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Chapter 1

Configurations, Forces, Moments, Equilibrium (Compact Sheet)

Geometry of a planar curve (arclength s)

Let the centerline be $\boldsymbol{r}(s) \in \mathbb{R}^2$, with unit tangent $\boldsymbol{t} = \boldsymbol{r}'(s)$, $|\boldsymbol{t}| = 1$, and unit normal $\boldsymbol{n}$ obtained by a $+90°$ rotation of $\boldsymbol{t}$. The tangent angle $\alpha(s)$ satisfies $\boldsymbol{t} = (\cos\alpha, \sin\alpha)$ and the signed curvature is $\kappa(s) = \alpha'(s)$.

Frenet relations

$$\boldsymbol{t}' = \kappa\,\boldsymbol{n}, \qquad \boldsymbol{n}' = -\kappa\,\boldsymbol{t}, \qquad \kappa = \alpha'(s).$$

Orientation and curvature sign

Reversing the parametrization (clockwise vs. counterclockwise) flips the sign of κ.

Forces and moments (Cosserat rod view)

Decomposition in n and t

$$\begin{aligned} N' - \kappa\,T + f_n &= 0, \\ T' + \kappa\,N + f_t &= 0, \\ M' + T + m_b &= 0, \end{aligned}$$

Internal resultants: contact force $\boldsymbol{F}(s)$ (decomposed as $\boldsymbol{F} = N\boldsymbol{n} + T\boldsymbol{t}$) and bending moment $M(s)$ (out of plane, scalar).
Distributed loads: body force per unit length $\boldsymbol{f}(s)$ and body couple per unit length $m_b(s)$.
Concentrated loads at $s = s_0$: force $\boldsymbol{P}$ and couple M_0.

Local equilibrium (no concentrated loads in the open interval)

Vector form

$$\boxed{\boldsymbol{F}'(s) + \boldsymbol{f}(s) = 0}, \qquad \boxed{M'(s) + \boldsymbol{r}'(s) \times \boldsymbol{F}(s) + m_b(s) = 0}.$$

In 2D ($\boldsymbol{r}' = \boldsymbol{t}$; the cross product is out-of-plane):
where $f_n = \boldsymbol{f}\cdot\boldsymbol{n}$ and $f_t = \boldsymbol{f}\cdot\boldsymbol{t}$.

Action–reaction and jump conditions

Across a point s_0 *with no* concentrated load:

$$\boldsymbol{F}(s_0^-) = \boldsymbol{F}(s_0^+), \quad M(s_0^-) = M(s_0^+).$$

If a concentrated force $\boldsymbol{P}$ and/or couple M_0 act at s_0:

Jump conditions at s_0

$$\boldsymbol{F}(s_0^+) - \boldsymbol{F}(s_0^-) + \boldsymbol{P} = 0,$$
$$M(s_0^+) - M(s_0^-) + \underbrace{(\boldsymbol{r}'(s_0) \times 0)}_{=0 \text{ for point load on centerline}} + M_0 = 0.$$

(If $\boldsymbol{P}$ acts off the centerline, include the appropriate *moment of* $\boldsymbol{P}$ about the section.)

Endpoint conditions

At the right end $s = L$ with applied $\boldsymbol{F}^{\text{ext}}, M^{\text{ext}}$:

$$\boldsymbol{F}(L) + \boldsymbol{F}^{\text{ext}} = 0, \qquad M(L) + M^{\text{ext}} = 0.$$

At the left end $s = 0$ with applied $\boldsymbol{F}_0, M_0$ (note sign due to orientation):

$$\boldsymbol{F}(0) - \boldsymbol{F}_0 = 0, \qquad M(0) - M_0 = 0.$$

A *free* end has $\boldsymbol{F} = \boldsymbol{0},\ M = 0$.

Worked examples

Circular vault / arch of radius R

Parametrize counterclockwise by angle θ; arclength $s = R\theta$. Then

$$\boldsymbol{t} = (-\sin\theta,\ \cos\theta), \quad \alpha = \theta, \quad \kappa = \alpha' = \frac{1}{R}.$$

Clockwise parametrization gives $\kappa = -1/R$.

Cantilever of length L with a tip load $P\,\hat{\boldsymbol{y}}$ at $s = L$ (weightless beam)

No distributed loads: $\boldsymbol{f} = \boldsymbol{0}$, $m_b = 0$.

$$\boldsymbol{F}' = 0 \;\Rightarrow\; \boldsymbol{F}(s) \equiv (0, P) \quad \text{(constant shear, no axial)},$$
$$M' + T = 0 \;\Rightarrow\; M'(s) + P = 0 \;\Rightarrow\; M(s) = -P(L - s).$$

Reactions at the wall ($s = 0$): $\boldsymbol{F}(0) = (0, P)$ and $M(0) = PL$.

