ESBOÇO DE CORRECÇÃO DO EXAME NORMAL

CII 2023

$\boxed{1,75}$

1. (a) $D_f = \{(x,y) \in \mathbb{R}^2 : -1 \le 2x+y \le 1\}$

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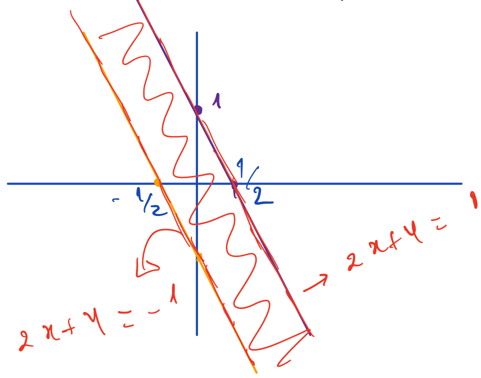

$2x+y=1$

$x=0 \rightarrow y=1$

$x=\frac{1}{2} \rightarrow y=0$

$2x+y=-1$

$x=0 \rightarrow y=-1$

$x=-\frac{1}{2} \rightarrow y=0$

(b) $f(x,y) = \text{arc sen}(2x+y)$

1 $\quad f(0,\frac{1}{2}) = \text{arc sen}(\frac{1}{2}) = \pi/6$ //

$$\frac{\partial f}{\partial x}(0,\tfrac{1}{2}) = \left.\frac{2}{\sqrt{1-(2x+y)^2}}\right|_{(0,\frac{1}{2})} = \frac{2}{\sqrt{1-\frac{1}{4}}} = \frac{2}{\frac{\sqrt{3}}{2}} = \frac{4\sqrt{3}}{3} \;//$$

$$\frac{\partial f}{\partial y}(0,\tfrac{1}{2}) = \left.\frac{1}{\sqrt{1-(2x+y)^2}}\right|_{(0,\frac{1}{2})} = \frac{2\sqrt{3}}{3} \;//$$

A eq. do plano pedido é

$$z = f(0,\tfrac{1}{2}) + \frac{\partial f}{\partial x}(0,\tfrac{1}{2})(x-0) + \frac{\partial f}{\partial y}(0,\tfrac{1}{2})(y-\tfrac{1}{2})$$

$$z = \frac{\pi}{6} + \frac{4\sqrt{3}}{3}x + \frac{2\sqrt{3}}{3}(y-\frac{1}{2})$$

$$\frac{4\sqrt{3}}{3}x + \frac{2\sqrt{3}}{3}y - z = \frac{\sqrt{3}}{3} - \pi/6 \;//$$

$\boxed{1,75}$

2. (a) Prove-se que $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

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$$\lim_{(x,y)\to(0,0)} \frac{2x^3 - y^3}{x^2+3y^2} =$$

$$= \lim_{(x,y)\to(0,0)} \underbrace{\underbrace{2x}_{\to 0} \underbrace{\left[\frac{x^2}{x^2+3y^2}\right]}_{0\leq \quad \leq 1}}_{\to 0} - \underbrace{\underbrace{y}_{\to 0} \underbrace{\left[\frac{y^2}{x^2+3y^2}\right]}_{0 \leq \quad \leq \frac{1}{3}}}_{\to 0}$$

$$= 0 = f(0,0).$$

Logo, f é contínua em $(0,0)$.

(b) 1

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{2h^3}{h^2}}{h}$$

$$= \lim_{h\to 0} \frac{2h^3}{h^3} = 2 //$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{-h^3}{3h^2}}{h} =$$

$$= -1/3 //$$

3. [1,75]

