Series Resonance

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Definition of Resonance

Resonance is defined as the condition in a circuit, containing at least one capacitor and one inductor when the supply voltage and supply current are in phase.

AC circuits made up of resistors, inductors and capacitors are said to be **resonant (Tuned) circuit** when the current drawn from the supply is in phase with the impressed sinusoidal voltage.

The voltage and current will be in phase if **the net imaginary part of impedance or admittance will be zero**.

Resonance circuit behaves as a **purely resistive circuit**.

For a resonance circuit the **power factor will be unity**.

Resonance circuit are two types:

(**i**) Series resonance or simply resonance, and

(**ii**) Parallel resonance or anti resonance

In a series resonance circuit, the voltage and current will be in phase if **the net imaginary part of impedance will be zero**.

In a parallel resonance circuit, the voltage and current will be in phase if **the net imaginary part of admittance will be zero**.

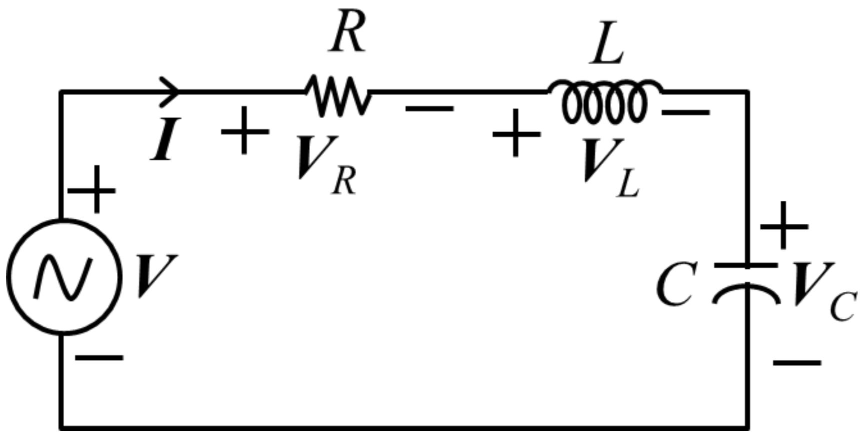
Series of Resonance

$$X_L = \omega L = 2\pi f L \qquad X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C} \qquad \boldsymbol{Z}_S = R + jX_L - jX_C = R + j(X_L - X_C)$$

$$\boldsymbol{I} = \frac{\boldsymbol{V}}{\boldsymbol{Z}_S} = \frac{\boldsymbol{V}}{R + j(X_L - X_C)} \qquad \boldsymbol{V}_R = R\boldsymbol{I} \qquad \boldsymbol{V}_L = jX_L\boldsymbol{I} \qquad \boldsymbol{V}_C = -jX_C\boldsymbol{I}$$

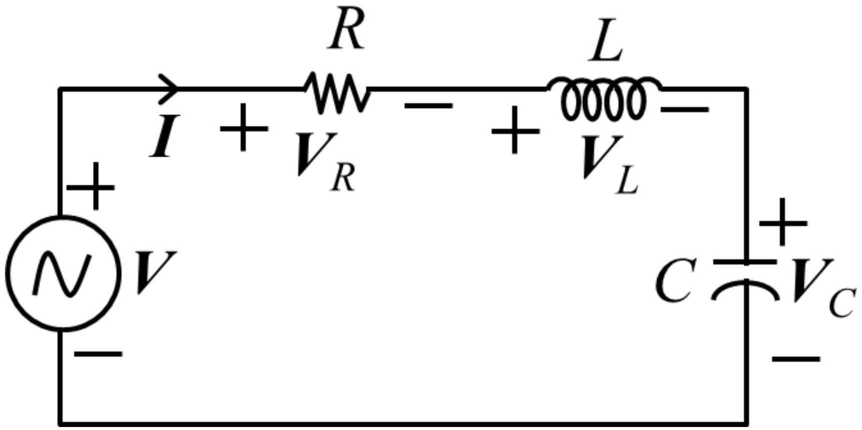
$$Z_S = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega L}\right)^2}$$

$$I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega L}\right)^2}}$$



$$V_R = IR = \frac{RV}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{RV}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega L}\right)^2}}$$

$$V_L = IX_L = \frac{X_L V}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{\omega L V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega L}\right)^2}}$$



$$V_C = IX_C = \frac{X_C V}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{\frac{1}{\omega C} V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega L}\right)^2}}$$

Resonance Condition

Resonance is occurred when the net reactance of impedance is zero. That means: $X_L - X_C = 0$

Condition for series resonance: $\boxed{X_L = X_C}$

At resonance condition the inductive reactance equals to capacitive reactance.

$\boldsymbol{Z_{sr}} = R$ $\quad \boldsymbol{I} = \frac{\boldsymbol{V}}{\boldsymbol{Z_{sr}}} = \frac{\boldsymbol{V}}{R}$ $\quad \boxed{Z_{sr} = R \ [\textbf{Minimum}]}$ $\quad \boxed{I = \frac{V}{R} \ [\textbf{Maximum}]}$

$\boxed{V_R = RI = V}$ $\quad V_L = X_L I = \frac{X_L V}{R}$ $\quad V_C = X_C I = \frac{X_C V}{R}$

Since $X_L = X_C$: $\boxed{V_L = V_C}$

At series resonance frequency the magnitude of voltage drops across the inductor and capacitor are equal but 180º out of phase.

Resonance Frequency

Let, f_{sr} is the resonance frequency and ω_{sr} is the angular frequency at resonance condition. Thus at resonance condition

$$\omega_{sr} = 2\pi f_{sr} \qquad f_{sr} = \frac{1}{2\pi}\omega_{sr} \qquad X_L = \omega_{sr} L = 2\pi f_{sr} L \qquad X_C = \frac{1}{\omega_{sr} C} = \frac{1}{2\pi f_{sr} C}$$

Since $X_L = X_C$:

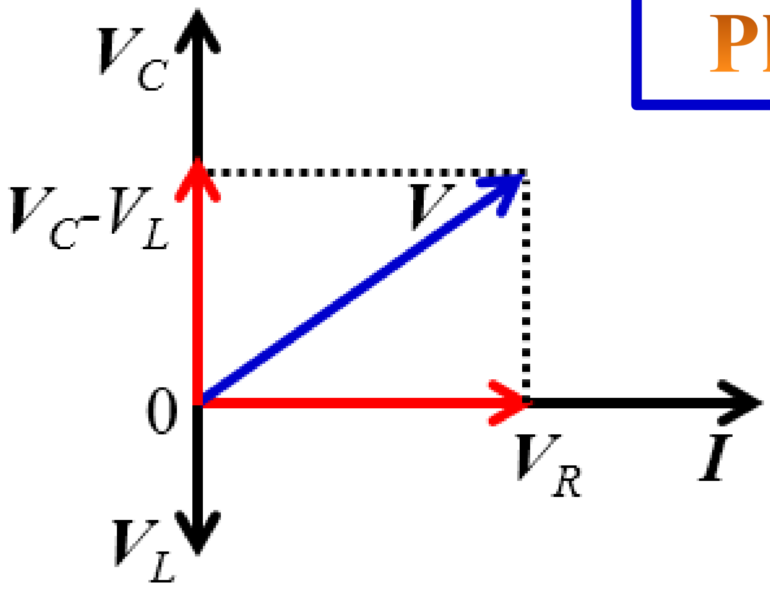
$$\omega_{sr} L = \frac{1}{\omega_{sr} C}$$

$$\omega_{sr}^2 = \frac{1}{LC}$$

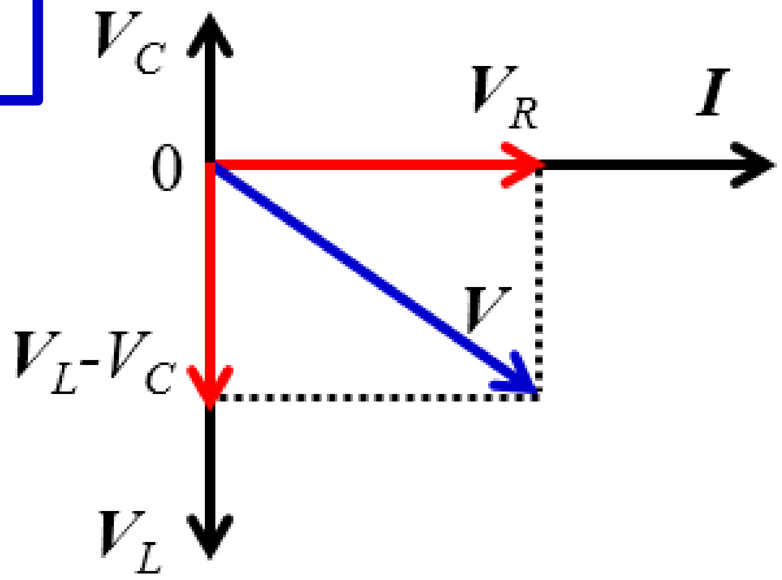
$$\boxed{\omega_{sr} = \frac{1}{\sqrt{LC}} \qquad (1)}$$

$$\boxed{f_{sr} = \frac{1}{2\pi\sqrt{LC}} \qquad (2)}$$

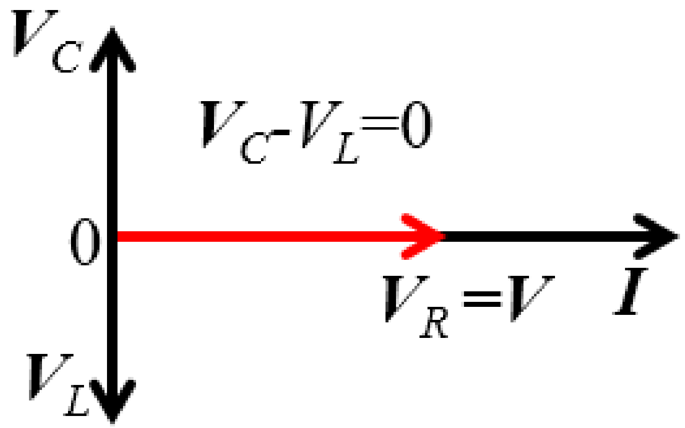
Phasor Diagram



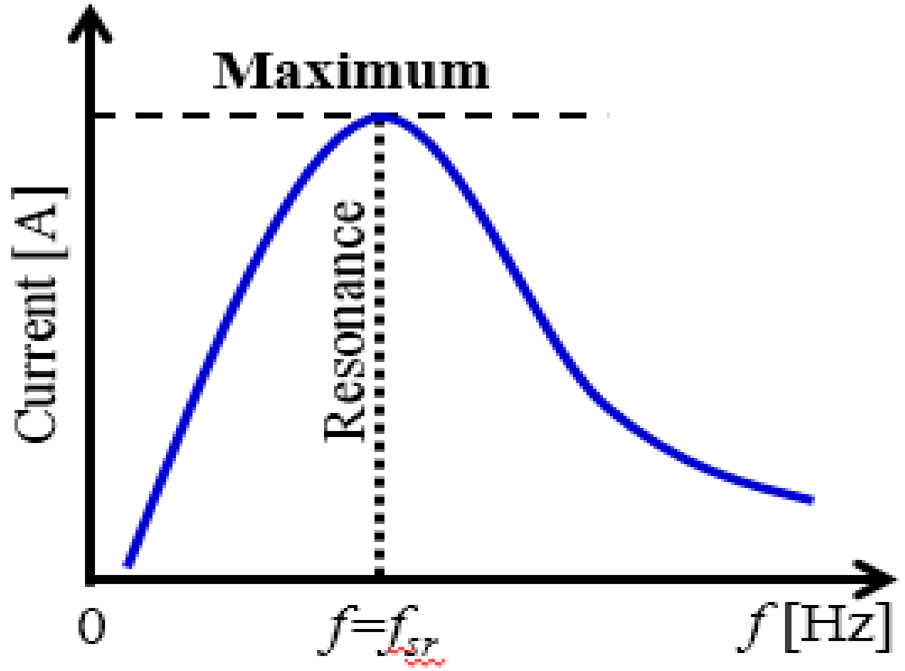
For $f < f_{sr}$; $(1/\omega C) > \omega L$; $\therefore$ $V_C > V_L$