Body couple from eccentric reinforcement (reinforced concrete column)

Concrete matrix area A_c, density ρ_c; steel area A_m, density ρ_m; gravitational $\boldsymbol{g} = g\,\hat{\boldsymbol{y}}$; steel centroid offset h from the centerline.

$$\text{Body force per length:} \quad \boldsymbol{f} = (\rho_c A_c + \rho_m A_m)\,\boldsymbol{g},$$
$$\text{Body couple per length:} \quad m_b = \rho_m A_m\, g\, h.$$

If $h = 0$ (reinforcement on centerline), then $m_b = 0$.

At-a-glance checklist

- Pick orientation and note κ sign.
- Resolve loads along $(\boldsymbol{n}, \boldsymbol{t})$; write N', T', M' equations.
- Apply jump/endpoint conditions (free, clamped, loaded).
- Integrate from boundary where reactions are known.

Minimal 3D→1D link (why the equations look like this)

Balancing 3D tractions and body forces over a thin control volume around a section and taking the slender limit yields the 1D balance: $\boldsymbol{F}' + \boldsymbol{f} = 0$ and $M' + \boldsymbol{t} \times \boldsymbol{F} + m_b = 0$. The action–reaction and jump conditions follow from the same control-volume argument.

Chapter 2

Deformation and Elasticity

Kinematics: reference vs. deformed configurations

Let S denote the arclength in the *reference* configuration and s the arclength in the *deformed* configuration. A material point is labeled by $S \in [0, L_0]$ and maps to a current position $\boldsymbol{r}(S)$ in the equilibrium configuration. Let $\alpha_0(S)$ be the tangent angle of the reference centerline and $\alpha(S)$ that of the deformed centerline. Define

$$\text{stretch} \quad E(S) := \frac{\mathrm{d}s}{\mathrm{d}S} - 1 = \|\boldsymbol{r}'(S)\| - 1, \qquad \text{flexural strain} \quad K(S) := \omega'(S), \; \omega(S) := \alpha(S) - \alpha_0(S).$$

Idea (measures of deformation).

E captures *in-plane stretching* (change of length); K captures *bending* (change of orientation relative to the reference). For a pure rigid motion: $E \equiv 0$ and $K \equiv 0$.

Useful relations.

$$\boldsymbol{t}_0 = (\cos\alpha_0, \sin\alpha_0), \;\; \boldsymbol{t} = (\cos\alpha, \sin\alpha), \;\; \kappa_0 = \frac{\mathrm{d}\alpha_0}{\mathrm{d}s_0}, \;\; \kappa = \frac{\mathrm{d}\alpha}{\mathrm{d}s}, \quad \text{and} \quad K = \frac{\mathrm{d}\alpha}{\mathrm{d}S} - \frac{\mathrm{d}\alpha_0}{\mathrm{d}S}.$$

Change of curvature vs. K

The geometric change of curvature $\bar{\kappa} := \kappa - \kappa_0$ is *not* the same as K in general (they coincide when $E \equiv 0$). This justifies using E and K as independent strain measures.

Examples (kinematics only)

Uniform stretching of a straight wire.

If $s' = \mathrm{d}s/\mathrm{d}S = 1 + \varepsilon$ (constant), then $E = \varepsilon$ (constant) and $K = 0$.

Uniform coiling of an unstretched string on a circle of radius R.

If $s' = 1$ (no stretch), the string winds into a circle: $\alpha'(S) = 1/R$ so $K = 1/R$ and $E = 0$.

Uniform dilation of a circular ring from R_0 to R.

The ring expands uniformly with no additional rotation relative to the reference centerline, giving $E = \dfrac{R - R_0}{R_0}$ and $K = 0$ (even though κ changes from $1/R_0$ to $1/R$).

Statics recap in Lagrangian form (per reference length)

Let $\boldsymbol{F}(S) = N\,\boldsymbol{n} + T\,\boldsymbol{t}$ be the internal force resultant and $M(S)$ the internal bending moment (out of plane). With body force $\boldsymbol{f}(S)$ and body couple $m_b(S)$ *per unit reference length*, the local balances read

Local equilibrium (no point loads in the open interval).

$$\boldsymbol{F}'(S) + \boldsymbol{f}(S) = 0, \qquad M'(S) + \boldsymbol{t}(S) \times \boldsymbol{F}(S) + m_b(S) = 0.$$

Jumps and endpoints.

Across S_0 with a concentrated force $\boldsymbol{P}$ and couple M_0: $\boldsymbol{F}^+ - \boldsymbol{F}^- + \boldsymbol{P} = 0$, $M^+ - M^- + M_0 = 0$. At a free end: $\boldsymbol{F} = \boldsymbol{0}$, $M = 0$.

