0 Quick constants & symbols

Speed of light c, Lorentz factor $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$, $\beta = v/c$. Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$; raise/lower with η.

1 Foundations of SR (kinematics)

Postulates, dilation, contraction, velocity addition

Time dilation: $\Delta t = \gamma\,\Delta\tau$. **Length contraction:** $L = \dfrac{L_0}{\gamma}$.

Standard Lorentz boost (x):
$$\begin{aligned} ct' &= \gamma\,(ct - \beta x), & x' &= \gamma\,(x - \beta ct),\\ y' &= y, & z' &= z. \end{aligned}$$

Velocity addition (colinear): $u' = \dfrac{u-v}{1-\dfrac{uv}{c^2}}$.

Invariant interval & proper quantities

Spacetime interval: $(\Delta s)^2 = (c\Delta t)^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$ (invariant).
Proper time: timelike separation with $\Delta\mathbf{x} = 0 \Rightarrow c\,\Delta\tau = \sqrt{(\Delta s)^2}$.
Proper length: spacelike separation with $\Delta t = 0 \Rightarrow L = \sqrt{-(\Delta s)^2}$.

2 Tensor formalism (covariant geometry)

Vectors, indices, and Einstein summation

Einstein summation: any index repeated once up and once down is summed. A vector in basis $\{e_\mu\}$:

$$X = x^\mu e_\mu, \qquad x'^\mu = \Lambda^\mu{}_\nu\, x^\nu.$$

The basis transforms inversely: $e'_\mu = (\Lambda^{-1})^\nu{}_\mu e_\nu$. *(Sess. 2, Lect. 2)*

Metric tensor and raising/lowering

Minkowski metric (flat spacetime):

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \eta^{\mu\nu} = (\eta_{\mu\nu})^{-1}.$$

Index operations:
$$x_\mu = \eta_{\mu\nu} x^\nu, \qquad x^\mu = \eta^{\mu\nu} x_\nu.$$

Scalar product (invariant): $X\cdot Y = \eta_{\mu\nu} x^\mu y^\nu = x_\mu y^\mu$. *(Sess. 2 §5–6, Summ. 2)*

Definition and transformation of tensors

A tensor of type (p,q) is a multilinear map

$$T : \underbrace{E^* \times \cdots \times E^*}_{p} \times \underbrace{E \times \cdots \times E}_{q} \to \mathbb{R}.$$

Its components:

$$T^{\nu_1 \ldots \nu_q}{}_{\mu_1 \ldots \mu_p} = T(e_{\mu_1}, \ldots, e_{\mu_p}, e^{\nu_1}, \ldots, e^{\nu_q}).$$

Under a change of basis Λ,

$$T'^{\nu_1 \ldots \nu_q}{}_{\mu_1 \ldots \mu_p} = (\Lambda^{-1})^{\rho_1}{}_{\mu_1} \cdots (\Lambda^{-1})^{\rho_p}{}_{\mu_p} \Lambda^{\nu_1}{}_{\sigma_1} \cdots \Lambda^{\nu_q}{}_{\sigma_q} T^{\sigma_1 \ldots \sigma_q}{}_{\rho_1 \ldots \rho_p}.$$

(Sess. 2 §7)

Special tensors

Metric $\eta_{\mu\nu}$: symmetric rank-2 $(0,2)$ tensor. Inverse metric $\eta^{\mu\nu}$: rank-2 $(2,0)$ tensor. Kronecker $\delta^\mu{}_\nu$: rank-2 $(1,1)$ tensor with same components in every basis. *(Sess. 2 §8)*

Tensor contractions and invariants

Contraction over one upper and one lower index yields a new tensor of rank $r-2$. Examples:

$$x^\mu T^\nu{}_\mu \text{ is a contravariant vector,} \qquad T^\nu{}_\mu x_\nu \text{ is a covector.}$$