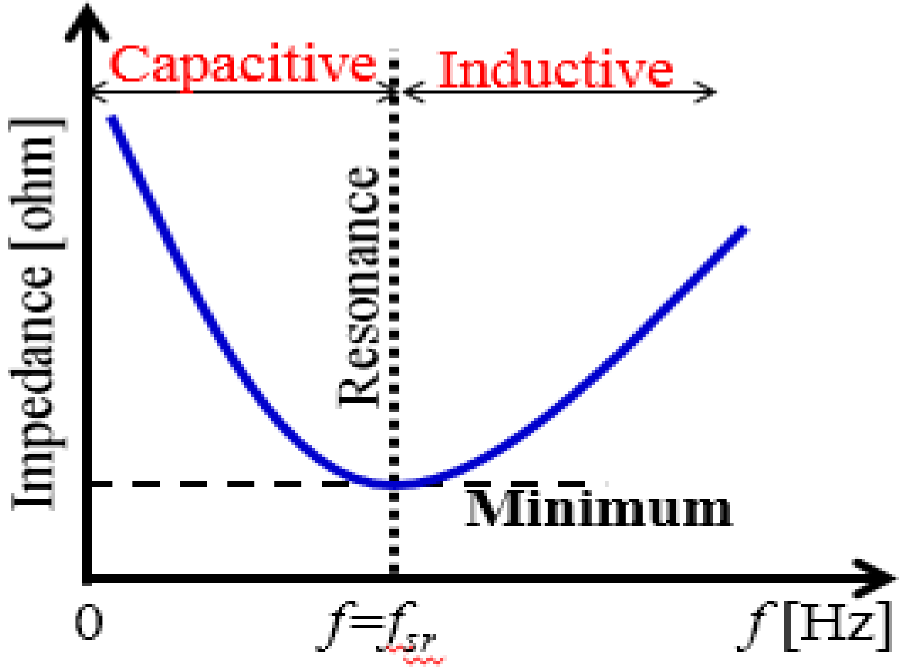


For $f > f_{sr}$; $\omega L > (1/\omega C)$; $\therefore$ $V_L > V_C$

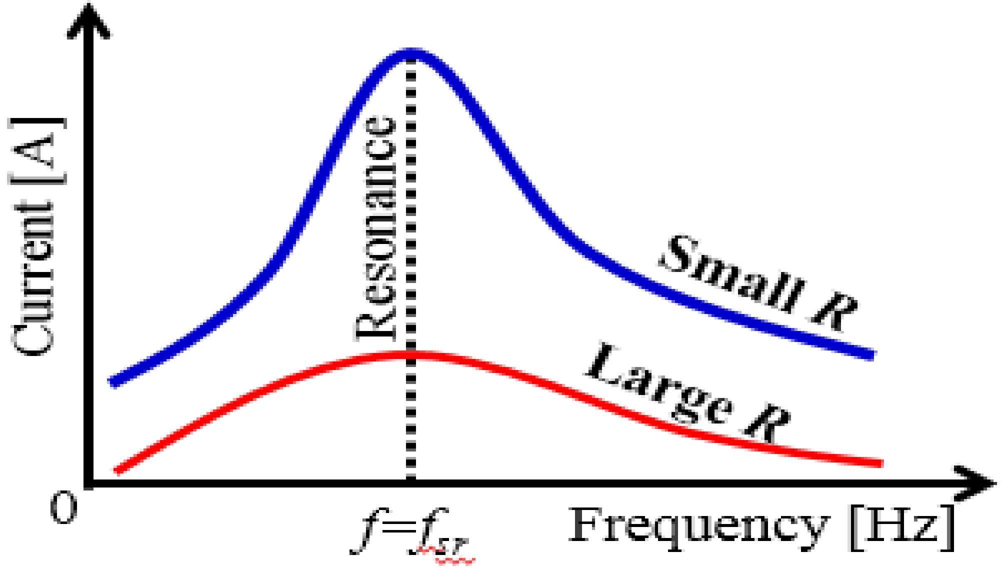


For $f = f_{sr}$; $\omega L = (1/\omega C)$; $\therefore$ $V_L = V_C$





Effect of resistance on current variation in the range of series resonance.



Variation of current and impedance with frequency in the range series resistance.

At $\omega < \omega_{sr}$, $X_C<X_L$ that means **the circuit under resonance frequency is capacitive**.

At $\omega > \omega_{sr}$, $X_L>X_C$ that means **the circuit above resonance frequency is inductive**.

At $\omega = \omega_{sr}$, $X_L=X_C$ that means **the circuit at resonance frequency is purely resistive**.

It is seen from Eq. (2) that the resonance can be achieved by:

1. **Varying the inductance *L*.**
2. **Varying the capacitance *C*.**
3. **Varying the frequency ω or *f*.**

Example

A coil of 5 mH inductance and 10 ohm resistance is connected in series with 5 μF capacitor. Determine frequency at which circuit resonates.

Solution: Given: R =10 ohm, L = 5 mH = 5×10^{-3} H, C = 5 μF = 5×10^{-6} F

The resonant frequency is given by: $$f_{sr} = \frac{1}{2\pi\sqrt{5\times10^{-3}\times5\times10^{-6}}} = 1006.58 \text{ Hz} = 1.0065 \text{ kHz}$$

Example

In a series *RLC* circuit with a sinusoidal a.c. voltage source, determined value of C required to achieve reactance in circuit at 5 kHz if value of resistance and inductance are 2 ohm and 1 mH respectively.

Solution: Given: R =2 ohm, L = 1 mH = 1×10^{-3} H, f_{sr} = 5 kHz = 5×10^{3} Hz

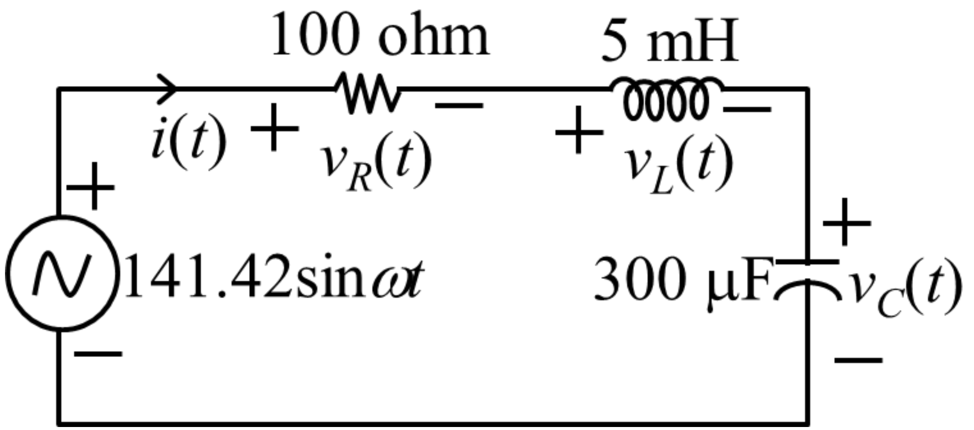
The resonant frequency is given by: $$f_{sr} = \frac{1}{2\pi\sqrt{LC}}$$

$$\therefore\ C = \frac{1}{(2\pi f_{sr})^2 L} = \frac{1}{\left(2\pi\times5\times10^{3}\right)\times1\times10^{-3}} = 1.0132\times10^{-6}\text{ F} = 1.0132\ \mu\text{F}$$

Example

For the series *RLC* combination as shown in the following figure, find the angular frequency to obtain the minimum impedance.

Solution: Given: R =100 ohm, L = 5 mH = 5×10^{-3} H, C = 300 μF = 300×10^{-6} F



The angular frequency at resonant condition is given by:

$$\boxed{\omega_{sr}=\frac{1}{\sqrt{5\times10^{-3}\times300\times10^{-6}}}=816.5\text{ rad/s}}$$

The resonance frequency is obtained by: $f_{sr}=\frac{1}{2\pi}\omega_{sr}==\frac{1}{2\pi}\times816.5=230\ \text{Hz}$

At resonance condition:

$$X_L=\omega_{sr}L=816.5\times5\times10^{-3}=4.08\ \Omega \qquad X_C=\frac{1}{\omega_{sr}C}=\frac{1}{816.5\times300\times10^{-6}}=4.08\ \Omega$$

$$\boldsymbol{Z}_{sr}=R=100\Omega \qquad \boldsymbol{I}=\frac{\boldsymbol{V}}{\boldsymbol{Z}_{sr}}=\frac{\boldsymbol{V}}{R}=\frac{141.42/\sqrt{2}}{100}=1\text{ A}$$

Example

For the series *RLC* combination as shown in the following figure, find the frequency, the power, the power factor, and the voltage drop across each part of the circuit at resonance.

Solution: Given: R =1 ohm, L = 0.1 H, C = 100 μF = 100×10^{-6} F

The resonance frequency is given by:

$$f_{sr} = \frac{1}{2\pi\sqrt{0.1\times100\times10^{-6}}} = 50.4 \text{ Hz}$$

$$I = \frac{V}{Z_{sr}} = \frac{V}{R} = \frac{100}{1} = 100 \text{ A}$$

$$P = VI = 100\times100 = 10 \text{ kW}$$

$$pf = \frac{P}{VI} = 1$$

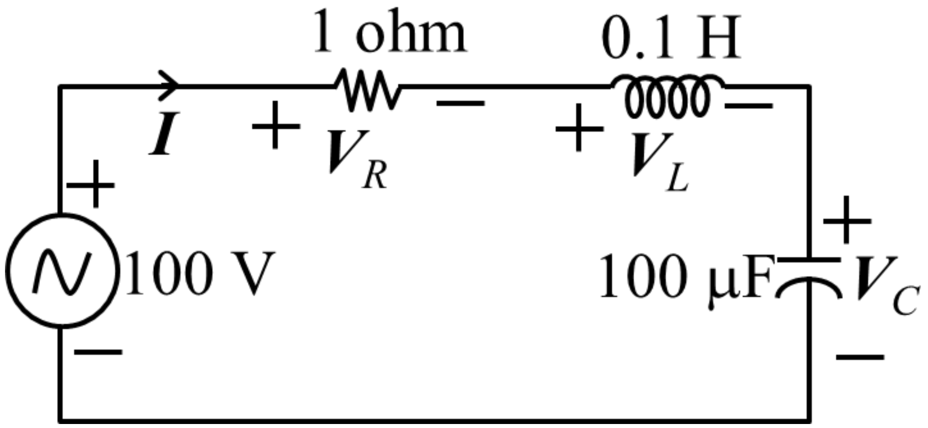
$$X_L = 2\pi\times50.4\times0.1 = 31.6 \text{ ohms}$$

