Electro-Griffiths

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Vector Analysis

Ver todos los resúmenes de otros temas.

The Dirac Delta Function

Consider the function $\vec{v} = \frac{1}{r^2}\hat{r}$.

We can calculate its divergence: $\nabla \cdot \vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0.$

On the other hand, we can calculate the flow over a sphere of radius R:

$$\begin{aligned}\int \vec{v} \cdot d\vec{a} &= \int \left(\frac{1}{R^2}\hat{r}\right) \cdot (R^2 \sin\theta d\theta d\phi \hat{r}) \\ &= \left(\int_0^{\pi} \sin\theta d\theta\right)\left(\int_0^{2\pi} d\phi\right) = 4\pi\end{aligned}$$

But the volume integral $\int \nabla \cdot \vec{v} d\tau$ is 0. Does this mean that the divergence theorem is false?

No. The problem is that the divergence is not exactly what we calculated:

$$\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$$

Or, since $\nabla \frac{1}{r} = -\frac{\hat{r}}{r^2}$, it follows that:

$$\nabla^2 \frac{1}{r} = -4\pi\delta^3(\vec{r})$$

Theorems

Potentials

Theorem 1: The following are equivalent:

- $\nabla \times \vec{F} = 0$ everywhere
- $\int_a^b \vec{F} \cdot d\vec{l}$ is independent of path for any given end points.
- $\oint \vec{F} \cdot d\vec{l} = 0$ for a closed loop
- $\vec{F}$ is the gradient of some scalar. $\vec{F} = -\nabla V$

Theorem 2: The following are equivalent:

- $\nabla \cdot \vec{F} = 0$ everywhere
- $\int \vec{F} \cdot d\vec{a}$ is independent of surface
- $\oint \vec{F} \cdot d\vec{a} = 0$ for any closed surface
- $\vec{F}$ is the curl of some vector $\vec{F} = \nabla \vec{A}$

Theorem: For any function $\vec{F}$, it can be written as:

$$\vec{F} = -\nabla V + \nabla \times \vec{A}$$

Electrostatics

The Electric Field

Coulomb's Law:
There is a point charge q at $\vec{r'}$ and a test charge Q at $\vec{r}$. We want to measure the electric force that Q produces on q.

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathfrak{r}}\vec{\mathfrak{r}}$$

Where $\vec{\mathfrak{r}} = \vec{r} - \vec{r'}$

Electric Field

If we have several point charges $q_1, q_2, \cdots, q_n$ at distances $\mathfrak{r}_1, \mathfrak{r}_2, \cdots, \mathfrak{r}_n$ from Q, the total force on Q is then (**superposition principle**):

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots = \frac{Q}{4\pi\epsilon_0}\left(\frac{q_1\vec{\mathfrak{r}}_1}{\mathfrak{r}_1^2} + \frac{q_2\vec{\mathfrak{r}}_2}{\mathfrak{r}_2^2} + \cdots\right)$$

That is:

$$\vec{F} = Q\vec{E}$$

Where:

$$\vec{E}(\vec{r}) := \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathfrak{r}_i^2}\vec{\mathfrak{r}}_i$$

Where $\vec{E}$ is called the **electric field**.

Continuous Charge Distribution

We have a continuous charge distribution, and want to measure the electric field at $\vec{r}$:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{1}{\mathfrak{r}^2}\vec{\mathfrak{r}}dq$$

Where $\vec{\mathfrak{r}} = \vec{r} - \vec{r}'$. Where $\vec{r}$ is the position of measurement and $\vec{r}'$ varies around the charge distribution.

Where:

$$dq \to \lambda dl' \sim \sigma da' \sim d\tau'$$

Thus, the electric field of a volume charge is:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r'})}{\vec{\mathfrak{r}}^2}\vec{\mathfrak{r}}d\tau' = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r'})}{|\vec{r}-\vec{r'}|^2}(\vec{r}-\vec{r'})d\tau'$$

Example 2.1: find the electric field a distance z from the midpoint of a line of length $2L$ and charge density λ

Solution: We chop the line into symmetrical placed pairs (at $\pm x$). The horizontal components cancel and the vertical ones contribute:

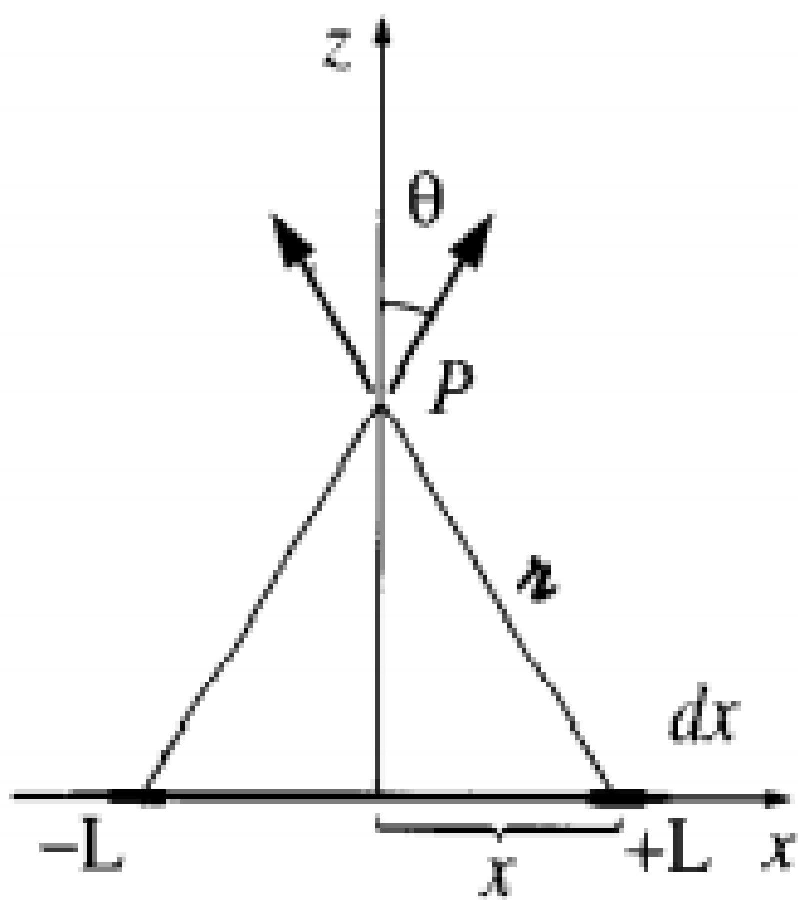


Figure 2.6

so:

$$d\vec{E} = 2\frac{1}{4\pi\epsilon_0}\left(\frac{\lambda dx}{\mathfrak{r}^2}\right)\cos\theta\hat{z}$$

Where $\mathfrak{r} = \sqrt{z^2+x^2}$ and $\cos\theta = z/\mathfrak{r}$ and x runs from 0 to L. Then:

$$\begin{aligned} E &= \frac{1}{4\pi\epsilon_0}\int_0^L \frac{2\lambda z}{(z^2+x^2)^{3/2}}dx \\ &= \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}} \end{aligned}$$

Problem 2.5: Find the electric field a distance z above the center of a circular loop of radius r, with a uniform line charge density λ:

Horizontal components cancel, leaving: $\vec{E} = \dfrac{1}{4\pi\epsilon_0}\int \dfrac{\lambda dl}{\mathfrak{r}^2}\cos\theta\hat{z}$.
Here $\mathfrak{r} = r^2 + z^2$ and $\cos\theta = \dfrac{z}{\mathfrak{r}}$ (both constants). So:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\lambda(2\pi r)z}{(r^2+z^2)^{3/2}}\hat{z}$$

Divergence and Curl

We define the flux of $\vec{E}$ through a surface S as:

$$\Phi_E := \int_S \vec{E}\cdot d\vec{a}$$

Where $d\vec{a}$ is a normal vector. If S is parametrized by $(x(u,v), y(u,v), z(u,v))$, then: $\int \vec{E}\cdot d\vec{a} = \int_u\int_v \vec{E}(x(u,v), y(u,v), z(u,v))\cdot\left(\dfrac{\partial\vec{r}}{\partial u}\times\dfrac{\partial\vec{r}}{\partial v}\right)dvdu$.

We begin with the simple case $\vec{E}(\vec{r}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^2}\hat{r}$:

$$\oint \vec{E}\cdot d\vec{a} = \int \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r^2}\hat{r}\right)\cdot(r^2\sin\theta d\theta d\phi\hat{r}) = \frac{1}{\epsilon_0}q$$

Now, for a group of charges, we use superposition and we get in general:

$$\boxed{\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}Q_{enc}}$$

We can turn it into a differential one, by applying the divergence theorem:

$$\oint_S \vec{E} \cdot d\vec{a} = \int_V (\nabla \cdot \vec{E}) d\tau$$

And we rewrite $Q_{enc} = \int_V \rho d\tau$. So Gauss's law becomes $\int_V (\nabla \cdot \vec{E}) d\tau = \int_V \frac{\rho}{\epsilon_0} d\tau$.
So finally:

$$\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}}$$

Divergence of $\vec{E}$

Alternative proof of Gauss's law:

We can calculate the divergence of $\vec{E}$ directly from the definition:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{all} \frac{\vec{\mathfrak{r}}}{\mathfrak{r}^2} \rho(\vec{r'}) d\tau'$$

We apply the divergence, so:

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\epsilon_0} \int \nabla \cdot \left(\frac{\widehat{\mathfrak{r}}}{\mathfrak{r}^2} \right) \rho(\vec{r'}) d\tau'$$

But, $\nabla \cdot \frac{\widehat{\mathfrak{r}}}{\mathfrak{r}^2} = 4\pi\delta^3(\vec{\mathfrak{r}})$.
So:

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\epsilon_0} \int 4\pi\delta^3(\vec{r} - \vec{r'})\rho(\vec{r'}) d\tau' = \frac{1}{\epsilon_0} \rho(\vec{r})$$

Applications of Gauss's law

When symmetry permits, we can use it to calculate fields:

Example 2.2: Find the field outside a uniformly charged solid sphere of radius R and charge q

We apply Gauss's Law for a sphere of radius $r > 0$:

$$\int_r \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon} q$$

But when integrating in the sphere of radius r, we have $\vec{E} = |\vec{E}|\widehat{r}$ by symmetry. So we can take it out of the integral. And the integral turns out to be $\int_S |\vec{E}| da = |\vec{E}| \int_S da = |\vec{E}| 4\pi r^2$.
So:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \widehat{r}$$

Gauss's law works for spherical symmetry, cylindrical or planar.

Example 2.3: A long cylinder carries a charge density proportional to the distance from the axis $\rho = ks$ for constant k. s is the radius variable:

Solution: Draw a Gaussian cylinder of length l and radius s, then:

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{enc}$$

So $Q_{enc} = \int \rho d\tau = \int (ks')(s' ds' d\phi dz) = 2\pi kl \int_0^s s'^2 ds' = \dfrac{2}{3}\pi kls^3$.
The field is symmetric, so $\int \vec{E} \cdot d\vec{a} = |\vec{E}| 2\pi sl$.
We equate this two parts and we get:

$$|\vec{E}| = \frac{1}{3\epsilon_0} ks^2 \widehat{s}$$

Example 2.4: An infinite plane carries a uniform surface charge σ, find the field

Solution: We use that $\vec{E}$ is symmetrical and use a pillbox, so $\int \vec{E} \cdot d\vec{a} = 2A|\vec{E}|$.
On the other hand, the charge is σA. So finally:

$$\vec{E} = \frac{\sigma}{2\epsilon_0} \widehat{n}$$

The Curl of $\vec{E}$

We begin by studying the simplest case:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \widehat{r}$$

We calculate the line integral between two random points:

$$\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l}$$

In spherical coordinates, $d\vec{l} = dr\widehat{r} + rd\theta\widehat{\theta} + r\sin\theta d\phi\widehat{\phi}$, so:

$$\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l} = \frac{1}{4\pi\epsilon_0} \int_{\vec{a}}^{\vec{b}} \frac{q}{r^2} dr = - \frac{q}{4\pi\epsilon_0 r}\bigg|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r_a} - \frac{q}{r_b} \right)$$

Then, the integral from one point to the other doesn't depend on path, and the integral on a closed cycle is 0, so:

$$\boxed{\nabla \times \vec{E} = 0}$$

The general solution is used by superposition and linearity.

Electric Potential

Introduction to Potential

The electric field $\vec{E}$ is not just any old function, it is independent of path as we saw earlier. This motivates us to define a well defined function. The **electric potential**:

$$\boxed{V(\vec{r}) = -\int_{O}^{\vec{r}} \vec{E} \cdot d\vec{l}}$$

Then, the difference of potential between two points is:

$$\boxed{V(\vec{b}) - V(\vec{a}) = -\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l}}$$

Now, the fundamental theorem of gradients states that $V(\vec{b}) - V(\vec{a}) = \int_{\vec{a}}^{\vec{b}} (\nabla V) \cdot d\vec{l}$. So $\int_a^b (\nabla V) \cdot d\vec{l} = -\int_a^b \vec{E} \cdot d\vec{l}$.
So:

$$\boxed{\vec{E} = -\nabla V}$$

The reference point O can be arbitrary. That doesn't affect the difference in potential.
Potential obeys the superposition.

Example 2.6: Find the potential inside and outside a spherical shell of radius R

Solution: The field outside is $\vec{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^2}\widehat{r}$. The field inside is zero.
For points outside the sphere:

$$V(r) = -\int_{O}^{\vec{r}} \vec{E} \cdot d\vec{l} = \frac{1}{4\pi\epsilon_0}\int_{\infty}^{r} \frac{q}{r'^2} dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

When $r < R$, we have:

$$V(r) = -\frac{1}{4\pi\epsilon_0}\int_{\infty}^{R} \frac{q}{r'^2} dr' - \int_{R}^{r} (0) dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{R}$$

Poisson's Equation and Laplace's

We found that $\vec{E} = -\nabla V$, so using Gauss's law, we get:

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}}$$

The potential of a localized Charge Distribution

We can calculate the potential of a point charge q as: $V(r) = -\frac{1}{4\pi\epsilon_0}\int_\infty^r \frac{q}{r'^2}dr' = \cdots = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$.
In general, the potential of a point charge q at $\vec{r'}$ measured in $\vec{r}$ is:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathfrak{r}}$$

For a volume distribution:

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r'})}{\mathfrak{r}}d\tau'}$$

Example 2.7: find the potential of a uniformly charged spherical shell of radius R

Solution: We use: $V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{\mathfrak{r}}da'$

We measure at a point P in $(0,0,z)$. Here $\mathfrak{r}^2 = R^2 + z^2 - 2Rz\cos\theta'$. An element of surface area on this sphere is $R^2\sin\theta' d\theta' d\phi'$, so:

$$4\pi\epsilon_0 V(z) = \sigma \int \frac{R^2 \sin\theta' d\theta' d\phi'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}} = \cdots = \frac{2\pi R\sigma}{z}\left(\sqrt{(R+z)^2} - \sqrt{(R-z)^2}\right)$$

So: $V(z) = \frac{R^2\sigma}{\epsilon_0 z}$ for point outside and $V(z) = \frac{R\sigma}{\epsilon_0}$ inside.

Summary

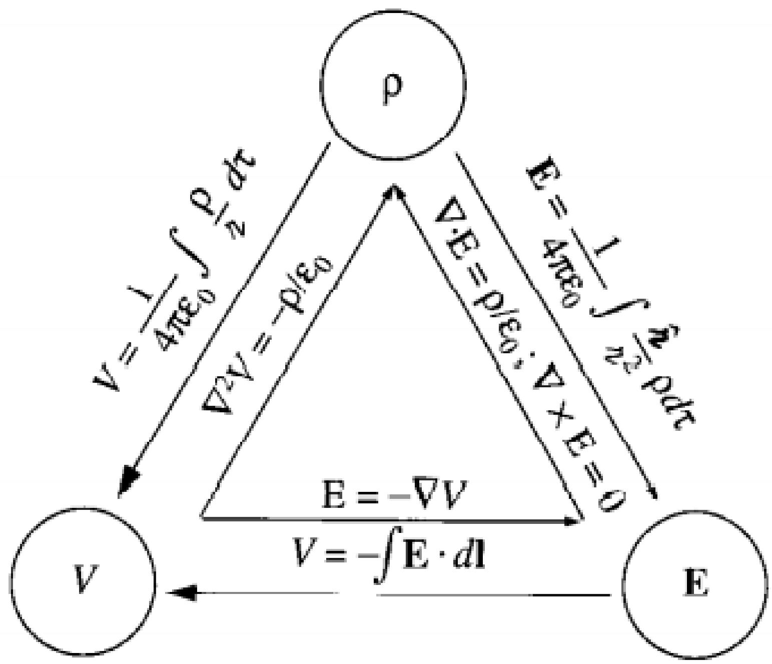


Figure 2.35

Discontinuity: Using the field of a plane, we can see that:

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{1}{\epsilon_0}\sigma$$

By contrast, the tangential component is:

$$E^{\parallel}_{above} = E^{\parallel}_{below}$$

In general:

$$\boxed{\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\widehat{n}}$$

Discontinuity for V:
We have $V_{above} - V_{below} = -\int_a^b \vec{E} d\vec{l}$. But for close points, this is 0.
So:

$$V_{above} = V_{below}$$

However, as $\vec{E} = -\nabla V$, the difference in $\vec{E}$ implies:

$$\nabla V_{above} - \nabla V_{below} = -\frac{1}{\epsilon_0}\sigma\widehat{n}$$

Or, pore conveniently:

$$\boxed{\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{1}{\epsilon_0}\sigma}$$

Where $\dfrac{\partial V}{\partial n} = \nabla V \cdot \widehat{n}$.

Work and Energy

Work done to take a charge Q from a point to another is:

$$W = \int_a^b \vec{F} \cdot d\vec{l} = -Q \int_a^b \vec{E} \cdot d\vec{l} = Q[V(\vec{b}) - V(\vec{a})]$$

In other words, the potential difference between two points is the work done per unit charge to carry a particle from $\vec{a}$ to $\vec{b}$.

Energy of a point charge distribution

Work in order to put two charges q_1, q_2 a distance $\mathfrak{r}$ coming from infinity:

$$W_2 = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{\mathfrak{r}_{12}}$$

The complete work for a system is:

$$W = \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}\frac{q_i q_j}{4\pi\epsilon_0 \mathfrak{r}_{ij}}$$

Energy of a continuous distribution

For a volume charge density, the equation becomes:

$$W = \frac{1}{2}\int \rho V d\tau$$

Which can be modified to:

$$W = \frac{\epsilon_0}{2}\int_{all} E^2 d\tau$$

Example 2.8: Find the energy of a uniformly charged spherical shell of charge q and radius R

Solution: We use $W = \frac{1}{2}\int \sigma V da$. The potential at the surface is $\frac{q}{4\pi\epsilon_0 R}$, so:

$$W = \frac{1}{8\pi\epsilon_0}\frac{q}{R}\int \sigma da = \frac{1}{8\pi\epsilon_0}\frac{q^2}{R}$$

We can use the other formula too, with E^2.

Where is the energy stored:
In the field itself:

$$\frac{\epsilon_0}{2}E^2 \qquad \text{energy per unit volume}$$

Conductors

Conductor: Is a material with totally free electrons.

Properties when equilibrium:

- $\vec{E} = 0$ inside the conductor. The charges go to the surface in such a way as to cancel the field in all points inside the conductor.
- $\rho = 0$ inside the conductor. (Gauss's law)
- Any net charge is at the surface
- A conductor is an equipotential
- $\vec{E}$ is perpendicular to the surface just outside a conductor and has magnitude $|E| = \frac{\sigma}{\epsilon_0}$

Induced Charges:

An electrical field outside a conductor causes a change in the distribution of charges in a way to cancel the field inside.

Example 2.9

An uncharged spherical conductor centered at the origin has a cavity of some weird shape carved out of it (Fig. 2.46). Somewhere within the cavity is a charge q. *Question:* What is the field outside the sphere?

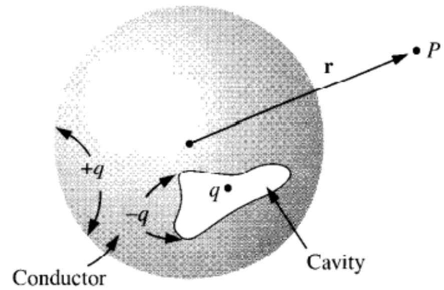


Figure 2.46

Solution: At first glance it would appear that the answer depends on the shape of the cavity and on the placement of the charge. But that's wrong: The answer is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

regardless. The conductor conceals from us all information concerning the nature of the cavity, revealing only the total charge it contains. How can this be? Well, the charge $+q$ induces an opposite charge $-q$ on the wall of the cavity, which distributes itself in such a way that its field cancels that of q, for all points exterior to the cavity. Since the conductor carries no net charge, this leaves $+q$ to distribute itself uniformly over the surface of the sphere. (It's *uniform* because the asymmetrical influence of the point charge $+q$ is negated by that of the induced charge $-q$ on the inner surface.) For points outside the sphere, then, the only thing that survives is the field of the leftover $+q$, uniformly distributed over the outer surface.

It may occur to you that in one respect this argument is open to challenge: There are actually *three* fields at work here, $\mathbf{E}_q$, $\mathbf{E}_{\text{induced}}$, and $\mathbf{E}_{\text{leftover}}$. All we know for certain is that the sum of the three is zero inside the conductor, yet I claimed that the first two *alone* cancel, while the third is separately zero there. Moreover, even if the first two cancel within the conductor, who is to say they still cancel for points outside? They do not, after all, cancel for points *inside* the cavity. I cannot give you a completely satisfactory answer at the moment, but this much at least is true: There *exists* a way of distributing $-q$ over the inner surface so as to cancel the field of q at all exterior points. For that same cavity could have been carved out of a *huge* spherical conductor with a radius of 27 miles or light years or whatever. In that case the leftover $+q$ on the outer surface is simply too far away to produce a significant field, and the other two fields would *have* to accomplish the cancellation by themselves. So we know they *can* do it ... but are we sure they *choose* to? Perhaps for small spheres nature prefers some complicated three-way cancellation. Nope: As we'll see in the uniqueness theorems of Chapter 3, electrostatics is very stingy with its options; there is always precisely one way—no more—of distributing the charge on a conductor so as to make the field inside zero. Having found a *possible* way, we are guaranteed that no alternative exists even in principle.

Surface charge

The surface charge just outside a conductor (Gauss's law) is:

$$\vec{E} = \frac{\sigma}{\epsilon_0}\widehat{n}$$

In terms of potential:

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$$

Capacitors

A capacitor is an arrangement of any two conductors. We put a charge Q at one and a charge $-Q$ at the other.
Then, the potential difference between any two points of them is:

$$V = V_+ - V_- = -\int_-^+ \vec{E} \cdot d\vec{l}$$

Since $\vec{E}$ is proportional to Q and also V, there will be a constant of proportionality for this arrengement:

$$C := \frac{Q}{V}$$

Example: find the capacitance of two parallel plates of area A at a distance d.

