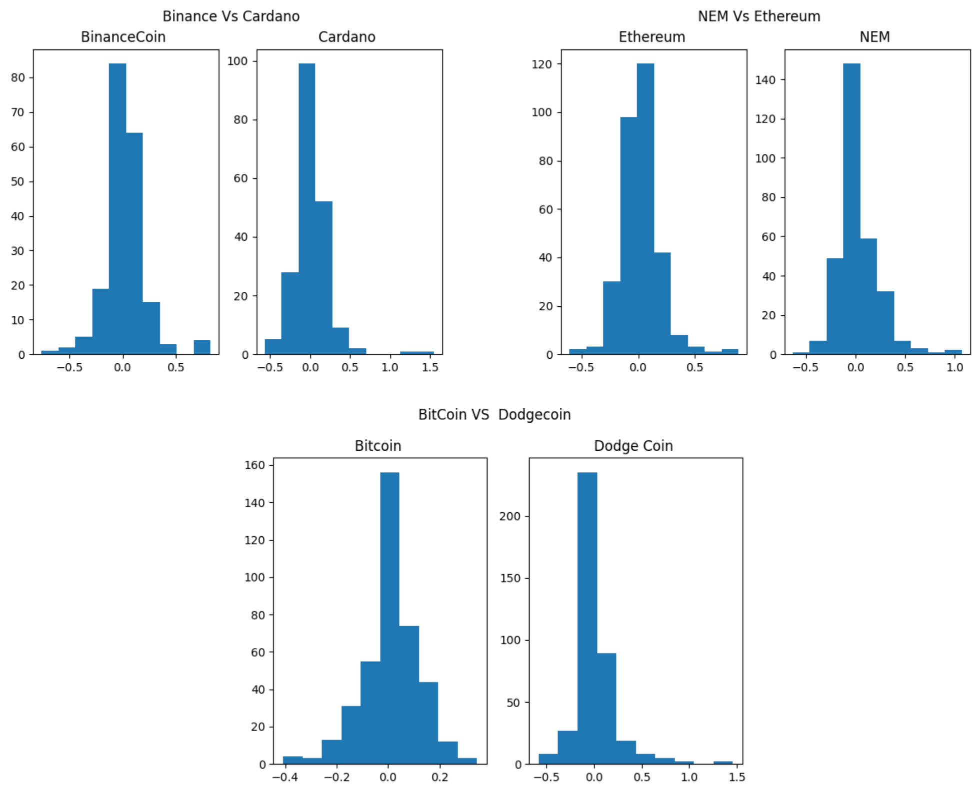

Figure 1: Histogram of Weekly Log Returns of Selected Cryptocurrencies

Fig 1 illustrates the attributes of the underlying distribution of weekly returns. It reveals the skewness and kurtosis of the data, which can then be interpreted probabilistically. As expected, the patterns shown in figure 1 accord with stylized fact five since all six cryptocurrency return series are leptokurtic. According to the histogram, weekly returns of both

BinanceCoin and Cardano assume a heavy tail distribution, but the latter is more likely to have a considerable positive return since it is more skewed to the right. Weekly returns of Ethereum and NEM behave similarly to each other–both have a heavy tail distribution, but NEM is more right-skewed and, hence, is more likely to have a large positive return. Bitcoin is less likely to have a large absolute return than Dogecoin as Dogecoin is skewed to the right and has a heavier tail.

