Foundations & Measurement

State: $|\psi\rangle = \sum_n c_n |n\rangle$, $\sum_n |c_n|^2 = 1$. Orthonormal basis $\{|n\rangle\}$, $\langle m \mid n\rangle = \delta_{mn}$.
Observable A Hermitian $\Rightarrow$ real eigenvalues, projectors $P_n = |a_n\rangle\langle a_n|$.
Born rule: $P(a_n) = |\langle a_n \mid \psi\rangle|^2 = \langle\psi| P_n |\psi\rangle$. Expectation: $\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$, variance $\Delta A^2 = \langle A^2\rangle - \langle A\rangle^2$.
Schrödinger: $\mathrm{i}\hbar\partial_t |\psi\rangle = \hat{H}|\psi\rangle$; solution $|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}\hat{H}(t-t_0)/\hbar}|\psi(t_0)\rangle$.
Commutator: $[A,B] = AB - BA$. If $[A,B]=0 \Rightarrow$ common eigenbasis.
Uncertainty: $\Delta A\,\Delta B \geq \frac{1}{2}|\langle[A,B]\rangle|$. In particular $[x,p_x] = \mathrm{i}\hbar \Rightarrow \Delta x\,\Delta p_x \geq \hbar/2$.
Resolution of identity: $\sum_n |n\rangle\langle n| = I$ (discrete), $\int \mathrm{d}x\, |x\rangle\langle x| = I$ (continuous).

Translations, Symmetries, Noether

Time evolution: $U(t) = \mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar}$ unitary, $U^\dagger U = I$.
Spatial translation: $\hat{T}_a = \mathrm{e}^{-\mathrm{i}\hat{p}a/\hbar}$, $(\hat{T}_a\psi)(x) = \psi(x-a)$; generator $\hat{p}$.
Rotation: $\hat{R}(\boldsymbol{\alpha}) = \mathrm{e}^{-\mathrm{i}\,\boldsymbol{\alpha}\cdot\hat{\boldsymbol{L}}/\hbar}$; generators $\hat{\boldsymbol{L}}$.
If $[\hat{H},\hat{G}]=0$ (continuous symmetry generated by $\hat{G}$) $\Rightarrow \hat{G}$ conserved (Noether).

Orbital Angular Momentum

$\hat{\boldsymbol{L}} = \hat{\boldsymbol{r}} \times \hat{\boldsymbol{p}}$; $[L_x, L_y] = \mathrm{i}\hbar L_z$ (cyclic), $[\hat{L}_i, \hat{L}^2] = 0$, $\hat{L}^2 = L_x^2 + L_y^2 + L_z^2$.
Eigenvalue equations on $Y_{\ell m}$:

$$\hat{L}^2 Y_{\ell m} = \hbar^2\ell(\ell+1)Y_{\ell m}, \quad \hat{L}_z Y_{\ell m} = m\hbar Y_{\ell m}.$$

Ladders: $L_\pm = L_x \pm \mathrm{i}L_y$, $L_\pm Y_{\ell m} = \hbar\sqrt{\ell(\ell+1) - m(m\pm1)}\,Y_{\ell,m\pm1}$.
Degeneracy: $2\ell+1$. Rotor rigid: $H = \hat{L}^2/(2I)$, $E_\ell = \frac{\hbar^2}{2I}\ell(\ell+1)$.

Harmonic Oscillator (1D, $a, a^\dagger$a,a†)

$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega(\hat{N} + \frac{1}{2})$, $\hat{N} = a^\dagger a$.
$a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{\mathrm{i}}{m\omega}\hat{p}\right)$, $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{\mathrm{i}}{m\omega}\hat{p}\right)$, $[a, a^\dagger] = 1$.
Spectrum: $E_n = \hbar\omega(n+\frac{1}{2})$, $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$. Actions: $a|n\rangle = \sqrt{n}\,|n-1\rangle$, $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$.
Ground state width: $\langle x\rangle = 0$, $\langle x^2\rangle = \hbar/(2m\omega)$, $\Delta x\,\Delta p = \hbar/2$.
2D HO: $H = \hbar\omega(n_x + n_y + 1)$, states $|n_x\rangle|n_y\rangle$, $E_{n_x,n_y} = \hbar\omega(n_x+n_y+1)$.

Bloch Theorem (Periodic Translation)

If $V(x+a) = V(x)$, $[\hat{H}, \hat{T}_a] = 0 \Rightarrow \hat{T}_a|\psi_k\rangle = \mathrm{e}^{-\mathrm{i}ka}|\psi_k\rangle$.
Bloch waves: $\psi_k(x) = \mathrm{e}^{\mathrm{i}kx}u_k(x)$ with $u_k(x+a) = u_k(x)$. Quasi-momentum k conserved.