Solution: If we put a charge Q and $-Q$, we have $\sigma = Q/A$. So the field is $\frac{1}{\epsilon_0}\frac{Q}{A}$. And the voltage is $V = \frac{Q}{A\epsilon_0}d$. Hence:

$$C = \frac{A\epsilon_0}{d}$$

Work to charge up a capacitor:

$$W = \int_0^Q V dq = \int_0^Q \frac{q}{C} dq = \frac{Q^2}{2C} = \frac{CV^2}{2}$$

Special Techniques

Laplace's Equation

The principal task of electrostatics is to find the electric field of a given distribution $\rho(\vec{r})$, technically this is done with:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\rho(\vec{r}')d\tau'$$

Or calculating first the potential:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathfrak{r}}\rho(\vec{r}')d\tau'$$

Or, by solving the differential equation:

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

Generally, we are interested in finding V in a region where $\rho = 0$ and there is charge elsewhere, that is, **laplace's** equation:

$$\nabla^2 V = 0$$

The Method of Images

Consider a charge q at $(0, 0, d)$ and an infinite grounded conducting plane at the xy plane. What is the field at this region above the plane?

From a mathematical point of view, we have to solve Poisson's equation in the region $z > 0$, with a single point charge q at (0,0,d), subject to boundary conditions:

- $V = 0$ when $Z = 0$
- $V \to 0$ far from the charge

We can do it by using a cool trick. Suppose there was a charge $-q$ at $(0, 0, -d)$ and no conducting plane, then the potential would be:

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+d)^2}}\right]$$

This potential fulfills the boundary conditions above the plane, so it must be the solution for $z \geq 0$ and we are done.

The surface charge, as we will see later is given by:

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial z}\bigg|_{z=0} = \frac{-qd}{2\pi(x^2 + y^2 + d^2)^{3/2}}$$

And the total induced charge is $Q = \int_{planoxy} \sigma da = \cdots = -q$ which makes sense.

Mathematical Problem:

Solve Laplace's equation subject to boundary conditions.

The possibility of finding a solution is guaranteed by:

First Uniqueness Theorem: The solution to Laplace's equation in some volume V is uniquely determined if V is specified on the boundary surface S.

Corollary: The potential in a volume V is uniquely determined if a) the charge density throughout the region and b) the value of V on all boundaries are specified.

Second Uniqueness Theorem: In a volume V surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the total charge on each conductor is given.

Continuity, Induced Charge, etc:

Continuity Condition for Electric Field crossing a surface:

$$\begin{aligned} \vec{E}_{above} - \vec{E}_{below} &= \frac{\sigma}{\epsilon_0}\widehat{n} \\ \Rightarrow \quad E^{\perp}_{above} - E^{\perp}_{below} &= \frac{1}{\epsilon_0}\sigma \\ \Rightarrow \quad E^{\parallel}_{above} &= E^{\parallel}_{below} \end{aligned}$$

Continuity Condition for Potential crossing a surface:

$$\begin{aligned} V_{above} &= V_{below} \\ \frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} &= -\frac{1}{\epsilon_0}\sigma \end{aligned}$$

Induced Surface Charge
If we have a potential V outside a conductor, then because the potential just inside is constant, it produces a surface charge of:

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$$

Separation of Variables

1D

- **Cartesian:** The equation is simply $\dfrac{\partial^2 V}{\partial x^2} = 0$. And the solution is:

$$V(x) = Ax + B$$

- **Spherical:** The equation is now $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0$ and the solution is:

$$V(r) = -\frac{a}{r} + b$$

- **Cylindrical:** The equation is $\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0$. And the solution is:

$$V(r) = a\log(r) + b$$

2 Dimensions

- **Spherical:** The laplacian is $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0$. Where θ is the polar angle in $\theta \in [0, \pi]$. After separation of variables, the solution is:

$$V(r, \theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) + B_n r^{-(n+1)} P_n(\cos\theta)$$

- **Cylindrical:** The equation is $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2} = 0$. And the solution is:

$$V(r, \theta) = a_0 + b_0\log(r) + \sum_{n=1}^{\infty}[r^n(a_n\cos n\theta + b_n\sin n\theta) + r^{-n}(c_n\cos n\theta + d_n\sin n\theta)]$$

- **Rectangular**
  The equation is $\nabla^2 V = \partial_x^2 V + \partial_y^2 V = 0$
  So, the separation of variables is:

  - $\frac{d^2X}{dx^2} = k^2 X \quad \Rightarrow \quad X(x) = Ae^{kx} + Be^{-kx}$
  - $\frac{d^2Y}{dy^2} = -k^2 Y \quad \Rightarrow \quad Y(y) = C\sin ky + D\cos ky$

  Then, the most general solution is:

$$V(x, y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$$

  We then apply the boundary conditions as we see fit.

3 Dimensions

- **Cylindrical:** As always, the equation is $\nabla^2 V = 0$. We use separation of variables.

  If we have boundary conditions of the form:
  $V(r,\theta,0) = 0 \quad , \quad V(a,\theta,z) = 0 \quad , \quad V(r,\theta,L) = f(r,\theta)$

  After the separation, we might get:

  - $\dfrac{d^2Z}{dz^2} - k^2 Z = 0 \quad \Rightarrow \quad Z(z) = e^{\pm kz} \;\; , \;\; A\cosh(kz) + B\sinh(kz)$
  - $\dfrac{d^2\Theta}{d\theta^2} + m^2\Theta \quad \Rightarrow \quad C\cos(m\theta) + D\sin(m\theta)$
  - $\dfrac{d^2R}{dp^2} + \dfrac{1}{p}\dfrac{dR}{dp} + \left(1 - \dfrac{m^2}{p^2}\right) R = 0 \quad (p = kr) \quad \Rightarrow \quad R(r) = J_m(kr)$

  Then, the most general solution is:

  $$V(r,\theta,z) = [A\sinh(kz) + B\cosh(kz)]J_m(kr)[C\cos(m\theta) + D\sin(m\theta)]$$

  When applying the boundary conditions, we have $V(r,\theta,z=0) = 0$. So we keep only the $\sinh(kz)$ part of Z. And $m \in \mathbb{Z}$ for Θ to be univalued.
  On the other hand, $V(a,\theta,z) = 0$, so $J_m(ka) = 0$ which means that the solution is of the type $J_m(\frac{j_{mn}}{a}r)$ where j_{mn} is the nth zero of J_m, so $k = j_{mn}/a$. This way, the complete solution is:

  $$V(r,\theta,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \sinh(j_{mn}z/a)J_m(j_{mn}r/a)[A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)]$$

- **Cylindrical 2:** Now let's say the initial conditions are:
  $V(r,\rho,0) = 0 \;\; , \;\; V(r,\theta,L) = 0 \;\; , \;\; V(a,\theta,z) = f(\theta,z)$
  In this case, we change a sign in the separation:

  - $\dfrac{d^2Z}{dz^2} + k^2 Z = 0 \quad \Rightarrow \quad Z(z) = A\cos(kz) + B\sin(kz)$
  - $\dfrac{d^2\Theta}{d\theta^2} + m^2\Theta = 0 \quad \Rightarrow \quad \Theta(\theta) = C\cos(m\theta) + D\sin(m\theta)$
  - $\dfrac{d^2R}{dp^2} + \dfrac{1}{p}\dfrac{dR}{dp} - \left(1 + \dfrac{m^2}{p^2}\right) R = 0 \quad \Rightarrow \quad R(r) = I_m(kr)$

  So the general solution is:

  $$V(r,\theta,\phi) = [A\cos(kz) + B\sin(kz)][C\cos(m\theta) + D\sin(m\theta)]I_m(kr)$$

If we apply the boundary conditions, they tell us to keep with the sine in Z, and it imposes that $k_n L = n\pi \quad \Rightarrow \quad k_n = \dfrac{n\pi}{L}$. So the complete solution is:

$$V(r,\theta,z) = \sin\left(\frac{n\pi}{L}z\right) Im\left(\frac{n\pi}{L}z\right)[A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)]$$

- **Cartesian:**
  the equation is $\partial_x^2 V + \partial_y^2 V + \partial_z^2 V = 0$. After separation of variables, we get:
  - $\dfrac{1}{X}\dfrac{d^2X}{dx^2} = -k_x^2 \quad \Rightarrow \quad X(x) = A\cos(k_x x) + B\sin(k_x x)$
  - $\dfrac{1}{Y}\dfrac{d^2Y}{dy^2} = k_y^2 \quad \Rightarrow \quad Y(y) = C\cosh(k_y y) + D\sinh(k_y y)$
  - $\dfrac{1}{Z}\dfrac{d^2Z}{dz^2} = -k_z^2 \quad \Rightarrow \quad Z(z) = E\cos(k_z z) + F\sin(k_z z)$

  Where $k_y^2 = k_x^2 + k_z^2$. And we then use boundary conditions.

- **Spherical Coordinates**
  The equation is $\nabla^2 V = 0$. We use separation of variables and find:
  - $\dfrac{d^2\Phi}{d\phi^2} = -m^2\Phi \quad \Rightarrow \quad \Phi(\phi) = A_m\cos(m\phi) + B_m\sin(m\phi)$
  - $r^2\dfrac{d^2R}{dr^2} + 2r\dfrac{dR}{dr} = l(l+1)R \quad \Rightarrow \quad R(r) = A_l r^l + B_l r^{-l-1}$
  - $l(l+1) - \dfrac{m^2}{\sin^2\theta} + \dfrac{\cos\theta}{\sin\theta}\dfrac{1}{\Theta}\dfrac{d\Theta}{d\theta} + \dfrac{1}{\Theta}\dfrac{d^2\Theta}{d\theta^2} = 0 \quad \Rightarrow \quad \Theta(\theta) = P_l^m(\cos\theta)$

  So the general solution is:

$$V(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=0}^{l}(A_l r^l + B_l r^{-l-1})P_l^m(\cos\theta)[S_m\sin(m\phi) + C_m\cos(m\phi)]$$

Exercises and Stuff

Example 3.3:

Two infinite metal plates lie parallel to the xz plane, one at $y = 0$ and one at $y = a$. The left end at $x = 0$ is closed off with an infinite strip and maintained at potential $V_0(y)$. Find the potential

The equation to solve is as always $\nabla^2 V = 0$. With the conditions
$V(y=0) = 0 \quad , \quad V(y=a) = 0 \quad , \quad V(x=0) = V_0(y) \quad , \quad V \to 0$ when $x \to \infty$

The general solution is:

$$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$$

For the final condition, $A = 0$. The first condition implies $D = 0$, the second solution implies $\sin ka = 0 \ \ \Rightarrow \ k = \dfrac{n\pi}{a}$. So the complete solution is:

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a}\sin(n\pi y/a)$$

We impose the final condition:

$$V(0,y) = \sum_{n=1}^{\infty} C_n \sin(n\pi y/a) = V_0(y)$$

To determine the coefficients C_n, we multiply by $\sin(n'\pi y/a)$ and integrate from 0 to a:

$$\sum_{n=1}^{\infty} C_n \int_0^{\infty} \sin(n\pi y/a)\sin(n'\pi y/a)dy = \int_0^a V_0(y)\sin(n'\pi y/a)dy$$
$$\Rightarrow C_n = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)dy$$

Once we know C_n, we have the complete solution.

Example 3.4:

Two infinitely long grounded metal plates at $y = 0$, $y = a$ are connected at $x = \pm b$ by metal strips maintained at potential V_0. Find the potential inside the rectangular pipe

The equation is $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$
And the boundary conditions are given by:
$V(y=0) = 0 \quad , \quad V(y=a) = 0 \quad , \quad V(x=b) = V_0 \quad , \quad V(x=-b) = V_0$
The solution is as before:

$$V(x,y) = (A\cosh kx + B\sinh kx)(C\sin ky + D\cos ky)$$

But the problem is symmetric with respect to x, so we disregard the sinh. As before, the conditions on y imply that $D = 0$ and that $k_n = n\pi/a$, so:

$$V(x,y) = C\cosh(n\pi x/a)\sin(n\pi y/a)$$

And the general solution is:

$$V(x,y) = \sum_{n=1}^{\infty} C_n \cosh(n\pi x/a)\sin(n\pi y/a)$$

And the final boundary condition says:

$$V(b,y) = \sum_{n=1}^{\infty} C_n \cosh(n\pi b/a)\sin(n\pi y/a) = V_0$$

So, we multiply by $\sin(n'\pi y/a)$ and integrate from $y=0$ to $y=a$ and:

$$\sum_{n=1}^{\infty}\int_0^a C_n \cosh(n\pi b/a)\sin(n\pi y/a)\sin(n'\pi y/a)dy = \int_0^a V_0 \sin(n'\pi y/a)dy$$

$$\Rightarrow \quad C_{n'}\cosh(n'\pi b/a)\frac{a}{2} = \frac{2a}{n'\pi} \quad \text{si n' impar}$$

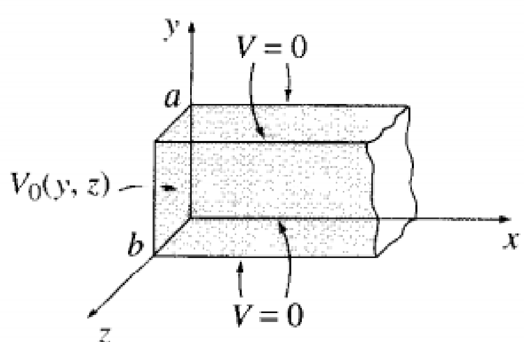
$$\Rightarrow \quad C_{n'} = \frac{4V_0}{n'\pi\cosh(n'\pi b/a)} \quad \text{si n' impar}$$

So the complete solution is:

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n \text{ impar}} \frac{1}{n}\frac{\cosh(n\pi x/a)}{\cosh(n\pi b/a)}\sin(n\pi y/a)$$

Example 3.5:

An infinitely long rectangular metal pipe (sides a and b) is grounded, but one end, at $x=0$, is maintained at a specific potential $V_0(y,z)$, find the potential inside the pipe



The equation is $\partial_x^2 V + \partial_y^2 V + \partial_z^2 V$ subject to the conditions:
i) $V(y=0)=0$, 2) $V(y=a)=0$, 3) $V(z=0)=0$, 4) $V(z=b)=0$, $V(x\to\infty)\to 0$, 5) $V(x=0)=V_0(y,z)$

The general solution is:

- $X(x) = Ae^{\sqrt{k^2+l^2}x} + Be^{-\sqrt{k^2+l^2}x}$
- $Y(y) = C\sin ky + D\cos ky$

- $Z(z) = E\sin lz + F\cos lz$

The condition 5) implies $A = 0$, condition i) implies $D = 0$, condition 3) implies $F = 0$ and condition 2) and 4) imply that $k = n\pi/a$ and $l = m\pi/b$, where n and m are positive integers. We are left with:

$$V(x, y, z) = Ce^{-pi\sqrt{(n/a)^2+(m/b)^2}x}\sin(n\pi y/a)\sin(m\pi z/b)$$

So the complete solution is:

$$V(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}e^{-pi\sqrt{(n/a)^2+(m/b)^2}x}\sin(n\pi y/a)\sin(m\pi z/b)$$

The final condition is:

$$V(0, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\sin(n\pi y/a)\sin(m\pi z/b) = V_0(y, z)$$

To determine the constants, we multiply by $\sin(n'\pi y/a)\sin(m'\pi z/b)$ and integrate y from 0 to a and z from 0 to b:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)dy\int_0^b \sin(m\pi z/b)\sin(m'\pi z/b)dz = \int_0^a\int_0^b V_0(y, z)\sin(n'\pi y/a)\sin(n$$

The left side is $(ab)/4C_{n'm'}$, so:

$$C_{nm} = \frac{4}{ab}\int_0^a\int_0^b V_0(y, z)\sin(n\pi y/a)\sin(m\pi z/b)dydz$$

Example 3.6

The potential $V_0(\theta)$ is specified on the surface of a hollow sphere, of radius R. Find the potential inside.

For a sphere with azimuthal symmetry, the general solution is:

$$V = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) + B_n r^{-n-1}P_n(\cos\theta)$$

But inside the sphere we must have $B_n = 0$, so:

$$V(r, \theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$

The boundary condition is:

$$V(R, \theta) = \sum_{n=0}^{\infty} A_n R^n P_n(\cos\theta) = V_0(\theta)$$

To find the constants, we multiply by $P_{n'}(\cos\theta)\sin\theta$ and integrate from 0 to π and we get:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\int_0^{\pi} A_n R^n P_n(\cos\theta)P_{n'}(\cos\theta)\sin\theta d\theta = \int_0^{\pi} V_0(\theta)P_{n'}(\cos\theta)\sin\theta d\theta \\
&\Rightarrow\quad A_{n'}R^{n'}\frac{2}{2n'+1} = \int_0^{\pi} V_0(\theta)P_{n'}(\cos\theta)\sin\theta d\theta \\
&\Rightarrow\quad A_n = \frac{2n+1}{2R^n}\int_0^{\pi} V_0(\theta)P_l(\cos\theta)\sin\theta d\theta
\end{aligned}$$

Without needing to do this, we can sometimes eye-ball the coefficients.

Example 3.7

Same problem as before but we are asked the potential outside:

Now the solution is:

$$V(r,\theta) = \sum_{n=0}^{\infty}\frac{B_n}{r^{n+1}}P_n(\cos\theta)$$

And the boundary condition is:

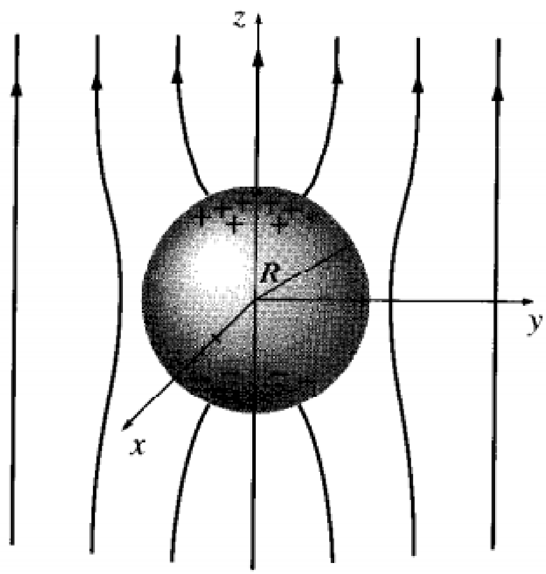
$$V(R,\theta) = \sum_{n=0}^{\infty}\frac{B_n}{R^{n+1}}P_n(\cos\theta) = V_0(\theta)$$

We multiply both sides by $P_{n'}(\cos\theta)\sin\theta$ and integrate from 0 to π. Exploiting orthogonality we find:

$$B_l = \frac{2l+1}{2}R^{l+1}\int_0^{\pi} V_0(\theta)P_l(\cos\theta)\sin\theta d\theta$$

Example 3.8

An uncharged metal (conducting) sphere of radius R is placed in a uniform electric field $\vec{E} = E_0\hat{z}$. Find the potential outside



The condition for large z is $V \to -E_0 z$.
Or, in spherical coordinates, the conditions are
1) $V(r = R) = 0$ and 2) $V(r >> R) = -E_0 r \cos\theta$

The general solution is $V = A_l r^l P_l(\cos\theta) + B_l r^{-l-1} P_l(\cos\theta)$.
Putting the first condition leads to:

$$A_l R^l + \frac{B_l}{R^{l+1}} = 0 \quad \Rightarrow \quad B_l = -A_l R^{2l+1}$$

So the general solution is:

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}} \right) P_l(\cos\theta)$$

We apply the second condition. For $r >> R$, the second term in parenthesis is negligible and the condition turns to:

$$\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r \cos\theta$$

So clearly, as $P_l(\cos\theta) = \cos\theta$, we have $A_1 = -E_0$ and all the others A_l are 0. Then:

$$V(r,\theta) = -E_0 \left(r - \frac{R^3}{r^2} \right) \cos\theta$$

Calculate the induced charge density:
According to some stupid formula:

$$\sigma(\theta) = -\epsilon_0 \frac{\partial V}{\partial r}\bigg|_{r=R} = \epsilon_0 E_0 \left(1 + 2\frac{R^3}{r^3} \right) \cos\theta \bigg|_{r=R} = 3\epsilon_0 E_0 \cos\theta$$

Example 3.9

A specified charge density $\sigma_0(\theta)$ is glued to a surface of a spherical shell of radius R. Find the resulting potential inside and outside the sphere

For the interior we have:

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

For the exterior:

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

The first boundary condition is continuity of V so:

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta)$$

It follows that $B_l = A_l R^{2l+1}$

Second condition says that:

$$\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\right)\bigg|_{r=R} = -\frac{1}{\epsilon_0}\sigma_0(\theta)$$
$$\Rightarrow \quad -\sum_{l=0}^{\infty}(l+1)\frac{B_l}{R^{l+2}} P_l(\cos\theta) - \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -\frac{1}{\epsilon_0}\sigma_0(\theta)$$

$$\Rightarrow \quad \sum_{l=0}^{\infty}(2l+1) A_l R^{l-1} P_l(\cos\theta) = \frac{1}{\epsilon_0}\sigma_0(\theta)$$

From here we can find the coefficients using Fourier's trick:

$$A_l = \frac{1}{2\epsilon_0 R^{l-1}} \int_0^{\pi} \sigma_0(\theta) P_l(\cos\theta)\sin\theta d\theta$$

For example, if $\sigma_0(\theta) = K\cos\theta = KP_1(\cos\theta)$, then clearly $A_1 = \dfrac{k}{3\epsilon_0}$.
With this, the potential inside is:

$$V = \frac{k}{3\epsilon_0} r\cos\theta$$

And outside:

$$V = \frac{kR^3}{3\epsilon_0}\frac{1}{r^2}\cos\theta$$

Multipole Expansion

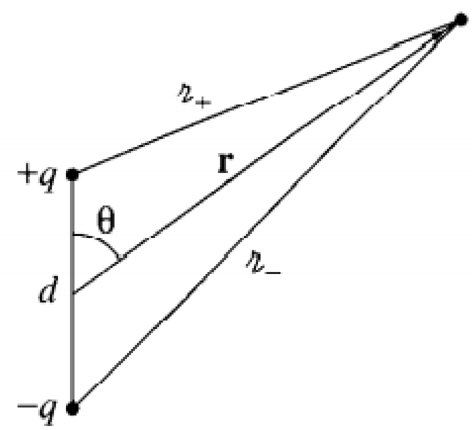
Approximate potential at large distances

Electric Dipole:
An electric dipole consists of charges $\pm q$ separated a distance d

Solution: Let $\mathfrak{r}_-$ be the distance from $-q$ and $\mathfrak{r}_+$ the distance from $+q$. Then, the potential is:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathfrak{r}_+} - \frac{q}{\mathfrak{r}_-}\right)$$

From the law of cosines:



we have:

$$\mathfrak{r}_\pm^2 = r^2 + (d/2)^2 \mp rd\cos\theta = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

We are interested in the régime $r >> f$, so the tgird term is negligible and:

$$\begin{aligned}\frac{1}{\mathfrak{r}_\pm} &\simeq \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \\ &\simeq \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)\end{aligned}$$

So

$$\frac{1}{\mathfrak{r}_+} - \frac{1}{\mathfrak{r}_-} \simeq \frac{d}{r^2}\cos\theta$$

Therefore, the **potential of a dipole is:**

$$V(\vec{r}) \simeq \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}$$

We now develop a systematic and more general approach to calculate the potential of an arbitrary distribution in powers of $1/r$.
To start, the potential is:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathfrak{r}}\rho(\vec{r'})d\tau'$$

We use the law of cosines:

$$\mathfrak{r}^2 = r^2 + (r')^2 - 2rr'\cos\theta' = r^2\left(1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta'\right)$$

Or:

$$\mathfrak{r} = r\sqrt{1+\epsilon}$$

Where:

$$\epsilon := \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right)$$

We can use a binomial expansion of $1/\mathfrak{r}$:

$$\frac{1}{\mathfrak{r}} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \cdots\right)$$

Or better yet, we want:

$$\frac{1}{\mathfrak{r}} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}(1 + \frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\theta)^{-1/2}$$

But this is just the generating function of Legendre polynomials, $(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$.
So actually:

$$\frac{1}{\mathfrak{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

So, the solution is:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\theta')\rho(\vec{r'})d\tau'$$

Or, more explicitly:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\vec{r'}d\tau' + \frac{1}{r^2}\int r'\cos\theta'\rho(\vec{r'})d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\vec{r'})d\tau' + \cdots\right]$$

This is the **Multipole expansion** of V in terms of power of $1/r$.

for example, the monopole term is simply $\dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r}\int \rho(\vec{r'})d\tau'$.