Virtual work and elastic energy

Consider a quasi-static adjacent equilibrium $\delta\boldsymbol{r}(S)$. The external virtual work equals the internal one:

Power-conjugate pairs.

$$\delta W = \int_0^{L_0} (N\,\delta E + M\,\delta K)\,\mathrm{d}S.$$

In an elastic rod, this derives from a *lineal strain-energy density* $W_e(E, K)$:

Constitutive in energetic form.
$$N = \frac{\partial W_e}{\partial E}, \qquad M = \frac{\partial W_e}{\partial K}.$$

Linear elasticity (Hooke-type rod)

For small strains and small curvature changes, a standard quadratic density is

$$W_e(E, K) = \tfrac{1}{2}\,EA\,E^2 + \tfrac{1}{2}\,EI\,K^2,$$

where EA is the *axial stiffness* and EI the *bending stiffness* (both possibly varying with S for inhomogeneous rods).

Linear constitutive laws.

$N = EA\,E, \qquad M = EI\,K.$

Section properties.

For homogeneous isotropic material with Young's modulus E_Y: $EA = E_Y A$, $EI = E_Y I$, where A is the cross-sectional area and I the (geometric) second moment of area about the out-of-plane axis of bending.

Order-of-magnitude (steel).

$E_Y \approx 200\,\mathrm{GPa}$. A $1\,\mathrm{m}$ steel rod with $A = 1\,\mathrm{cm}^2$ under $P = 1000\,\mathrm{N}$ in uniform tension stretches by $\Delta L \approx \dfrac{PL}{EA} \sim 5 \times 10^{-6}\,\mathrm{m}$.

Special limiting models

Summary of idealizations.

- **Inextensible rod:** $E \equiv 0$ (constraint). N becomes a reaction (not constitutively prescribed). Bending elastic with $M = EI\,K$.
- **Inflexible rod (rigid bar):** $K \equiv 0$ (constraint). M becomes a reaction. Axial elastic with $N = EA\,E$.
- **String (perfectly flexible):** $M \equiv 0$; only axial response $N = EA\,E$ (or inextensible string: $E \equiv 0$, N reactive).

Checklist for problems

1. Choose reference $\to$ current orientation; note whether E or K are constrained (idealizations).
2. Write local equilibrium in S; decompose along $(\boldsymbol{t}, \boldsymbol{n})$ if needed.
3. Use constitutive laws ($N = EA\,E$, $M = EI\,K$) or constraints (inextensible/inflexible).
4. Apply jumps/endpoint conditions; integrate from a boundary with known reactions.

Chapter 3

Equilibrium Boundary-Value Problems

Overview

A static boundary-value problem (BVP) combines:

1. geometric relations (kinematics),
2. equilibrium equations (force and moment balance),
3. constitutive relations ($N(E,K)$, $M(E,K)$),
4. and boundary conditions (BCs) at endpoints.

At each endpoint we impose either **position** or **force**, and either **orientation** or **moment**, but not both. $\Rightarrow$ Three BCs total: two translational, one rotational.

Typical boundary conditions

Common boundary conditions

Clamped (built-in): both position and tangent angle α prescribed.
Pinned (hinged): position prescribed, rotation free $\Rightarrow M = 0$.
Roller: point slides on a known curve $y_c(x)$ (one geometric constraint), rotation free $\Rightarrow M = 0$.
Loaded end: applied resultant $\boldsymbol{F}$ and/or moment M specified.
Periodic: for a closed ring, $\boldsymbol{r}(0) = \boldsymbol{r}(L)$, $\alpha(0) = \alpha(L)$, $N(0) = N(L)$, $M(0) = M(L)$.

Heuristic rule for endpoints

At each end: impose (either E or $\boldsymbol{r}$) and (either M or α).

Summary of governing equations (Lagrangian form)

Governing equations

$$
\begin{aligned}
\text{Geometry:}\quad & \boldsymbol{r}' = \boldsymbol{t} = (\cos\alpha, \sin\alpha), \qquad \alpha' = K,\\
\text{Force balance:}\quad & \boldsymbol{F}' + \boldsymbol{f} = 0, \qquad \boldsymbol{F} = N\boldsymbol{n} + T\boldsymbol{t},\\
\text{Moment balance:}\quad & M' + \boldsymbol{t} \times \boldsymbol{F} + m_b = 0,\\
\text{Constitutive:}\quad & N = EA\,E, \qquad M = EI\,K \quad \text{(or degenerate models)},\\
\text{BCs:}\quad & \text{as above (position/force + angle/moment).}
\end{aligned}
$$

Rigid-body model

Rigid $E = 0$, $K = 0$. Only reactions are determined by equilibrium.

Rigid cantilever with uniform weight

Length L, uniform load qg per length, clamped at $S = 0$, free at $S = L$.
Force balance: $T'(S) + qg = 0 \Rightarrow T(S) = qg(L - S)$, $N(S) = 0$.
Moment balance: $M'(S) + T(S) = 0 \Rightarrow M(S) = \frac{1}{2}qg(L - S)^2$.
At wall ($S = 0$): $T(0) = qgL$, $M(0) = \frac{1}{2}qgL^2$.