$$X_C = \frac{1}{2\pi\times50.4\times0.0001} = 31.6 \text{ ohms}$$

$$V_R = IR = 100\times1 = 100 \text{ A}$$

$$V_L = IX_L = 100\times31.6 = 3160 \text{ V}$$

$$V_C = IX_C = 100\times31.6 = 3160 \text{ V}$$



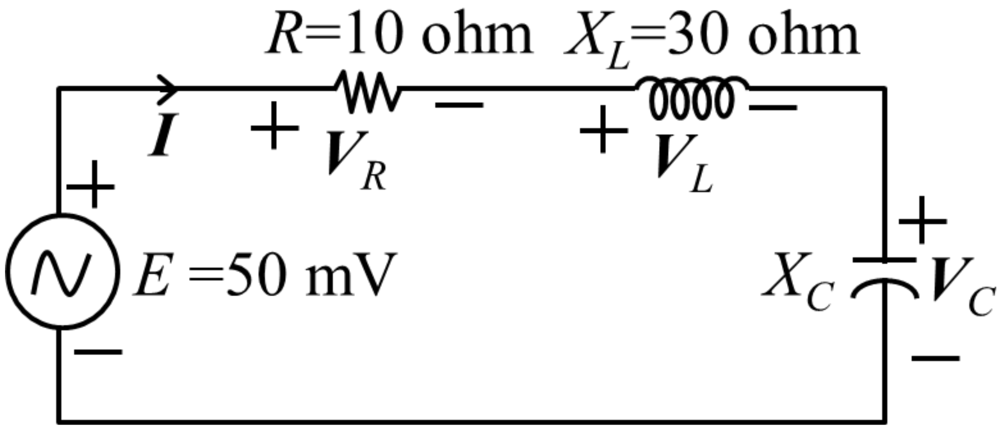
Example

For the following series *RLC* circuit at resonance condition: (**a**) find (i) the value of capacitive reactance (X_C), (ii) the total impedance, (iii) the rms value of current (I), voltage drop across the resistance (V_R), voltage drop across the inductance (V_L) and voltage drop across the capacitance (V_C), and (iv) the power dissipated by the circuit. (**b**) If the resonance frequency is 5 kHz, calculate the value of L and C.

Solution: Given: R =10 ohm, X_L = 30 ohm, E = 50 mV.

(a) At resonance condition **(i)** $X_C = X_L = 30$ ohm

(ii) $Z_{sr} = R = 10$ ohm



(iii) $I = \dfrac{E}{R} = \dfrac{50\text{ mV}}{10\ \Omega} = 5\,\text{mA}$

$$V_R = IR = 5\,\text{mA} \times 10\,\Omega = 50\text{ mV}$$

$$V_C = IX_C = 5\,\text{mA} \times 30\,\Omega = 150\text{ mV}$$

$$V_L = IX_L = 5\,\text{mA} \times 30\,\Omega = 150\text{ mV}$$

(iv) $P = I^2 R = (5\,\text{mA})^2 \times 10\,\Omega = 0.25\,\text{mW}$

(b) At 5 kHz:

$$X_L = 2\pi f_{sr} L$$

$$\therefore \quad L = \frac{X_L}{2\pi f_{sr}} = \frac{30}{2\pi \times 5 \times 10^3} = 9.55 \times 10^{-4} \text{ H} = 0.955 \text{ mH}$$

$$X_C = \frac{1}{2\pi f_{sr} C}$$

$$\therefore C = \frac{1}{2\pi \times 5 \times 10^3 \times 30} = 1.06 \times 10^{-6} \text{ F} = 1.06 \ \mu\text{F}$$

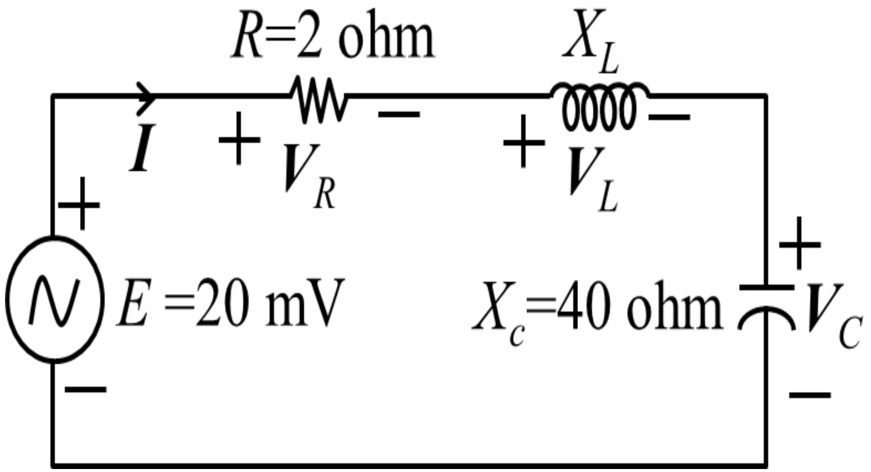
Home Work 1.1

Problem 1: Find the resonance angular frequency (ω_{sr}) and resonance frequency (f_{sr}) for the following parameters: (**i**) R = 10 ohm, L =1 H, C = 16 μF; (**ii**) R = 300 ohm, L =0.5 H, C = 0.16 μF; (**iii**) R = 20 ohm, L = 0.28 mH, C = 7.46 μF.

Problem 2: In a series *RLC* circuit with a sinusoidal a.c. voltage source, determined value of inductance (L) required to achieve reactance in circuit at 1.8 kHz if value of resistance (R) and capacitance (C) are 4.7 ohm and 2 μF respectively.

Problem 3: A series *RLC* circuit is connected to 230 V ac supply. The current drawn the circuit at the resonant is 25 A. The voltage drop across the capacitor is 4000 V at the series resonance. Calculate the resistance, inductance if capacitance is C = 5 μF. Also calculate the resonant frequency.

Problem 4: For the following series *RLC* circuit at resonance condition: (**a**) find (i) the value of inductive reactance (X_L), (ii) the total impedance, (iii) the rms value of current (I), voltage drop across the resistance (V_R), voltage drop across the inductance (V_L) and voltage drop across the capacitance (V_C), and (iv) the power dissipated by the circuit. (**b**) If the resonance frequency is 5 kHz, calculate the value of L and C.



Problem 5: (**a**) What is the resonant frequency of a series circuit consisting of R = 2 ohm, L =150 μH, C = 200 μμF? (**b**) What is the resonant frequency of a series circuit consisting of R = 3 ohm, L =300 μH, C = 100 μμF? (**c**) What is the impedance of each of the combination at 1000 kilocycles?

Series Resonance By Varying Inductance

When L=0, X_L=0, inductor behaves as a short-circuit as a result V_L=0 but I, V_R and V_C are not zero.

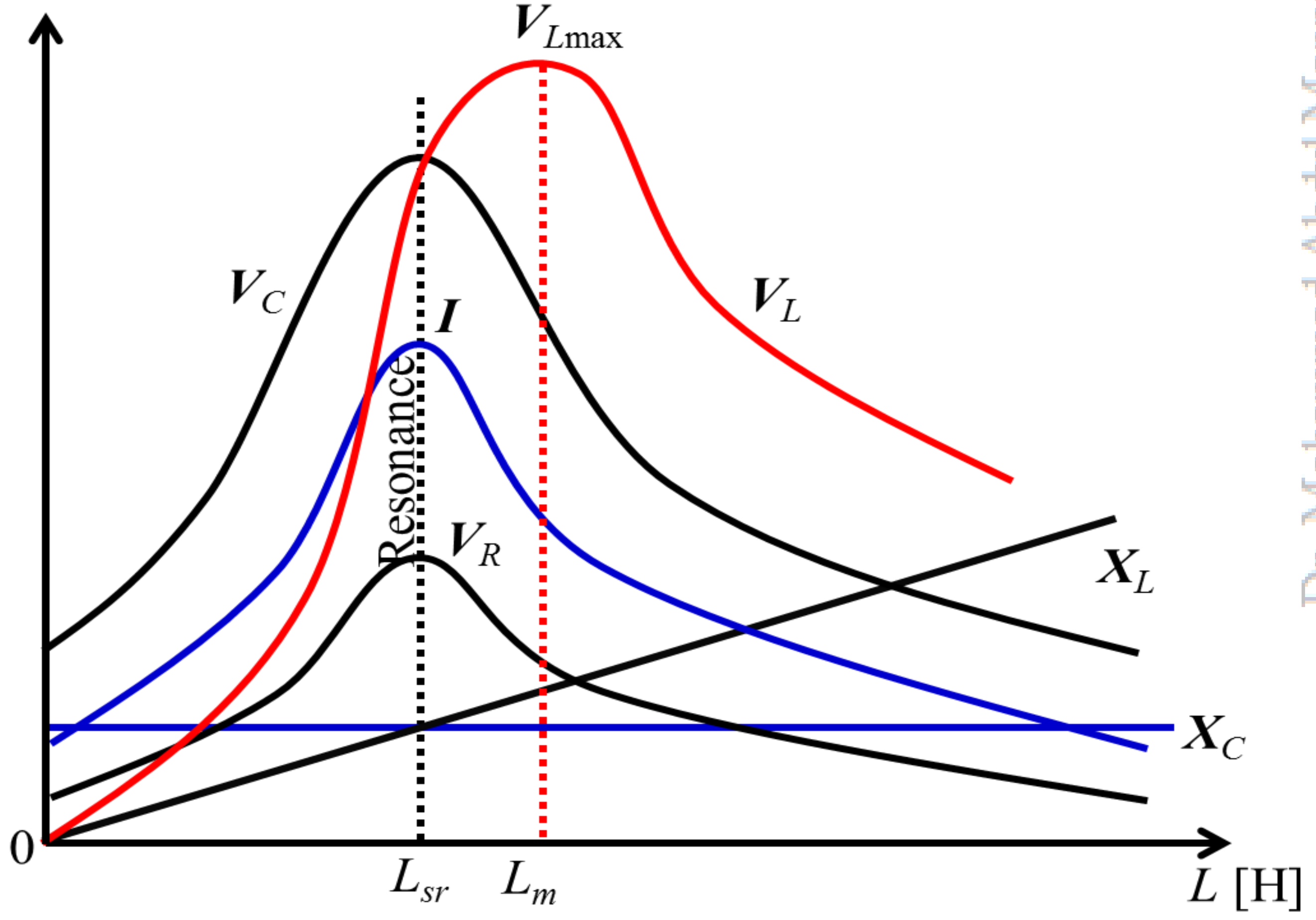
The resonance is occurred when $L=L_{sr}$, i.e. $X_L=X_C$, $V_L=V_C$, I , V_R and V_C are maximum.

V_L is maximum after the resonance. It is assumed that when $L=L_m$ the voltage across the inductor is maximum i.e. $V_L=V_{L\max}$.

The value of L_m is obtained by setting:

$$\frac{dV_L}{dX_L} = \frac{d}{dX_L}\left[\frac{VX_L}{\sqrt{R^2+(X_L-X_C)^2}}\right] = 0$$

$$X_L = \frac{R^2+X_C^2}{X_C} \qquad L_m = C\left(R^2+X_C^2\right)$$



Series Resonance By Varying Capacitance

When C=0, X_C=infinite, capacitor behaves as an open-circuit as a result V_C=supply voltage and I, V_R and V_L are zero.