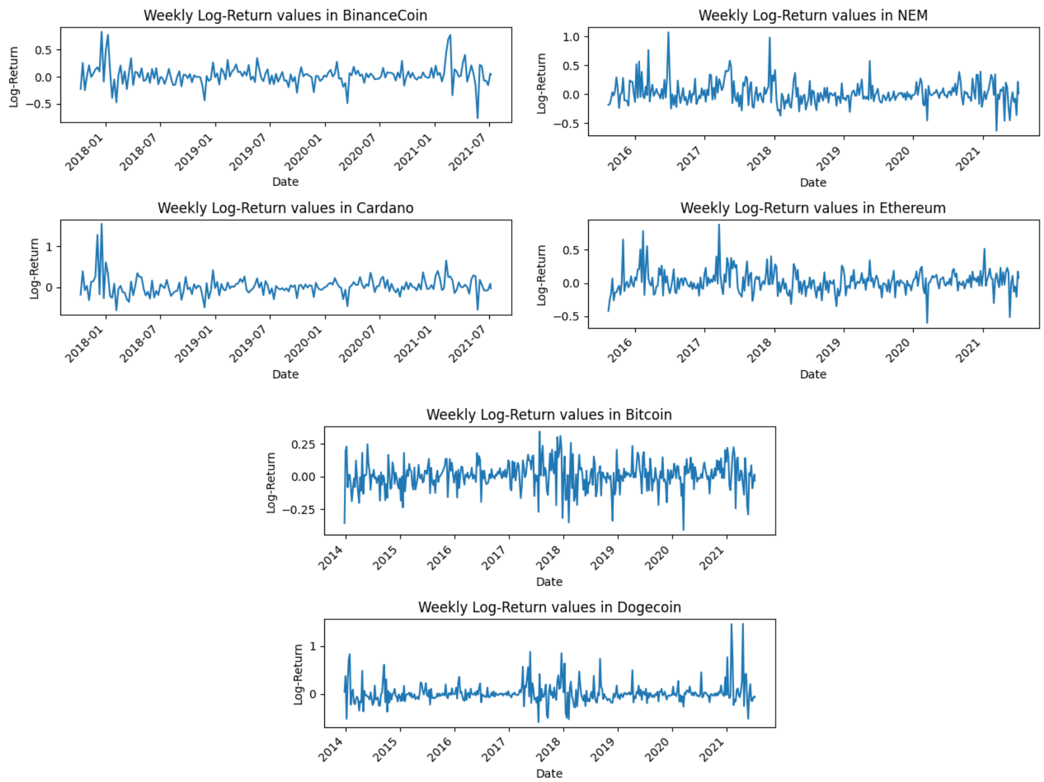

Figure 2: Line Graphs of Weekly Log Returns of Selected Cryptocurrencies over Time

Fig 2 portrays the evolution of the weekly log returns of each cryptocurrency. All six line graphs in the figure show signs of heteroscedasticity and non-stationarity. The fluctuation in weekly log returns varies as a function of time, and clusters of extreme returns can be observed. Therefore, the variance and mean of the returns of six cryptocurrencies are not constant and should be calculated using conditional measurements. In addition, the line graphs suggest that cryptocurrency behaves similarly to traditional currency. The patterns of the cryptocurrency weekly log return series agree with stylized facts one, four, and six. Although cryptocurrency is a decentralized financial currency, it has a strong relationship with traditional centralized currency.

3.2.3 ARMA + GARCH Model Fitting

ARMA(p_1, q_1) + GARCH(p_2, q_2) models approximate the conditional mean and variance of a time series. An optimal order of the model can be found by choosing the model with

Cryptocurrecy	p_1	q_1	p_2	q_2
Binance	3	4	2	2
Cardano	3	4	2	2
NEM	3	4	2	2
Ethereum	3	3	4	4
Bitcoin	10	10	9	10
Dogecoin	5	5	4	4

Table 1: Fitted ARMA+GARCH Models

the lowest AIC and BIC values. Table 1 summarizes the ARMA(p_1, q_1) + GARCH(p_2, q_2) selected to fit the data. The model fitting procedure is completed using the *garchFit* function from the *fGarch* package.

3.2.4 VaR Analysis

Recall that VaR is a statistical measure that quantifies the risk of losses of financial assets. It provides a probabilistic view of hazardous events that may incur extreme losses. In this study, VaRs are computed according to:

$$VaR_\alpha = \mu_t + \sigma_t t_{n-2}^{-1}(\alpha)$$

where $\alpha = 0.05$, n is number of observations, μ_t and σ_t denotes the mean and variance measured at time t using ARMA + GARCH models. In this study, an α level of 0.05 is selected because the left tail of the log return distributions is unfavourable as it represents drops in the value of a cryptocurrency. Therefore, a lower VaR means a higher risk. The t distribution is chosen so that the model can account for the thickness of the tails.