When the total charge is 0, the interesting part is the dipole:

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\theta'\rho(\vec{r'})d\tau'$$

But $r'\cos\theta' = \widehat{r}\cdot\vec{r'}$. So:

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\widehat{r}\cdot\int \vec{r'}\rho(\vec{r'})d\tau'$$

We define the **dipole moment** of the distribution as:

$$\vec{p} := \int \vec{r'}\rho(\vec{r'})d\tau'$$

And so:

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\widehat{r}}{r^2}$$

The electric field of a dipole:

If we choose coordinates such that $\vec{p}$ lies at the origin and point in the z direction, then:

$$V_{dip}(r,\theta) = \frac{\widehat{r}\cdot\vec{p}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

By taking the gradient, we obtain:

$$\vec{E}_{dip}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\widehat{r} + \sin\theta\widehat{\theta})$$

Electric Fields in Matter

Polarization

Dielectrics

Most everyday objects belong to conductors or dielectrics.

Induced Dipole:
When we have a neutral molecule and put it in a field $\vec{E}$, the molecule can become polarized. Typically, the induced polarization is approximately proportional to the field, so:

$$\vec{p} = \alpha\vec{E}$$

For complicated molecules, this model is not that simple and α should be changed to a **polarizability tensor**

Polar Molecules

Some molecules are already polar by intrinsic properties. Suppose we have a simple dipole with dipole moment $\vec{p}$.
We put the dipole in a uniform electric field $\vec{E}$.
Then, the positive end feels a force $\vec{F}_+ = q\vec{E}$ and the negative a force $\vec{F}_- = -q\vec{E}$. There is no net force on the molecule.

There will be a **torque:**

$$\begin{aligned}\vec{N} &= (\vec{r}_+ \times \vec{F}_+) + (\vec{r}_- \times \vec{F}_-)\\ &= [(\vec{d}/2) \times (q\vec{E})] + [(-\vec{d}/2) \times (-q\vec{E})] = q\vec{d} \times \vec{E}\end{aligned}$$

Thus, a dipole $\vec{p} = q\vec{d}$ in a uniform electric field $\vec{E}$ feels a torque of:

$$\vec{N} = \vec{p} \times \vec{E}$$

If the field is non-uniform, the forces do not cancel and there is a net force:

$$\vec{F} = \vec{F}_+ + \vec{F}_- = q(\vec{E}_+ - \vec{E}_-) = q(\Delta\vec{E})$$

Where $\Delta\vec{E}$ represents the difference in field in the plus and minus end. We can approximate it as $\Delta E_x := (\nabla E_x) \cdot \vec{d}$, or more compactly as $\Delta\vec{E} = (\vec{d} \cdot \nabla)\vec{E}$.
Thus, the force on a non uniform field is:

$$\vec{F} = (\vec{p} \cdot \nabla)\vec{E}$$

Exercise 4.7: Show that the energy of an ideal dipole $\vec{p}$ in an electric field $\vec{E}$ is given by:

$$U = -\vec{p} \cdot \vec{E}$$

Polarization

We can now say that when a material is put into an electric field, if the atoms are neutral, they are polarized and if they are polar, they experience a torque that tends to align them to the field.
Anyways, the result is a lot of dipoles pointing in the same direction of the field, and the material is **polarized**. We define:

$$\vec{P} = \quad \text{dipole moment per unit volume}$$

The Field of a Polarized Object

Suppose we have a piece of polarized material with polarization $\vec{P}$, we want to know the field. For a single dipole $\vec{p}$ the potential would be:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\widehat{\mathfrak{r}}\cdot\vec{p}}{\mathfrak{r}^2}$$

Where $\mathfrak{r}$ is the vector from the dipole to the point where we are evaluating the potential. In the present context, $\vec{p} = \vec{P}d\tau'$, so the total potential is:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\widehat{\mathfrak{r}}\cdot\vec{P}(\vec{r'})}{\mathfrak{r}^2}d\tau'$$

In principle this does it, but we can simplify it by using $\nabla'\left(\frac{1}{\mathfrak{r}}\right) = \frac{\widehat{\mathfrak{r}}}{\mathfrak{r}^2}$ (differentiation respect to the source coordinates r'). So we have:

$$V = \frac{1}{4\pi\epsilon_0}\int_V \vec{P}\cdot\nabla'\left(\frac{1}{\mathfrak{r}}\right)d\tau'$$

We integrate by parts and get:

$$\begin{aligned}
V &= \frac{1}{4\pi\epsilon_0}\left[\int_V \nabla'\cdot\left(\frac{\vec{P}}{\mathfrak{r}}\right)d\tau' - \int_V \frac{1}{\mathfrak{r}}(\nabla'\cdot\vec{P})d\tau'\right] \\
&= \frac{1}{4\pi\epsilon_0}\oint_S \frac{1}{\mathfrak{r}}\vec{P}\cdot d\vec{a'} - \frac{1}{4\pi\epsilon_0}\int_V \frac{1}{\mathfrak{r}}(\nabla'\cdot\vec{P})d\tau'
\end{aligned}$$

This terms lead us to define a surface bound charge:

$$\sigma_b := \vec{P}\cdot\widehat{n}$$

And a volume density bound charge:

$$\rho_b := -\nabla\cdot\vec{P}$$

With these definitions, the potential is:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\oint_S \frac{\sigma_b}{\mathfrak{r}} da' + \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho_b}{\mathfrak{r}} d\tau'$$

So the potential is equal to what we would get if the polarized object was replaced by a surface density charge $\sigma_b = \vec{P}\cdot\widehat{n}$ and a volume density charge of $\rho_b := -\nabla\cdot\vec{P}$.
These are called **bound charges**, so we can calculate the field by using these charges.

Example 4.2: Find the electric field produced by a uniformly polarized sphere of radius R

Solution: We choose the z axis to coincide with the direction of polarization. The volume bound charge density ρ_b is 0 (since $\vec{P}$) is uniform. Buy:

$$\sigma_b = \vec{P}\cdot\widehat{n} = P\cos\theta$$

We want the field produced by a surface charge as the one given $\sigma_b = P\cos\theta$.
For the interior we have $V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$ and for the exterior $V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$.
We first use continuity of V at $r = R$ to get $B_l = A_l R^{2l+1}$.
Secondly, we use $\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\right)\Big|_{r=R} = -\frac{1}{\epsilon_0}\sigma(\theta) = P\cos\theta$.
Using this condition, we get $\sum_{l=0}^{\infty}(2l+1)A_l R^{l-1} P_l(\cos\theta) = \frac{P}{\epsilon_0}\cos\theta$
So clearly $A_1 = \frac{P}{3\epsilon_0}$ and all the other A are 0.
Therefore, the field is:

$$V(r,\theta) = \begin{cases} \frac{P}{3\epsilon_0} r\cos\theta & , \quad r \le R \\ \frac{P}{3\epsilon_0}\frac{R^3}{r^2}\cos\theta & , \quad r \ge R \end{cases}$$

Since $r\cos\theta = z$, the field inside the sphere is uniform:

$$\vec{E} = -\nabla V = -\nabla\left(\frac{P}{3\epsilon_0} z\right) = -\frac{P}{3\epsilon_0}\widehat{z} = -\frac{1}{3\epsilon_0}\vec{P}$$

Also impressively, the potential outside the sphere is equal to one of a dipole with $\vec{p} = \frac{4}{3}\pi R^3 \vec{P}$.
So the field outside is also that of a dipole with that moment and aligned to $\widehat{z}$.

Physical interpretation of Bound charges

ρ_b and σ_b are not just mathematical imagination, they are physical.

The field inside a dielectric

Inside the dielectric things are more complicated. We ignore the microscopic bumps and wrinkles in the electric field inside matter and concentrate on the average field, and we will always refer to this field when inside matter.

The inside field is the same as the one generated by the σ_b and ρ_b we saw earlier, it doesn't matter we are inside the material.

To prove this, consider a dielectric and we want to calculate the field at $\vec{r}$. First, we take a small sphere inside the dielectric over which we will average. We see that the macroscopic field in the sphere (the average field) consists of two parts, the average due to the charges outside plus the average due to charges inside:

$$\vec{E} = \vec{E}_{out} + \vec{E}_{in}$$

Now, we know that the average field over a sphere produced by charges outside is equal to the field they produce at the center. So $\vec{E}_{out}$ is the at $\vec{r}$ due to the dipoles exterior to the sphere. These are far enough away that we can use:

$$V_{out} = \frac{1}{4\pi\epsilon_0}\int_{outside} \frac{\widehat{r}\cdot\vec{P}(\vec{r'})}{\mathfrak{r}^2} d\tau'$$

The dipoles inside are too close to use this approximation. But all we need is the average field due to charges inside, according to example 4.2, the field due to charges in a uniformly polarized (we can assume this because it is small) sphere inside the sphere is:

$$\vec{E}_{in} = -\frac{1}{3\epsilon_0}\vec{P}$$

So, the total field is:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{out} \frac{\widehat{\mathfrak{r}}\cdot\vec{P}(\vec{r'})}{\mathfrak{r}^2} d\tau' - \frac{1}{3\epsilon_0}\vec{P}$$

But, of the sphere is small enough, $-\frac{1}{3\epsilon_0}\vec{P} = \frac{1}{4\pi\epsilon_0}\int_{in} \frac{\widehat{\mathfrak{r}}\cdot\vec{P}(\vec{r'})}{\mathfrak{r}^2} d\tau'$. So the total potential:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{all} \frac{\widehat{\mathfrak{r}}\cdot\vec{P}(\vec{r'})}{\mathfrak{r}^2} d\tau'$$

The Electric Displacement

Gauss's Law in the presence of Dielectrics

We found that the effect of polarization is to cause a bound charge $\rho_b = -\nabla\cdot\vec{P}$ and a surface charge $\sigma_b = \vec{P}\cdot\widehat{n}$. The field due to polarization is just the field of this bound charge.

There may be other types of charge in the dielectric, **free charge** ρ_f, it is any charge that is not a result of polarization. Then, within the dielectric, the total charge density is:

$$\rho = \rho_b + \rho_f$$

And the, Gauss's law gives:

$$\epsilon_0 \nabla \cdot \vec{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \vec{P} + \rho_f$$

Where $\vec{E}$ is the **total field** generated by polarization and free charges.
Then, we may write:

$$\nabla \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_f$$

That way, we define the **electric Displacement** as:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

So that now Gauss's law is:

$$\nabla \cdot \vec{D} = \rho_f$$
$$\Rightarrow \oint \vec{D} \cdot d\vec{a} = Q_{f_{enc}}$$

Where $Q_{f_{enc}}$ is the total free charge enclosed in the volume. The usefulness of this is that it only makes reference to free charges, in a typical problem we know ρ_f but not necessarily ρ_b.

Example 4.4: A long straight wire, carrying uniform line charge λ, is surrounded by rubber insulation out to a radious a, find the electric displacement

We may draw a cylindrical Gaussian surface of radius s and length L inside the rubber cylinder and enclosing the wire in its center. Then, applying Gauss's law for free charges we get:

$$\oint \vec{D} \cdot d\vec{a} = Q_{f_{enc}}$$
$$\Rightarrow D(2\pi s L) = \lambda L$$
$$\Rightarrow \vec{D} = \frac{\lambda}{2\pi s}\hat{s}$$

We notice that this formula holds both inside the insulation and outside of it. In the outside region, $\vec{P} = 0$, so we can calculate $\vec{E}$ easily:

$$\vec{E} = \frac{1}{\epsilon_0}(\vec{D} - \vec{P}) = \frac{1}{\epsilon_0}\vec{D} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s}$$

.

Problem 4.14: When you polarize a neutral dielectric, prove that the total charge is 0:

The total charge is:

$$\begin{aligned} Q_{tos} &= \oint_S \sigma_b da + \int_V \rho_b d\tau \\ &= \oint_S \vec{P} \cdot d\vec{a} - \int_V \nabla \cdot \vec{P} d\tau \\ &= \int_V \nabla \cdot \vec{P} d\tau - \int_V \nabla \cdot \vec{P} d\tau = 0 \end{aligned}$$

.

Problem 4.15: A thick spherical shell (inner radius a and outer radius b) is made of dielectric material with a frozen-in polarization

$$\vec{P}(\vec{r}) = \frac{k}{r}\widehat{r}$$

where k is constant, there is no free charge. Find the electric field in all three regions

a) Locate all the bound charge, and use Gauss's law to calculate the field it produces:

The bound charge (which is the total charge) is $\rho_b = -\nabla \cdot \vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{k}{r}\right) = -\frac{k}{r^2}$. And we may calculate the surface charge density:

$$\sigma_b = \vec{P} \cdot \widehat{n} = \begin{cases} +\vec{P} \cdot \widehat{r} = k/b & (r = b) \\ -\vec{P} \cdot \widehat{r} = -k/a & (r = a) \end{cases}$$

Then we use Gauss's law (and spherical symmetry) with spherical Gauss surfaces:

- $(r < a) \quad Q_{enc} = 0 \Rightarrow \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q_{enc}}{r^2}\widehat{r} = 0$
- $(r > b) \quad Q_{enc} = 0$ (because of last problem) So $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q_{enc}}{r^2}\widehat{r} = 0$
- $(a < r < b) \quad Q_{enc} = \left(\frac{-k}{a}\right)(4\pi a^2) + \int_a^r \left(\frac{-k}{r^2}\right) 4\pi r^2 dr = -4\pi ka - 4\pi k(r-a) = -4\pi kr$.
  So the field is $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q_{enc}}{r^2}\widehat{r} = \frac{1}{4\pi\epsilon_0}\frac{-4\pi kr}{r^2}\widehat{r} = -(k/\epsilon_0 r)\widehat{r}$

b) Now solve it by finding $\vec{D}$ first
We know that $\oint \vec{D} \cdot d\vec{a} = Q_{f_{enc}}$, but as there is no $Q_{f_{enc}}$, then $\oint \vec{D} \cdot d\vec{a} = 0$, so by symmetry we have $\vec{D} = 0$ everywhere.
Then, using $\vec{D} = \epsilon_0 \vec{E} + \vec{P} = 0 \quad \Rightarrow \quad \vec{E} = (-1/\epsilon_0)\vec{P} \quad \Rightarrow$
$\vec{E} = 0$ for $(r < a$ or $r > b)$
$\vec{E} = -(k/\epsilon_0 r)\hat{r}$ for $(a < r < b)$

.

Problem 4.16: Suppose the field inside a large piece of dielectric is $\vec{E}_0$, so that the electric displacement is $\vec{D}_0 = \epsilon_0 \vec{E}_0 + \vec{P}$

a) **A small spherical cavity is hollowed out, find the field at the center of the cavity**
   We must take out the field caused by a polarized sphere with uniform polarization $\vec{P}$, which is $-\vec{P}/3\epsilon_0$, so $\vec{E} = \vec{E}_0 + \dfrac{1}{3\epsilon_0}\vec{P}$
   The displacement vector is $\vec{D} = \epsilon_0 \vec{E}$ (polarization inside is 0) $= \epsilon_0 \vec{E}_0 + \dfrac{1}{3}\vec{P} = \vec{D}_0 - \vec{P} + \dfrac{1}{3}\vec{P}$. So $\vec{D} = \vec{D}_0 - \dfrac{2}{3}\vec{P}$

A Deceptive Parallel

The equation $\nabla \cdot \vec{D} = \rho_f$ looks like Gauss's law. You may be tempted to say that $\vec{D}$ is just like $\vec{E}$. That would lead us to:

$$\vec{D}(\vec{r}) \neq \frac{1}{4\pi} \int \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} \rho_f(\vec{r'}) d\tau'$$

This is not true, because this law requires the curl of $\vec{E}$ to be 0, but the curl of $\vec{D}$ isn't always 0 because:

$$\nabla \times \vec{D} = \epsilon_0(\nabla \times \vec{E}) + (\nabla \times \vec{P}) = \nabla \times \vec{P}$$

And there is no reason in general for $\vec{P}$ to vanish.
And similarly there is no potential for $\vec{D}$

Boundary Conditions

The electrostatic boundary conditions can be recast in terms of $\vec{D}$, so:

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$$

And the discontinuity in parallel components is:

$$\vec{D}^{\parallel}_{above} - \vec{D}^{\parallel}_{below} = \vec{P}^{\parallel}_{above} - \vec{P}^{\parallel}_{below}$$

Linear Dielectrics

Susceptibility, permittivity, Dielectric constant

For many substances, the polarization is proportional to the electric field:

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

We call χ_e the **electric susceptibility** of the medium. The factor ϵ_0 is extracted to make χ_e dimensionless.
These materials are called **linear dielectrics**.

Note that $\vec{E}$ is the **total** field, due to in part polarization and free charge.

In linear media we have:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E} = \epsilon_0 (1 + \chi_e) \vec{E}$$

So $\vec{D}$ is also proportional to $\vec{E}$:

$$\vec{D} = \epsilon \vec{E}$$

Where:

$$\epsilon := \epsilon_0 (1 + \chi_e)$$

The constant ϵ is called the **permittivity** of the material.
If we remove the ϵ_0, we call the remaining dimensionless quantity the **dielectric constant:**

$$\epsilon_r = 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$$

Example 4.5: A metal sphere of radius a carries a charge Q. It is surrounded, out to radius b, by linear dielectric material of permittivity ϵ. Find the potential at the center (relative to infinity)

Solution: To compute V we need to know $\vec{E}$. We know the free charge Q due to the sphere, then, we can calculate $\vec{D}$ directly due to spherical symmetry:

$$\vec{D} = \frac{Q}{4\pi r^2} \hat{r} \quad , \quad r > a$$

Inside the metal sphere, of course $\vec{E} = \vec{P} = \vec{D} = 0$. Once we know $\vec{E}$, we can calculate $\vec{E}$ by using $\vec{D} = \epsilon \vec{E}$ (because it is a linear dielectric). So

$$\vec{E} = \begin{cases} \dfrac{Q}{4\pi \epsilon r^2} \hat{r} & , \quad a < r < b \\ \dfrac{Q}{4\pi \epsilon_0 r^2} \hat{r} & , \quad r > b \end{cases}$$

The potential is therefore:

$$V = -\int_{\infty}^{0} \vec{E} \cdot d\vec{l} = -\int_{\infty}^{b} \left(\frac{Q}{4\pi\epsilon_0 r^2} \right) dt - \int_{b}^{a} \left(\frac{Q}{4\pi\epsilon r^2} \right) dr - \int_{a}^{0} 0dr$$
$$= \frac{Q}{4\pi} \left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b} \right)$$

We could know the polarization using:

$$\vec{P} = \epsilon_0 \chi_e \vec{E} = \frac{\epsilon_0 \chi_e Q}{4\pi\epsilon r^2} \widehat{r}$$

And hence:

$$\rho_b = -\nabla \cdot \vec{P} = 0$$

While:

$$\sigma_b = \vec{P} \cdot \widehat{n} = \begin{cases} \dfrac{\epsilon_0 \chi_e Q}{4\pi\epsilon b^2} & , \quad \text{outer surface} \\ \dfrac{-\epsilon_0 \chi_e Q}{4\pi\epsilon a^2} & , \quad \text{at the inner surface} \end{cases}$$

.

Full space with linear dielectric

You might suppose that linear dielectrics do fulfill $\nabla \cdot \vec{D} = 0$, but this is not true, for if we have different materials, $\vec{P}$ will jump.

If the space is completely filled with a homogeneous linear dielectric, then we do have:

$$\nabla \cdot \vec{D} = \rho_f \quad , \quad \nabla \times \vec{D} = 0$$

So $\vec{D}$ can be found from the free charge just as though the dielectric was not there, but with ϵ instead of ϵ_0. So:

$$\vec{D} = \epsilon_0 \vec{E}_{vac}$$

Where $\vec{E}_{vac}$ is the field due to the same distribution in the absence of any dielectric. Therefore:

$$\vec{E} = \frac{1}{\epsilon} \vec{D} = \frac{1}{\epsilon_r} \vec{E}_{vac}$$

Conclusion: When all space is filled with a homogeneous linear dielectric, the field everywhere is simply reduced by the dielectric constant $\frac{\epsilon}{\epsilon_0}$, so the field of one charge is:

$$\vec{E} = \frac{1}{4\pi\epsilon} \frac{q}{r^2} \widehat{r}$$

Example 4.6: A parallel plate capacitor is filled with insulating material of dielectric constant ϵ_r, what effect does this have on its capacitance?