Roller at free end

Replacing the free condition by a roller removes one reaction: $M(L) = 0$, but the tangential reaction remains unknown—consistent with one less BC.

Rigid ring under uniform pressure p

Equilibrium of a rigid ring

Normal force: $N(S) = pR\,\cos(\theta - \theta_0)$,
Shear force: $T(S) = pR\,\sin(\theta - \theta_0)$,
Bending moment: $M(S) = pR^2 \sin(\theta - \theta_0)$.

Interpretation

The rigid model leaves three arbitrary constants (global translation and rotation). Elasticity removes this indeterminacy.

Elastic ring

Uniform contraction under pressure

With $N = EA\,E$, $M = EI\,K$, assume uniform contraction so $K = 0$, E constant. Equilibrium $\Rightarrow N = pR = EA\,E \;\Rightarrow\; E = \frac{pR}{EA}$.

Remarks

For $p > 0$ (compression), large enough p produces buckling (critical $p_{\mathrm{cr}} \sim 3EI/R^3$). For $p < 0$, expansion is uniform.

Strings

For a string, $EI = 0 \Rightarrow M = 0$.

String equilibrium

$N' + f_t = 0, \qquad T \equiv 0, \qquad \boldsymbol{F} = N\boldsymbol{t}.$

Tangent discontinuities

When $EI = 0$, bending energy vanishes and α may be discontinuous. Therefore α cannot be prescribed at endpoints.

Extensible string pulled by a tip force F

Weightless, pinned at $S = 0$, tension $N(L) = F$. Force balance $\Rightarrow N(S) = F$. From $N = EA\,E$ we get $E = \frac{F}{EA}$ constant. For an inextensible string ($E = 0$), N is not given constitutively but adjusts to satisfy geometry and BCs.

Summary of models

Model	Assumptions	Constitutive laws	Remarks
Rigid bar	$E = 0,\ K = 0$	none (reactions only)	No deformation
Elastic rod	small E, K	$N = EA\,E,\ M = EI\,K$	Standard elastic
Inextensible rod	$E = 0$	$M = EI\,K$	N reactive
String	$EI = 0$	$N = EA\,E,\ M = 0$	No bending
Inextensible string	$E = 0,\ EI = 0$	none (N reactive)	Perfectly flexible

Chapter 4

Model Comparison in Statics

Introduction

We now compare the various models introduced so far — from the **rigid body** (no deformation) to the **extensible string** (perfectly flexible). The central one is the **elastic rod**, characterized by both axial and bending stiffnesses (EA, EI). While the elastic model is the most accurate, it can be mathematically demanding; simpler models may reproduce its behavior accurately under specific load regimes.

Goal

Find which simplified model best approximates the full elastic model for a given structure, material, and load magnitude.

Galileo's beam setup

A beam of stiffness (EA, EI), length L, clamped at $S = 0$, and loaded by a vertical force F at its tip.

Dimensionless form

Introducing $\xi = S/L$ and the critical Euler load $F_c = \pi^2 EI/L^2$, we obtain:

$$\alpha'' + \lambda^2 \sin\alpha = 0, \qquad \alpha(0) = 0,\ \alpha(1) = 0,$$

where $\lambda^2 = \frac{FL^2}{EI}$ measures load magnitude relative to bending stiffness.

Load regimes

Regimes by order of magnitude

Regime	Dominant features	Appropriate model
$F \ll EI/L^2$	small rotations, small curvature	linearized (Hookean) beam
$F \sim EI/L^2$	finite rotations, inextensible	nonlinear elastic rod
$F \gg EI/L^2$	large deflections, negligible bending	string (flexible) model

Small load regime

For $F/EI \ll 1$, $\sin\alpha \approx \alpha$ and the problem linearizes:

$$EI\,\alpha'' + F\,\alpha = 0, \qquad \alpha(0) = \alpha(L) = 0.$$

Linearized solution

$$\alpha(S) = \frac{FL^2}{2EI}\left(\frac{S}{L}\right)\left(1 - \frac{S}{L}\right), \quad M(S) = EI\,\alpha'(S) = F(L-S), \quad T(S) = F, \quad N(S) \approx 0.$$

Interpretation

The linearized beam behaves almost rigidly — internal forces and moments match those of the rigid model to first order. Deformations, though small, can be computed analytically.

Moderate loads: nonlinear elastic rod

For $F/EI = \mathcal{O}(1)$, extensibility remains negligible ($E \simeq 0$), but the sine term must be retained.

Phase-space formulation

Multiplying by α', integrate once:

$$\tfrac{1}{2}EI(\alpha')^2 = F(1 - \cos\alpha) + C.$$

This defines trajectories in (α, α') phase space corresponding to different load levels and boundary angles.