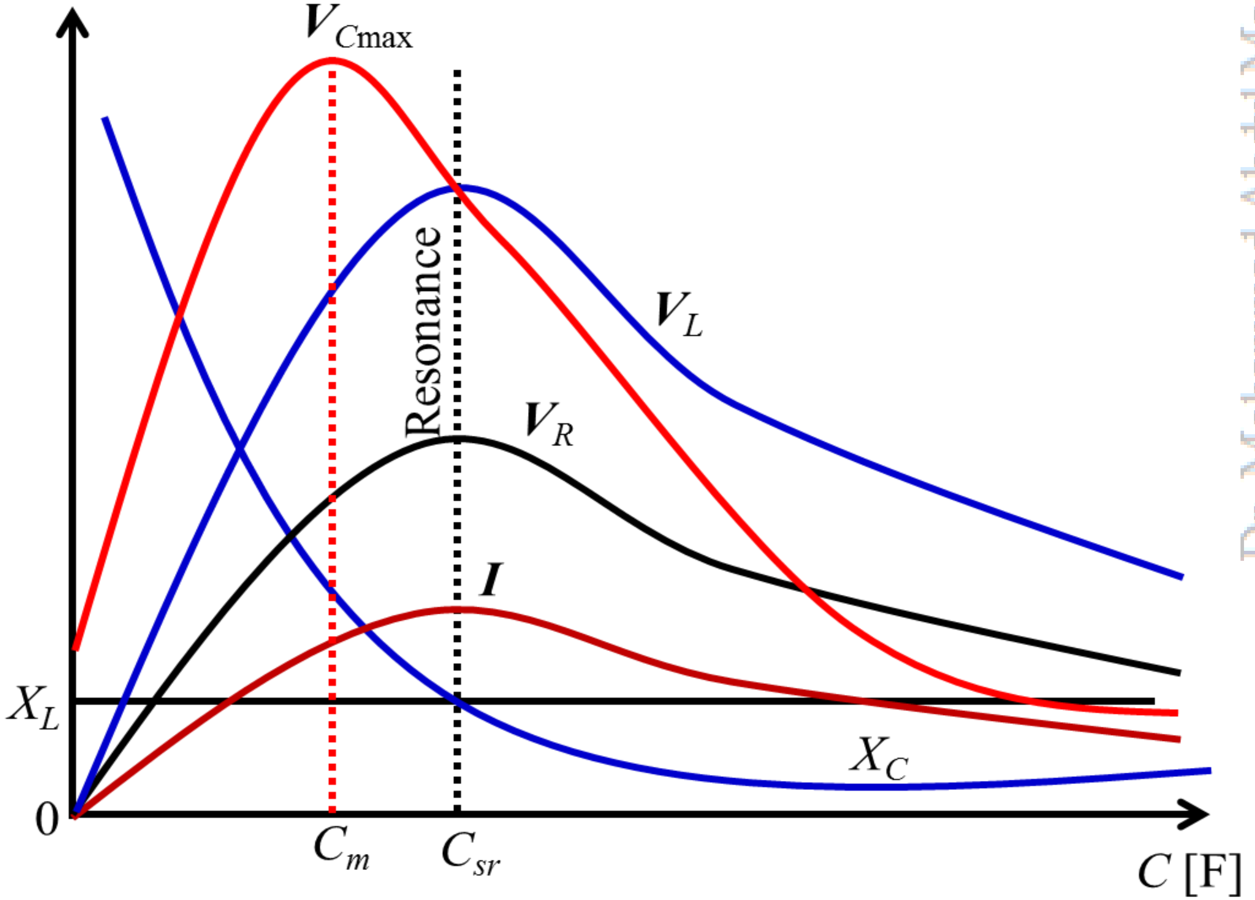
The resonance is occurred when $C=C_{sr}$, i.e. $X_L=X_C$, $V_L=V_C$, I , V_R and V_L are maximum.

V_C is maximum before the resonance. It is assumed that when $C=C_m$ the voltage across the capacitor is maximum i.e. $V_C=V_{C\max}$.

The value of C_m is obtained by setting:

$$\frac{dV_C}{dX_C}=\frac{d}{dX_C}\left[\frac{VX_C}{\sqrt{R^2+(X_L-X_C)^2}}\right]=0$$

$$X_C=\frac{R^2+X_L^2}{X_L} \qquad C_m=\frac{R^2+X_L^2}{L}$$



Example

As inductor (L) is varied to produce resonance in a series circuit containing R=100 ohms, X_C = 200 ohms, and f=60 Hz, (**i**) find the voltage drop across inductor (L) at resonance, and (**ii**) find the voltage drop across the inductor (L) and the value of inductance when the drop across inductor (L) is a maximum if 1000 V are impressed.

Solution: **At Resonance Condition**

$$X_L = X_C = 200\ \Omega \qquad \boldsymbol{Z} = 100 + j(200-200) = 100\ \Omega \qquad I = \frac{V}{Z} = \frac{1000}{100} = 10\ \text{A}$$

$$V_L(\text{at resonance}) = IX_L = 10 \times 200 = 2000\ \text{V}$$

For Maximum Voltage Drop Across Inductor $V_{L\max}$

$$X_L = \frac{R^2 + X_C^2}{X_C} = \frac{100^2 + 200^2}{200} = 250\ \Omega \qquad L_m = \frac{X_L}{2\pi f} = \frac{250}{2\pi \times 60} = 0.6631\ \text{H}$$

$$I = \frac{V}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{1000}{\sqrt{100^2 + (250-200)^2}} = 8.94\ \text{A}$$

$$V_{L\max} = IX_L = 8.94 \times 250 = 2235\ \text{V}$$

Example

If the impressed voltage on a series circuit containing 100 ohms resistance, 100 ohms inductive reactance at 60 cycles, and a variable capacitance is 100 V, (**i**) find the voltage drop across capacitor (C) at resonance, and (**ii**) find the voltage drop across the capacitor (C) and the value of capacitance when the drop across capacitor (C) is maximum.

Solution: **At Resonance Condition**

$$X_C = X_L = 100\ \Omega \qquad \boldsymbol{Z} = 100 + j(100-100) = 5\ \Omega \qquad I = \frac{V}{Z} = \frac{100}{100} = 1\ \text{A}$$

$$V_C(\text{at resonance}) = IX_C = 1\times 100 = 100\ \text{V}$$

For Maximum Voltage Drop Across Inductor $V_{L\max}$

$$X_C = \frac{R^2 + X_L^2}{X_L} = \frac{100^2 + 100^2}{100} = 200\ \Omega \qquad Cm = \frac{1}{2\pi\times 60\times 200} = 13.26\ \mu\text{F}$$

$$I = \frac{V}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{100}{\sqrt{100^2 + (100-200)^2}} = 0.7071\ \text{A}$$

$$V_{C\max} = IX_C = 0.7071\times 200 = 141.42\,\text{V}$$

Home Work 1.2

Problem 1: As inductor (L) is varied to produce resonance in a series circuit containing R=150 ohms, X_C = 250 ohms, and f=50 Hz, (**i**) find the voltage drop across inductor (L) at resonance, and (**ii**) find the voltage drop across the inductor (L) and the value of inductance when the drop across inductor (L) is a maximum if 1500 V are impressed.

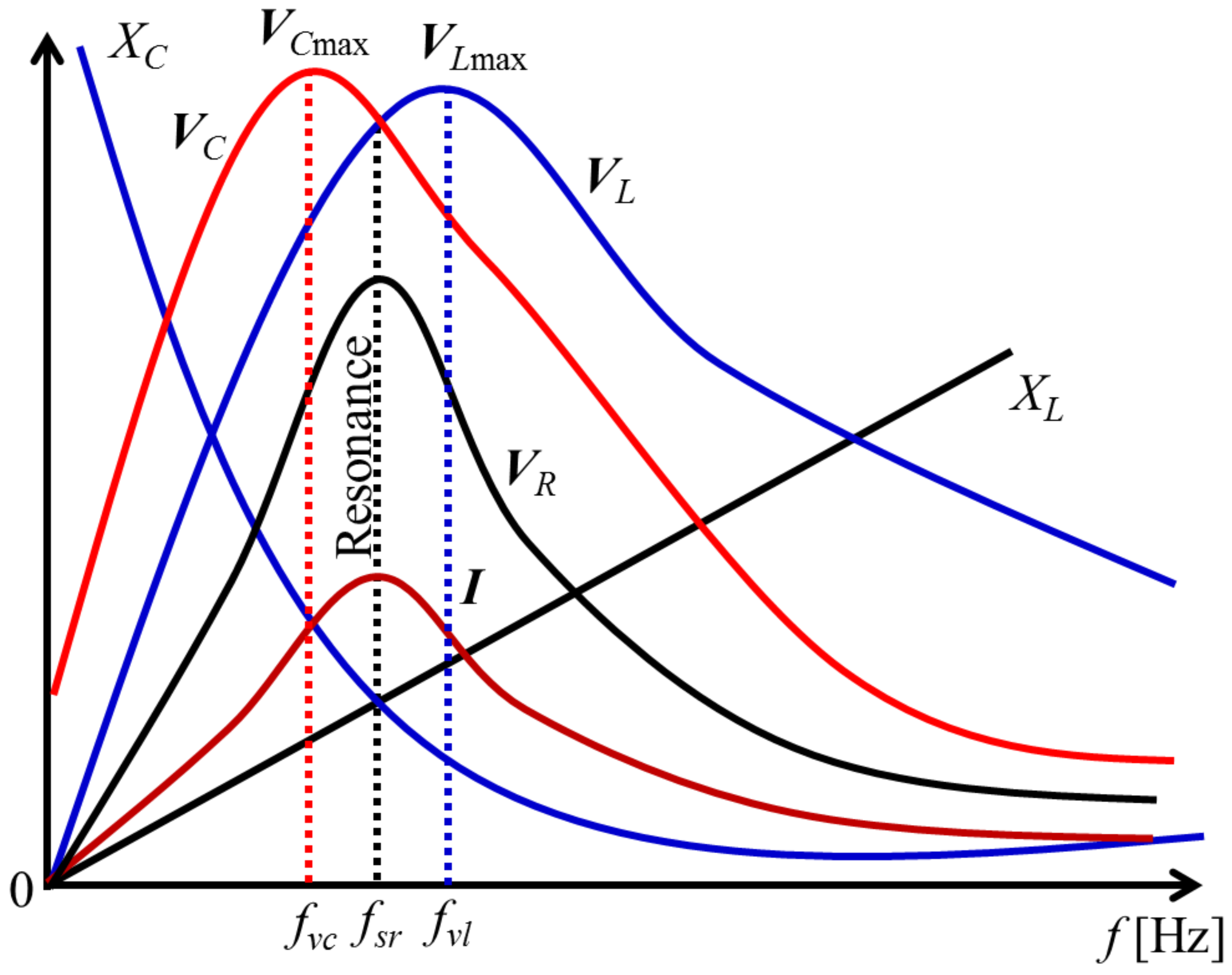
Problem 2: If the impressed voltage on a series circuit containing 100 ohms resistance, 100 ohms inductive reactance at 60 cycles, and a variable capacitance is 100 V, (**i**) find the voltage drop across capacitor (C) at resonance, and (**ii**) find the voltage drop across the capacitor (C) and the value of capacitance when the drop across capacitor (C) is maximum.