Scalars (Lorentz invariants): $T^\mu{}_\mu$, $x^\mu T^\nu{}_\mu x_\nu$. *(Sess. 2 §9)*

Application: Minkowski spacetime

Four-vector coordinates:

$$x^\mu = (ct, x, y, z), \quad x_\mu = \eta_{\mu\nu} x^\nu = (ct, -x, -y, -z).$$

Invariant norm:

$$X^2 = \eta_{\mu\nu} x^\mu x^\nu = (ct)^2 - x^2 - y^2 - z^2.$$

Scalar product of $A^\mu = (A^0, \boldsymbol{A})$ and $B^\mu = (B^0, \boldsymbol{B})$:

$$A \cdot B = A^0 B^0 - \boldsymbol{A} \cdot \boldsymbol{B}.$$

Transformations preserving $\eta_{\mu\nu}$ are Lorentz transformations satisfying $\Lambda^T \eta \, \Lambda = \eta$. *(Sess. 2 §2, 6; Summ. 2)*

3 Four-vectors & tensors (covariant formalism)

Basics

Position 4-vector: $X^\mu = (ct, x, y, z)$. 4-velocity: $U^\mu = \dfrac{dX^\mu}{d\tau}$ with $U^2 = \eta_{\mu\nu} U^\mu U^\nu = c^2$.
Index gymnastics: $x_\mu = \eta_{\mu\nu} x^\nu$, $x^\mu = \eta^{\mu\nu} x_\nu$.
Scalar product: $X \cdot Y = \eta_{\mu\nu} X^\mu Y^\nu = X_\mu Y^\mu$ (Lorentz invariant). *(Lect. 2, Sess. 2)*

Classification

Timelike: $X^2 > 0$ (proper time exists). Spacelike: $X^2 < 0$ (proper length). Null: $X^2 = 0$. *(Lect. 2)*

4 Relativistic phenomena (exam regulars)

Twin paradox (resolution cue)

Asymmetry arises from acceleration/frame switch of the traveling twin $\Rightarrow$ worldline lengths differ; compare proper times via $d\tau = dt/\gamma$. *(Lect. 2–3)*

Relativistic Doppler & aberration

Doppler (general): $\nu'_r = \dfrac{\nu_e}{\gamma(1-\beta\cos\theta')}$; transverse ($\theta' = \pi/2$): $\nu'_r = \nu_e/\gamma$.

Aberration: $\cos\theta' = \dfrac{\cos\theta + \beta}{1+\beta\cos\theta}$, $\sin\theta' = \dfrac{\sin\theta}{\gamma(1+\beta\cos\theta)}$. *(Lect. 2–3, Sum. 2)*

Apparent superluminal motion

$\beta_{\text{app}} = \dfrac{\beta\sin\theta}{1-\beta\cos\theta}$, $\beta_{\text{app,max}} = \beta\gamma$ at $\cos\theta \simeq \beta$.
Condition for any $\beta_{\text{app}} > 1$: $\beta \gtrsim 1/\sqrt{2} \approx 0.71$. *(Note)*

5 Electromagnetism in covariant form

Sources & continuity

Charge is Lorentz scalar Q. In moving frame: $\rho = \gamma\rho^*$. 4-current $J^\mu = (\rho c,\ \mathbf{j}) = \rho^* U^\mu$.
Continuity: $\partial_\mu J^\mu = 0$. *(Lect. 3, Sum. 3)*

Potentials, field tensor, Lorentz force

4-potential $A^\mu = (\phi/c,\ \mathbf{A})$, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

Explicitly: $F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$.

Lorentz force (4-form): $f^\mu = q\,F^{\mu\nu}u_\nu$. *(Lect. 3, Sess. 3–4, Sum. 3)*

Maxwell in tensor form & invariants

$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$, $\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$.