Plots of VaR as a function of time are given in figure 3. The grey dots are the weekly log returns, the red lines represent the values of VaR computed by ARMA+GARCH models, and the dark green lines indicate the VaR calculated through the delta-normal approach. A comprehensive analysis of the features of VaR plots for each pair of cryptocurrencies is provided as follows:

1. **BinanceCoin vs Cardano**: The VaR of Cardano is more volatile than the VaR of BinanceCoin since the former has more fluctuations that are rapid and vigorous. In general, Cardano has a lower VaR and thus has a greater chance of devaluation.

2. **NEM vs Ethereum**: The VaR of NEM is more volatile than the VaR of Ethereum, and several extreme negative-return are noticeable in the plot. The former fluctuates more aggressively and tends to have a lower VaR. In other words, NEM has higher volatility and risk than Ethereum.

3. **Bitcoin vs Dogecoin**: The VaR of Dogecoin exhibits higher variability as it shows more fluctuations with longer peaks. In comparison, the VaR of Dogecoin is smoother but fluctuates more rapidly. Furthermore, it can be seen that Dogecoin has more VaR with lower values. In conclusion, although Dogecoin is more volatile than Bitcoin, Bitcoin is riskier because it has a higher chance of devaluation.

VaR is a good indicator of underlying risks and volatility. From the behaviour of VaR, one can deduce whether the subject of interest is an asset of low returns. Usually, high