Series Resonance By Varying Frequency

When f=0, X_L=0 (short-circuit), and X_C=infinity (open-circuit), thus I, V_R and V_C are zero and V_C=supply voltage.

The resonance is occurred when X_L=X_C, V_L=V_C, I , V_R are maximum.

V_L is maximum after the resonance and V_C is maximum before the resonance.



Frequency for Maximum Voltage Drop Across Inductor

It is assumed that when $f=f_{vl}$ and $\omega=\omega_{vl}$ the voltage across the inductor is maximum i.e. $V_L=V_{L\max}$. The value of ω_{vl} is obtained by setting:

$$\frac{dV_L}{d\omega_{vl}}=\frac{d}{\omega_{vl}}\left[\frac{\omega_{vl}LV}{\sqrt{R^2+\left(\omega_{vl}L-\dfrac{1}{\omega_{vl}L}\right)^2}}\right]=0$$

$$\boxed{\omega_{vl}=\frac{1}{\sqrt{LC}\sqrt{1-\dfrac{R^2C}{2L}}}=\frac{1}{\sqrt{LC}}\frac{1}{\sqrt{1-\dfrac{R^2C}{2L}}}=\frac{\omega_{sr}}{\sqrt{1-\dfrac{R^2C}{2L}}}}$$

$$\boxed{f_{vl}=\frac{1}{2\pi\sqrt{LC}\sqrt{1-\dfrac{R^2C}{2L}}}=\frac{1}{2\pi\sqrt{LC}}\frac{1}{\sqrt{1-\dfrac{R^2C}{2L}}}=\frac{f_{sr}}{\sqrt{1-\dfrac{R^2C}{2L}}}}$$

Frequency for Maximum Voltage Drop Across Capacitor

It is assumed that when $f = f_{vc}$ and $\omega = \omega_{vc}$ the voltage across the capacitor is maximum i.e. $V_C = V_{C\max}$. The value of ω_{vc} is obtained by setting:

$$\frac{dV_C}{d\omega_{vc}} = \frac{d}{\omega_{vc}}\left[\frac{\frac{1}{\omega_{vc}L}V}{\sqrt{R^2 + \left(\omega_{vc}L - \frac{1}{\omega_{vc}L}\right)^2}}\right] = 0$$

$$\boxed{\omega_{vc} = \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{R^2C}{2L}} = \omega_{sr}\sqrt{1 - \frac{R^2C}{2L}}}$$

$$\boxed{f_{vc} = \frac{1}{2\pi\sqrt{LC}}\sqrt{1 - \frac{R^2C}{2L}} = f_{sr}\sqrt{1 - \frac{R^2C}{2L}}}$$

Example

A series *RLC* circuit with resistance of 10 Ω, inductance of 0.1 H, and capacitance of 50 μF is supplied voltage source of 50 V. Find resonant frequency and the frequencies at which maximum voltage appears across *L* and *C*.

Solution: **Resonance frequency is obtained by:**

$$f_{sr} = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(0.1)(50\times10^{-6})}} = 71.18\ \text{Hz}$$

Frequency when the voltage drop across the inductor is maximum is obtained by:

$$f_{vl} = \frac{f_{sr}}{\sqrt{1-\frac{R^2C}{2L}}} = \frac{71.18}{\sqrt{1-\frac{(10)^2(50\times10^{-6})}{2(0.1)}}} = 72.08\ \text{Hz}$$

Frequency when the voltage drop across the capacitor is maximum is obtained by:

$$f_{vc} = f_{sr}\sqrt{1-\frac{R^2C}{2L}} = 71.18\times\sqrt{1-\frac{(10)^2(50\times10^{-6})}{2(0.1)}} = 70.28\ \text{Hz}$$

Home Work 1.3

Problem 1: A series *RLC* circuit consists of R = 100 ohms, L = 0.02 H and C =0.02 μF. Calculate frequency of resonance. Find the frequency at which voltage across inductor and capacitor is maximum.

$$f_{sr} = 7.957 \text{ Hz} \qquad f_{vl} = 7.977 \text{ Hz} \qquad f_{vc} = 7.937 \text{ Hz}$$

Problem 2: A series *RLC* circuit consists of R = 50 ohms, L = 0.05 H and C = 20 μF. Calculate frequency of resonance. Find the frequency at which voltage across inductor and capacitor is maximum.

Problem 3: In a series *RLC* circuit has R = 20 ohms, L = 0.005 H and C = 0.2 μF. The circuit is excited by a 100 V source. Calculate (i) the frequency at which current is maximum, (ii) the impedance at resonance frequency, (iii) the voltage across the inductance and capacitor at resonance frequency, (iv) the maximum value of voltage across the inductor and the frequency at which this occurs, and (v) the maximum value of voltage across the capacitor and the frequency at which this occurs.

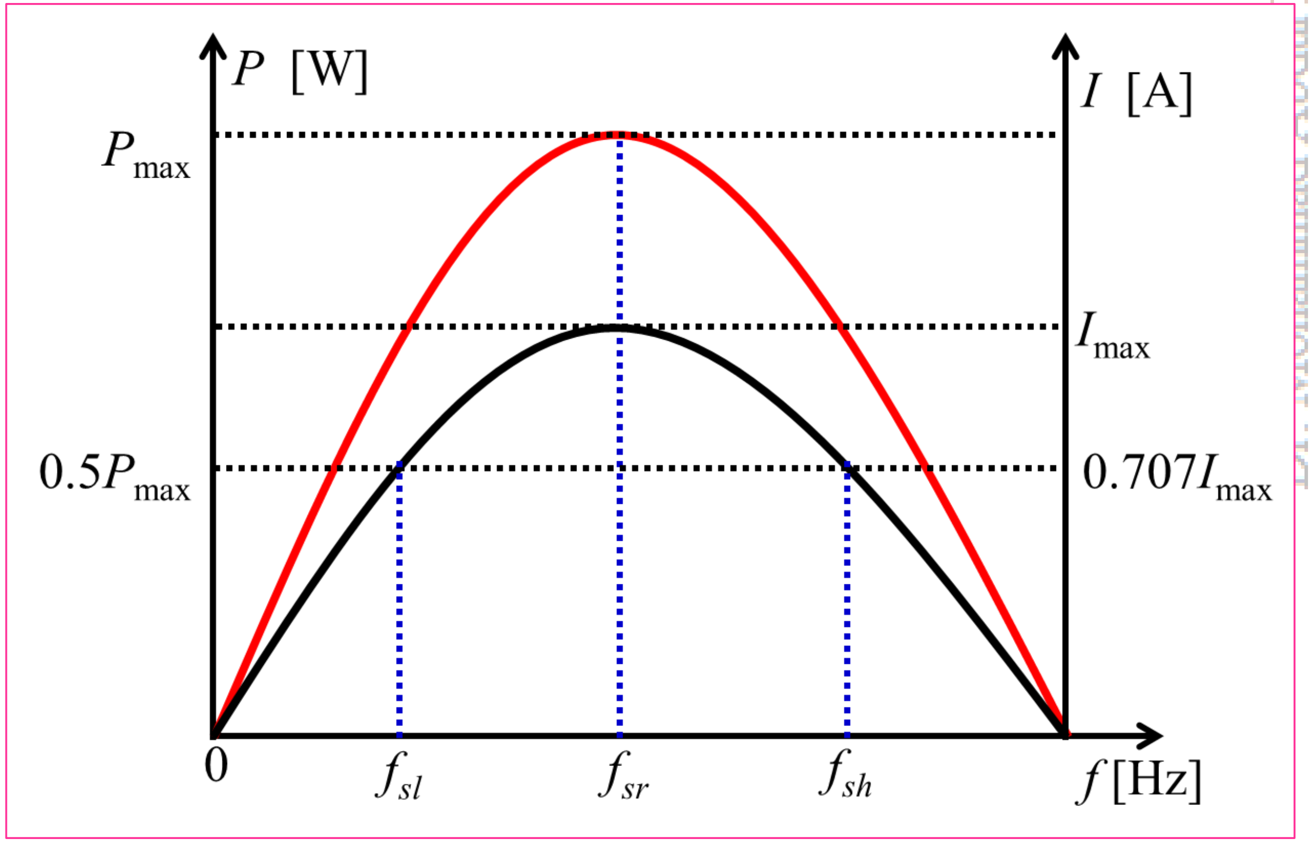
The Series RLC Circuit as a Selector

Even though the *RLC* circuit passes all waves of finite frequency to some extent, it has been shown to have the **maximum current/power** and **minimum impedance** for the resonant frequency. As shown in the following figure, the *RLC* circuit passes frequencies near the resonant frequency more readily than other frequencies. The circuit thus has selective properties.

Half-Power Frequencies:

It can be observed that at two frequencies f_{sl} and f_{sh} the power is half of the maximum value. These frequencies are called **half-power frequencies**.

These frequencies also called **band/ cutoff/half-power/corner/break/-3dB frequencies**.



Current and Impedance at Cut-off or Half-power Frequencies

The current/power at resonance frequency is the maximum, thus

$$I_{max} = \frac{V}{R} \qquad\qquad P_{max} = I_{max}^2 R$$

At cut-off frequency: $$P_{HPF} = \frac{1}{2} P_{max} = \frac{1}{2} I_{max}^2 R = \left(\frac{I_{max}}{2}\right)^2 R = I_{HPF}^2 R$$

Current at cut-off frequency: $$\boxed{I_{HPF} = \frac{I_{max}}{\sqrt{2}} = 0.707 I_{max}}$$

Impedance at cut-off frequency:

$$I_{HPF} = \frac{I_{max}}{\sqrt{2}} = \frac{V}{\sqrt{2}R} \qquad\qquad \boxed{Z_{HPF} = \frac{V}{I_{HPF}} = \sqrt{2}R}$$

Derivation of Cut-off or Half-power frequencies

The impedance of cut-off frequency can be obtained as follows:

$$Z_{HPF} = \sqrt{2}R = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 = 2R^2$$

$$\left(\omega L - \frac{1}{\omega C}\right)^2 = R^2$$

$$\omega L - \frac{1}{\omega C} = \pm R \qquad (1)$$

Let, ω_{sl} and ω_{sh} two angular frequencies corresponding the lower cut-off and higher cut-off frequencies of f_{sl} and f_{sh}, respectively. From Eq. (1) we have:

$$\omega_{sh} L - \frac{1}{\omega_{sh} C} = R \qquad (2)$$

$$\omega_{sl} L - \frac{1}{\omega_{sl} C} = -R \qquad (3)$$

From Eqs. (2) and (3) we have:

$$\omega_{sh}^2 - \frac{R}{L}\omega_{sh} - \frac{1}{LC} = 0 \qquad (4)$$

$$\omega_{sh}^2 + \frac{R}{L}\omega_{sh} - \frac{1}{LC} = 0 \qquad (5)$$

The solution of Eq. (4) as follows:

$$\omega_{sh} = \frac{-\left(-\frac{R}{L}\right) \pm \sqrt{\left(-\frac{R}{L}\right)^2 - 4\left(-\frac{1}{LC}\right)}}{2} = \frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