Invariants: $\dfrac{1}{2}F_{\mu\nu}F^{\mu\nu} = B^2 - \dfrac{E^2}{c^2}$, $\dfrac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} = \dfrac{1}{c}\mathbf{E}\cdot\mathbf{B}$. *(Lect. 3, Sess. 3, Sum. 3)*

Field transformations (boost along x)

$E'_x = E_x$, $\mathbf{B}'_x = B_x$.
$\mathbf{E}'_\perp = \gamma\left(\mathbf{E}_\perp + \mathbf{v}\times\mathbf{B}\right)$, $\mathbf{B}'_\perp = \gamma\left(\mathbf{B}_\perp - \frac{\mathbf{v}\times\mathbf{E}}{c^2}\right)$.
Parity (space inversion): $\mathbf{E} \to -\mathbf{E}$, $\mathbf{B} \to \mathbf{B}$. Rotations: $(\mathbf{E},\mathbf{B})$ rotate as vectors. *(Sess. 3–4, Midterm Corr.)*

Plane wave seen from a moving frame

Wave in O: $\mathbf{E} = E_0 \cos(kx - \omega t)\,\hat{y}$, $\mathbf{B} = \frac{E_0}{c} \cos(kx - \omega t)\,\hat{z}$.

Boost $+x$ with v: amplitude/frequency redshift $E_0' = \gamma(1-\beta)E_0$, $\quad \omega' = \omega\sqrt{\dfrac{1-\beta}{1+\beta}}, \quad k' = \omega'/c$.

Speed remains c in all inertial frames. *(Sess. 4)*

6 Worked patterns you can copy fast

Relative speed (spaceships A vs B)

Given v_A and v_B in Earth frame (opposite directions), relative speed: $v_{A/B} = \dfrac{v_A + v_B}{1 + \frac{v_A v_B}{c^2}}$.
Example (midterm): $v_A \simeq 0.732c$, $v_B = 0.60c \Rightarrow v_{A/B} \approx 0.926c$.
Proper vs coordinate times: relate via corresponding γ; if needed, transform with Lorentz directly. *(Midterm Corr.)*

Aberration+Doppler to color

Source rest frame wavelength λ' at emission angle θ'; observer frame:

$$\lambda = \lambda'\,\gamma\,(1 - \beta\cos\theta') \quad (\text{equivalently } \nu = \nu'/[\gamma(1-\beta\cos\theta')]).$$

Midterm plug: $\lambda' = 530\,\text{nm}$, $\beta = 0.4$, $\theta' = 60^\circ \Rightarrow$ redshift to longer λ. *(Midterm Corr.)*

Ladder paradox (fit condition)

Basic fit (garage rest frame): $L' = \dfrac{L_0}{\gamma} \le G_0 \Rightarrow \beta \ge \sqrt{1 - \left(\dfrac{G_0}{L_0}\right)^2}$.
Example: $G_0 = 4\,\text{m}, L_0 = 5\,\text{m} \Rightarrow \beta \ge 0.6$.
Trap case with shock speed w (ladder rest frame):

$$\beta \;\ge\; \frac{1 - \sqrt{1 - (1 + (w/c)^2 f^2)\,(1 - f^2)}}{1 + (w/c)^2 f^2}\,\frac{w}{c}, \qquad f = \frac{G_0}{L_0}.$$

Worst case $w = c \Rightarrow \beta \ge \dfrac{1 - f^2}{1 + f^2}$; for $f = 4/5$, $\beta \ge 0.22$. *(Sess. 4 Extra, Midterm Corr.)*

EM tensor quick transforms

Rotation about x by φ: $\begin{aligned} &E_x' = E_x, \quad E_y' = \cos\varphi\,E_y + \sin\varphi\,E_z, \quad E_z' = -\sin\varphi\,E_y + \cos\varphi\,E_z, \\ &B_x' = B_x, \quad B_y' = \cos\varphi\,B_y + \sin\varphi\,B_z, \quad B_z' = -\sin\varphi\,B_y + \cos\varphi\,B_z. \end{aligned}$
Spatial reflection (parity): $E_i'' = -E_i$, $\quad B_i'' = B_i$. *(Midterm Corr.)*