Solution: Since the field is confined in the space between the plates, the dielectric will reduce $\vec{E}$, and hence also the potential difference V by a factor $1/\epsilon_r$. Accordingly, the capacitance $C = Q/V$ is increased by a factor of the dielectric constant:

$$C = \epsilon_r C_{vac}$$

Boundary Value Problems with linear Dielectrics

In a homogeneous linear dielectric the bound charge density ρ_b is proportional to the free charge density ρ_f:

$$\rho_b = -\nabla \cdot \vec{P} = -\nabla \cdot \left(\epsilon_0 \frac{\chi_e}{\epsilon} \vec{D} \right) = -\left(\frac{\chi_e}{1+\chi_e} \right) \rho_f$$

In particular, unless free charge is embedded in the material, $\rho = 0$, and any net charge is at the surface. Then, the potential obeys Laplace's equation. We can rewrite the boundary conditions:

$$\epsilon_{above} E^{\perp}_{above} - \epsilon_{below} E^{\perp}_{below} = \sigma_f$$

Or:

$$\epsilon_{above} \frac{\partial V_{above}}{\partial n} - \epsilon_{below} \frac{\partial V_{below}}{\partial n} = -\sigma_f$$

Whereas the potential itself is continuous:

$$V_{above} = V_{below}$$

Example 4.7: A sphere of homogeneous linear dielectric material is placed in an otherwise uniform electric field $\vec{E}_0$. Find the electric field inside the sphere:

Solution: We have solved a similar problem but for a conducting sphere, for a dielectric we need to solve Laplace equation with:

- $V_{in} = V_{out} \quad , \quad r = R$
- $\epsilon \frac{\partial V_{in}}{\partial r} = \epsilon_0 \frac{\partial V_{out}}{\partial r} \quad , \quad r = R$
- $V_{out} \to -E_0 r \cos\theta \quad , \quad r >> R$

Inside the sphere the field is $V_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$
Outside the sphere, we have $V_{out} = -E_0 r\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$
Boundary condition i) implies that:

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = -E_0 R\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta)$$

So:

- $A_l R^l = \frac{B_l}{R^{l+1}}$
- $A_1 R = -E_0 R + \frac{B_1}{R^2}$

Meanwhile, condition ii) yields:

$$\epsilon_r \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -E_0\cos\theta - \sum_{l=0}^{\infty} \frac{(l+1)B_l}{R^{l+2}} P_l(\cos\theta)$$

So $\epsilon_r l A_l R^{l-1} = -\frac{(l+1)B_l}{R^{l+2}} \quad , \quad l \neq 1$
$\epsilon_r A_1 = -E_0 - \frac{2B_1}{R^3}$
It follows that $A_l = B_l = 0$ for $l \neq 1$
$A_1 = -\frac{3}{\epsilon_r + 2} E_0$ and $B_1 = \frac{\epsilon_r - 1}{\epsilon_r + 2} R^3 E_0$.
Evidently then:

$$V_{in} = -\frac{3E_0}{\epsilon_r + 2} r\cos\theta = -\frac{3E_0}{\epsilon_r + 2} z$$

And hence, the field inside the sphere is uniform:

$$\vec{E} = \frac{3}{\epsilon_r + 2}\vec{E}_0$$

Energy in Dielectric Systems

It takes work to charge up a capacitor:

$$W = \frac{1}{2} C V^2$$

If the capacitor is filled with linear dielectric, its capacitance is:

$$C = \epsilon_r C_{vac}$$

In the case of a dielectric filled capacitor, the energy stored is:

$$W = \frac{1}{2}\int \vec{D}\cdot\vec{E} d\tau$$

0.1. Summary

- If a capacitor is filled with a dielectric, the capacitance increases by a factor k known as the dielectric constant
- The potential of a charge distribution can be written in powers of $1/r$. A net 0 charge has no monopole moment. The potential and field due to a dipole:

$$\phi(r,\theta) = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} \quad , \quad \vec{E}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\widehat{r} + \sin\theta\widehat{\theta}\right)$$

- The torque on an electric dipole is $\vec{N} = \vec{p}\times\vec{E}$. The force is $\vec{F} = (\vec{p}\cdot\nabla)\vec{E}$
- An external electric field will cause an atom to become polarized. The atomic polarizability α is defined by $\vec{p} = \alpha\vec{E}$
- Some molecules have a permanent dipole moment.
- The **polarization** per unit volume is given by $\vec{P} = \vec{p}N$.
- A polarized object produced inside and outside a field as if was caused by surface density $\sigma_b = \vec{P}\cdot\widehat{n}$ and a volume density $\rho_b = -\nabla\cdot\vec{P}$
- When we talk about the electric field inside matter, we mean the spatial average $\langle\vec{E}\rangle = (1/V)\int\vec{E}d\tau$. Inside a uniformly polarized slab, the average is $-\vec{P}/\epsilon_0$.
  Inside a uniformly polarized sphere, the field is $-\vec{P}/3\epsilon_0$
- The **electric susceptibility is defined by**

$$\chi_e := \frac{P}{\epsilon_0 E} \quad \Rightarrow \quad \chi_e = k-1$$

- For any material, $\nabla\vec{P} = -\rho_b$. Combine it with Gauss's law $\div\vec{E}_T = \rho_T/\epsilon_0$, so:

$$\div\vec{D} = \rho_f \quad , \quad \vec{D} := \epsilon_0\vec{E} + \vec{P}$$

  If we are dealing with a **linear dielectric**:

$$\vec{D} = \epsilon\vec{E}$$

  Where $\epsilon = k\epsilon_0$

Magnetostatics

The Lorenz Force

Magnetic Fields

Remember the basic problem of classical electrodynamics: We have a collection of charges $q_1, q_2, \cdots$ as a source and we want to know the force they exert on a test charge Q. Up to now, we have solved the problem for electrostatics.

With some experiments, we find that if we have two wires with charges flowing, they will attract or repel each other.
To account for this, we define a **magnetic field** $\vec{B}$ caused by moving particles.

Magnetic Forces

The magnetic field $\vec{B}$ is defined in such a way that the force in a charge Q with velocity $\vec{v}$ in the magnetic field $\vec{B}$ has a force:

$$\boxed{\vec{F}_{mag} = Q(\vec{v} \times \vec{B})}$$

This is known as the **Lorentz force law**.
In the present of an electric and magnetic field, the force is:

$$\vec{F} = Q[\vec{E} + (\vec{v} \times \vec{B})]$$

This is a fundamental axiom of the theorem.

Example 5.1 Cyclotron Motion: We have a particle Q initially moving with speed $\vec{v}\widehat{i}$ in a magnetic field $\vec{B} = B_0\widehat{z}$. Clearly, there will always be a force on the particle perpendicular to the motion and of fixed magnitude QvB. So we will have:

$$QvB = m\frac{v^2}{R}$$

Then, the radius of the cyclotron movement is:

$$R = \frac{mv}{qB}$$

To actually solve it, we will solve a more general one

Example 5.2 Cycloid: A charge moves in a field $\vec{B} = B\widehat{x}$ and $\vec{E} = E\widehat{z}$

There will not be speed in the x, so the speed is always:

$$\vec{v} = (0, \dot{y}, \dot{z})$$

So we have:

$$\vec{v} \times \vec{B} = B\dot{z}\widehat{y} - B\widehat{y}\widehat{z}$$

And hence:

$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B}) = Q(E\widehat{z} + B\dot{z}\widehat{y} - B\dot{y}\widehat{z}) = m\vec{a} = m(\ddot{y}\widehat{y} + \ddot{z}\widehat{z})$$

So:

$$QB\dot{z} = m\ddot{y} \quad , \quad QE - QB\dot{y} = m\ddot{z}$$

We define: $\omega := \dfrac{QB}{m}$.
Therefore:

$$\ddot{y} = \omega\dot{z} \quad , \quad \ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right)$$

Their general solution is:

$$\begin{aligned} y(t) = C_1 \cos\omega t + C_2 \sin\omega t + (E/B)t + C_3 \\ z(t) = C_2 \cos\omega t - C_1 \sin\omega t + C_4 \end{aligned}$$

But the particle started from rest $\dot{y}(0) = \dot{z}(0) = 0$ and $y(0) = z(0) = 0$ Then, we can determine the constraints and get:

$$y(t) = \frac{E}{\omega B}(\omega t - \sin\omega t) \quad , \quad z(t) = \frac{E}{\omega B}(1 - \cos\omega t)$$

This is a cycloid.

Magnetic Forces do no Work

If Q moves an amount $d\vec{l} = \vec{v}dt$, the work done is:

$$dW_{mag} = \vec{F}_{mag} \cdot d\vec{l} = Q(\vec{v} \times \vec{B}) \cdot \vec{v}dt = 0$$

Magnetic forces alter the direction of a particle but not its speed.

Currents

The **current** in a wire is the charge per unit time passing a given point and it is measured in Amperes, where $1A = 1C/s$
A line charge λ traveling down a wire at speed v constitutes a current:

$$\vec{I} = \lambda\vec{v}$$

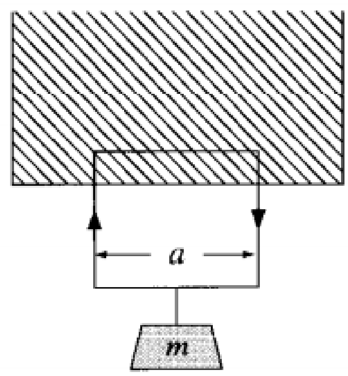
The **magnetic force on a segment of current-carrying wire** is evidently:

$$\vec{F}_{mag} = \int (\vec{v} \times \vec{B}) dq = \int (\vec{v} \times \vec{B}) \lambda dl = \int (\vec{I} \times \vec{B}) dl$$

As $\vec{I}$ and $d\vec{l}$ point in the same direction, we may write it as:

$$\boxed{\vec{F}_{mag} = \int I(d\vec{l} \times \vec{B})}$$

Example: 5.3: A rectangular loop of wire, supporting a mass m, hangs vertically with one end in a uniform magnetic field $\vec{B}$, which point into the page. for what current I in the loop would the magnetic force exactly sustain the mass?



The only vertical force is due to the top wire, of an amount $\int I(d\vec{l} \times \vec{B}) = IaB$ pointing upwards.
For this force to match the weight, we need $IBa = mg \quad \Rightarrow \quad I = \dfrac{mg}{Ba}$

Even if the magnetic force is higher than mg ant if lifts the box, there is still no work. That is because the charges are now moving slightly vertically and so, the magnetic force on them tilts back perpendicular to the speed still. The work done is done by the force needed to overcome this magnetic field in order to pump up the charges. The magnetic field merely redirects this work to the upward direction.

Surface current density
If charge flows over a surface, we define a surface current density vector $\vec{K}$ as follows: Consider a ' ribbon ' of infinitesimal width $dl_\perp$ running parallel to the flow. If the current in this ribbon is $d\vec{I}$, the surface current density is:

$$\vec{K} := \frac{d\vec{I}}{dl_\perp}$$

In other words, $\vec{K}$ is the current per unit width perpendicular to flow. In particular, if the mobile surface charge density is σ and its velocity is $\vec{v}$, then:

$$\vec{K} = \sigma \vec{v}$$

The magnetic force on the surface current is:

$$\vec{F}_{mag} = \int (\vec{v} \times \vec{B})\sigma da = \int (\vec{K} \times \vec{B}) da$$

Volume current density J
When the flow of charge is distributed throughout a three dimensional region, we describe it by the **volume current density** $\vec{J}$, defined as follows.
Consider a tube of infinitesimal cross section $da_{\perp}$ running parallel to the flow, if the current passing though is $d\vec{I}$, the volume current density is:

$$\vec{J} := \frac{d\vec{I}}{da_{\perp}}$$

So J is the current per unit area perpendicular to flow. If the mobile volume charge density is ρ and the velocity $\vec{v}$, then:

$$\vec{J} = \rho\vec{v}$$

Then, the total current flowing through an area A is:

$$I = \int_A \vec{J} \cdot d\vec{a}$$

In particular, the total charge per unit time (current) leaving a volume V is:

$$\oint_S \vec{J} \cdot d\vec{a} = \int_V (\nabla \cdot \vec{J}) d\tau$$

Because charge is conserved, whatever flows out should come from what is inside:

$$\int_V (\nabla \cdot \vec{J}) d\tau = -\frac{d}{dt} \int_V \rho d\tau = -\int_V \left(\frac{\partial \rho}{\partial t}\right) d\tau$$

Or, in differential form, the **continuity equation** says:

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

The magnetic force on a volume current is then:

$$\vec{F}_{mag} = \int (\vec{v} \times \vec{B})\rho d\tau = \int (\vec{J} \times \vec{B}) d\tau$$

Example 5.4: A current I is uniformly distributed over a wire of circular cross section, with radius a. Find the volume current density J.

Solution: The area perpendicular to flow is πa^2, so:

$$J = \frac{I}{\pi a^2}$$

b) Suppose the current density in the wire is $J = ks$ (s is the distance to the axis) Find the total current.
We need a surface integral:

$$I = \int J \cdot da = \int (ks)(sdsd\phi) = 2\pi k \int_0^a s^2 ds = \frac{2\pi k a^3}{3}$$

In general, the different current densities are related by:

$$\sum_{i=1}^{n} ()q_i\vec{v}_i \sim \int_{line} ()\vec{I}dl \sim \int_{surface} ()\vec{K}da \sim \int_{vol} ()\vec{J}d\tau$$

Biot Savart Law

Stationary charges produce constat electric field. Similarly, steady currents produce constant magnetic fields.
Notice that a point charge cannot possibly be a steady current.

When a steady current flows in a wire, then $\partial p/\partial t = 0$, hence by continuity equation:

$$\nabla \cdot \vec{J} = 0$$

Magnetic Field of a Steady Current

The magnetic field of a steady line current is given by the **Biot Savart Law**

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\hat{\mathfrak{r}}}{\mathfrak{r}^2}dl' = \frac{\mu_0}{4\pi}I\int \frac{d\vec{l'}\times\hat{\mathfrak{r}}}{\mathfrak{r}^2}}$$

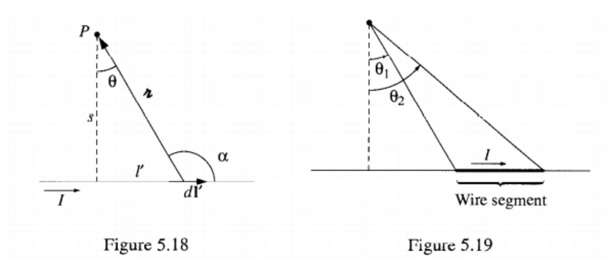
The integration is along the current path, in the direction of flow. $d\vec{l'}$ is an element of length along the wire and $\mathfrak{r}$ is the vector $\vec{r}-\vec{r'}$ from the source to $\vec{r}$.
We can parametrize the curve of the flow with $\vec{r'}(s) = (r_1'(s), r_2'(s), r_3'(s))$ and so, $d\vec{l'} - \frac{d\vec{r'}}{ds}$ is the element of line flow.

The units of $\vec{B}$ is **Teslas** (T):

$$1T = 1N/(A\cdot m)$$

Example 5.5: Find the magnetic field a distance s from a long straight wire carrying a current I

Solution:



The amount $d\vec{l'}\times\hat{\mathfrak{r}}$ point out of page and has a magnitude:

$$dl'\sin\alpha = dl'\cos\theta$$

Also, $l' = s\tan\theta$, so:

$$dl' = \frac{s}{\cos^2\theta}d\theta$$

And $s = \mathfrak{r}\cos\theta$, so:

$$\frac{1}{\mathfrak{r}^2} = \frac{\cos^2\theta}{s^2}$$

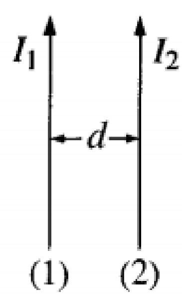
Thus:

$$\begin{aligned} B &= \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2}\left(\frac{\cos^2\theta}{s^2}\right)\left(\frac{s}{\cos^2\theta}\right)\cos\theta d\theta \\ &= \frac{\mu_0 I}{4\pi s}\int_{\theta_1}^{\theta_2}\cos\theta d\theta = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1) \end{aligned}$$

If the wire is infinite, then the field would be:

$$B = \frac{\mu_0 I}{2\pi s}$$

Example 5.5.5: Find the force of attraction between two long, parallel wires a distance d apart carrying currents I_1, I_2:



Solution: The field at (2) due to (1) is:

$$B = \frac{\mu_0 I_1}{2\pi d}$$

And it points into the page. The Lorentz force predicts a force in (2) due to (1) of magnitude:

$$F = I_2\int d\vec{l}\times\vec{B} = \frac{I_2\mu_0 I_1}{2\pi d}\int dl$$

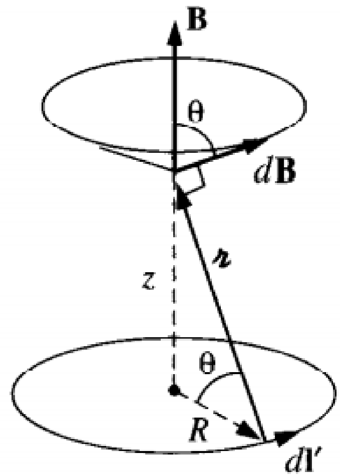
Pointing to attraction (to the left). The force per unit length is:

$$f = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}$$

If the currents are antiparallel, the force is repulsive, if they are parallel, the force is attractive.

Example 5.6: Find the magnetic field a distance z above the center of a circular loop of radius R with steady current I:

Solution



The component $d\vec{B}$ due to a segment $d\vec{l'}$ is shown and has the value $d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{d\vec{l'} \times \hat{\mathfrak{r}}}{\mathfrak{r}^2}$
This vector has a magnitude $\frac{dl'}{\mathfrak{r}^2}$ and points as shown. Only the vertical components count, and they have a magnitude $\frac{\mu_0 I}{4\pi}\frac{dl'}{\mathfrak{r}^2}\cos\theta$.
$\cos\theta = \frac{R}{\mathfrak{r}}$ is constant and $\mathfrak{r}$ is constant, so the integral is:

$$\begin{aligned} B(z) &= \frac{\mu_0 I}{4\pi}\int \frac{dl'}{\mathfrak{r}^2}\cos\theta = \frac{\mu_0 I}{4\pi}\frac{\cos\theta}{\mathfrak{r}^2}\int dl' = \frac{\mu_0 I}{4\pi}\frac{R}{(R^2+z^2)^{3/2}}2\pi R \\ &= \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}} \end{aligned}$$

Another way of solving it:
We parameterize the points in the curve with θ as $\vec{r'} = (R\cos\theta, R\sin\theta, 0)$.
And the point of measurement is $\vec{r} = (0, 0, z)$
The differential part of curve is $d\vec{l'} = \frac{dr'}{d\theta} = (-R\sin\theta, R\cos\theta, 0)$
Then, we have $\vec{\mathfrak{r}} = (-R\cos\theta, -R\sin\theta, z)$
So $\mathfrak{r} = \sqrt{R^2+z^2}$.
Then, the field is:

$$\begin{aligned} B(z) &= \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l'} \times \vec{\mathfrak{r}}}{\mathfrak{r}^3} \\ &= \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{(zR\cos\theta, zR\sin\theta, R^2)}{(R^2+z^2)^{3/2}}d\theta \\ &= \frac{\mu_0 I}{4\pi}\frac{(0,0,2\pi R^2)}{(R^2+z^2)^{3/2}} = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}} \end{aligned}$$

Biot Savart for volume currents and surface currents

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r'}) \times \hat{\mathfrak{r}}}{\mathfrak{r}^2} da'$$
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r'}) \times \hat{\mathfrak{r}}}{\mathfrak{r}^2} d\tau'$$

Problem 5.11

Find the magnetic field at point P on the axis of a tightly wound solenoid consisting of n turns per unit length of radius a and current I. Express your answer in terms of θ_1, θ_2.

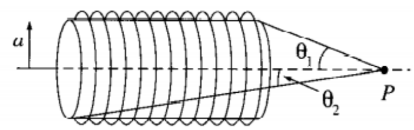
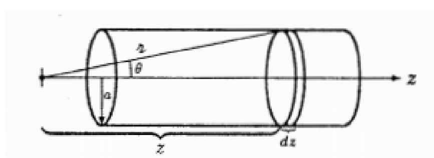


Figure 5.25

We begin by taking a ring of width dz with $I \to nIdz$



This ring causes a field of:

$$dB = \frac{\mu_0 nI}{2}\frac{a^2}{(a^2+z^2)^{3/2}}dz$$

Then, the complete field is:

$$B = \frac{\mu_0 nI}{2}\int \frac{a^2}{(a^2+z^2)^{3/2}}dz$$

But $z = a\cot\theta$, so $dz = -\dfrac{a}{\sin^2\theta}$ and $\dfrac{1}{(a^2+z^2)^{3/2}} = \dfrac{\sin^3\theta}{a^3}$

So:

$$B = \frac{\mu_0 nI}{2}\int \frac{a^2\sin^3\theta}{a^3\sin^2\theta}(-ad\theta) = -\frac{\mu_0 nI}{2}\int \sin\theta d\theta = \frac{\mu_0 nI}{2}\cos\theta\Big|_{\theta_1}^{\theta_2} = \frac{\mu_0 nI}{2}(\cos\theta_2 - \cos\theta_1)$$

For an infinite solenoid, $\theta_2 = 0$, $\theta_1 = \pi$, so $(\cos\theta_2 - \cos\theta_1) = 1 - (-1) = 2$

So $B = \mu_0 nI$

Divergence and Curl of B

Suppose we have a wire of current I. Then, the field $\vec{B}$ goes around the wire and has magnitude $\frac{\mu_0 I}{2\pi s}$
That is, the field is:

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\widehat{\phi}$$

And $d\vec{l} = ds\widehat{s} + sd\phi\widehat{\phi} + dz\widehat{z}$
So the line integral of the field around a loop is:

$$\oint \vec{B} \cdot d\vec{l} = \frac{\mu_0 I}{2\pi} \oint \frac{1}{s} sd\phi = \frac{\mu_0 I}{2\pi} \int_0^{2\pi} d\phi = \mu_0 I$$

If the path didn't enclose the wire, then the integral would be 0.

Therefore, we conclude that:

$$\boxed{\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}}$$

Where I_{enc} is the current that goes through the path.

If $\vec{J}$ is the volume current density $\vec{J}$, then $I_{enc} = \int \vec{J} \cdot d\vec{a}$.
Therefore, we get:

$$\int (\nabla \times \vec{B}) \cdot d\vec{a} = \mu_0 \int \vec{J} \cdot d\vec{a}$$
$$\Rightarrow \boxed{\nabla \times \vec{B} = \mu_0 \vec{J}}$$

We derived this only for infinite wires, we now derive it more in general.