Bifurcation and branches

When F exceeds the Euler critical value F_c, multiple equilibrium shapes (buckled branches) appear. These are symmetric, periodic solutions $\alpha(S)$ corresponding to arcs AB, CD, etc. in phase space.

Large load regime

For $F/EI \gg 1$, bending stiffness becomes negligible except near the clamped end. The rod behaves as a **string** almost everywhere.

Asymptotic behavior

Outer region: $EI\alpha'' \approx 0 \Rightarrow \cos\alpha \approx 0 \Rightarrow$ string-like.
Inner region: $EI\alpha''$ balances $F\cos\alpha$, forming a narrow **boundary layer** near $S = 0$.

Boundary layer correction

Let $x = S/\delta$, with $\delta \sim (EI/F)^{1/2}$ the thickness where bending and tension balance. The boundary-layer equation reads:

$$\alpha'' = \cos\alpha, \quad \alpha(0) = 0, \quad \alpha(+\infty) = \alpha_{\text{string}}.$$

Its unique smooth solution connects the clamped boundary to the outer (string) region.

Boundary-layer solution

$$\alpha(x) = 4\arctan\big[\tanh(x/2)\big],$$

with rotation rapidly changing over $x = \mathcal{O}(1)$ then matching the string angle $\alpha \to \alpha_{\text{string}}$.

Model validity summary

Summary diagram

Load range	Dominant physics	Best model
$F \ll 0.25\,F_c$	small deflection	Linearized beam
$0.25F_c \lesssim F \lesssim 5F_c$	finite deflection	Nonlinear elastic rod
$F \gg 5F_c$	near-string behavior	String + boundary layer

Takeaway

The most appropriate model depends not only on material and geometry (EA, EI), but also on the *magnitude of the applied load*:

$$\frac{FL^2}{EI} \ll 1 \Rightarrow \text{linear}, \quad \frac{FL^2}{EI} \sim 1 \Rightarrow \text{elastic}, \quad \frac{FL^2}{EI} \gg 1 \Rightarrow \text{string with boundary layer}.$$

Chapter 5

Linear Elasticity

Prelude: Galileo's beam in the small-load regime

For small Euler number $F/F_c \ll 1$, the nonlinear elastica reduces to a linear theory. The equilibrium configuration coincides (to first order) with the reference configuration predicted by the rigid-body model.

Observation

Internal forces (N, T, M) are the same as in the rigid-body model, but now the small *deformation* field (u, w) can be explicitly computed.

Apparent paradox

Inextensibility implies $u = 0$, yet a transverse displacement $w(S)$ changes the length. Resolution: going to higher order in the asymptotic expansion introduces a small longitudinal correction $u \neq 0$.

Assumptions of linear elasticity

We assume:

- Small external loads $\Rightarrow$ small internal forces and moments;
- Small displacements and rotations: $|u'|, |w'| \ll 1$;
- Small strain and curvature: $|E|, |K| \ll 1$.

Physical meaning

"Small" means relative to the **critical buckling load**, which measures the intrinsic bending stiffness EI of the structure. For such small loads, nonlinear geometric effects are negligible.

Linearized kinematics

Let the reference and deformed configurations be parameterized by S and

$$\boldsymbol{r}(S) = \boldsymbol{r}_0(S) + u(S)\,\boldsymbol{t}_0 + w(S)\,\boldsymbol{n}_0.$$

Compatibility relation

Differentiating $\boldsymbol{r}' = (1+E)\boldsymbol{t}$ and projecting along $\boldsymbol{n}$ yields:

$$w''(S) = K(S), \qquad u'(S) = E(S).$$

Thus smallness of E and K follows directly from the smallness of the displacement gradients.

Linearized equilibrium equations

Starting from $\boldsymbol{F}' + \boldsymbol{f} = 0$ and $M' + \boldsymbol{t} \times \boldsymbol{F} + m_b = 0$:

Simplification

These equations are evaluated on the reference configuration (undeformed geometry). Body forces and couples are also taken with respect to this configuration, making the system linear.

Linearized constitutive relations

Hooke-type constitutive laws

$$N = EA\,E, \qquad M = EI\,K,$$

derived from the quadratic strain-energy density:

$$W_e(E, K) = \tfrac{1}{2}EA\,E^2 + \tfrac{1}{2}EI\,K^2.$$

Remarks

For a homogeneous material, $EA = E_Y A$ and $EI = E_Y I$, where E_Y is Young's modulus. Linearization is valid about the natural (unstressed) configuration where $N = M = 0$.