6 Formula index (one-liners)

Lorentz boost (x)	$ct' = \gamma(ct - \beta x),\ x' = \gamma(x - \beta ct)$
Velocity add.	$u' = (u - v)/(1 - uv/c^2)$
Interval	$(\Delta s)^2 = c^2 \Delta t^2 - \Delta \mathbf{x}^2$
Proper time	$c\,\Delta\tau = \sqrt{(\Delta s)^2}$
4-velocity	$U^\mu = dX^\mu/d\tau,\ U^2 = c^2$
Doppler	$\nu' = \nu/[\gamma(1 - \beta\cos\theta')]$
Aberration	$\cos\theta' = (\cos\theta + \beta)/(1 + \beta\cos\theta)$
β_{app}	$\beta_{\text{app}} = \beta\sin\theta/(1 - \beta\cos\theta)$
4-current	$J^\mu = (\rho c,\ \mathbf{j}),\ \partial_\mu J^\mu = 0$
EM tensor	$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$
Invariants	$B^2 - E^2/c^2, \quad \mathbf{E}\cdot\mathbf{B}/c$
Lorentz force	$f^\mu = qF^{\mu\nu}u_\nu$
Energy–momentum	$P^\mu = (E/c,\ \mathbf{p}),\ E^2 = p^2c^2 + m^2c^4$
CM energy	$s = (P_1 + P_2)^2$

Optional: Covariant derivative (flat spacetime)

Covariant derivative in flat space (Cartesian coords): $\nabla_\mu T^\alpha{}_\beta = \partial_\mu T^\alpha{}_\beta$. In general coordinates: $\nabla_\mu T^\alpha{}_\beta = \partial_\mu T^\alpha{}_\beta + \Gamma^\alpha_{\mu\nu} T^\nu{}_\beta - \Gamma^\nu_{\mu\beta} T^\alpha{}_\nu$.
Christoffel symbols (Levi-Civita): $\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$.

Orthogonality of 4-force and 4-velocity

$f^\mu = qF^{\mu\nu}u_\nu, \quad u_\mu f^\mu = 0.$
Hence, the 4-force changes direction (momentum), not the norm $U^2 = c^2$.

If $F_{\mu\nu}F^{\mu\nu} > 0 \to$ magnetic-dominated field; If $F_{\mu\nu}F^{\mu\nu} < 0 \to$ electric-dominated; If $F_{\mu\nu}F^{\mu\nu} = 0$ and $F_{\mu\nu}\tilde{F}^{\mu\nu} = 0 \to$ pure radiation field.

6 Units sanity (SI)

Fundamental: $[c] = \mathrm{m\,s^{-1}}, \quad [\varepsilon_0] = \mathrm{A^2\,s^4\,kg^{-1}\,m^{-3}}, \quad [\mu_0] = \mathrm{N\,A^{-2}}$ with $c^2 = \dfrac{1}{\varepsilon_0\mu_0}$.

Fields & potentials: $[\mathbf{E}] = \mathrm{V\,m^{-1}} = \mathrm{N\,C^{-1}}, \quad [\mathbf{B}] = \mathrm{T} = \mathrm{N\,s\,C^{-1}\,m^{-1}}$,
$[\phi] = \mathrm{V}, \quad [\mathbf{A}] = \mathrm{V\,s\,m^{-1}}$ (since $\mathbf{E} = -\nabla\phi - \partial_t\mathbf{A},\ \mathbf{B} = \nabla\times\mathbf{A}$).

Sources: $[\rho] = \mathrm{C\,m^{-3}}, \quad [\mathbf{j}] = \mathrm{A\,m^{-2}}, \quad [J^\mu] = (\rho c, \mathbf{j}) = \left[\mathrm{A\,m^{-2}}\right]$.