$h(x,y,z) = x^{y \ln z}$ $\quad P = (e,2,e) \quad Q = (e+2, 1, e-2)$

$$\vec{PQ} = Q - P = (e+2, 1, e-2) - (e, 2, e) = (2, -1, -2)$$

$$\|\vec{PQ}\| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3 //$$

$$\vec{u} = \text{vers } \vec{PQ} = (2/3, -1/3, -2/3)$$

$$\frac{\partial h}{\partial x}(P) = y \ln z \; x^{y \ln z - 1} \Big|_{(e,2,e)} = 2 . \ln e . e^{2-1} = 2e //$$

$$\frac{\partial h}{\partial y}(P) = x^{y \ln z} \ln z \ln x \Big|_{(e,2,e)} = e^2 //$$

$$\frac{\partial h}{\partial z}(P) = x^{y \ln z} \, \frac{y}{z} \ln x \Big|_{(e,2,e)} = e^2 \, \frac{2}{e} = 2e$$

A derivada direccional pedida é

$$D_{\vec{u}} h(P) = \nabla_h(P) \cdot \vec{u} = (2e, e^2, 2e) \cdot \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right) =$$

$$= \frac{4e}{3} - \frac{e^2}{3} - \frac{4e}{3} = -e^2/3$$

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4. $f(x,y) = x \operatorname{sen} y$ (Dif. em $\mathbb{R}^2$)

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \quad \begin{cases} \operatorname{sen} y = 0 \\ x \cos y = 0 \end{cases} \quad \begin{cases} y = n\pi, \; n \in \mathbb{Z} \\ x \cos(n\pi) = 0 \end{cases}$$

$$\begin{cases} - \\ (-1)^n x = 0 \end{cases} \quad \begin{cases} y = n\pi, & n \in \mathbb{Z} \\ x = 0 \end{cases}$$

Pontos críticos: $P_n = (0, n\pi), \; n \in \mathbb{Z}$

$$\frac{\partial^2 f}{\partial x^2} = 0 \; ; \; \frac{\partial^2 f}{\partial y \partial x} = \cos y \; ; \; \frac{\partial^2 f}{\partial y^2} = -x \operatorname{sen} y$$

$$\Delta(P_n) = \begin{vmatrix} 0 & (-1)^n \\ (-1)^n & 0 \end{vmatrix} = -(-1)^{2n} = -1 < 0$$

$\therefore$ Os P_n são pontos de sela.

5. $\boxed{2}$? Máximo e mínimo de

$$f(x,y,z) = 4 - z \quad \text{s.a} \quad x^2+y^2 = 8 \quad \text{e} \quad x+y+z = 1$$

$$L(x,y,z,\lambda_1,\lambda_2) = 4 - z + \lambda_1(x^2+y^2) + \lambda_2(x+y+z)$$

$$\begin{cases} \dfrac{\partial L}{\partial x} = 0 & (1) \\ \dfrac{\partial L}{\partial y} = 0 & (2) \\ \dfrac{\partial L}{\partial z} = 0 & (3) \\ x^2+y^2 = 8 & (4) \\ x+y+z = 1 & (5) \end{cases} \quad \begin{cases} 2\lambda_1 x + \lambda_2 = 0 \\ 2\lambda_1 y + \lambda_2 = 0 \\ -1 + \lambda_2 = 0 \\ \text{———} \\ \text{———} \end{cases} \quad \begin{cases} \lambda_1 x - \lambda_1 y = 0 \\ - \\ \lambda_2 = 1 \\ - \\ - \end{cases} \quad \begin{cases} \lambda_1 = 0 \vee x = y \\ \\ \\ \\ \end{cases}$$

se $\lambda_1 = 0$, $\lambda_2 = 0$, o q é imp. por (3).

se $x = y$, por (4) $2x^2 = 8$, isto é, $x = 2$ ou $x = -2$.

se $x = 2 = y$, vem por (5) $z = 1 - x - y = -3$

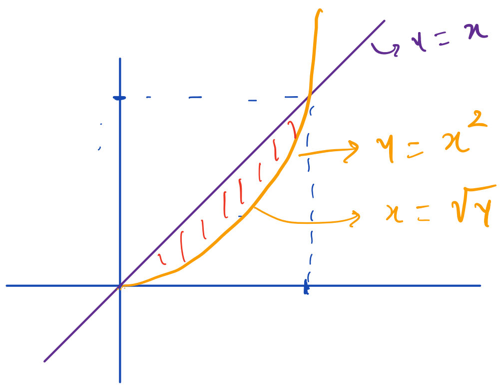
se $x = y = -2$, vem por (5) $z = 1 + 2 + 2 = 5$