Divergence and Curl of B

The Biot-Savart law for the general case of volume current is:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'}) \times \widehat{\mathfrak{r}}}{\mathfrak{r}^2} d\tau'$$

This formula gives the magnetic field at point $\vec{r} = (x, y, z)$ in terms of the integral of $\vec{J}(x', y', z')$ and $d\tau' = dx'dy'dz'$.
We now take the divergence:

$$\begin{aligned} \nabla \cdot \vec{B} &= \frac{\mu_0}{4\pi} \int \nabla \left(\vec{J} \times \frac{\widehat{\mathfrak{r}}}{\mathfrak{r}^2} \right) d\tau' \\ &= \frac{\mu_0}{4\pi} \int \frac{\widehat{\mathfrak{r}}}{\mathfrak{r}^2} \cdot (\nabla \times \vec{J}) - \vec{J} \cdot \left(\nabla \times \frac{\widehat{\mathfrak{r}}}{\mathfrak{r}^2} \right) \end{aligned}$$

But $\nabla \times \vec{J} = 0$ because $\vec{J}$ doesn't depende on the unprimed variables (x, y, z), whereas $\nabla \times (\hat{\mathfrak{r}}/\mathfrak{r}^2) = 0$, so:

$$\boxed{\nabla \cdot \vec{B} = 0}$$

Curl
On the other hand, we calculate the curl:

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \int \nabla \left(\vec{J} \times \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} \right) d\tau'$$
$$= \frac{\mu_0}{4\pi} \int \vec{J} \left(\nabla \cdot \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} \right) - (\vec{J} \cdot \nabla) \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} d\tau'$$

We have dropped the terms involving derivatives of $\vec{J}$ because $\vec{J}$ does not depend on (x, y, z). The second term integrates to 0 as we will see later.
And the first term involves the amount $\nabla \cdot \left(\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} \right) = 4\pi\delta^3(\vec{\mathfrak{r}})$
Thus:

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r'}) 4\pi\delta^3(\vec{r} - \vec{r'}) d\tau' = \mu_0 \vec{J}(\vec{r})$$

We need to check that the second term integrates to 0. Becauses ∇ acts only on $\hat{\mathfrak{r}}/\mathfrak{r}^2$ we can switch ∇ for ∇' at the cost of a minus sign.

$$-(\vec{J} \cdot \nabla) \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} = (\vec{J} \cdot \nabla') \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}$$

For example, the x componente in particular is:

$$(\vec{J} \cdot \nabla') \left(\frac{x - x'}{\mathfrak{r}^3} \right) = \nabla' \cdot \left[\frac{(x - x')}{\mathfrak{r}^3} \vec{J} \right] - \left(\frac{x - x'}{\mathfrak{r}^3} \right) (\nabla' \cdot \vec{J})$$

But $\nabla \vec{J} = 0$ for steady currents, so:

$$\left[-(\vec{J} \cdot \nabla) \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} \right]_x = \nabla' \cdot \left[\frac{(x - x')}{\mathfrak{r}^3} \vec{J} \right]$$

And therefore the contribution to the integral can be written as:

$$\int_V \nabla' \cdot \left[\frac{(x - x')}{\mathfrak{r}^3} \vec{J} \right] d\tau' = \oint_S \frac{(x - x')}{\mathfrak{r}^3} \vec{J} \cdot d\vec{a'}$$

(By parts)
But if the volume is large enough, $\vec{J}$ is 0 and the integral is 0.

Therefore, we get **Apere's Law:**

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{J}}$$

Or, in integral form:

$$\boxed{\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}}$$

So Biot Savart law leads to Ampere's law (and the other way around).

Example 5.7
Find the magnetic field a distance s from a long straight wire, carrying a steady current I

Solution: We know $\vec{B}$ is circunferential circling around the wire. By symmetry, the magnitude of $\vec{B}$ should be constant B so:

$$\oint \vec{B} \cdot d\vec{l} = B \oint dl = B2\pi s = \mu_0 I_{enc} = \mu_0 I$$

So that:

$$B = \frac{\mu_0 I}{2\pi s}$$

Example 5.8:
Find the magnetic field of an infinite uniform surface current $\vec{K} = K\widehat{x}$ over the xy plane.

Solution: First of all by Biot savart, it is clear that $\vec{B}$ should point in the y direction and should change direction on the other side of the plane.
Then, we make a square wire that is half above and half below the plane and we have:

$$\oint \vec{B} \cdot d\vec{l} = 2Bl = \mu_0 I_{enc} = \mu_o K l$$

Then:

$$\vec{B} = \begin{cases} +(\mu_0/2)K\widehat{y} \quad , \quad z < 0 \\ -(\mu_0/2)K\widehat{y} \quad , \quad z > 0 \end{cases}$$

Solenoid
For a solenoid of current I and n turns per unit length, the field is found to be:

$$\vec{B} = \begin{cases} \mu_0 n I \widehat{z} \quad \text{inside} \\ 0 \quad , \quad \text{outside} \end{cases}$$

Maxwell Static Laws

Electrostatic field:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$
$$\nabla \times \vec{E} = 0$$

Magnetostatic Field:

$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

Force Law:

$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B}$$

Coulumb's law:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\widehat{\mathfrak{r}}}{\mathfrak{r}^2} \rho d\tau$$

Biot Savart Law:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int I \frac{d\vec{l'} \times \widehat{\mathfrak{r}}}{\mathfrak{r}^2}$$

Magnetic Vector potential

Because $\nabla \cdot \vec{B} = 0$, there most exist a vector potential $\vec{A}$ such that:

$$\vec{B} = \nabla \times \vec{A}$$

And we ask this $\vec{A}$ to fulfill:

$$\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

But $\vec{A}$ has a built in ambiguity, we can add any curless field to $\vec{A}$, so we can add such a field that makes:

$$\nabla \cdot \vec{A} = 0$$

Therefore, the potential $\vec{A}$ is a field such that:

$$\nabla \times \vec{A} = \vec{B}$$

And that also fulfill $\nabla \cdot \vec{A} = 0$, so:

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

This is like having three Poisson's equations, and we can solve it as we know:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'})}{\mathfrak{r}} d\tau'$$

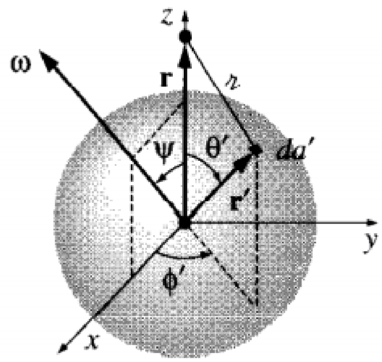
For line and surface currents, we have:

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{I}}{\mathfrak{r}} dl' = \frac{\mu_0 I}{4\pi} \int \frac{1}{\mathfrak{r}} d\vec{l'}$$
$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{K}}{\mathfrak{r}} da'$$

$\vec{A}$ is usually not as useful as V for electrostatics.

Example 5.11:
A spherical shell, of radius R, carrying a uniform surface charge density σ, is set spinning at angular velocity $\vec{\omega}$. Find the vector potential it produces at a point $\vec{r}$.

Solution:
The best thing to do is to align $\vec{r}$ in the z axis, with $\vec{\omega}$ tilted at an angle ψ (we tilt it in a way such that $\vec{\omega}$ is in the xz plane).



The potential is:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r'})}{\mathfrak{r}}da'$$

Where $\vec{K} = \sigma\vec{v}$, $\mathfrak{r} = \sqrt{R^2 + r^2 - 2Rr\cos\theta'}$ and $da' = R^2\sin\theta' d\theta' d\phi'$. Now, the velocity at a point $\vec{r'}$ is $\vec{\omega}\times\vec{r'}$. In this case:

$$\vec{v} = \vec{\omega}\times\vec{r'} = \left|\begin{pmatrix} \widehat{x} & \widehat{y} & \widehat{z} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{pmatrix}\right|$$
$$= R\omega[-(\cos\psi\sin\theta'\sin\phi')\widehat{x} + (\cos\psi\sin\theta'\cos\phi' - \sin\psi\cos\theta')\widehat{y} + (\sin\psi\sin\theta'\sin\phi')\widehat{z}]$$

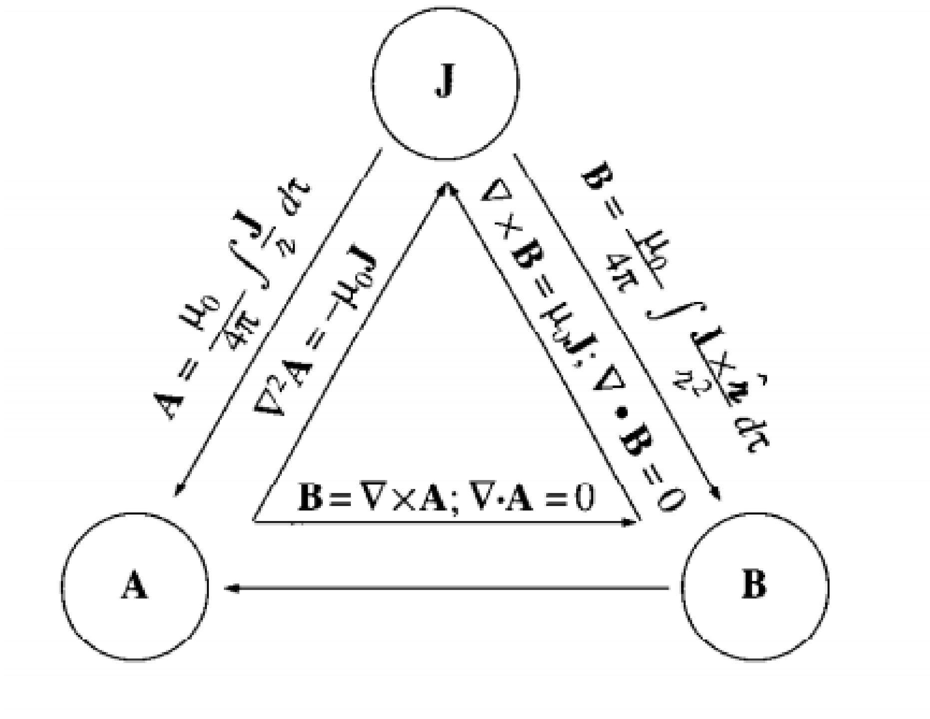
Then, when we make the integration, The $\widehat{x}, \widehat{y}$ parts disappear when integrating ϕ' from 0 to 2π. Therefore, we only get:

$$\vec{A}(\vec{r}) = -\frac{\mu_0 R^3\sigma\omega\sin\psi}{2}\left(\int_0^\pi \frac{\cos\theta'\sin\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}}d\theta'\right)\widehat{y}$$

When we integrate, we get:

$$\vec{A}(\vec{r}) = \begin{cases} \dfrac{\mu_0 R\sigma}{3}(\vec{\omega}\times\vec{r}) & \text{, inside sphere} \\ \dfrac{\mu_0 R^4\sigma}{3r^3}(\vec{\omega}\times\vec{r}) & \text{, outside sphere} \end{cases}$$

Summary



Discontinuity

$$B^{\perp}_{above} = B^{\perp}_{below}$$

$$B^{\parallel}_{above} = B^{parallel}_{below} = \mu_0 K$$

For the potential:

$$\vec{A}_{above} = \vec{A}_{below}$$

$$\frac{\partial A_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{K}$$

Multipole Expansion of the vector Potential

As before, we can write $\frac{1}{\mathfrak{r}}$ in terms of Legendre polynomials:

$$\frac{1}{\mathfrak{r}} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr'\cos\theta'}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

Accordingly, the vector potential of a current loop can be written as:

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{1}{\mathfrak{r}} d\vec{l'} = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\theta') d\vec{l'}$$

So:

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\vec{l'} + \frac{1}{r^2}\oint r'\cos\theta' d\vec{l'} + \frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right) d\vec{l'} + \cdots\right]$$

It happens that the monopole term is always 0 , because $\oint d\vec{l'} = 0$

The dominant term is then the dipole:

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r'\cos\theta' d\vec{l'} = \frac{\mu_0 I}{4\pi r^2}\oint (\hat{r}\cdot\vec{r'}) d\vec{l'}$$

We can rewrite it in a more iluminating way using $\oint(\vec{c}\cdot\vec{r}) d\vec{l} = \vec{a}\times\vec{c}$ where $\vec{a}$ is the vector area.
therefore, we can write:

$$\oint (\hat{r}\cdot\vec{r'}) d\vec{l'} = -\hat{r}\times\int d\vec{a'}$$

Then:

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

Where $\vec{m}$ is the **magnetic dipole moment**:

$$\vec{m} := I\int d\vec{a} = I\vec{a}$$

where $\vec{a}$ is the vector area of the loop.
It can then be shown that the magnetic field due to a dipole of momentum $\vec{m}$ is:

$$\boxed{\vec{B}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}]}$$

Pure Dipole

A pure dipole is a ring with current I, it has a vector potential of:

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}$$

And hence:

$$\vec{B}_{dip}(\vec{r}) = \nabla\times\vec{A} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$$

.

Magnetic Fields in Matter

Magnetization

Diamagnets, Paramagnets, Ferromagnets

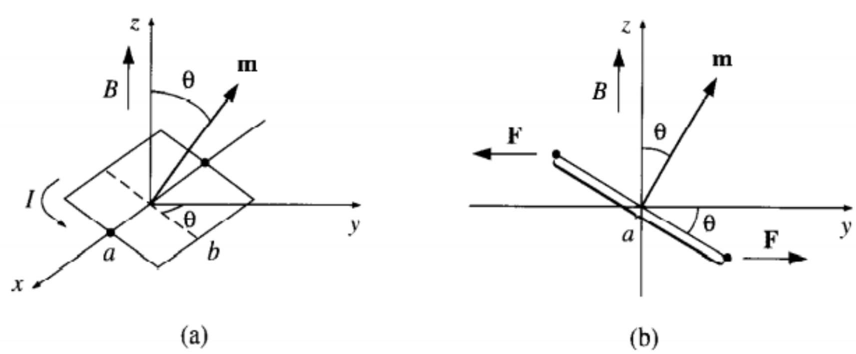
All magnetic effects are ultimately due to charges in motion.
A piece of matter usually has lots of electric charges in motion but whose magnetic effects cancel out.
Yet, when a magnetic field is applied $\vec{B}$, a net alignment of the dipoles in the material may occur, and the medium becomes magnetically polarized, or **magnetized**.

Some materials acquire a magnetization:

- A magnetization parallel to $\vec{B}$ (**paramagnet**)
- A magnetization opposite to $\vec{B}$ (**diamagnets)**
- A few substances retain their magnetization even after the external field is removed (**ferromagnet**)

Torques and Forces on Magnetic Dipoles

A magnetic dipole experiences a torque in a magnetic field. We can calculate the torque on a rectangular current loop in a uniform field $\vec{B}$
We center the loop at the origin, and tilt it an angle θ form the z axis towards the y axis.



The forces on the two sloping sides cancel (they tend to stretch the loop but not rotate). The forces on the horizontal sides are opposite buy they generate a torque:

$$\vec{N} = aF \sin\theta \widehat{x}$$

The magnitude of the force on each segment is $F = IbB$
And therefore:

$$\vec{N} = IabB \sin\theta \widehat{x} = mB \sin\theta \widehat{x}$$

Or:

$$\boxed{\vec{N} = \vec{m} \times \vec{B}}$$

Where $m = Iab$ is the magnetic dipole moment.
This is the exact torque on a dipole due to a uniform $\vec{B}$ field.

We notice that the torque is in a direction as to line the dipole up parallel to the field. It is this torque that accounts for **paramagnetism**.
On a atom, every electron spin acts like a small dipole, but in some atoms, these dipoles cancel out due to pair of electrons with opposite spin and in other atoms (those with a predominant spin), the torques align and there is a paramagnetic phenomenon.

Force on a Magnetic Dipole
In a uniform field, the net force on a current loop is zero since:

$$\vec{F} = I \oint (d\vec{l} \times \vec{B}) = I \left(\oint d\vec{l} \right) \times \vec{B} = 0$$

In a **nonuniform** field, this is no longer the case.
For example, suppose a circular wire of radius R, carrying a current I, is suspended above a short solenoid just above. Here there is a radial component of $\vec{B}$ and it is not uniform, there is a net downward force.

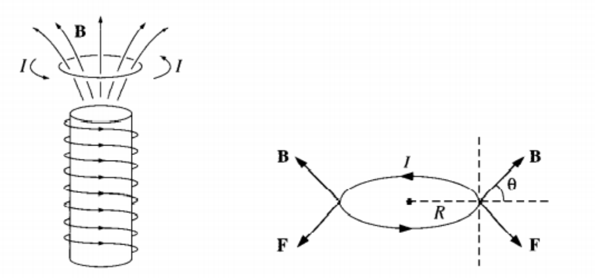


Figure 6.3 Figure 6.4

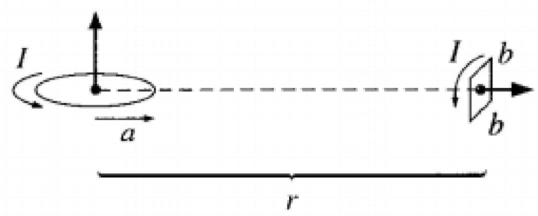
The force is $F = 2\pi IRB \cos\theta$.

In general, for an **infinitesimal** loop, with dipole moment $\vec{m}$ in a field $\vec{B}$ there is a force:

$$\vec{F} = \nabla(\vec{m} \cdot \vec{B})$$

It can be proven in a more formal way.

Problem 6.1: Calculate the torque exerted on the square loop shown in the figure due to the circular loop ($r >> a, b$). What is the equilibrium orientation of the loop?



The magnetic field due to the circle, evaluated at the square is:

$$\begin{aligned}\vec{B}_{dip}(\vec{r}) &= \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(m_1 \cdot \hat{r})\hat{r} - \vec{m}_1] \\ &= \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(0)\hat{r} - \vec{m}_1] \\ &= -\frac{\mu_0}{4\pi}\frac{m_1}{r^3}\hat{z}\end{aligned}$$

Where $\vec{m}_1 = m_1\hat{z} = \pi a^2 I\hat{z}$
Then, the torque on the second dipole $\vec{m}_2 = m_2\hat{y} = b^2 I$ is:

$$\begin{aligned}\vec{N} &= \vec{m}_2 \times \vec{B}_1 = m_2\hat{y} \times (-\frac{\mu_0}{4\pi}\frac{m_1}{r^3}\hat{z}) = -\frac{\mu_0}{4\pi}\frac{m_1 m_2}{r^3}(\hat{y} \times \hat{z}) \\ &= -\frac{\mu_0}{4\pi}\frac{m_1 m_2}{r^3}\hat{x} \\ &= -\frac{\mu_0}{4}\frac{(abI)^2}{r^3}\hat{x}\end{aligned}$$

Seeing that the field $\vec{B}$ points in $-\hat{z}$, that is the final orientation for the square, and there it will have 0 torque.

Effect of a Magnetic Field on Atomic Orbitals

Electrons revolve around the nucleus at (say) a circle of radius R. The period is $T = 2\pi R/v$, which is so fast that it looks as a steady current of:

$$I = \frac{e}{T} = \frac{ev}{2\pi R}$$

Accordingly, the orbital dipole moment $(I\pi R^2)$ is:

$$\vec{m} = -\frac{1}{2}evR\hat{z}$$

(minus sign because of charge of e).
Like any other dipole, it experiences a torque $\vec{m} \times \vec{B}$ when placed in a magnetic field. But it is a lot harder to tilt the entire orbit than it is the spin, so the orbital contribution to paramagnetism is small.
There is a more significant effect:

The electron speeds up or slows down depending on the orientation of $\vec{B}$.
For, without a field, the centripetal acceleration is $\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} = m_e\frac{v^2}{R}$.
In the presence of a magnetic field, there is an additional force $-e\vec{v} \times \vec{B}$. Let's say that $\vec{B}$ is perpendicular to the plane of the orbit, in such a way that it causes an inward force on the electron, then:

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} + ev'B = m_e\frac{v'^2}{R}$$

The new speed v' is greater than v:

$$ev'B = \frac{m_e}{R}(v'^2 - v^2) = \frac{m_e}{R}(v' + v)(v' - v)$$

Or, assuming the change $\Delta v = v' - v$ is small, we get:

$$\Delta v = \frac{eRB}{2m_e}$$

So, when B is turned on, the electron speeds up by this amount.
Therefore, the dipole moment increases by:

$$\Delta\vec{m} = -\frac{1}{2}e(\Delta v)R\widehat{z} = -\frac{e^2R^2}{4m_e}\vec{B}$$

The change in $\vec{m}$ is opposite to the direction of $\vec{B}$ (an electron circling the other way would have a dipole pointing upward, but now the field would slow it down, anyway, the dipole moment now points a little bit more opposite)

Diamagnetism: Although the electron orbits are randomly oriented, when putting a field $\vec{B}$, each atom picks up a little extra dipole moment opposite to the field. So the general moment of the material points a little bit antiparallel to the field.
it is typically much weaker than paramagnetism, and is therefore observed mainly in atoms with even number of electrons where paramagnetismo is absent.

Magnetization

In the presence of a magnetic field, matter becomes magnetized, so it will be found to contain many tiny dipoles, with a net alignment along some direction.
We have discussed two mechanisms that account for this magnetic polarization:

- **Paramagnetism:** The dipoles associated with the spins of unpaired electrons experience a torque tending to line them up parallel to the field.
  When put above a solenoid, the induced magnetization would be parallel to the field $\vec{B}$. So the force $\vec{F} = \nabla(\vec{m} \cdot \vec{B})$ points in the direction of increasing $\vec{m} \cdot \vec{B}$, that is, the direction of increasing field (attracted by a solenoid).

- **Diamagnetism:** The orbital speed of the electrons is altered in such a way that the orbital dipole moment tends to oppose the field. So there is a force $\vec{F} = \nabla(\vec{m} \cdot \vec{B})$ points in the direction of increasing $\vec{m} \cdot \vec{B}$ (which is negative), so the force points in direction of decreasing field (repelled by a solenoid)

We define the **magnetization** of a piece of matter as:

$$\vec{M} : \text{Magnetic dipole moment per unit volume}$$

The magnetization due to paramagnetism and diamagnetism is usually so weak it is very hard to see.

Field of a Magnetized Object

We will measure the field of a magnetized object without caring for the origin of the magnetization.