Linearized boundary conditions

Boundary conditions summary

- **Clamped end:** $u = w = w' = 0$ (both position and rotation fixed).
- **Pinned end:** $u = w = 0$, $M = 0$ (rotation free).
- **Roller:** u constrained along tangent of support, $M = 0$.
- **Free end:** $N = T = M = 0$.
- **Loaded end:** $N = N_{\text{ext}}$, $T = T_{\text{ext}}$, $M = M_{\text{ext}}$ prescribed.

Solving the linearized BVP

Integrate successively:

1. From force balance $N' + f_t = 0$, obtain $N(S)$;
2. Then $T(S)$ from $T' + f_n = 0$;
3. From $M' + T = 0$ and BCs, compute $M(S)$;
4. Use $M = EI\,w''$ to integrate for $w(S)$ (and $E = u'$ for $u(S)$).

Example: clamped–roller beam under uniform weight

Weight qg per length, clamped at $S = 0$, roller at $S = L$.

$$qg = f_n, \quad f_t = 0.$$

Then:

$$N = 0, \quad T(S) = qg(L - S), \quad M(S) = \tfrac{1}{2}qg(L - S)^2.$$

Deflection:

$$EI\,w'''' = qg, \quad w(0) = w'(0) = 0,\ w''(L) = w'''(L) = 0.$$

Solution:

$$w(S) = \frac{qg}{24EI}\,S^2(6L^2 - 4LS + S^2).$$

Validation of assumptions

The linear theory is consistent if $|w'| \ll 1$ and $|M|/(EI/L) \ll 1$, which translates to small load-to-stiffness ratio $qgL^3/(EI) \ll 1$.

Relation with Galileo's asymptotic expansion

Comparison with asymptotic expansion

- Both yield identical internal force and moment distributions as the rigid model (to first order).
- Linear elasticity allows explicit computation of u, w and checks of smallness assumptions.
- Validity breaks when rotations are no longer small, i.e. $F/F_c \gtrsim 0.25$.

Key takeaway

Linear elasticity is the simplest, self-consistent model for small displacements and rotations. It bridges the gap between rigid-body statics and nonlinear elasticity, forming the analytical foundation for most engineering beam and frame analyses.

Chapter 6

Stability of Conservative Systems with N Degrees of Freedom (Part 1)

Conservative forces

A force $\boldsymbol{F}(\boldsymbol{x})$ is **conservative** if it derives from a potential $U(\boldsymbol{x})$ such that

$$\boldsymbol{F}(\boldsymbol{x}) = -\nabla U(\boldsymbol{x}).$$

Key property

The work done by a conservative force is path-independent:

$$W_{A\to B} = -\int_A^B \nabla U \cdot \mathrm{d}\boldsymbol{x} = U(A) - U(B).$$

Examples and counterexamples

- Gravity: $\boldsymbol{F} = m\boldsymbol{g}$, $U = mgz$.
- Spring: $\boldsymbol{F} = -k\boldsymbol{x}$, $U = \frac{1}{2}k|\boldsymbol{x}|^2$.
- Inertial (centrifugal) force in rotating frame: $\boldsymbol{F} = m\omega^2\boldsymbol{r}$, $U = -\frac{1}{2}m\omega^2 r^2$.
- *Non-conservative:* a velocity-dependent or non-curl-free field (e.g. $\boldsymbol{F} = (k\,y, 0)$) for which $\nabla \times \boldsymbol{F} \neq 0$.

Conservative elastic systems

A conservative system has no dissipation; its total potential energy is

$$\Pi[\boldsymbol{r}] = \Pi_{\text{int}} + \Pi_{\text{ext}},$$

where

$$\Pi_{\text{int}} = \int W_e(E, K)\,\mathrm{d}S, \qquad \Pi_{\text{ext}} = -\int \boldsymbol{f}\cdot\boldsymbol{r}\,\mathrm{d}S - \boldsymbol{F}\cdot\boldsymbol{r}_{\text{end}}.$$

Typical internal energies

- Rigid body: $\Pi_{\text{int}} = 0$.
- Inextensible string: $\Pi_{\text{int}} = 0$.
- Elastic string: $\Pi_{\text{int}} = \frac{1}{2}EA\int E^2\,\mathrm{d}S$.

- Elastic rod: $\Pi_{\text{int}} = \frac{1}{2} EI \int K^2 \,\mathrm{d}S$.
- Linear elastic rod: $\Pi_{\text{int}} = \frac{1}{2} \int (EA\, E^2 + EI\, K^2) \,\mathrm{d}S$.

Examples

Elastic string: pinned at one end, force F at the other: $\Pi = \frac{1}{2} EA \int_0^L (s' - 1)^2 \,\mathrm{d}S - F\, y(L)$.

Elastic rod: clamped at one end, tip load F: $\Pi = \frac{1}{2} \int_0^L (EA\, E^2 + EI\, K^2) \,\mathrm{d}S - F\, y(L)$.