$P_0 = (2,2,-3)$ $\quad f(P_0) = 4 - (-3) = 7$ (MÁXIMO)

$P_1 = (-2,-2,5)$ $\quad f(P_1) = 4 - 5 = -1$ (MÍNIMO)

6. $\boxed{2}$ $\quad 0 \leq x \leq 1$

$x^2 \leq y \leq x$



$$\int_0^1 \int_{x^2}^{x} \frac{x}{y} e^y \, dy \, dx =$$

$$= \int_0^1 \int_y^{\sqrt{y}} \frac{x}{y} e^y \, dx \, dy = \frac{1}{2}\int_0^1 \frac{e^y}{y} \left[x^2 \right]_{x=y}^{x=\sqrt{y}} dy$$

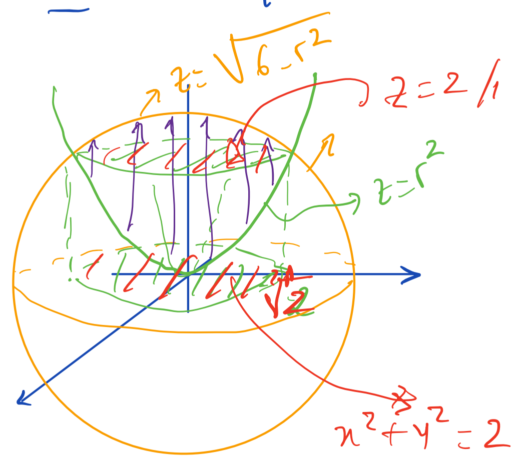
$$= \frac{1}{2}\int_0^1 \frac{e^y}{y}(y-y^2)\,dy = \frac{1}{2}\int_0^1 e^y - y e^y \, dy =$$

$$= \frac{1}{2}\left(\left[e^y\right]_0^1 - \left(\left[y e^y\right]_0^1 - \left[e^y\right]_0^1\right)\right) =$$

$$= \frac{1}{2}(2e - 2 - e) = \frac{e}{2} - 1 \;//$$

7. $\boxed{2}$

$$\begin{cases} x^2+y^2+z^2=6 \\ z = x^2+y^2 \end{cases} \quad \begin{cases} z^2+z-6=0 \\ — \end{cases} \quad \begin{cases} z = \dfrac{-1\pm 5}{2} \\ — \end{cases} \quad \begin{cases} \cancel{z=-3} \vee z=2 \\ \end{cases}$$



$$V = \iiint 1 \, dV =$$

$$= \int_0^{\sqrt{2}} \int_0^{2\pi} \int_{r^2}^{\sqrt{6-r^2}} r \, dz \, d\theta \, dr$$

$$= 2\pi \int_0 r(6-r^2)^{1/2} - r^3 \, dr =$$

$$= 2\pi \left(-\frac{1}{\cancel{2}} \frac{\cancel{2}}{3} \left[(6-r^2)^{3/2}\right]_0^{\sqrt{2}} - \left[\frac{r^4}{4}\right]_0^{\sqrt{2}} \right)$$

$$= 2\pi \left(-\frac{1}{3}\left(8 - 6^{3/2}\right) - 1 \right)$$

$$= \frac{2\pi}{3}\left(6\sqrt{6} - 11\right) \text{ u.v.}$$

$z = x^2+y^2$

$\boxed{z = r^2}$

$x^2+y^2+z^2=6$

$z^2 = 6 - x^2 - y^2$

$z^2 = 6 - r^2$

P $z \geq 0$

$z = \sqrt{6 - r^2}$

8. $\boxed{2}$ $$\int_1^2\int_0^{2\pi}\int_0^{\pi/6} \frac{\cancel{\rho^2}\cos^2\theta\,\text{sen}^2\phi\;\rho\cos\phi\;\cancel{\rho^2}\,\text{sen}\,\phi}{\cancel{\rho^4}}\,d\phi\,d\theta\,d\rho$$