Interferometer (Mach–Zehnder)

Beam splitter $S = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & \mathrm{i}\\ \mathrm{i} & 1\end{pmatrix}$, mirror $M = \begin{pmatrix}0 & \mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}$, phase $\Phi = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\phi}, 1)$.
Output probabilities: $P(D_x) = \cos^2(\phi/2)$, $P(D_y) = \sin^2(\phi/2)$.

Spin-$\frac{1}{2}$ (Pauli)

$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $\sigma_y = \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}$, $\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$.
$\hat{S}_i = \frac{\hbar}{2}\sigma_i$, $[S_x, S_y] = \mathrm{i}\hbar S_z$ (cyclic), $[\hat{S}_i, \hat{S}^2] = 0$, $S^2 = \frac{3}{4}\hbar^2$.
Eigenstates: $S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$; $|+\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$, $|+\rangle_y = \frac{1}{\sqrt{2}}(|+\rangle + \mathrm{i}\,|-\rangle)$.
Arbitrary axis $\boldsymbol{u} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$: $\hat{S}_{\boldsymbol{u}} = \frac{\hbar}{2}\boldsymbol{u}\cdot\boldsymbol{\sigma}$.
Eigenkets: $|+\rangle_{\boldsymbol{u}} = \mathrm{e}^{-\mathrm{i}\phi/2}\cos\frac{\theta}{2}|+\rangle + \mathrm{e}^{\mathrm{i}\phi/2}\sin\frac{\theta}{2}|-\rangle$; $\langle\boldsymbol{S}\rangle = \frac{\hbar}{2}\boldsymbol{u}$.

Larmor: $\hat{H} = -\boldsymbol{\mu}\cdot\boldsymbol{B} = -\gamma\boldsymbol{B}\cdot\hat{\boldsymbol{S}}$; precession $\omega_L = -\gamma B$.

Spin-1 Matrices

$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$, $S_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -\mathrm{i} & 0\\ \mathrm{i} & 0 & -\mathrm{i}\\ 0 & \mathrm{i} & 0\end{pmatrix}$, $S_z = \hbar\,\mathrm{diag}(1, 0, -1)$.
Eigenvalues for S_x, S_y, S_z: $\{+\hbar, 0, -\hbar\}$. If $H = \Omega S_z$, component m acquires phase $\mathrm{e}^{-\mathrm{i}m\Omega t}$.

Two Spins $\frac{1}{2}\oplus\frac{1}{2}$1/2+1/2

Total $\hat{\boldsymbol{S}} = \hat{\boldsymbol{S}}_1 + \hat{\boldsymbol{S}}_2$; S^2 eigenvalues $S=1$ (triplet), $S=0$ (singlet).
Triplet: $|1,1\rangle = |{+}{+}\rangle$, $|1,0\rangle = \frac{1}{\sqrt{2}}(|{+}{-}\rangle + |{-}{+}\rangle)$, $|1,-1\rangle = |{-}{-}\rangle$.
Singlet: $|0,0\rangle = \frac{1}{\sqrt{2}}(|{+}{-}\rangle - |{-}{+}\rangle)$.
Heisenberg coupling: $\hat{H} = \frac{E_0}{2\hbar^2}\left(S^2 - S_1^2 - S_2^2\right) \Rightarrow E_{S=1} = +E_0/4$, $E_{S=0} = -3E_0/4$.
Commutation: $[\hat{H}, S^2] = [\hat{H}, S_z] = 0$. Ladders: $S_\pm = S_{\pm,1} + S_{\pm,2}$.
Product basis vs. total-spin basis conversion via Clebsch–Gordan above.

Tensor/Kronecker Product Rules

For operators on $E\otimes F$: $(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$, $(A\otimes B)^\dagger = A^\dagger\otimes B^\dagger$.
$\mathrm{Tr}(A\otimes B) = \mathrm{Tr}A\,\mathrm{Tr}B$; $\det(A\otimes B) = (\det A)^m(\det B)^n$ (if A is $n\times n$, B $m\times m$).
Eigenvalues: $\lambda_i(A\otimes B) = \lambda_i(A)\mu_j(B)$. If invertible, $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$.