Kinematically admissible configurations

A configuration $\mathcal{C}$ is **kinematically admissible** (K.A.) if:

- It satisfies all kinematic BCs and constraints (e.g. rigidity, inextensibility);
- It yields a finite total potential energy $\Pi[\mathcal{C}]$.

Example: inextensible string

$$\mathcal{C}_{\text{KA}} = \{ \boldsymbol{r}(S) : |\boldsymbol{r}'(S)| = 1,\ \boldsymbol{r}(0) = \boldsymbol{r}_0 \}.$$

If bending rigidity vanishes ($EI = 0$), $\alpha(S)$ may jump, but $\boldsymbol{r}$ remains continuous.

Stability criterion

Definition of stability

An equilibrium configuration $\boldsymbol{r}_e$ is **stable** if every nearby admissible configuration $\boldsymbol{r}_e + \delta\boldsymbol{r}$ yields higher (or equal) potential energy:

$$\Pi[\boldsymbol{r}_e + \delta\boldsymbol{r}] \geq \Pi[\boldsymbol{r}_e].$$

Equivalently, $\boldsymbol{r}_e$ is a *local minimum* of Π.

Finite-dimensional form

For N generalized coordinates $\boldsymbol{q} = (q_1, \ldots, q_N)$,

$$\Pi = \Pi(\boldsymbol{q}), \qquad \text{equilibrium: } \frac{\partial \Pi}{\partial q_i} = 0.$$

Stability test: analyze the Hessian matrix

$$H_{ij} = \frac{\partial^2 \Pi}{\partial q_i \partial q_j}.$$

Stability conditions

- **Necessary:** $\dfrac{\partial \Pi}{\partial q_i} = 0$ at equilibrium.
- **Sufficient for stability:** Hessian H positive definite ($\lambda_i > 0$).
- **Sufficient for instability:** at least one negative eigenvalue ($\lambda_i < 0$).

Example 1: Rotating rigid bar

Equilibrium and stability

$$\frac{\mathrm{d}\Pi}{\mathrm{d}\alpha} = 0 \;\Rightarrow\; mg \sin\alpha = I\omega^2 \sin\alpha\cos\alpha.$$

Solutions:

$$\sin\alpha = 0 \quad (\alpha = 0, \pi), \qquad \cos\alpha = \frac{mg}{I\omega^2}.$$

Stability: $\Pi''(\alpha) > 0$ stable. Thus the downward configuration ($\alpha = 0$) is stable if ω is small; beyond a critical ω_c, instability (bifurcation) occurs.

Interpretation

At equilibrium, the stationarity of Π coincides with the force and moment balances. A loss of stability (change of sign in Π'') signals buckling or dynamic reversal.

Example 2: Rotating rigid beam on a roller

The beam can slide freely along the horizontal axis while rotating about a roller at its lower end.

Result

The equilibrium manifold includes $\beta = 0$ and $\sin\alpha = 0$ (vertical) or $\cos\alpha = \frac{mg}{I\omega^2}$ (tilted). The Hessian has one zero eigenvalue (due to free translation), implying that every equilibrium is marginally unstable.

Energy interpretation

For the pinned bar, $\alpha = 0$ is a *minimum* of Π (stable). For the roller-supported bar, the same configuration is a *saddle point* — potential energy is stationary but not minimal.

General conclusions

Summary

- Stability of a conservative system Π has a local minimum.
- Equilibrium configurations satisfy $\nabla\Pi = 0$.
- Positive definite Hessian stable; indefinite unstable.
- The energy criterion is equivalent to the force–moment balance laws.

Takeaway

In conservative mechanics, equilibrium and stationarity of total potential energy coincide. The **nature of equilibrium** (minimum, maximum, saddle) dictates **stability**.

$$\text{Minimum} \quad \text{Stable}, \qquad \text{Maximum/Saddle} \quad \text{Unstable}.$$

Stability of the Rubber Balloon

We worked on Recitation 7, on the problem of the Rubber Balloons, and here are the main results

Prelude – Fluid–structure interaction

Inflating a rubber balloon couples a **fluid** (air) and a **deformable membrane** (rubber). The internal pressure stretches the elastic membrane, whose material exhibits a **nonlinear constitutive law**. Because the membrane is thin and soft, its bending stiffness is negligible.

Instability of coupled balloons

When two identical balloons are connected by a tube, one may inflate while the other deflates — a spontaneous **symmetry breaking** due to nonlinear elasticity and fluid coupling.

Equilibrium of an elastic circular membrane

Neglecting gravity, consider a closed elastic string enclosing a fluid at pressure $p > 0$. In its reference configuration, the string has radius R_0 and length $\ell_R = 2\pi R_0$.

Let ε denote the stretch of the string, with constitutive law

$$N = N_e(\varepsilon) = W_e'(\varepsilon),$$

where $W_e(\varepsilon)$ is the strain energy per unit reference length.