Only + sign is taken before the square root. This is done to ensure that cut-off frequency is always positive. Hence,

$$\omega_{sh} = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \qquad (6) \qquad f_{sh} = \frac{1}{2\pi}\left[\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right] \qquad (6.1)$$

Similarly, the solution of Eq. (5) is obtained as follows:

$$\omega_{sl} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \qquad (7) \qquad f_{sl} = \frac{1}{2\pi}\left[-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right] \qquad (7.1)$$

f_{sh}: Higher Cut-off frequency; **ω_{sh}: Angular frequency at higher Cut-off frequency**

f_{sl}: Lower Cut-off frequency; **ω_{sl}: Angular frequency at lower Cut-off frequency**

$$\because \omega_{sr} = \frac{1}{\sqrt{LC}} \qquad i.e.\ \omega_{sr}^2 = \frac{1}{LC}$$

$$\omega_{sh} = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \omega_{sr}^2} \qquad (6.2)$$

$$f_{sh} = \frac{1}{2\pi}\left[\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \omega_{sr}^2}\right] \qquad (6.3)$$

$$\omega_{sl} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \omega_{sr}^2} \qquad (7.2)$$

$$f_{sl} = \frac{1}{2\pi}\left[-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \omega_{sr}^2}\right] \qquad (7.3)$$

In a selective *RLC* branch, $\left(\frac{R}{2L}\right)^2 << \omega_{sr}^2$ then $\sqrt{\left(\frac{R}{2L}\right)^2 + \omega_{sr}^2} \approx \omega_{sr}$ thus

$$\omega_{sh} = \frac{R}{2L} + \omega_{sr} \qquad (6.4)$$

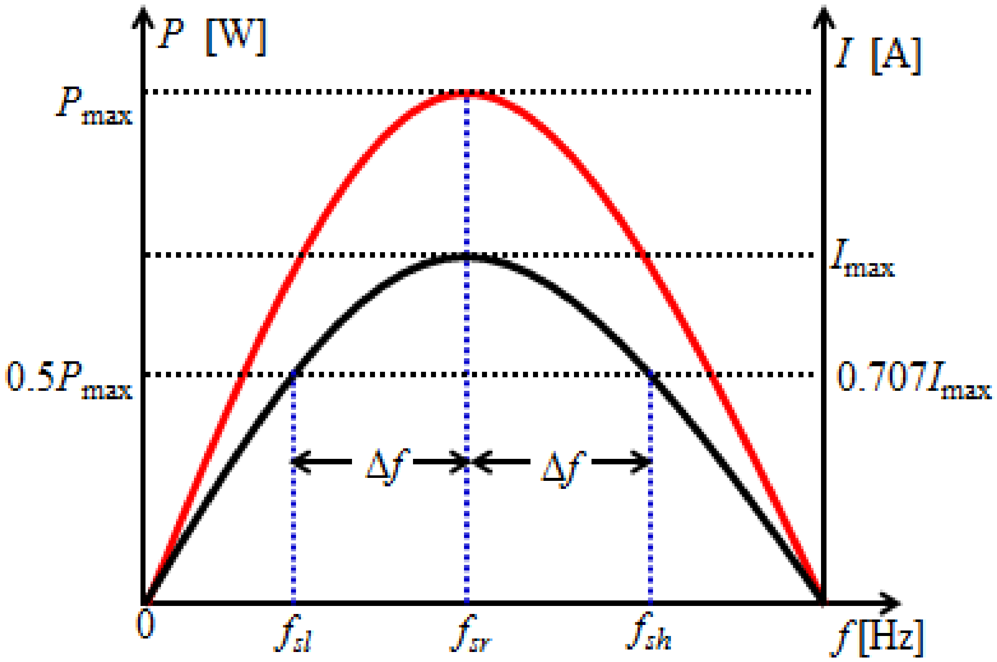
$$f_{sh} = \frac{1}{2\pi}\left[\frac{R}{2L} + \omega_{sr}\right] = f_{sr} + \frac{R}{4\pi L} \qquad (6.5)$$

$$\omega_{sl} = -\frac{R}{2L} + \omega_{sr} \qquad (7.4)$$

$$f_{sl} = \frac{1}{2\pi}\left[-\frac{R}{2L} + \omega_{sr}\right] = f_{sr} - \frac{R}{4\pi L} \qquad (7.5)$$

Bandwidth:

The difference between the half-power frequencies f_{sh} and f_{sl} at which power is half of its maximum is called bandwidth of the series *RLC* circuit.



$$BW = 2\Delta f = f_{sh} - f_{sl} = \frac{R}{2\pi L} \qquad (8)$$

$$\Delta f = \frac{BW}{2} = \frac{R}{4\pi L} \qquad (8.1)$$

$$2\Delta\omega = \omega_{sh} - \omega_{sl} = \frac{R}{L} \qquad (9)$$

$$\Delta\omega = \frac{\omega_{sh} - \omega_{sl}}{2} = \frac{R}{2L} \qquad (9.1)$$

Combining Eq. (8.1) with Eqs. (6.5) and (7.5) we have:

$$f_{sh} = f_{sr} + \frac{BW}{2} = f_{sr} + \Delta f \qquad (6.6)$$

$$f_{sl} = f_{sr} - \frac{BW}{2} = f_{sr} - \Delta f \qquad (7.6)$$

Combining Eq. (9.1) with Eqs. (6.4) and (7.4) we have:

$$\omega_{sh} = \omega_{sr} + \Delta\omega \qquad (6.7)$$

$$\omega_{sl} = \omega_{sr} - \Delta\omega \qquad (7.7)$$

Relation Among Cut-off or Half-Power Frequencies and Resonance Frequency

Multiply Eq. (6.2) with Eq. (7.2) we have:

$$\omega_{sh}\omega_{sl} = \left[\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \omega_{sr}^2}\right]\left[-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \omega_{sr}^2}\right]$$

$$\omega_{sh}\omega_{sl} = \left[\sqrt{\left(\frac{R}{2L}\right)^2 + \omega_{sr}^2}\right]^2 - \left[\frac{R}{2L}\right]^2 = \left(\frac{R}{2L}\right)^2 + \omega_{sr}^2 - \left(\frac{R}{2L}\right)^2 = \omega_{sr}^2$$

$$\omega_{sr}^2 = \omega_{sh}\omega_{sl} \qquad (10)$$

$$\omega_{sr} = \sqrt{\omega_{sh}\omega_{sl}} \qquad (10.1)$$

$$f_{sr}^2 = f_{sh} f_{sl} \qquad (11)$$

$$f_{sr} = \sqrt{f_{sh} f_{sl}} \qquad (11.1)$$

The Eq. (11.1) shows that the resonant frequency is the geometric mean of the two half-power (or cut-off) frequencies.

Selectivity:

In the bandwidth, the power is more than half the maximum value. The bandwidth decides the selectivity.

The **selectivity** is defined as the ratio of the resonant frequency to the bandwidth.

$$\textbf{Selectivity} = \frac{f_{sr}}{BW} = \frac{f_{sr}}{f_{sh} - f_{sl}} \qquad (12)$$

If the bandwidth is more, the selectivity of the circuit is less.

Example

A series *RLC* circuit has resonance frequency of 150 kHz and bandwidth of 75 kHz. Determine its half-power frequencies.

Solution: $f_{sh} - f_{sl} = BW = 75 \qquad (1)$ $\qquad f_{sh} f_{sl} = f_{sr}^2 = 150^2 = 22500 \qquad (2)$

From (2) $\quad f_{sh} = \dfrac{22500}{f_{sl}} \qquad (3)$ $\qquad$ With (1) and (3) $\dfrac{22500}{f_{sl}} - f_{sl} = 75$

$$f_{sl}^2 + 75 f_{sl} - 22500 = 0$$

$$f_{sl} = 117.1\ \text{kHz} \ \text{ or } \ -192.1\ \text{kHz}$$

Ignoring the negative value, we have f_{sl}=**171.1** kHz.

Hence From (1) $\qquad f_{sh} = 75 + f_{sl} = \mathbf{192.1}\ \text{kHz}$

Example

Determine the parameters of a *RLC* series circuit that will resonant at 10 kHz, have a bandwidth of 1 kHz, and draw 15.3 W from 200 V source operating at the resonance frequency of the circuit.

Solution: At resonance, $V_R = V$ = 200 V. The power drawn is given as

$$P = \frac{V_R^2}{R} \qquad R = \frac{V_R^2}{P} = \frac{200^2}{15.3} = \mathbf{2.61}\ \text{k}\Omega$$

$$BW = \frac{R}{2\pi L} \qquad \therefore\ L = \frac{R}{2\pi(BW)} = \frac{2610}{2\pi \times 1000} = \mathbf{415.4}\ \text{mH}$$

The resonant frequency is given by: $$f_{sr} = \frac{1}{2\pi\sqrt{LC}}$$

$$\therefore\ C = \frac{1}{(2\pi f_{sr})^2 L} = \frac{1}{\left(2\pi \times 10 \times 10^3\right) \times 415.4 \times 10^{-3}} = \mathbf{610}\ \text{pF}$$

Example

In a series *RLC* resonant circuit, the resonant frequency is 796 Hz and the quality factor had a value of 4. Determine the bandwidth, lower cut-off frequency and higher cut-off frequency.

Solution:

$$BW = \frac{f_{sr}}{Q_{sr}} = \frac{796}{5} = \mathbf{159.2}\ \text{Hz}$$

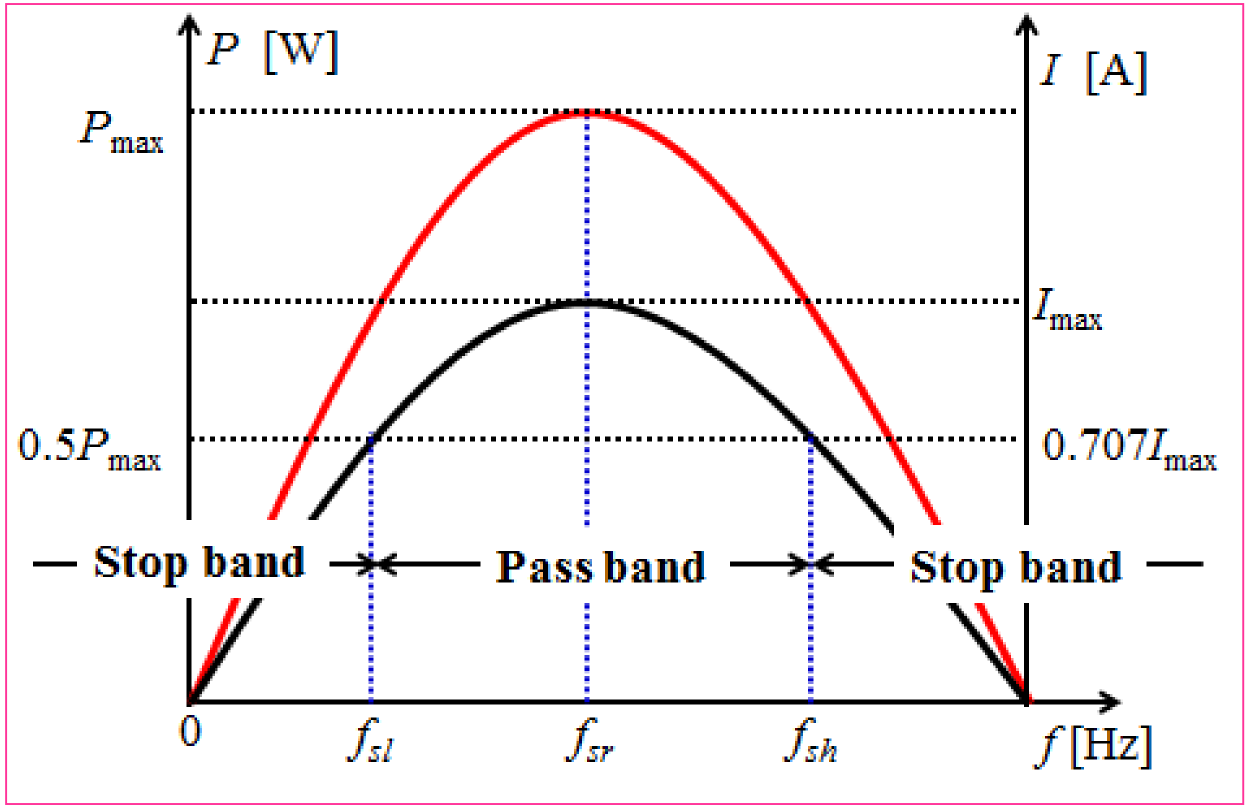
$$f_{sh} = f_{sr} + \frac{BW}{2} = 796 + \frac{159.2}{2} = \mathbf{875.6}\ \text{Hz}$$

$$f_{sl} = f_{sr} - \frac{BW}{2} = 796 - \frac{159.2}{2} = \mathbf{716.4}\ \text{Hz}$$

Pass Band: The range of frequencies which is quiet readily allowed to pass through a circuit i.e. the power is more than half the maximum value.