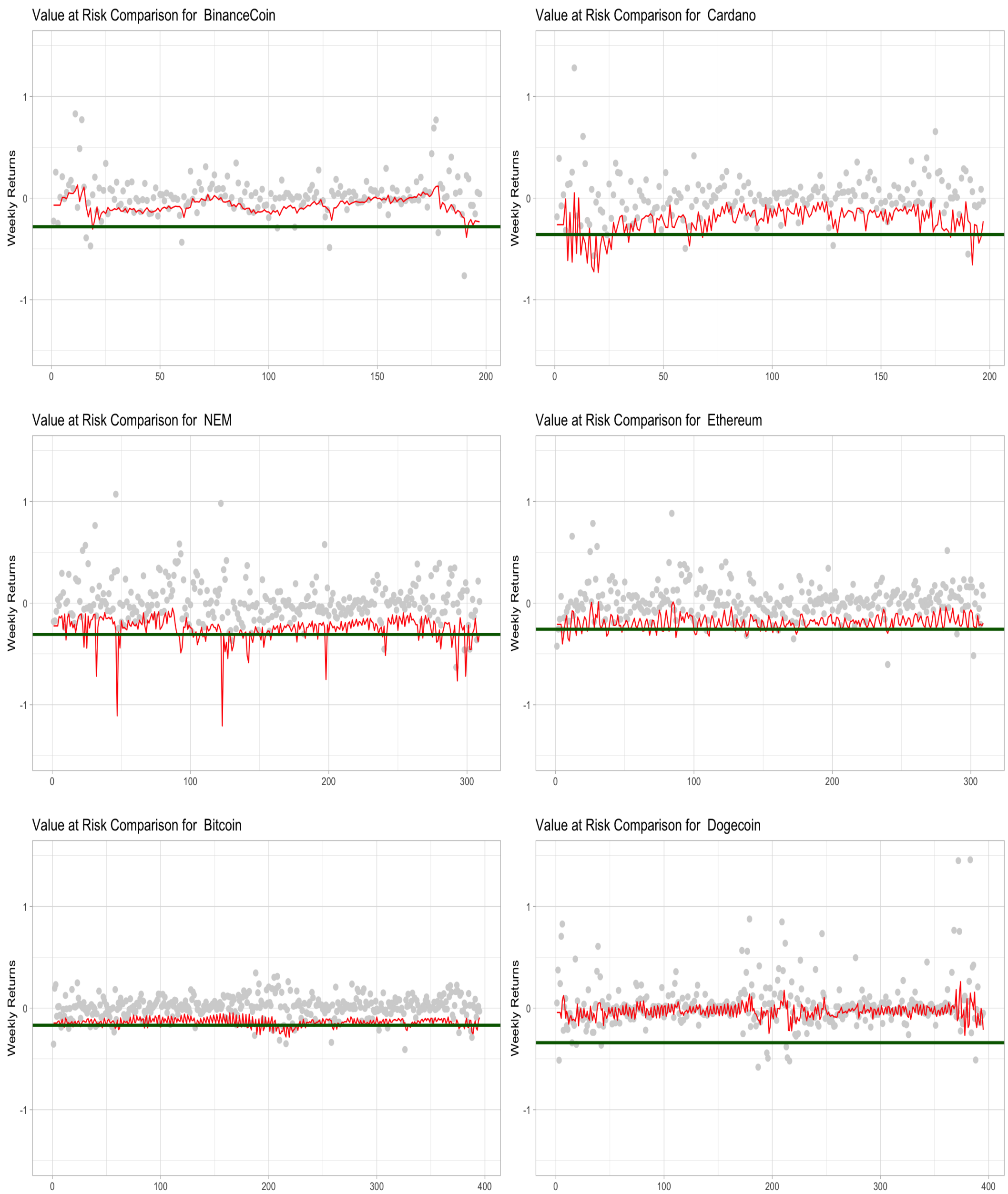

Figure 3: VaR(Red Lines) as a Function of Time

risk correlates with high volatility and low returns, and vice versa. For instance, while Dogecoin has a lower VaR than Bitcoin most of the time, it also has more VaR with large positive returns. Nevertheless, high VaR may insinuate signs of potentially high returns. For example, while the VaR line suggests that NEM is a higher-risk cryptocurrency, it reaches large positive returns more frequently than Ethereum. VaR should be interpreted comprehensively and carefully. The same pattern of VaR may reveal different phenomena under different contexts.

3.2.5 Correlation Structure

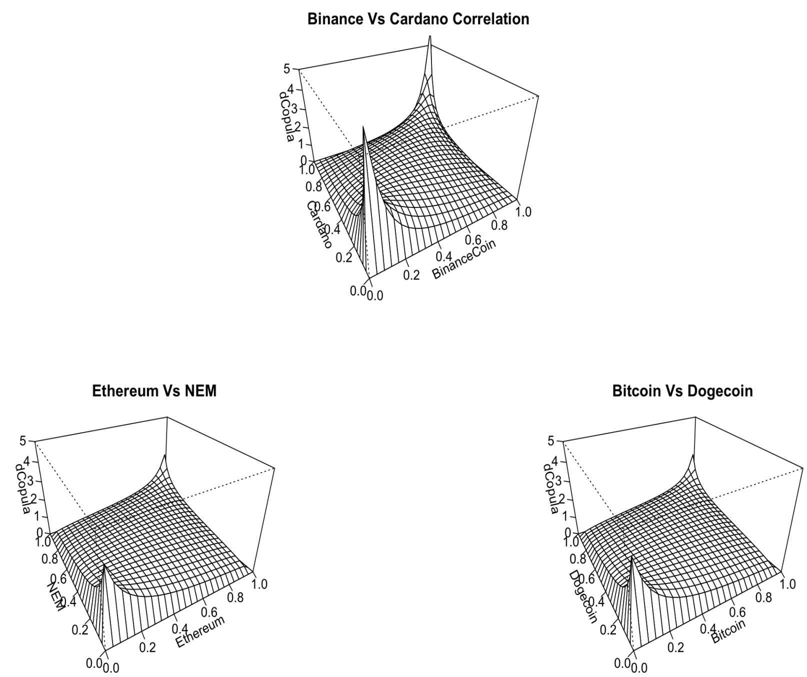

Figure 4: Correlation Structure of Cryptocurrencies

A copula interprets the correlation structure in the data by computing the probability of d covariates being smaller than d quantiles. The optimal copula model for each cryptocurrency pair is found by choosing the model with the least AIC and BIC values using the *BiCopSelect()* function from the *VineCopula* package in R. Table 4 shows an overview of the copula model used in this study.