Uniform equilibrium condition

For circular equilibrium with uniform strain:

$$\boxed{\frac{N_e(\varepsilon)}{1+\varepsilon} = p\,R_0.}$$

Interpretation

The ratio $\dfrac{N_e}{1+\varepsilon}$ represents the internal stress per unit deformed radius; equilibrium is reached when it balances the fluid pressure.

Linear elasticity check

For a linearly elastic string $N_e = EA\,\varepsilon$:

$$\frac{EA\,\varepsilon}{1+\varepsilon} = pR_0,$$

which reaches a maximum at a finite ε. Therefore, there exists a **critical pressure** p_c beyond which no equilibrium is possible.

Pressure–controlled experiment

In this setting, the applied pressure p is prescribed, and we seek equilibrium radii $R = (1+\varepsilon)R_0$.

Total potential energy

$$P_{\text{tot}}(\varepsilon) = 2\pi R_0\, W_e(\varepsilon) - \pi R_0^2\, p\,(1+\varepsilon)^2.$$

Equilibrium equation

$$\frac{dP_{\text{tot}}}{d\varepsilon} = 0 \quad \Longrightarrow \quad \boxed{\frac{W_e'(\varepsilon)}{1+\varepsilon} = pR_0.}$$

Define

$$f(\varepsilon) = \frac{W_e'(\varepsilon)}{1+\varepsilon}.$$

Stability criterion

$$f'(\varepsilon) > 0 \;\Rightarrow\; \text{stable}, \qquad f'(\varepsilon) < 0 \;\Rightarrow\; \text{unstable}.$$

Physical meaning

As p increases, the function $f(\varepsilon)$ may intersect pR_0 multiple times:

- At low p: single small-ε stable equilibrium.
- At intermediate p: three equilibria (two stable, one unstable).
- At high p: single large-ε stable equilibrium.

This leads to the characteristic **snap-through inflation curve**.

Pressure cycle

Hysteresis

When p increases from 0 to a critical p^* and then decreases, the balloon inflates and deflates along different paths — a **hysteresis loop** typical of nonlinear instabilities.

Mass–controlled experiment

Now the total mass m of air (rather than p) is fixed. At constant temperature, the gas behaves ideally:

$$p = \frac{km}{A}, \qquad A = \pi R_0^2(1+\varepsilon)^2.$$

Pressure work at fixed mass

$$\delta W_p = p\,\delta A = km\,\frac{\delta A}{A} \quad \Rightarrow \quad W_p = km\ln\frac{A}{A_0} = 2km\ln(1+\varepsilon).$$

Total potential energy

$$P_{\text{tot}}(\varepsilon) = 2\pi R_0\,W_e(\varepsilon) - 2km\ln(1+\varepsilon).$$

Equilibrium condition

$$\frac{dP_{\text{tot}}}{d\varepsilon} = 0 \quad \Longrightarrow \quad \boxed{\frac{W_e'(\varepsilon)}{1+\varepsilon} = \frac{km}{\pi R_0^2(1+\varepsilon)^2}.}$$

Single–balloon stability

This equation admits a **unique solution** $\varepsilon^*(m)$ which is always stable and increases monotonically with m.

Two coupled balloons

Two identical balloons are connected by a small pipe, with total gas mass $2m$.

Closed tap

Each balloon independently contains mass m and has the same equilibrium stretch ε^*.

Open tap

When the tap opens, the pressures equalize, but the balloons may deform differently. Let $\varepsilon_1, \varepsilon_2$ be their stretches.

Total potential energy of the system

$$P_{\text{tot}}(\varepsilon_1,\varepsilon_2) = 2\pi R_0\left[W_e(\varepsilon_1) + W_e(\varepsilon_2)\right] - 2km\,\ln\left[\frac{(1+\varepsilon_1)^2 + (1+\varepsilon_2)^2}{2}\right].$$

Symmetric equilibrium

$$\boxed{\varepsilon_1 = \varepsilon_2 = \varepsilon^*.}$$

Stability condition

The symmetric configuration is

$$\begin{cases} \text{stable if } (1+\varepsilon^*)W_e''(\varepsilon^*) > W_e'(\varepsilon^*), \\ \text{unstable if } (1+\varepsilon^*)W_e''(\varepsilon^*) < W_e'(\varepsilon^*). \end{cases}$$

Physical interpretation

At intermediate inflation levels, the symmetric state loses stability. One balloon inflates the other — precisely the behavior observed in the laboratory.

Summary

- Nonlinear elasticity of rubber leads to multiple equilibrium branches.
- Under **pressure control**, stability follows the slope $f'(\varepsilon)$ of $f(\varepsilon) = W_e'/(1+\varepsilon)$.
- Under **mass control**, the system can exchange air and exhibit symmetry breaking.
- Coupled balloons illustrate a clean example of **bifurcation and instability** in soft elastic structures.