Stop Band: The range of frequencies which is stopped to pass through a circuit i.e. the power is less than half the maximum value.

The series resonant circuit is often described as an *acceptor circuit* since it has its minimum impedance, and thus maximum current/power at the resonant frequency.



Quality factor or Figure of Merit (*Q*)

The resonance phenomenon is observed in ac circuits consisting reactive elements such as inductor and capacitor. These two elements are basic passive elements of energy storing type. It is found convenient to express the **efficiency** with these elements store energy. It is also found simpler to compare various inductors and capacitors inn terms of efficiency while designing such circuits. Such efficiency is measured as ***quality factor*** (*Q*-factor). It is also called ***figure of merit***.

Q-factor refers to the "***goodness***" or "***efficiency***" of a reactive components (inductor or capacitor).

The quality factor is an indication of "***how much energy is placed in storage compared to that dissipated***".

Quality factor is calculated at resonance condition.

The ***quality factor*** (Q-factor) or ***figure of merit*** is defined as:

$$Q = 2\pi\left[\frac{\text{Maximum energy stored}}{\text{Energy loss per cycle}}\right]$$

The quality factor (***Q***-factor) of a series resonant circuit (Q_{sr}) is defined as ***the ratio of the reactive power of either the inductor or the capacitor to the average power of the resistor at resonance***; that is,

$$Q_{sr} = \left[\frac{\text{Reactive Power}}{\text{Real Power}}\right] = \frac{Q_L}{P} = \frac{Q_C}{P} = \frac{X_L}{R} = \frac{X_C}{R} = \frac{\omega_{sr} L}{R} = \frac{1}{R\omega_{sr} C}$$

$$Q_{sr} = \frac{2\pi f_{sr} L}{R} = \frac{1}{2\pi f_{sr} RC} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

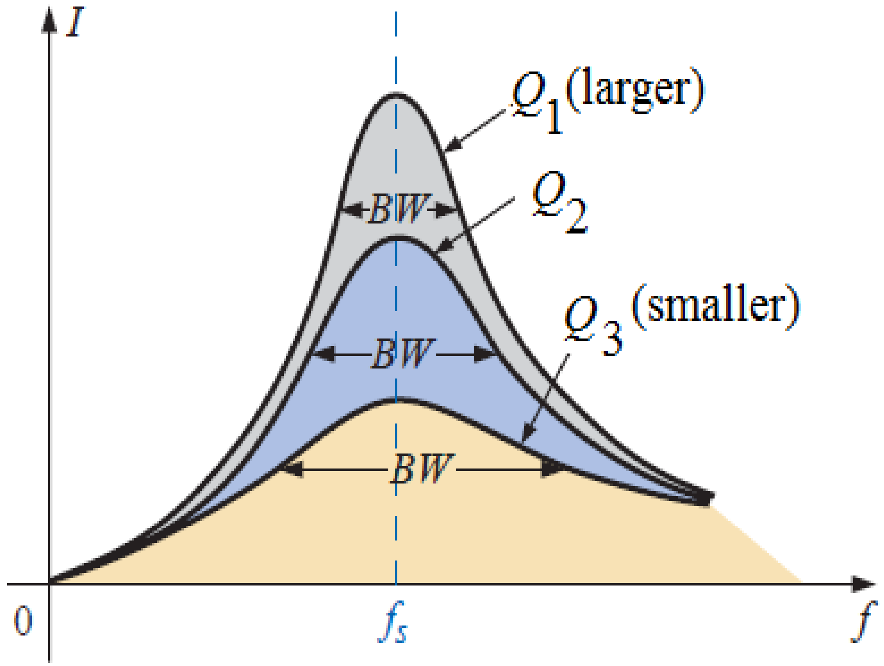
Relation of Quality Factor with Selectivity

$$\because \text{BW} = \frac{R}{2\pi L} \qquad Q_{sr} = \frac{2\pi f_{sr} L}{R} = \frac{f_{sr}}{R/2\pi L} = \frac{f_{sr}}{BW} = \text{Selectivity} \qquad BW = \frac{f_{sr}}{Q_{sr}}$$

A small Q_{sr}, therefore, is associated with a resonant curve having a large bandwidth and a small selectivity, while a large Q_{sr} indicates the opposite.

The significance of quality factor can be stated as:

1. It indicates the selectivity or sharpness of the tuning of a series circuit.
2. It gives the correct indication of the selectivity of such series *RLC* circuit which are used in many radio circuits.



Voltage Magnification

At resonance frequency: $Z_{sr} = R$ $\quad I = \dfrac{V}{Z_{sr}} = \dfrac{V}{R}$ $\quad V_L = IX_L = \dfrac{X_L}{R}V = Q_{sr}V$

$$V_C = IX_C = \frac{X_C}{R}V = Q_{sr}V$$

$$Q_{sr} = \frac{V_L}{V} = \frac{V_C}{V}$$

The voltage across the inductor V_L (and voltage across the capacitor V_C) is Q_{sr} (Q-factor) times the total current at resonance frequency.

Since, the voltage increases Q_{sr} times in series resonance circuit, it is also known as a ***voltage magnification of resonant circuit***.

Example

(**a**) For the series resonant circuit of following Figure, find ***I***, $\boldsymbol{V}_R$, $\boldsymbol{V}_L$, and $\boldsymbol{V}_C$ at resonance. (**b**) What is the Q_{sr} of the circuit? (**c**) If the resonant frequency is 5000 Hz, find the bandwidth. (**d**) What is the power dissipated in the circuit at the half-power frequencies?

$$\boldsymbol{Z}_{sr} = R = 2\ \Omega \qquad \boldsymbol{I} = \frac{\boldsymbol{E}}{\boldsymbol{Z}_{sr}} = \frac{10V\angle 0^\circ}{2\Omega\angle 0^\circ} = 5A\angle 0^\circ \qquad \boldsymbol{V}_R = \boldsymbol{I}R = \boldsymbol{E} = 10V\angle 0^\circ$$

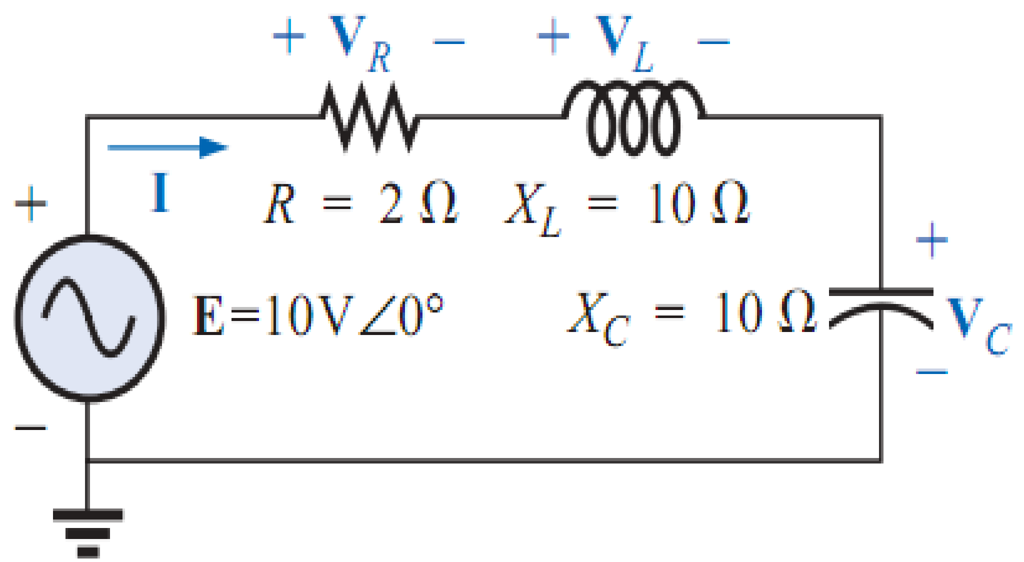
$$\boldsymbol{V}_L = \boldsymbol{I}(X_L\angle 90^\circ) = (5A\angle 0^\circ)(10\Omega\angle 90^\circ) = 50V\angle 90^\circ = j50V$$

$$\boldsymbol{V}_C = \boldsymbol{I}(X_C\angle -90^\circ) = (5A\angle 0^\circ)(10\Omega\angle -90^\circ) = 50V\angle -90^\circ = j50V$$

$$Q_{sr} = \frac{X_L}{R} = \frac{10\Omega}{2\Omega} = 5$$

$$\text{BW} = \frac{f_{sr}}{Q_{sr}} = \frac{5000}{5} = 1000\,Hz$$

$$P_{HPF} = \frac{1}{2}P_{\max} = \frac{I_{\max}^2}{2}R = \frac{(5A)^2}{2R}(2\Omega) = 25W$$



Example

The bandwidth of a series resonant circuit is 400 Hz. (**a**) If the resonant frequency is 4000 Hz, what is the value of Q_{sr}? (**b**) If R = 10 Ω, what is the value of X_L at resonance? (**c**) Find the inductance L and capacitance C of the circuit.

$$Q_{sr} = \frac{f_{sr}}{BW} = \frac{4000\,Hz}{400\,Hz} = 10 \qquad Q_{sr} = \frac{X_L}{R} \qquad X_L = Q_{sr} R = 10 \times 10\,\Omega = 100\,\Omega$$

$$X_L = 2\pi f_{sr} L$$

$$L = \frac{X_L}{2\pi f_{sr}} = \frac{100\,\Omega}{2\pi(4000\,Hz)} = 3.98\ mH$$

$$X_C = \frac{1}{2\pi f_{sr} C}$$

$$C = \frac{1}{2\pi f_{sr} X_C} = \frac{1}{2\pi(4000\,Hz)(100\,\Omega)} = 0.398\ \mu F$$