Bound currents

The vector potentia of a single dipole $\vec{m}$ is:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{\mathfrak{r}}}{\mathfrak{r}^2}$$

In a magnetized body, each piece of volume $d\tau'$ carries a dipole moment $\vec{M} d\tau'$, so the total potential is:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{M}(\vec{r'}) \times \hat{\mathfrak{r}}}{\mathfrak{r}^2} d\tau'$$

We can make a little trick to simplify things. We use $\nabla' \frac{1}{\mathfrak{r}} = \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}$, with this:

$$\begin{aligned}
\vec{A}(\vec{r}) &= \frac{\mu_0}{4\pi} \int \left(\vec{M}(\vec{r'}) \times \left(\nabla' \frac{1}{\mathfrak{r}} \right) \right) d\tau' \\
&= \frac{\mu_0}{4\pi} \left[\int \frac{1}{\mathfrak{r}} [\nabla' \times \vec{M}(\vec{r'})] d\tau' - \int \nabla' \times \left[\frac{\vec{M}(\vec{r'})}{\mathfrak{r}} \right] d\tau' \right] \\
&= \frac{\mu_0}{4\pi} \int \frac{1}{\mathfrak{r}} [\nabla' \times \vec{M}(\vec{r'})] d\tau' + \frac{\mu_0}{4\pi} \oint \frac{1}{\mathfrak{r}} [\vec{M}(\vec{r'}) \times d\vec{a'}]
\end{aligned}$$

The first integral looks like the potential of a volume current of:

$$\vec{J}_b = \nabla \times \vec{M}$$

While the second looks like the potential of a surface current:

$$\vec{K}_b = \vec{M} \times \widehat{n}$$

With this definitions:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}_b(\vec{r'})}{\mathfrak{r}} d\tau' + \frac{\mu_0}{4\pi} \oint_S \frac{\vec{K}_b(\vec{r'})}{\mathfrak{r}} da'$$

So the potential (and hence the field) is the same that would be produced by a
Volume current $\vec{J}_b = \nabla \times \vec{M}$
Plus a surface current $\vec{K}_b = \vec{M} \times \widehat{n}$ on the boundary. These are called **bound currents**.

Example 6.1 Find the magnetic field of a uniformly magnetized sphere

Solution: We choose the z axis as the direction of $\vec{M}$, we have:

$$\vec{J}_b = \nabla \times \vec{M} = 0 \quad , \quad \vec{K}_b = \vec{M} \times \widehat{n} = M \sin\theta \widehat{\phi}$$

Now, a rotating spherical shell, of uniform surface charge σ, corresponds to a surface current density of:

$$\vec{K} = \sigma \vec{v} = \sigma \omega R \sin\theta \widehat{\phi}$$

So the field of a uniformly magnetized sphere is identical to the field of a spinning spherical shell with the identification $\sigma R \omega \to M$. So, by exercise 5.11, we conclude that

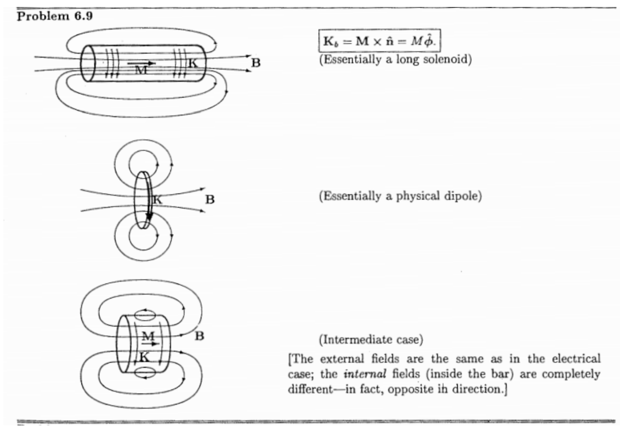
$$\vec{B} = \frac{2}{3} \mu_0 \vec{M}$$

inside the sphere, and a field outside the sphere which is the same as that of a pure dipole $\vec{m} = \frac{4}{3} \pi R^3 \vec{M}$

Problem 6.7: An infinitely long circular cylinder carries a uniform magnetization M parallel to its axis. Find the magnetic field inside and outside the cylinder

$J_b = \nabla \times M = 0 \quad , \quad \vec{K}_b = \vec{M} \times \widehat{n} = M \widehat{\phi}$
The field is that of a surface current $\vec{K}_b = M \widehat{\phi}$. But that is a solenoid.
So the field outside is 0 and inside $B = \mu_0 K_b = \mu_0 M$ pointing upward.

Problem 6.9: A short circular cylinder of radius a and Length L carries a frozen in uniform magnetization $\vec{M}$ parallel to the axis. Find the bound current and sketch the magnetic field



Physical interpretation

Bound current $\vec{J}_b$ and $\vec{K}_b$ arise physically.

If we have a slab of thickness t of material with dipoles $Md\tau$ (M uniform), then we can see each one as a small rectangular current I with area a and dipole moment $m = Mat$.
However, $m = Ia$, so $I = Mt$.
All the currents cancel out except the ones at the surface, with a surface current $K_b = I/t = M$.
And we get $\vec{K}_b = \vec{M} \times \widehat{n}$

We can do something similar to account for the bound volume current $\vec{J}_b = \nabla \times \vec{M}$ for a non uniform magnetization M.

The magnetic Field Inside matter

We mean the macroscopic field taken as an average.
We can prove that the magnetic field inside polarized matter is also the one caused by $\vec{K}_b = \vec{M} \times \widehat{n}$ and $\vec{J}_b = \nabla \times \vec{M}$.

The auxiliary field H

The effect of polarization it to establish a bound current $\vec{J}_b = \nabla \times \vec{M}$ and $\vec{K}_b = \vec{M} \times \widehat{n}$ on the surface.
The field due to magnetization is just the field produced by these bound charges.

We now aggregate the field due to **free current** (those currents that are not bound duh). The field current might flow through wires imbedded in the magnetized substance or in the material itself.
In any event, the total currentis:

$$\vec{J} = \vec{J}_b + \vec{J}_f$$

We can use Ampere's law as:

$$\frac{1}{\mu_0}(\nabla \times \vec{B}) = \vec{J} = \vec{J}_f + \vec{J}_b = \vec{J}_f + \nabla \times \vec{M}$$

Therefore:

$$\nabla \times \left(\frac{1}{\mu_0}\vec{B} - \vec{M} \right) = \vec{J}_f$$

We define the quantity in parenthesis as $\vec{H}$:

$$\vec{H} := \frac{1}{\mu_0}\vec{B} - \vec{M}$$

and Ampere's law says:

$$\nabla \times \vec{H} = \vec{J}_f$$

Or, in integral form:

$$\oint \vec{H} \cdot d\vec{l} = I_{f\ enc}$$

So $\vec{H}$ allows us to express the law in terms of free current alone (and free current is what we control directly). Bound current comes for the ride when we magnetize the object.

Example 6.2: A long copper rod of radius R carries a uniformly distributed free current I. Find H inside and outside the rod

Solution: Copper is weaklydiamagnetic, so the dipoles will line up opposite to the field. This results in bound current running antiparallel to I within the wire and parallel to I along the surface (we don't know how great these bound charges are).

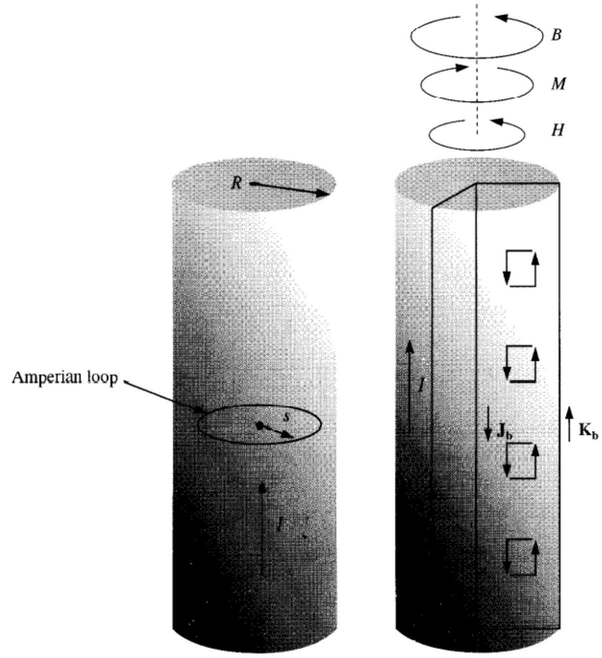


Figure 6.19 Figure 6.20

All currents are longitudinal , so $\vec{B}, \vec{M}, \vec{H}$ are circunferential.
We apply Ampere law to a loop of radius $s < R$:

$$H(2\pi s) = I_{fenc} = I\frac{\pi s^2}{\pi R^2}$$

So:

$$\vec{H} = \frac{I}{2\pi R^2} s\widehat{\phi} \quad (s \leq R)$$

within the wire. Meanwhile, outside the wire:

$$\vec{H} = \frac{1}{2\pi s}\widehat{\phi}$$

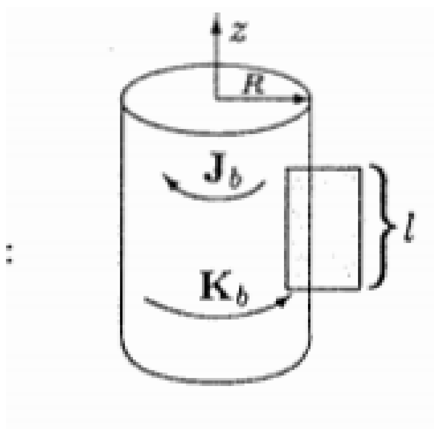
The outside region is empty, so:

$$\vec{B} = \mu_0 \vec{H} = \frac{\mu_0 I}{2\pi s}\widehat{\phi} \qquad (s \geq R)$$

$\vec{H}$ is more useful than $\vec{D}$.
That is because when creating a field, we control the free current through a coil and that determines $\vec{H}$.
But when creating an electric field, we control the voltage (not the charge) and that determines $\vec{E}$

Problem 6.12: An infinitely long cylinder of radius R carries a frozen in magnetization parallel to the axis of $M = ks\hat{z}$.
Where k is a constant and s is the distance form the axis; there is no free current anywhere. Find the magnetic field inside and outside the cylinder by two methods.

- a) We locate the bound charges to be $\vec{J}_b = \nabla \times \vec{M} = -k\hat{\phi}$, $\vec{K}_b = \vec{M} \times \hat{n} = kR\hat{\phi}$.
  $\vec{B}$ is essentially a superposition of solenoid, so it points in the z direction and $\vec{B} = 0$ outside
  We use an amperian loop as typical in a solenoid and find $\oint B \cdot dl = Bl = \mu_0 I_{enc} = \mu_0[\int J_b da + K_b l] = \mu_0[-kl(R-s) + kRl = \mu_o kls$
  So $\vec{B} = \mu_0 ks\hat{z}$ inside the solenoid.



b) By symmetry, $\vec{H}$ points in the z direction. That same amperian loop gives $\oint H \cdot dl = Hl = \mu_0 I_{fenc} = 0$, since there is no free current.
So $\vec{H} = 0$ and therefore $\vec{B} = \mu_0 \vec{M}$
So inside $\vec{B} = \mu_0 ks\hat{z}$ and outside $\vec{B} = 0$

Problem 6.13: Suppose the field inside a large piece of magnetic material is $\vec{B}_0$, so that $\vec{H}_0 = (1/\mu_0)\vec{B}_0 - \vec{M}$

- **a) A small spherical cavity is hollowed out of the material. Find the field at the center of the cavity**

  The field of a magnetized sphere is $\frac{2}{3}\mu_0\vec{M}$, so we need to take this out. $\vec{B} = \vec{B}_0 - \frac{2}{3}\mu_0\vec{M}$ in the cavity
  In the cavity, $\vec{H} = \frac{1}{\mu_0}\vec{B} = \frac{\vec{B}_0}{\mu_0} - \frac{2}{3}\mu_0\vec{M} = \vec{H}_0 + \vec{M} - \frac{2}{3}\vec{M} = \vec{H}_0 + \frac{1}{3}\vec{M}$

A deceptive parallel

$\mu_0\vec{H}$ is not just like $\vec{B}$ but with source $\vec{J}_f$ instead of $\vec{J}$, for we don't have that $\nabla \cdot H = 0$. Actually:

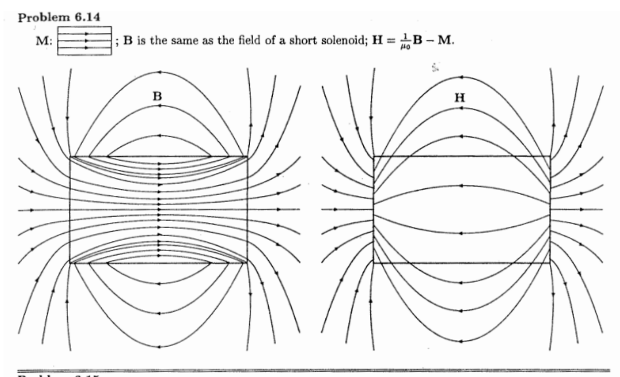
$$\nabla \cdot \vec{H} = \nabla(1/\mu_0 B - M) = -\nabla\vec{M}$$

Still, if there is symmetry, we might be able to find H with ampere.

Boundary

$$\vec{H}^{\parallel}_{above} - \vec{H}^{\parallel}_{below} = \vec{K}_f \times \widehat{n}$$
$$\vec{B}^{\parallel}_{above} - \vec{B}^{\parallel}_{below} = \mu_0(\vec{K} \times \widehat{n})$$

Problem: Sketch the field of a bar magnet with uniform polarization



Linear and Nonlinear Media

Susceptibility and Permeability

In paramagnetic and diamagnetic materials, the magnetization is sustained by the field, when $\vec{B}$ is removed, $\vec{M}$ disappears. And in fact, for small field, the magnetic polarization is proportional to the field.
We would like to write this as:

$$\vec{M} = \frac{1}{\mu_0}\chi_m\vec{B} \quad \text{incorrect!!!}$$

But by convention, we actually write it as:

$$\vec{M} = \chi_m\vec{H}$$

where χ_m is the **Magnetic susceptibility**

Materials that obey this law are called **Linear media**

It is a dimensionless quantity negative for diamagnets and positive for paramagnets

Material	Susceptibility	Material	Susceptibility
Diamagnetic:		*Paramagnetic:*	
Bismuth	-1.6×10^{-4}	Oxygen	1.9×10^{-6}
Gold	-3.4×10^{-5}	Sodium	8.5×10^{-6}
Silver	-2.4×10^{-5}	Aluminum	2.1×10^{-5}
Copper	-9.7×10^{-6}	Tungsten	7.8×10^{-5}
Water	-9.0×10^{-6}	Platinum	2.8×10^{-4}
Carbon Dioxide	-1.2×10^{-8}	Liquid Oxygen ($-200°$ C)	3.9×10^{-3}
Hydrogen	-2.2×10^{-9}	Gadolinium	4.8×10^{-1}

For this **linear media** we have:

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu_0(1 + \chi_m)\vec{H}$$

So $\vec{B}$ is also proportional to $\vec{H}$ and therefore:

$$\vec{B} = \mu\vec{H}$$
$$\mu := \mu_0(1 + \chi_m)$$

Where $\mu := \mu_0(1 + \chi_m)$ is the **permeability** of the material.
In a vacuum $\chi_m = 0$, and the permeability is μ_0.

Example 6.3: An infinite solenoid (n turns per unlit length, current I) is filled with linear material of susceptibility χ_m. Find the magnetic field inside

Solution: Since $\vec{B}$ is due to bound currents (which we don't know), we cannot compute it directly. However, by symmetry we can get $\vec{H}$ by free current alone. Using Ampere's law, we get:

$$\vec{H} = nI\hat{z}$$

Then, the magnetic field is:

$$\vec{B} = \mu_0(1 + \chi_m)nI\hat{z}$$

If the medium is paramagnetic, the field is slightly enhanced, if it's diamagnetic, the field is somewhat reduced.

You might suppose that linear media avoid the parallel between $\vec{B}$ and $\vec{H}$ since now they are always proportional. Unfortunately, this does not happen as there are problems in the boundary between materials

Ferromagnetism

Ferromagnets (which are emphatically non linear) require no external field to sustain magnetization, the alignment becomes frozen in once set.
Ferromagnetism involves the magnetic dipoles associated with the spins of unpaired electrons.

In ferromagnetism, the interaction between nearby dipoles causes each dipole to point in the same direction as its neighbors. The reason for this preference is essentially quantum mechanical.

The alignment usually occurs in small patches called **domains**.
Each domain contains millions of dipoles all lined up, but the domains themselves are randomly oriented and their fields cancel.

Now, to produce a **permanent magnet**, we introduce an outside magnetic field which tends to align the domains and when the field is switched off, most of the alignment remains

Summary

- The magnetic moment of a current loop is $\vec{m} = I\vec{a}$. The vector potential is $\vec{A} = (\mu_0/4\pi)\vec{m} \times \widehat{r}/r^2$ and the magnetic dipole field in spherical coordinates is:
$$B_r = \frac{\mu_0 m}{2\pi r^3}\cos\theta \quad , \quad B_\theta = \frac{\mu_0 m}{4\pi r^3}\sin\theta$$
- The force on a magnetic dipole due to a non uniform field is:
$$\vec{F} = \nabla(\vec{m} \cdot \vec{B})$$
- The torque due to a uniform magnetic field in a dipole is:
$$\vec{N} = \vec{m} \times \vec{B}$$
And it tends to align the moment to the field
- The magnetic moment due to the orbital motion of an electron is $\vec{m} = -(e/2m_e)\vec{L}$ where $\vec{L}$ is the orbital angular momentum
- There are 3 types of magnetism:
  - Diamagnetism: Arises from the orbital angular momentum of electrons and points antiparallel to the external magnetic field. It experiences a force pointing in decreasing magnetic field strength
  - Paramagnetism: Arises from the spin angular momentum of electrons and points parallel to the external field. It experiences a force pointing in increasing magnetic field strength
  - Ferromagnetism; Arises from the spin angular momentum combined with quantum interaction that allows for a permanent magnetization.
- We define a magnetization $\vec{M}$ which gives the magnetization per unit volume.
A magnetized object produces a field as if due to $\vec{J}_b = \nabla \times \vec{M}$ and $\vec{K}_b = \vec{M} \times \widehat{n}$
- The magnetic field $\vec{H}$ is defined by:
$$\vec{H} := \frac{\vec{B}}{\mu_0} - \vec{M}$$
And satisfies:
$$\nabla \times \vec{H} = \vec{J}_{free} \qquad \int_C \vec{H} \cdot d\vec{l} = I_{free}$$
- **Linear:** If $\vec{M}$ is proportional to $\vec{B}$ we define the magentic permeability by:
$$\vec{M} = \chi_m \vec{H}$$
- Then, we have:
$$\vec{B} = \mu_0(1 + \chi_m)\vec{H} = \mu\vec{H}$$
Where μ is the **magnetic permeability**

Electrodynamics

Electromotive Forve

Ohm's Law

To make a current flow, you have to push the charges, for most substances, the current density $\vec{J}$ is proportional to the force per unit charge $\vec{f}$:

$$\vec{J} = \sigma \vec{f}$$

The proportionality factor σ is called the **conductivity** and is measured empirically.
On the other hand, the **resistivity** is defined as $\rho = 1/\sigma$

We will consider the driving force as being purely electric (neglecting magnetic because of low velocity), so:

$$\vec{J} = \sigma \vec{E}$$

Which is **Ohm's Law**

We said that $\vec{E} = 0$ inside a conductor, but that is for stationary charges $\vec{J} = 0$.

Example 7.1: A cylindrical resistor of cross sectional area A and length L is made from material with conductivity σ. If the potential is constant over each end, and the potential difference is V, what current flows?

The field is uniform and has a magnitude $EL = V \;\Rightarrow\; E = V/L$.
And the flow is $I = JA$
So $J = \sigma E \;\Rightarrow\; \dfrac{I}{A} = \sigma \dfrac{V}{L} \;\Rightarrow\; I = \dfrac{\sigma A}{L} V$

Example 7.2: Two long cylinders (radii a and b) are separated by a material of conductivity σ. If they are maintained at a potential difference V, what current flows?

The field between the cylinders is caused by the inner one and is:

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s}$$

With λ the charge per unit length in the inner cylinder. The current is therefore:

$$I = \int \vec{J} \cdot d\vec{a} = \sigma \int \vec{E} \cdot d\vec{a} = \frac{\sigma}{\epsilon_0} \lambda L$$

Meanwhile, the potential difference is $V = -\int_b^a \vec{E} \cdot d\vec{l} = \dfrac{\lambda}{2\pi\epsilon_0} \log \dfrac{b}{a}$, so

$$I = \frac{2\pi\sigma L}{\log(b/a)} V$$

As these examples illustrate, the total current flowing from one **electrode** to the other is proportional to the potential difference:

$$V = IR$$

Where R is called the **resistance**

Example 7.3: Prove that the field in problem 7.1 is really uniform

Within the cylinder, V satisfies Laplace's equation. It is constant at one face (say it is 0) and V_0 at the other. Also we have that $\vec{J} \cdot \widehat{n} = 0$ so $\vec{E} \cdot \widehat{n} = 0$, so $\partial V / \partial n = 0$.
we can guess one potential that fulfills this conditions: $V(z) = \dfrac{V_0 z}{L}$
And by uniqueness, it has to be the potential.
Therefore, the field is $\vec{E} = -\nabla V = -\dfrac{V_0}{L}\widehat{z}$

Problem 7.1: Two concentric metal spherical shells of radius a, b respectively, are separated by weakly conducting material of conductivity σ

- **If they are maintained at potential difference V, what current flows?**
  If Q is the charge of the inner shell, then $\vec{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r^2}\widehat{r}$ in the space between them. So the potential difference is $-\int_a^b \vec{E} \cdot d\vec{r} = \dfrac{Q}{4\pi\epsilon}\left(\dfrac{1}{a} - \dfrac{1}{b}\right)$
  Then, the current is:
  $$I = \int \vec{J} \cdot d\vec{a} = \sigma \int \vec{E} \cdot d\vec{a} = \sigma \frac{Q}{\epsilon_0} = 4\pi\sigma \frac{V_a - V_b}{1/a - 1/b}$$

- **Find the resistance:**
  $$R = \frac{V_a - V_b}{I} = \frac{1}{4\pi\sigma}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Problem 7.2: A capacitor C has been charged up to a potential V_0; at time $t = 0$ it is connected to a resistor R and begins to discharge

- **Determine the charge on the capacitor as a function of time $Q(t)$, and the current through the resistor $I(t)$**

We have $V = Q/C = IR$. And positive I means that charge on the capacitor is decreasing, therefore
$\frac{dQ}{dt} = -I = -\frac{1}{RC}Q$
So $Q(t) = Q_0 e^{-t/RC}$ but $Q(0) = CV_0$ so $Q(t) = CV_0 e^{-t/RC}$
Hence $I(t) = -\frac{dQ}{dt} = CV_0 \frac{1}{RC} e^{-t/RC} = \frac{V_0}{R} e^{-t/RC}$

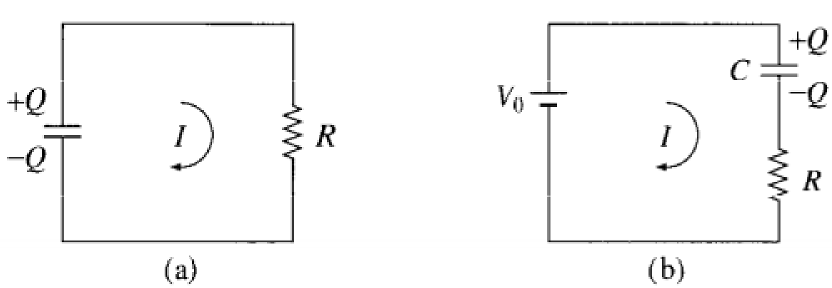
- **What was the original energy stored in the capacitor? Confirm that the heat delivered is equal to the energy loss by the capacitor.**

$W = \frac{1}{2}CV_0^2$.
And the energy delivered to the resistor is
$\int_0^\infty P dt = \int_0^\infty I^2 R dt = \cdots = \frac{1}{2}CV_0^2$

- **Charging up the capacitor by connecting it to a fixed voltage V_0 at time $t = 0$**

Now the equation is $V_0 = Q/C + IR$.
This time positive I means that Q is increasing, so $\frac{dQ}{dt} = I = \frac{1}{RC}(CV_0 - Q)$
We can solve it by separating and get $Q(t) = CV_0 + Ke^{-t/RC}$
But $Q(0) = 0$, so $k = -CV_0$, therefore:
$Q(t) = CV_0 \left(1 - e^{-t/RC}\right)$
And finally $I(t) = \frac{dQ}{dt} = \frac{V_0}{R} e^{-t/RC}$

We can calculate the energy given: $\int_0^\infty V_0 I dt = \cdots = CV_0^2$
So half the energy needed goes to the capacitor to give it energy $CV_0^2/2$ and the other half $CV_0^2/2$ is lost in the resistor.