$$= \int_1^2 \rho\,d\rho \int_0^{2\pi} \frac{1+\cos 2\theta}{2}\,d\theta \int_0^{\pi/6} \text{sen}^3\phi\,\cos\phi\,d\phi$$

$$= \left[\frac{\rho^2}{2}\right]_1^2 \frac{1}{2}\left[\theta + \frac{1}{2}\,\text{sen}\,2\theta\right]_0^{2\pi} \left[\frac{\text{sen}^4\phi}{4}\right]_0^{\pi/6} =$$

$$= \frac{3}{2}\cdot\frac{1}{\cancel{2}}\cancel{2}\pi\cdot\left(\frac{1}{2}\right)^4\frac{1}{4} = \frac{3\pi}{2^7} = \frac{3\pi}{128} \;//$$

9. $1{,}75$ $$\frac{dx}{dt} = (1-t) + x(1-t)$$

$$\frac{dx}{dt} = (1-t)(1+x)$$

$x \neq -1$ $$\frac{1}{1+x}\,dx = (1-t)\,dt$$

$$\ln|1+x| = t - \frac{t^2}{2} + c$$

$$|1+x| = e^{t-\frac{t^2}{2}+c}$$

$$|1+x| = e^c\cdot e^{t-t^2/2}$$

$$1+x = \underbrace{\pm e^c}_{k}\, e^{t-t^2/2}$$

$$1+x = k\,e^{t-t^2/2}$$

$$x = K\, e^{t - t^2/2} - 1 \qquad (K, c^{te})$$

Nota: $x = -1$ é também solução já que, neste caso, $\frac{dx}{dt} = 0 = (1-t) \cdot 0$.

$\boxed{1{,}75}$

10.
$$x y' + 2y = \frac{\operatorname{sen} x}{x}$$

$$y' + \frac{2}{x} y = \frac{\operatorname{sen} x}{x^2}$$

$$I(x) = e^{2\int \frac{1}{x}\, dx} = e^{\ln x^2} = x^2$$

$$(y \cdot x^2)' = \frac{\operatorname{sen} x}{\cancel{x^2}} \cdot \cancel{x^2}$$

$$y \cdot x^2 = \int \operatorname{sen} x \, dx$$

$$y \cdot x^2 = -\cos x + C$$

$$y = -\frac{\cos x}{x^2} + \frac{C}{x^2}$$

E, p.to,
$$\lim_{x \to +\infty} \Big(\underbrace{-\frac{1}{x^2}}_{\to 0} \cdot \underbrace{\cos x}_{\text{limitada}} + \underbrace{\frac{C}{x^2}}_{\to 0} \Big) = 0$$

$\boxed{2}$

11.
$$y'' + 2y' + (1+\pi^2)\, y = 0$$

Eq. caract.:
$$r^2 + 2r + (1+\pi^2) = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(1+\pi^2)}}{2}$$

$$r = \frac{-2 \pm 2\pi i}{2} \iff r = -1 \pm \pi i \ //$$

$$y(x) = e^{-x}\left(c_1 \cos(\pi x) + c_2 \operatorname{sen} \pi x\right)$$

$$y'(x) = -e^{-x}\left(c_1 \cos \pi x + c_2 \operatorname{sen}(\pi x)\right) + e^{-x}\left(-\pi c_1 \operatorname{sen}(\pi x) + \pi c_2 \cos(\pi x)\right)$$

$$\begin{cases} 0 = e^{-1}\left(c_1 \cos \pi + \overbrace{c_2 \operatorname{sen} \pi}^{=0}\right) \\ \pi = -e^{-1}(-c_1) + e^{-1}(-\pi c_2) \end{cases} \qquad \begin{cases} c_1 = 0 \ // \\ \pi = -\pi e^{-1} c_2 \end{cases}$$

$$\begin{cases} — \\ c_2 = -e \ // \end{cases}$$

$$\therefore \ y(x) = -e^{-x}\left(e \operatorname{sen}(\pi x)\right) \ //$$

$$= -e^{-x+1} \operatorname{sen}(\pi x) \ //$$