Example

A 240 V, 100 Hz ac source is connected to a series *RLC* circuit consisting of a coil and a variable capacitor. The coil has a resistance of 55 mΩ and inductance of 7 mH. The capacitor is varied so as to achieve resonance. Determine (a) the value of the capacitance, (b) the circuit quality factor, and (c) the half-power (or cut-off) frequencies.

$$f_{sr} = \frac{1}{2\pi\sqrt{LC}} \qquad C = \frac{1}{(2\pi f_{sr})^2 L} = \frac{1}{(2\pi \times 100)^2 \times 7 \times 10^{-3}} = 361.86\ \mu F$$

$$Q = \frac{2\pi f_{sr} L}{R} = \frac{2\pi \times 100 \times \left(7 \times 10^{-3}\right)}{55 \times 10^{-3}} = 79.93 \qquad BW = \frac{f_{sr}}{Q} = \frac{100}{79.93} = 1.251 \quad \text{Hz}$$

$$f_{sh} = f_{sr} + \frac{BW}{2} = 100 + \frac{1.251}{2} = 100.6255\ Hz$$

$$f_{sl} = f_{sr} - \frac{BW}{2} = 100 - \frac{1.251}{2} = 99.3745\ Hz$$

Example

A *RLC* series circuit with resistance of 10 ohms, a inductance of 0.2 H and a capacitance of 40 μF is supplied with 100 V at variable frequency. (***a***) Find the following (**i**) the resonance frequency, (**ii**) the current, (**iii**) the power, (**iv**) the voltage across *R*, *L*, *C* at that time, (**v**) the quality factor of the circuit, (**vi**) the half-power or cut-off frequencies, (**vii**) the bandwidth. (***b***) Calculate (**i**) the frequency at which the voltage across the inductance is maximum and the maximum voltage across *L*, (**ii**) the frequency a which the voltage across the capacitance is maximum and the maximum voltage across *C*.

(***a***) (**i**) $$f_{sr} = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.2\times 40\times 10^{-6}}} = 56.2697 \text{ Hz}$$

(**ii**) $$I_{\max} = \frac{V}{R} = \frac{100}{10} = 10 \text{ A}$$

(**iii**) $$P_{\max} = I_{\max}^2 R = 10^2 \times 10 = 1000 \text{ W}$$

(**iv**) $$V_R = I_{\max} R = V = 100 \text{ V}$$

$$X_L = 2\pi\times 56.2697\times 0.2 = 70.7105\ \Omega$$

$$X_C = \frac{1}{2\pi\times 56.2697\times 40\times 10^{-6}} = 70.7105\ \Omega$$

$$V_L = IX_L = 10\times 70.7105 = 707.105 \text{ V}$$

$$V_C = IX_C = 10\times 70.7105 = 707.105 \text{ V}$$

$$(\mathbf{v})\ Q_{sr} = \frac{X_L}{R} = \frac{70.7105}{10} = 7.071$$

$$(\mathbf{vi})\ f_{sh} = f_{sr} + \frac{R}{4\pi L} = 56.2697 + \frac{10}{4\pi \times 0.2} = 56.2697 + 3.9789 = 60.25\ \text{Hz}$$

$$f_{sl} = f_{sr} - \frac{R}{4\pi L} = 56.2697 - \frac{10}{4\pi \times 0.2} = 56.2697 - 3.9789 = 52.3\ \text{Hz}$$

$$(\mathbf{vii})\ BW = f_{sh} - f_{sl} = 60.25 - 52.3 = 7.95\ \text{Hz}$$

$$(\boldsymbol{b})\ (\mathbf{i})\ f_{vl} = \frac{f_{sr}}{\sqrt{1 - \frac{R^2 C}{2L}}} = \frac{56.2697}{\sqrt{1 - \frac{10^2 \times 40 \times 10^{-6}}{2 \times 0.2}}} = \frac{56.2697}{\sqrt{0.99}} = \frac{56.2697}{0.995} = 56.55\ \text{Hz}$$

$$X_L = 2\pi \times 56.55 \times 0.2 = 71.07\ \Omega$$

$$X_C = \frac{1}{2\pi \times 56.55 \times 40 \times 10^{-6}} = 70.37\ \Omega$$

$$Z_{vl} = \sqrt{10^2 + (71.07 - 70.37)^2} = 10.03\ \Omega$$

$$I = \frac{V}{Z_{vl}} = \frac{100}{10.03} = 9.97\ \text{A}$$

$$V_{L\max} = IX_L = 9.97 \times 71.07 = 708.57\ \text{V}$$

(ii) $$f_{vc} = f_{sr}\sqrt{1 - \frac{R^2 C}{2L}} = 56.2697\sqrt{1 - \frac{10^2 \times 40 \times 10^{-6}}{2 \times 0.2}} = 56.2697\sqrt{0.99} = \frac{56.2697}{0.995} = 55.99\ \text{Hz}$$

$$X_L = 2\pi \times 55.99 \times 0.2 = 70.36\ \Omega$$

$$X_C = \frac{1}{2\pi \times 55.99 \times 40 \times 10^{-6}} = 71.06\ \Omega$$

$$Z_{vc} = \sqrt{10^2 + (70.36 - 71.06)^2} = 10.025\ \Omega$$

$$I = \frac{V}{Z_{vl}} = \frac{100}{10.025} = 9.98\ \text{A}$$

$$V_{C\max} = IX_c = 9.98 \times 71.06 = 709.18\ \text{V}$$

Properties of Series Resonance Circuit

Under Resonance:

- Applied ac voltage and resulting ac current are in phase.
- The power factor is unity as a result the real power and apparent power are equal.
- The net reactance is zero as a result the net reactive power is also zero.
- The impedance below resonance frequency is capacitive while above the resonance frequency it is inductive in nature.
- The impedance is resistive.
- The impedance is minimum.
- The current is maximum and hence power is maximum.
- Voltage drop across the inductor and across the capacitor are equal and 180^o degree out of phase.
- The voltage drop across the resistance equal to supply voltage.

Under Resonance:

- The energy stored by inductor and capacitor is equal value.
- The series resonant circuit acts as **voltage magnification or amplifier** with the quality factor i.e. Q_{sr} acting as amplification of the magnification factor.
- The series resonant circuit is often described as an **acceptor circuit** since it has its minimum impedance, and thus maximum current/power at the resonant frequency.
- The quality factor of the circuit decides selectivity of the circuit. Its required value must be very large enough. It decides how much the resonant circuit is selective.
- Series resonance circuit allows to pass a specific band of frequency.
- Series resonance circuit is also called **tune circuit**.

Home Work 1.4

Problem 1: A *RLC* series circuit has R = 10 ohms, L = 0.1 H and C = 8 μF. Calculate (***a***) the resonance frequency, (***b***) the quality factor at resonance, and (***c***) the half-power frequencies, (***d***) the bandwidth.

Problem 2: A *RLC* series circuit with R = 4 ohms, L = 0.5 H and a variable capacitance C is connected across a 100 V, 50 Hz supply. Calculate (***a***) the value of capacitance for which resonance will occur, (***b***) the voltage across the capacitance at resonance, and (***c***) the quality factor.

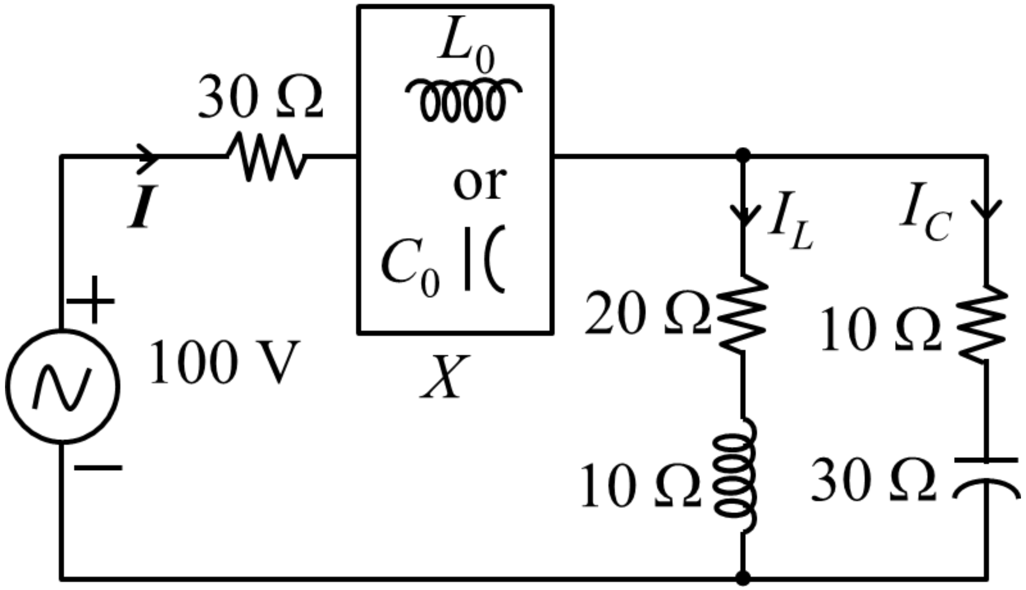
Problem 3: A resistor, a variable inductor and a capacitor are connected across a 230 V, 50 Hz supply. Maximum current of 1.5 A was obtained in the circuit by changing the inductance. At that time the voltage across the capacitor was measured as 600 V. Calculate the values of circuit parameters.

Problem 4: A *RLC* series circuit has R = 12 ohms, L = 0.15 H and C = 100 μF. It is connected to an ac source voltage 100 V, where frequency can be varied. Determine (i) the frequency of the source at which the current supplied by its maximum, (ii) the value of this current, (iii) the frequency at which the voltage across the capacitor is maximum, 9iv) the frequency at which the voltage across the inductor is maximum, (v) the reactance of inductor and capacitor at resonance, (vi) the voltage drop across each elements at resonance, (vii) the quality factor, and (viii) the bandwidth.

Example

Two impedances $\boldsymbol{Z}_1 = 20 + j10$ and $\boldsymbol{Z}_2 = 10 - j30$ are connected in parallel and this combination is connected in series with $\boldsymbol{Z}_3 = 30 + jX$. Find whither inductor or capacitor is required which reactance is X to pass a specified frequency.

$$\boldsymbol{Z}_1 = 20 + j10\ \Omega \qquad \boldsymbol{Z}_2 = 10 - j30\ \Omega$$



The impedance of parallel combination of $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ is obtained by:

$$\boldsymbol{Z}_p = \frac{(20 + j10)(10 - j30)}{20 + j10 + 10 - j30} = 19.23 - j3.85\ \Omega$$

Since $\boldsymbol{Z}_p$ is capacitive an inductor is required to pass a specified frequency. Now the total impedance is given by:

$$\boldsymbol{Z}_T = \boldsymbol{Z}_p + \boldsymbol{Z}_3 = 19.23 - j3.85 + 30 + jX = 49.23 + j(X - 3.85)\ \Omega$$

For resonance the imaginary part of total impedance should be zero that means:

$$X - 3.85 = 0 \qquad X = 3.85\ \Omega$$

Home Work 1.5

Problem 1: Two impedances are connected in parallel and this combination is connected in series another impedance as shown in the following figure. Find whither inductor (L_0) or capacitor (C_0) is required which to pass a 1 kHz frequency. Calculate the value of L_0 or C_0.