Power:
Since the work done per unit charge is V and the charge flowing per unit time is I, the power delivered is:

$$P = VI = I^2 R$$

This is **Joule Heating law**

Electromotive Force

The potential difference between two terminals a, b is:

$$V = -\int_a^b \vec{E} \cdot d\vec{l} = \int_a^b \vec{f_s} \cdot d\vec{l} = \oint \vec{f_s} \cdot d\vec{l} := \varepsilon$$

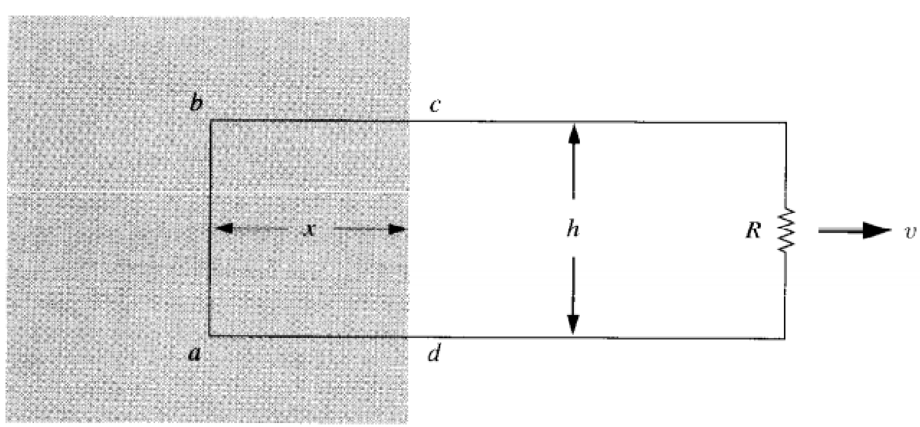
The function of a battery is to establish and maintain a voltage difference equal to the electromotive force. The resulting electrostatic field drives current around the rest of the circuit. ε can be interpreted as the work done, per unit charge, by the source.

Problem 7.5: A battery of emf ε and internal resistance r is hooked to a variable load resistance R. It you want to deliver the maximum possible power, what resistance R must you choose:

$I = \dfrac{\varepsilon}{r+R}$, so
$P = I^2 R = \dfrac{\varepsilon^2 R}{(r+R)^2}$
Therefore, we see $\dfrac{dP}{dR} = 0$ and we get that the maximum is obtained when $R = r$

Motional emf

Motional emf happens when you move a wire through a magnetic field.



The charges in the segment ab experience a magnetic force whose vertical component is qvB drives current around the loop. The emf is:

$$\varepsilon = \oint f_{mag} \cdot d\vec{l} = vBh$$

The integral is done at one instant of time.

Although the magnetic force is responsible for establishing the emf, it is not doing any work, the work is done by the person moving the circuit

Let Φ be the flux of B through the loop:

$$\Phi = \int \vec{B} \cdot d\vec{a}$$

For the rectangular loop:

$$\Phi = Bhx$$

As the loop moves, the flux decreases:

$$\frac{d\Phi}{dt} = Bh\frac{dx}{dt} = -Bhv$$

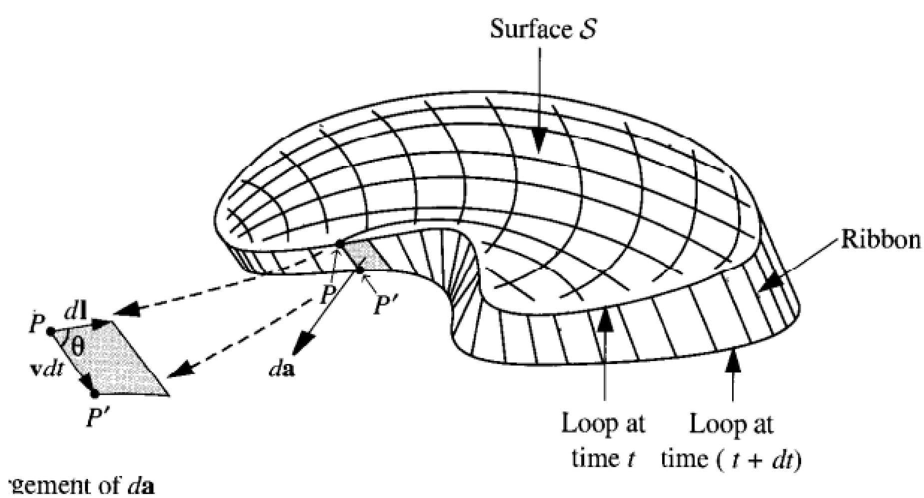
Therefore:

$$\boxed{\varepsilon = -\frac{d\Phi}{dt}}$$

Flux Rule: For any loop moving arbitrarily in a field B, the electromotive force is

$$\boxed{\varepsilon = -\frac{d\Phi}{dt}}$$

- **Proof:**



We show a loop of wire at time t and also a short time dt later.
Suppose we compute the flux at time t, using surface S, and the flux at time $t+dt$ using surface $S+$ ribbon.The change in flux is therefore:

$$d\Phi = \Phi(t+dt) - \Phi(t) = \Phi_{ribbon} = \int_{ribbon} \vec{B} \cdot d\vec{a}$$

Now we focus on point P which moves to P' in time dt. Let $\vec{v}$ be the velocity of the wire and $\vec{u}$ the velocity of charge down the wire, $\vec{w} = \vec{v} + \vec{u}$ is the resultant velocity of a charge at P.
The infintesimal element of area on the ribbon is $d\vec{a} = (\vec{v} \times d\vec{l})dt$
Therefore: $\frac{d\Phi}{dt} = \oint \vec{B} \cdot (\vec{v} \times d\vec{l})$
And since $\vec{w} = \vec{u} + \vec{v}$ and $\vec{u}$ is parallel to $d\vec{l}$, then $\frac{d\Phi}{dt} = \oint \vec{B} \cdot (\vec{w} \times d\vec{l}) = -\oint (\vec{w} \times \vec{B}) \cdot d\vec{l} = -\oint f_{mag} \cdot d\vec{l}$
Where f_{mag} is the force per unit charge.
But this last thing is the emf. QED

Example 7.4: A metal disk of radius a rotate with angular velocity ω about a vertical axis, through a uniform field $\vec{B}$ pointing up. A circuit is made by connecting a resistor to the axle and the other end to a sliding contact, which touches the outer edge of the disc, find the current.

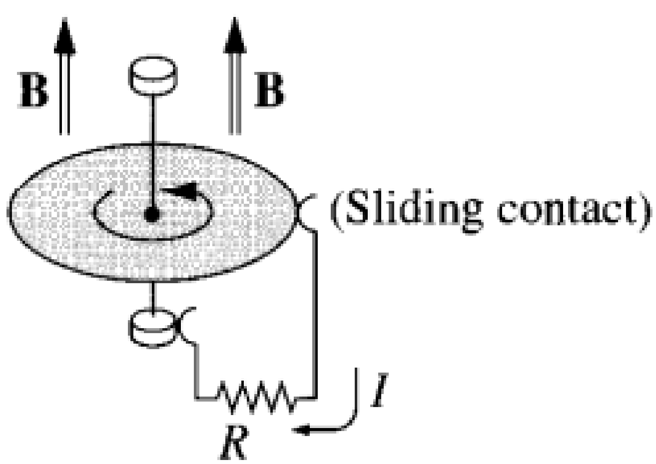


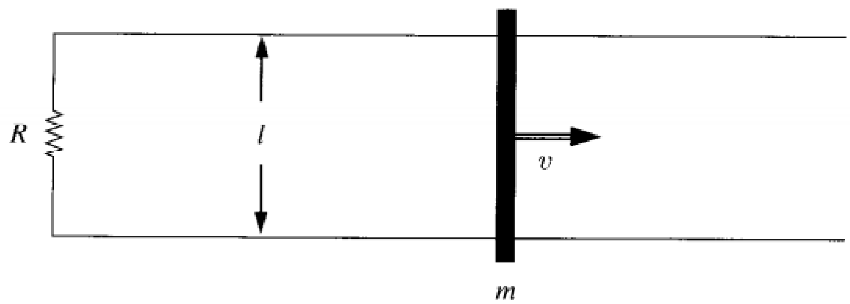
Figure 7.14

The speed of a point on the disk a distance s is $v = \omega s$, so the force per unit charge is $\vec{f}_{mag} = \vec{v} \times \vec{B} = \omega s \vec{B} \hat{s}$. Therefore, the Emf is:

$$\varepsilon = \int_0^a f_{mag} ds = \omega B \int_0^a s ds = \frac{\omega B a^2}{2}$$

And the current is $I = \dfrac{\varepsilon}{R} = \dfrac{\omega B a^2}{2R}$

Problem 7.7: A metal bar of mass m slides frictionlessly on two parallel conducting rails a distance l apart. A resistor R is connected across the rails and a uniform magnetic field $\vec{B}$, pointing into the page, fills the entire region. If the bar moves to the right at speed v, what is the current in the resistor?



The flux at any time is:

$$\Phi(t) = \Phi_0 + vtlB_0$$
$$\Rightarrow \frac{d\Phi(t)}{dt} = vlB_0$$

And finally:

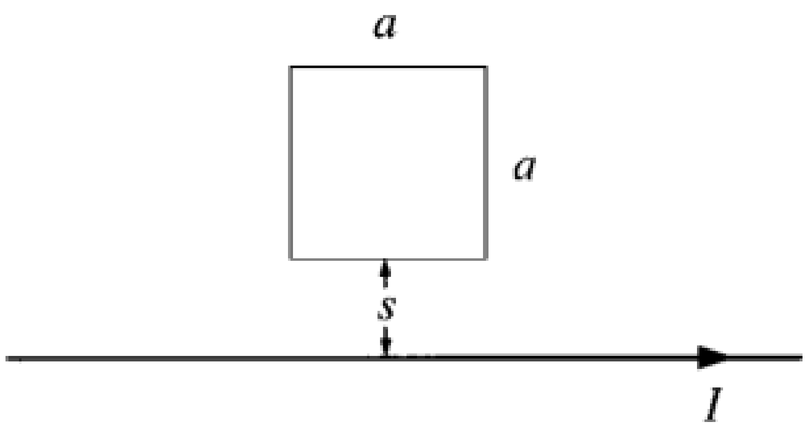
$$\varepsilon = -\frac{d\Phi}{dt} = -vlB_0$$

The generated current will have a field opposing the increase in B, so it will point out of the page, therefore, the direction is counterclockwise. And the current is $I = \dfrac{\varepsilon}{R} = \dfrac{vlB_0}{R}$.

- **b) What is the magnetic force on the bar**
  It has a upward current $I = \dfrac{vlB_0}{R}$, so a force $\vec{F} = \int I d\vec{l} \times \vec{B} = IlB = \dfrac{B^2 l^2 v}{R}$ to the left.

- **c) If the bar starts up with speed v_0 at time $t = 0$, what is the peed after time t**
  $F = ma \;\; \Rightarrow \;\; \dfrac{dv}{dt} = -\dfrac{B^2 l^2}{Rm} v \;\Rightarrow\; v = v_0 e^{-B^2 l^2/(mR)t}$

Problem 7.8: A square loop of wire (side a) lies on a table, a distance s from a very long straight wire, which carries a current I, find the flux



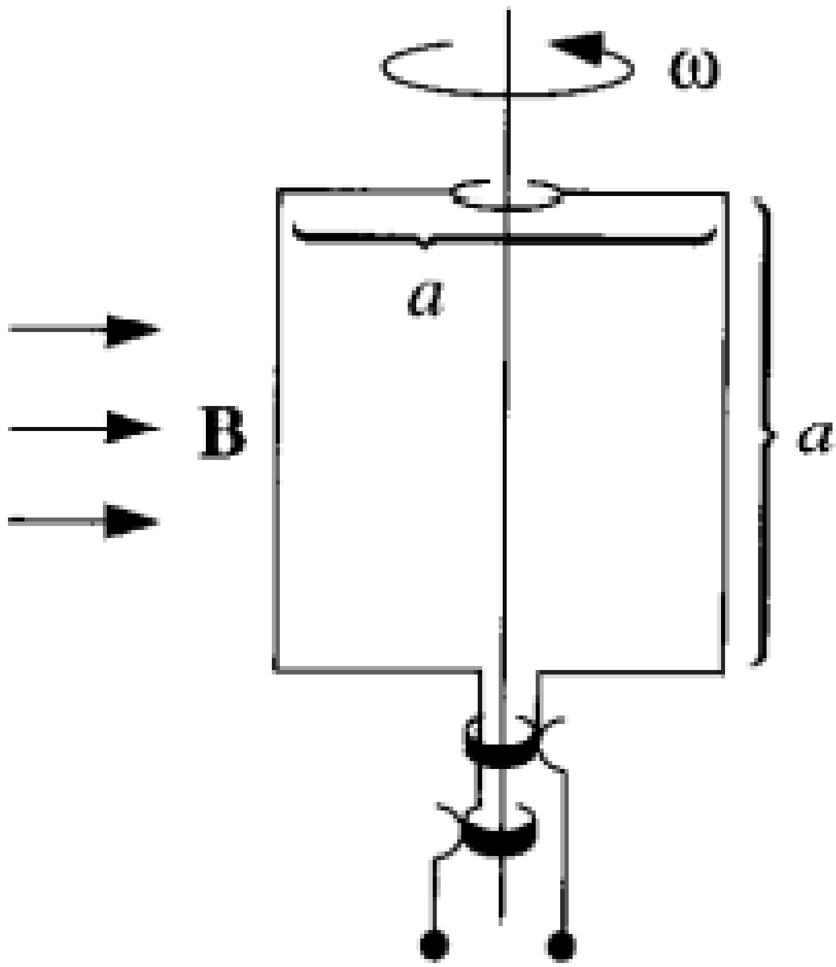
The field is $\vec{B} = \dfrac{\mu_0 I}{2\pi s}\widehat{\phi}$.
So the flux is $\Phi = \int \vec{B} \cdot d\vec{a} = \dfrac{\mu_0 I}{2\pi} \int_s^{s+a} \dfrac{1}{s}(a ds) = \dfrac{\mu_0 I a}{2\pi} \log\left(\dfrac{s+a}{s}\right)$

If womeone pulls the loop away at speed v, what is the Emf?
Say the distance is $s = vt$ so that $ds/dt = v$, therefore:

$$\varepsilon = -\frac{d\Phi}{dt} = -\frac{\mu_0 I a}{2\pi}\frac{d}{dt}\log\left(\frac{s+a}{s}\right) = \cdots = \frac{\mu_0 I a^2 v}{2\pi s(s+a)}$$

The direction is counterclockwise to contrarrest the diference in flow.

Problem 7.10: A square loop of side a is mounted on a vertical shaft and rotated at angular velocity ω. A uniform magnetic field $\vec{B}$ points to the right, calculate the $\varepsilon(t)$



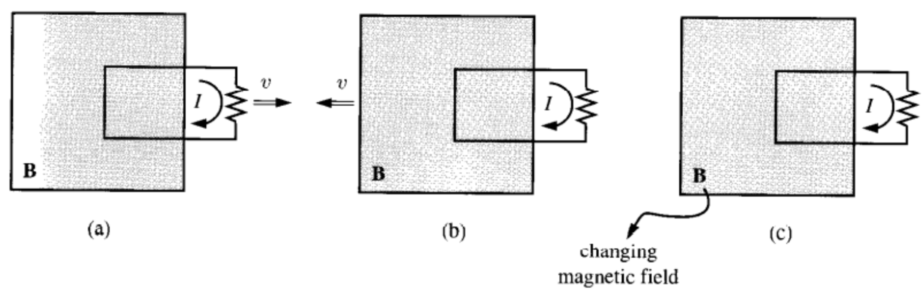
If θ is the angle respect to the direction of $\vec{B}$, then the flow is $\Phi = Ba^2\cos(\theta)$
So $\dfrac{d\Phi}{dt} = -Ba^2\sin(\omega t)\omega$
Therefore: $\varepsilon = -\dfrac{\partial B}{\partial t} = Ba^2\sin(\omega t)\omega$

Electromagnetic Induction

Faraday's law

In 1831 Faraday reported three similar experiments:

- **Experiment 1:** He pulled a loop of wire to the right through a magnetic field and saw a current flow
- **Experiment 2:** He moved a magnet to the left, holding the loop still, again a current flowed
- **Experiment 3:** With both the loop and the magnet at rest, he changed the strength of the field. Once again, current flowed in the loop.



The first experiment is an example of motional emf and produces a voltage difference:

$$\varepsilon = -\frac{d\Phi}{dt}$$

By relativity, clearly the second result should be the same (and it is)

But without using relativity, the result isn't obvious, because the charges are stationary and magnetic field don't affect them. What does affect them are electric fields, therefore:

A changing magnetic field induces an electric field

It is this induced field that produces the Emf in experiment 2, therefore:

$$\varepsilon = \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}$$

$$\Rightarrow \quad \oint \vec{E} \cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$$

This is **Faraday's law in integral form:**

$$\boxed{\oint \vec{E} \cdot d\vec{l} = -\frac{\partial \Phi}{\partial t}}$$

And the differential form is:

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

In experiment three, the field changes by a different reason, but it still works

Many people call both laws Faraday's law, but they aren't the same.
In the first experiment, the electromotive force is due to Lorentz force, while in the other experiments it is due to the induced electric field.
That they have the same equation is an incredible fact and part of what lead to special relativity.

A long cylindrical magnet of length L and radius a carries a uniform magnetization M parallel to the axis. It passes at constant velocity v through a circular wire ring of slighty larger diameter, graph the induced emf as function of time
The magnetic field is the same as that of a long solenoid with surface current $\vec{K}_b = M\widehat{\phi}$. So the field is $\vec{B} = \mu_0\vec{M}$.
The flux starts at zero, then bulilds up to a max of $\mu_0 M\pi a^2$ and stays there until the magnet starts to get out. So the emf is a downward spike and then a upward one.

Lenz Rule: The induced current points in a way that it creates a magnetic field against the external changing external field.

The jumping ring: If you wind a solenoidal coil around an iron core (to beef up the field), place a ring on top and plug it in, the ring jumps, why?

Before turning it on, the flux through the ring is 0, afterward a big flux appeared and that generates a big current in the ring that tries to counteract the change in flux.
So the ring generates a magnetic field opposing the solenoid and ass opposite currents repel, the ring flies off.

The induced electric field

There are two types of electric fields, the ones created by electric charges and those associated with changing magnetic fields.

So far, we have $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ and $\nabla \times \vec{B} = \mu_0 \vec{J}$
If the field $\vec{E}$ is purely due to Faraday and $\rho = 0$, then $\nabla \cdot \vec{E} = \nabla \cdot \vec{B} = 0$

So Faraday induced electric fields are determined by $-\partial\vec{B}/\partial t$ the same way as magnetostatic fields are determined by $\mu_0\vec{J}$

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textbfProblem 7.15: A long solenoid with radius a and n turns per unit length carries a time dependent current $I(t)$ in the $\widehat{\phi}$ direction. Find the electric field a distance s from the axis

In the Quasistatic approximation, we suppose $\vec{B}$ is magnetostatic, so $\vec{B} = \begin{cases} \mu_0 nI\widehat{z} & s < a \\ 0 \quad , & s > a \end{cases}$
.
Clearly, as the $\vec{B}$ is changing in the $\widehat{z}$ direction, the $\vec{E}$ field will circle this direction.

Inside: For an amperian loop of radius $s < a$ (circle with center $\widehat{z}$)

$$\Phi = B\pi s^2 = \mu_0 nI\pi s^2$$

Therefore, if we integrate the $\vec{E}$ field:

$$\oint \vec{E} \cdot d\vec{l} = E2\pi s = -\frac{d\Phi}{dt} = -\mu_0 n\pi s^2 \frac{dI}{dt}$$

Therefore:

$$\vec{E} = -\frac{\mu_0 ns}{2}\frac{dI}{ds}\widehat{\phi}$$

Outside: We make an amperian loop of radius $s > a$ so now the flux is $\Phi = B\pi a^2$. Then the procedure is the same and we obtain:

$$\vec{E} = -\frac{\mu_0 na^2}{2s}\frac{dI}{ds}\widehat{\phi}$$

Inductance

Suppose we have two loops of wire at rest. If you run a current I_1 along loop 1, it produces a magnetic field $\vec{B}_1$.
Some of the field lines pass through loop 2, let Ψ_2 be the flux of $\vec{B}_1$ through 2.