EM tensor: $[F^{0i}] = [E_i/c] = \mathrm{V\,m^{-1}}/\mathrm{m\,s^{-1}} = \mathrm{V\,s\,m^{-2}}, \quad [F^{ij}] = [B_k] = \mathrm{T}$.

Energy–momentum: $[E] = \mathrm{J}, \quad [\mathbf{p}] = \mathrm{kg\,m\,s^{-1}}, \quad [P^\mu] = (E/c, \mathbf{p})$.

Handy: 1 eV $= 1.602 \times 10^{-19}$ J; $[\nabla] = \mathrm{m^{-1}}$; curl/div preserve units expected by Maxwell.

6 Relativity problem playbooks (fast templates)

1) Atmospheric muon decay (two-frame solution)

Earth frame (time dilation): Proper lifetime τ_0 (muon rest), observed lifetime $\tau = \gamma\tau_0$. Travel distance $L = \beta c\,\tau = \beta\gamma c\,\tau_0$.

Muon rest frame (length contraction): Atmosphere thickness H contracts to $H' = \dfrac{H}{\gamma}$. Transit time $t' = H'/(\beta c) = \dfrac{H}{\gamma\beta c}$.

Survival fraction: $P = \exp\left(-\dfrac{t'}{\tau_0}\right) = \exp\left(-\dfrac{H}{\gamma\beta c\,\tau_0}\right) = \exp\left(-\dfrac{H}{\beta\gamma c\,\tau_0}\right)$ (same either way).

Tip: for "how far before decaying?", use $L = \beta\gamma c\,\tau_0$ directly.

(Matches the Session 1 treatment style for decay/travel setups.)

2) Relative velocity: Galilean vs relativistic

Same line, same direction: Galilean: $u_{\text{rel}} = u - v$. Relativistic: $u_{\text{rel}} = \dfrac{u - v}{1 - \dfrac{uv}{c^2}}$.

Head-on (opposite directions): Galilean: $u_{\text{rel}} = u + v$. Relativistic: $u_{\text{rel}} = \dfrac{u + v}{1 + \dfrac{uv}{c^2}}$.

Procedure: (i) pick Earth (lab) frame, (ii) compute u_{rel} with the correct sign, (iii) if you need times/distances seen by one ship, Lorentz-transform intervals (don't mix proper with coordinate intervals).

(Exactly what's used in the "spaceships" exercises and midterm correction.)

3) Spaceship A & B template (like Session 1 & Midterm)

Step 1 — Speed from Earth data: $v_A = \dfrac{\Delta x_E}{\Delta t_E}$ (both measured in Earth frame).

Step 2 — Relative speed seen from B: If A and B move in opposite directions wrt Earth, $v_{A/B} = \dfrac{v_A + v_B}{1 + \dfrac{v_A v_B}{c^2}}$.

Step 3 — Intervals as seen from B: If you know A's proper time $\Delta\tau_A$, then $\Delta t_B = \gamma_{A/B}\,\Delta\tau_A$ with $\gamma_{A/B} = \dfrac{1}{\sqrt{1 - (v_{A/B}/c)^2}}$; distance from B: $\Delta x_B = v_{A/B}\,\Delta t_B$.

Direct Lorentz method (safer when unsure): Choose consistent origins (all $t = 0$ aligned). Use $\begin{cases} \Delta t' = \gamma\left(\Delta t + \dfrac{v\,\Delta x}{c^2}\right), \\ \Delta x' = \gamma\,(\Delta x + v\,\Delta t), \end{cases}$ with v the velocity of B relative to Earth, to map Earth intervals $(\Delta t, \Delta x)$ to B-frame $(\Delta t', \Delta x')$.

(Matches the worked solution in the midterm correction: compute v_A, then $v_{A/B}$, then transform intervals.)

Tip: mark go-to pages with tabs (Doppler, EM invariants, ladder). Keep a non-programmable calculator; pre-compute common $\gamma(\beta)$ values.