Entanglement, EPR, Bell

Separable: $c_{mn} = a_m b_n \Rightarrow |\Psi\rangle = |e\rangle\otimes|f\rangle$; else *entangled.*
Singlet $|\Sigma_s\rangle = \frac{1}{\sqrt{2}}(|{+}{-}\rangle - |{-}{+}\rangle)$ is rotationally invariant; perfect anti-correlation along any common axis.
Correlation for singlet: $E(\boldsymbol{a},\boldsymbol{b}) = \frac{4}{\hbar^2}\langle\Sigma_s|\,S_{\boldsymbol{a}}\otimes S_{\boldsymbol{b}}\,|\Sigma_s\rangle = -\boldsymbol{a}\cdot\boldsymbol{b}$.
CHSH: $S = E(\boldsymbol{a},\boldsymbol{b}) + E(\boldsymbol{a},\boldsymbol{b}') + E(\boldsymbol{a}',\boldsymbol{b}') - E(\boldsymbol{a}',\boldsymbol{b})$; local realism $\Rightarrow |S|\leq 2$, QM max $2\sqrt{2}$.
Coincidence prob. (axes angle $\Delta\theta$): $P(\text{same}) = \cos^2(\Delta\theta/2)$.

Deutsch–Jozsa (Outline)

Hadamard $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$; $H^{\otimes n}|b\rangle = \frac{1}{\sqrt{2^n}}\sum_{s\in\{0,1\}^n}(-1)^{b\cdot s}|s\rangle$.
Oracle $U_f|x\rangle|y\rangle = |x\rangle|y\oplus f(x)\rangle$. Standard start $|0^n\rangle|1\rangle$; apply $H^{\otimes(n+1)}$, then U_f, then $H^{\otimes n}$ on first register.
Outcome: if f is constant $\Rightarrow \Pr(|0^n\rangle) = 1$; if balanced $\Rightarrow \Pr(|0^n\rangle) = 0$ (one query).

Spin in Magnetic Fields & Rabi (quick)

Zeeman: $\hat{H} = -\gamma\boldsymbol{B}\cdot\hat{\boldsymbol{S}}$. For $\boldsymbol{B} = B_0\hat{z}$, $\hat{H} = \omega_0 S_z$ with $\omega_0 = -\gamma B_0$.
Time evolution: each m picks phase $\mathrm{e}^{-\mathrm{i}m\omega_0 t}$. Expectation precesses at ω_0.
Driven (on-resonance) two-level: Ω_R Rabi freq., $P_{\text{flip}}(t) = \sin^2(\Omega_R t/2)$; detuned: $\Omega = \sqrt{\Delta^2 + \Omega_1^2}$, $P_-(t) = \frac{\Omega_1^2}{\Omega^2}\sin^2(\Omega t/2)$.

Probability, Expectation (continuous)

$\langle x\rangle = \int_{-\infty}^{\infty}\psi^*(x)\,x\,\psi(x)\,\mathrm{d}x$, $\langle p\rangle = \int\psi^*(x)\,(-\mathrm{i}\hbar\partial_x)\,\psi(x)\,\mathrm{d}x$.
Projector method: $\Pr(\text{outcome in subspace } \mathcal{S}) = \langle\psi|P_{\mathcal{S}}|\psi\rangle$.

How to Attack Midterm Problems (Recipes)

Spin-1 in $B \parallel z$:
1. Write $H = \Omega S_z$ and eigenbasis $\{|m_z\rangle\}$.

2. Expand $|\psi(0)\rangle = \sum_m c_m |m_z\rangle$ (often after measuring S_x: convert using eigenvectors).
3. Evolve: $c_m(t) = c_m(0)\,\mathrm{e}^{-im\Omega t}$.
4. Compute $\langle S_i(t)\rangle = \langle\psi(t)|\, S_i\, |\psi(t)\rangle$ (precession).

Two spins, Heisenberg $S_1 \cdot S_2$:

1. Switch to $|S, M\rangle$ basis (singlet/triplet).
2. Use $H = \frac{E_0}{2\hbar^2}(S^2 - \frac{3}{2}\hbar^2) \Rightarrow$ energies.
3. For measurements on one spin, project in product basis; use CG relations above.

CHSH/Bell:

1. Use singlet; $E(\boldsymbol{a}, \boldsymbol{b}) = -\boldsymbol{a}\cdot\boldsymbol{b} = -\cos\theta$.
2. Choose standard angles (e.g. 45° geometry) to show $|S| = 2\sqrt{2}$.

Deutsch–Jozsa:

1. Track only amplitude of $|0^n\rangle$; phases from $H^{\otimes n}$ and U_f cancel (balanced) or add (constant).

Extra Identities (handy)

$[L_i, S_j] = 0$ (orbital vs. spin). Degeneracy $2j + 1$.
$S_\pm |j, m\rangle = \hbar\sqrt{j(j+1) - m(m \pm 1)}\,|j, m \pm 1\rangle$.
Projectors: $P_{S=0} = |0,0\rangle\langle 0,0|$, $P_{S=1} = \sum_{m=-1}^{1} |1,m\rangle\langle 1,m|$.
Density matrix basics (if needed): $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$, $\langle A\rangle = \mathrm{Tr}(\rho A)$.