Figure 4 consists of three 3D plots of the copula probabilistic correlation measure(z-axis) against pairs of cryptocurrencies(x-axis & y-axis). The grid's height and thickness reflect the correlation strength and pattern. In this study, the correlation of cryptocurrencies is evaluated as follows:

1. **BinanceCoin vs Cardano**:Two high peaks can be observed in the bottom-to-top diagonal line of the graph, and the rest of the area is relatively flat, suggesting a high

Cryptocurrency Pair	Copula Family	θ	δ
Binance vs Cardano	Survival BB7	1.5864821	0.8189832
Ethereum vs NEM	BB1	0.3338055	1.1697278
Bitcoin vs Dogecoin	Survival BB7	1.2007029	0.5064552

Table 2: Summary of Copula Fitted to each of the Three Cryptocurrency Pairs

probability that: the extreme financial gain of the two cryptocurrencies will happen simultaneously and the severe loss will occur at the same time.

2. **Ethereum vs NEM**: The two peaks in this graph are smaller, and most of the region is flat. One can deduce that there is a smaller but non-negligible chance of detecting two simultaneous extreme financial gains or devaluations.

3. **Bitcoin vs Dogecoiin**: The copula correlation structure between Bitcoin and Dogecoin is similar to that between Ethereum and NEM; only the heights of the two peaks of the former are slightly more significant. Therefore, the graph implies a non-trivial probability of simultaneous appreciation or declines in cryptocurrency prices.

Cryptocurrency Pair	Spearman	Kendall	Pearson
Binance vs Cardano	0.573505	0.4107772	0.6200122
Ethereum vs NEM	0.3937281	0.2740141	0.3404415
Bitcoin vs Dogecoin	0.3620769	0.2532026	0.3442538

Table 3: Summary of Correlation Measurements of the Three Cryptocurrency Pairs

Furthermore, from table 3, one can see that the Binance and Cardano have the highest absolute correlation measurements and hence are the most correlated. The Ethereum and NEM pair have similar correlation measures to the Bitcoin and Dogecoin pair. The three pairs of cryptocurrencies are all positively correlated.

3.3 Discussion

Using time series models and copulas, the student analyzed the risk and correlation of six popular cryptocurrencies. It has been shown that Cardano is riskier and more volatile than BinanceCoin, NEM is riskier and more volatile than Ethereum, and Bitcoin is riskier but less volatile than Dogecoin. The result also suggests that Cardano and BinanceCoin are the most correlated cryptocurrency pair. On the other hand, it is interesting to notice the similarity between cryptocurrency and traditional currency. The stylized facts established by finance experts can be a reference for people working in decentralized finance.

Although this study focuses on historical prices, the principles and methodologies can be extended to forecast future return series and risks. Moreover, statistical analysis tools such as the lack of fit test and leverage/influential point identification can be performed to improve the robustness and accuracy of the fitted models. Altogether, it is necessary to realize that the quantitative risk management skills discussed in this project can be applied to other data-science-related fields such as environmental science and social science.

4 Conclusion

The project explores the principles and techniques of quantitative risk management. The textbook used by the student is Quantitative Risk Management - Concepts, Techniques, and Tools [2]. The book introduces basic notions of risks and describes multiple analysis methodologies, including Time Series Analysis and Copula Correlation Analysis. The student conducted a quantitative risk analysis on historical prices of six cryptocurrencies: BinanceCoin, Cardano, NEM, Ethereum, Dogecoin, and Bitcoin. Using VaR, GARCH + ARMA models, and Copulas, the student was able to analyze the underlying risk and correlation of the cryptocurrencies. More importantly, the approaches used in the project can be easily generalized to many other fields of study. Through the project, the student has gained more understanding of Finance and Statistics, learned new programming tools, and familiarized himself with proper data science techniques.

References

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