The field Φ_1 is proportional to the current I_1 and therefore, the flux Φ_2 is too. Thus, there is a relation:

$$\Phi_2 = M_{21} I_1$$

Neumann Formula:
We can express $\Phi_2 = \int \vec{B}_1 \cdot d\vec{a}_2 = \int (\nabla \times \vec{A}_1) \cdot d\vec{a}_2 = \oint \vec{A}_1 \cdot d\vec{l}_2$
But according to Biot savaart:

$$\vec{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\vec{l}_1}{\mathfrak{r}}$$

Hence:

$$\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \left(\oint \frac{d\vec{l_1}}{\mathfrak{r}} \right) \cdot d\vec{l_2}$$

Therefore:

$$\boxed{M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\vec{l_1} \cdot d\vec{l_2}}{\mathfrak{r}}}$$

With this formula we can conclude:

- M_{21} is a purely geometrical quantity
- $M_{21} = M_{12}$

So the flux through 2 when we run a current I around 1 is equal to the flux through 1 when we send the same current I around 2.

Example 7.10: A short Solenoid (length l, radius a with n_1 turns per unit length) lies on the axis of a very long solenoid (b, n_2). Current I runs through the short solenoid, What is the flux through the long one?

We exploit the equality of inductances, since this side of the problem is difficult.
If we put a current I in the long one, it produces a uniform field $B = \mu_0 n_2 I$ inside.
So it produces a flux of $B\pi a^2 = \mu_0 n_2 I \pi a^2$
through every single loop in the short solenoid.
There are $n_1 l$ turns in all, so the total flux through the inner solenoid is:

$$\Phi = \mu_0 \pi a^2 n_1 n_2 l I$$

Then, $M = \mu_0 \pi s^2 n_1 n_2 l$
And this is the same in the original problem.

Suppose we now vary the current in loop 1. The flux through loop 2 will vary accordingly, and Faraday's law says that this changing flux will induce an emf in loop 2:

$$\varepsilon_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt}$$

So, every time you change the current in loop 1, an induced current flows in loop 2.

Self inductance
A changing current in a loop also produces an emf on itself. Once again, the field (and also flux) is proportional to I so:

$$\Phi = LI$$

Where L is the **self inductance** and depends on the geometry of the loop. If the current changes, the emf induced in the loop is:

$$\varepsilon = -L\frac{dI}{dt}$$

Inductance opposes the emf, so it acts like inertia on the system.

Suppose current I is flowing around a loop when someone cuts the wire. The current drops instantaneously to zero, this generates a whopping back emf, for dI/dT is enormous, that's why you often draw a spark when unplugging something with an inductor.

Example 7.11: Find the self inductance of a toroifal coil with rectangular cross sections (inner radius a, outer b, height h), which carries a total of N turns.

Solution: The magnetic field inside the toroid is: $B = \dfrac{\mu_0 N I}{2\pi s}$
The flux through a single turn is:

$$\int \vec{B}\cdot d\vec{a} = \frac{\mu_0 N I}{2\pi} h \int_a^b \frac{1}{s} ds = \frac{\mu_0 N I h}{2\pi} \log\left(\frac{b}{a}\right)$$

The total flux is N times this. So the self inductance is:

$$L = \frac{\mu_0 N^2 h}{2\pi} \log\left(\frac{b}{a}\right)$$

Example 7.12: Suppose a battery of constant ε_0 is connected to a circuit of resistance R and inductance L, what current flows?

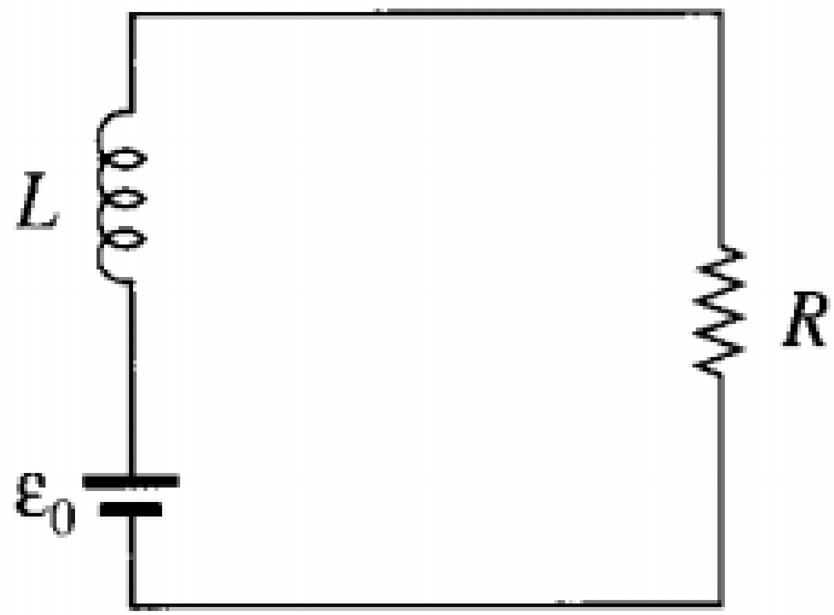


Figure 7.34

Solution: The total emf in this circuit is that provided by the battery plus that resulting from self inductance, then:

$$\varepsilon_0 - L\frac{dI}{dt} = IR$$

The general solution is $I(t) = \frac{\varepsilon_0}{R} + ke^{(-R/L)t}$

And if $I(0) = 0$, it is $I(t) = \frac{\varepsilon_0}{R}\left(1 - e^{-(R/L)t}\right)$

Problem 7.20: A small loop of wire (radius a) lies a distance z above the center of a large loop (radius b)

- **Suppose a current I flows in the big one, what is the flux in the small one**

  We may consider just the field in the axis, which is $\vec{B} = \frac{\mu_0 I}{2}\frac{b^2}{(b^2+z^2)^{3/2}}\widehat{z}$, so the flux through the little one is $\Phi = \frac{\mu_0 \pi I a^2 b^2}{2(b^2+z^2)3/2}$

- **Suppose Current I flows in the little loops, calculate the flux in the big one**
  We approximate as a dipole, so $\vec{B} = \frac{\mu_0}{4\pi}\frac{m}{r^3}(2\cos\theta\widehat{r} + \sin\theta\widehat{\theta})$

Problem 7.22: Find the self inductance per unit length of a long solenoid

The field is $B = n\mu_0 I$, so the flux in a single turn is $\Phi_1 = \mu_0 n I \pi R^2$.
In a length l, there are nl turns, so the total flux is $\Phi = \mu_0 n^2 \pi R^2 I l$. Then, the self inductance per unit length is $L = \mu_0 n^2 \pi R^2$

Problem 7.25: A capacitor C is charged up to a potential V and connected to an inductor L. At time $t = 0$ the circuit is closed. Find the current as a function of time. How does your answer change if we include a resistor in series?

There is no extra emf, so $\varepsilon = -L\dfrac{dI}{dt} = Q/C$, where Q is the charge of the capacitor and $I = dQ/dt$.
So $\dfrac{d^2Q}{dt^2} = -\dfrac{1}{LC}Q$.
Therefore the general solution is $Q(t) = A\cos\omega t + B\sin\omega t$ with $\omega = \dfrac{1}{\sqrt{LC}}$. Then we adapt it to the particular conditions.
If we put a resistor, the equation becomes damped $-L\dfrac{dI}{dt} = \dfrac{Q}{C} + IR$ and it is a damped oscillation.

Energy in a Magnetic Field

The amount of work to go from $I = 0$ to I in an inductor is:

$$W = \frac{1}{2}LI^2$$

Because the power is $dW/dt = -\varepsilon I = LI dI/dt$

There is a nicer way to write W using some properties and things similar as we did for electric fields:

$$W = \frac{1}{2\mu_0}\int_{allspace} B^2 d\tau$$

So energy is ' stored in the magnetic field.

Problem 7.29: suppose the circuit has been connected for a long time when at time $t = 0$ it is changed, what is the current, total energy delivered to the resistor?

The initial current is $I_0 = \varepsilon_0/R$ because it was connected a long time and inductor doesn't matter any more.
The equation is $-L\dfrac{dI}{dt} = IR \;\Rightarrow\; \dfrac{dI}{dt} = -\dfrac{R}{L}I \;\Rightarrow\; I = I_0 e^{-Rt/L}$

So $I(t) = \frac{\varepsilon_0}{R} e^{-Rt/L}$
The power is $P = I^2 R = \frac{\varepsilon_0}{R} e^{-2Rt/L} = \frac{dW}{dt}$
And the total power delivered after a while is $W = \int_0^\infty P dt = \cdots = \frac{1}{2} L(\varepsilon_0/R)^2$
Which is clearly the initial energy on the inductor.

Example 7.13: A long coaxial cable carries a current I (flows through the surface of the inner cylinder of radius a, outer of radius b) Find the magnetic energy stored in a section of length l

The field between cylinders is (according to ampere's law $\vec{B} = \frac{\mu_0 I}{2\pi s} \widehat{\phi}$
so the energy per unit volume is $\frac{1}{2\mu_0} \frac{\mu_0^2 I^2}{4\pi^2 s^2} = \frac{\mu_0 I^2}{8\pi^2 s^2}$
The energy in a shell of length l, radius s and thickness ds is then:

$$\left(\frac{\mu_0 I^2}{8\pi^2 s^2}\right) 2\pi l s ds = \frac{\mu_0 I^2 l}{4\pi} \left(\frac{ds}{s}\right)$$

And integrating along shells from a to b we get: $W = \frac{\mu_0 I^2 l}{4\pi} \log\left(\frac{b}{a}\right)$
According to $W = 1/2LI^2$, we can get the inductance from this, $L = \frac{\mu_0 l}{2\pi} \log \frac{b}{a}$

Maxwell's Equations

Equations Before Maxwell

- $\nabla \cdot \vec{E} = \frac{1}{\epsilon_0}\rho$
- $\nabla \cdot \vec{B} = 0$
- $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
- $\nabla \times \vec{B} = \mu_0 \vec{J}$

There is something wrong, since $\nabla \cdot \nabla \times \vec{B} = \mu_0 \nabla \cdot \vec{J} \neq 0$
But it should be zero because of divergence of curl.
For steady currents $\nabla \cdot \vec{J} = 0$, but beyond magnetostatics there must be something else.

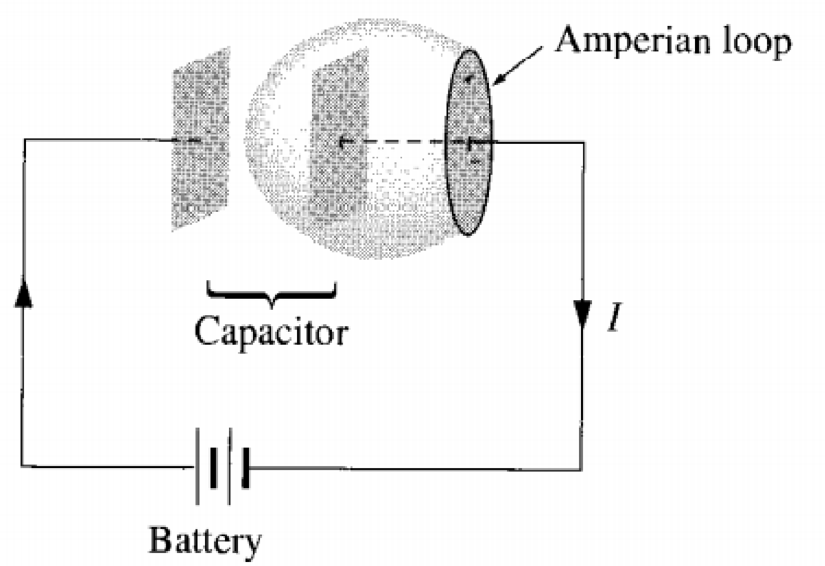


Figure 7.42

We want to apply $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$ to the loop shown. When we draw the surface like shown, the total I_{enc} is 0 , but if we draw a plain circle, it is I.
So, the changing electric field between the plates must be what causes the field $\vec{B}$

How Maxwell fixed Ampere

We have as before:

$$\begin{aligned}\nabla \cdot \nabla \times \vec{B} &= \mu_0 \nabla \cdot \vec{J} \\ &= -\mu_0 \frac{\partial \rho}{\partial t} \\ &= -\mu_0 \frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \vec{E}) \\ &= -\nabla \cdot \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)\end{aligned}$$

Therefore, to make this 0, we must add something to the curl of B:

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

This extra term is hard to detect and it does nothing for still charges. But either way:

A changing electric field induces a magnetic field

Maxwell called this extra term the **displacement current:**

$$\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Though it has nothing to do with current

We can now solve the Capacitor paradox.
If the plates are close together, $E = \frac{1}{\epsilon_0} \frac{Q}{A} 4$
Thus, between the plates there is a changing electric field:

$$\frac{\partial E}{\partial t} = \frac{1}{\epsilon_0 A} \frac{dQ}{dt} = \frac{1}{\epsilon_0 A} I$$

Therefore, now the integral form of the equation reads:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \int \left(\frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{a}$$

If we choose the plane circle surface, then $E = 0$ and $I_{enc} = I$, so the result is $\mu_0 I$
If we choose the balloon, then $I_{enc} = 0$ but $\int (\partial \vec{E}/\partial t) \cdot d\vec{a} = \frac{1}{\epsilon_0 A} IA = \frac{I}{\epsilon_0}$ so the result is again $\mu_0 I$

Maxwell's Equations

- $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
- $\nabla \cdot \vec{B} = 0$
- $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
- $\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

And the force:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Maxwell's equations in matter

We have learned that an electric polarization $\vec{P}$ produced a bound charge:

$$\rho_b = -\nabla \cdot \vec{P}$$

Likewise, a magnetic polarization $\vec{M}$ results in a current:

$$\vec{J}_b = \nabla \times \vec{M}$$

Then, in terms of free charges and currents, the equations are:

- $\nabla \cdot \vec{D} = \rho_f$
- $\nabla \cdot \vec{B} = 0$
- $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
- $\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$

Where $\vec{D} := \epsilon_0 \vec{E} + \vec{P}$
And $\vec{H} := \frac{1}{\mu_0}\vec{B} - \vec{M}$

And the **constitutive relations** that relate $\vec{D}, \vec{H}$ to $\vec{E}, \vec{B}$:

- $\vec{P} = \epsilon_0 \chi_e \vec{E}$
- $\vec{D} = \epsilon \vec{E}$
- $\vec{M} = \chi_m \vec{H}$
- $\vec{H} = \frac{1}{\mu}\vec{B}$

Where $\epsilon = \epsilon_0(1 + \chi_e)$ and $\mu = \mu_0(1 + \chi_m)$

Boundart conditions

- $\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f$
- $B_1^{\perp} - B_2^{\perp} = 0$
- $\vec{E}_1^{\perp} - \vec{E}_2^{\perp} = 0$
- $\dfrac{1}{\mu_1}\vec{B}_1^{\parallel} - \dfrac{1}{\mu_2}\vec{B}_2^{\parallel} = \vec{K}_f \times \widehat{n}$

Conservation Laws

Charge and Energy

Formally, the charge in a volume V is:

$$Q(t) = \int_V \rho(\vec{r}, t) d\tau$$

And the current flowing through the boundary S, is $\int_S \vec{J} \cdot d\vec{a}$, so **local conservation of charge says**:

$$\frac{dQ}{dt} = -\int_S \vec{J} \cdot d\vec{a}$$

We can rewrite it using the div theorem to get:

$$\boxed{\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{J}}$$

This is the **Continuity equation**. As we saw earlier, it can be derived from Maxwell's equations.

Poynting's Theorem

Before we found that the work necessary to assemble a static charge distribution is:

$$W_e = \frac{\epsilon_0}{2} \int E^2 d\tau$$

And the work required to get currents going is:

$$W_m = \frac{1}{2\mu_0} \int B^2 d\tau$$

Therefore, the **total energy in a electromagnetic field** is:

$$\boxed{U_{em} = \frac{1}{2} \int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau}$$

Suppose we have some charge and current configuration that at time t produces a field $\vec{E}$ and $\vec{B}$. In the next instant, dt, the charges move around a bit, how much work, dW is done by electromagnetic forces acting on these charges in the interval dt?
According to the Lorentz force law, the work done on a charge q is $\vec{F} \cdot d\vec{l} = q(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{v} dt = q\vec{E} \cdot \vec{v} dt$

Now, $q = \rho d\tau$ and $\rho \vec{v} = \vec{J}$, so the rate at which work is done on all the charges in a volume V is $\frac{dW}{dt} = \int_V (\vec{E} \cdot \vec{J}) d\tau$

Then, evidently $\vec{E}\cdot\vec{J}$ is the work done per unit time, per unit volume, that is, the **power** delivered per unit volume. We can use the Ampere Maxwell law to get rid of the $\vec{J}$:

$$\vec{E}\cdot\vec{J}=\frac{1}{\mu_0}\vec{E}\cdot(\nabla\times\vec{B})-\epsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$$

Then, using some product rules and so on, we can find that:

$$\vec{E}\cdot\vec{J}=-\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0E^2+\frac{1}{\mu_0}B^2\right)-\frac{1}{\mu_0}\nabla\cdot(\vec{E}\times\vec{B})$$

Then, putting this into the formula of dW/dt and using div theorem, we get:

$$\boxed{\frac{dW}{dt}=-\frac{d}{dt}\int_V\frac{1}{2}\left(\epsilon_0E^2+\frac{1}{\mu_0}B^2\right)d\tau-\frac{1}{\mu_0}\oint_S(\vec{E}\times\vec{B})\cdot d\vec{a}}$$

Where S is the surface bounding V. This is the **Poynting theorem** or Work-Energy theorem of electrodynamics.

The first integral is the total energy of the fields, U_{em}.
The second integral represents the rate at which energy is carried out of V across the boundary surface.
So the work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, less the energy that flowed out through the surface.

The *Energy per unit time, per unit area, transported by the fields* is the **Poynting vector:**

$$\boxed{\vec{S}:=\frac{1}{\mu_0}(\vec{E}\times\vec{B})}$$

Then, $\vec{S}\cdot d\vec{a}$ is the energy per unit time crossing the infinitesimal surface $d\vec{a}$-the energy flux-. So S is the **energy flux density**.
Then, we can express **Poynting's theorem** more compactly:

$$\boxed{\frac{dW}{dt}=-\frac{dU_{em}}{dt}-\oint_S\vec{S}\cdot d\vec{a}}$$

.

The energy density of the fields is defines as:

$$\boxed{u_{em}=\frac{1}{2}\left(\epsilon_0E^2+\frac{1}{\mu_0}B^2\right)}$$

We can also define the **mechanical energy density as** $\frac{dW}{dt}=\frac{d}{dt}\int_V u_{mech}d\tau$
Then, the **differential form of Poynting's theorem** is:

$$\boxed{\frac{\partial}{\partial t}(u_{mech}+u_{em})=-\nabla\cdot\vec{S}}$$

.

Example 8.1

When current flows down a wire, work is done, which shows up as Joule hearing of the wire, calculate it

Assuming it is uniform, the electric field parallel to the wire is $E = \dfrac{V}{L}$
With V the potential difference between the ends and L the length.
The magnetic field is circumferencial and with value:

$$B = \frac{\mu_0}{2\pi a}$$

Then, the magnitude of the Poynting vector is:

$$S = -\frac{1}{\mu_0}\frac{V}{L}\frac{\mu_0 I}{2\pi a} = \frac{VI}{2\pi aL}$$

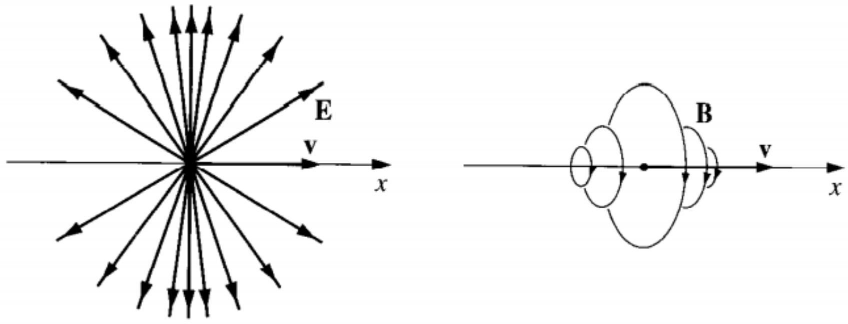
And it point Radially inward. The energy per unit time passing through the surface of the wire is therefore:

$$\int \vec{S} \cdot d\vec{a} = S(2\pi aL) = VI$$

Momentum

Newton's third law

Imagine a charge q traveling along the x axis at constant speed v. Because it is moving, its electric field is not given by Coulumb's law, nevertheless $\vec{E}$ still points radially outward at any instantaneous position of the charge.
And since a single charge doesn't constitute a stationary current, we cannot use the Biot Savaart law to find the field, nevertheless, the field always circles around the axis as suggested by the RHR.
We will prove this in chapter 10.

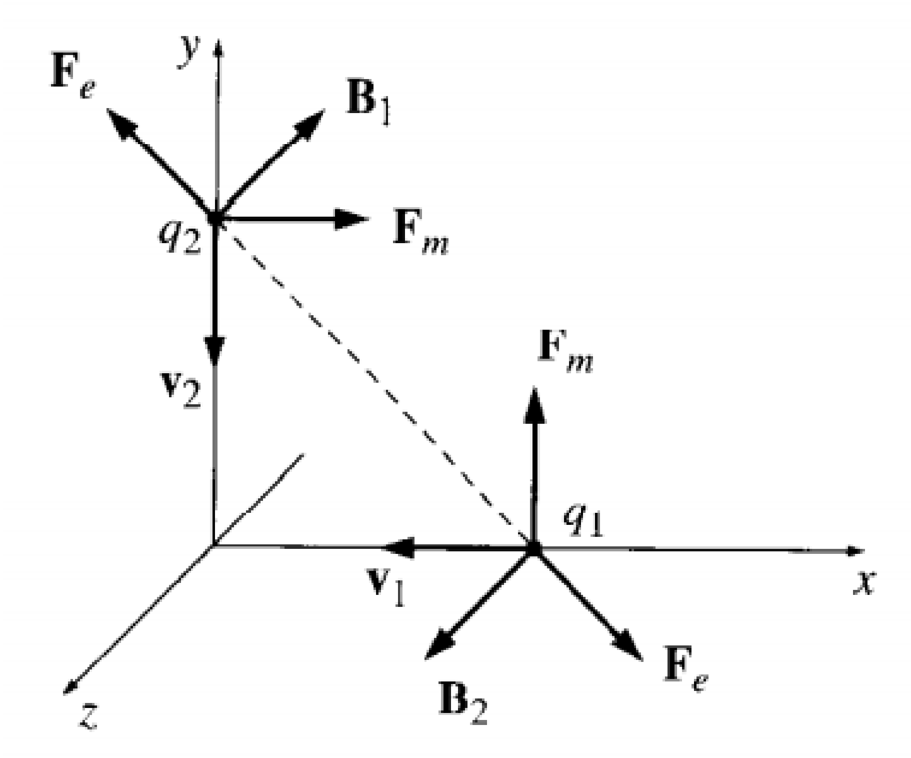


(a) (b)

Now suppose this charge encounters an identical one, proceeding in at the same speed along the y axis.
Let's assume that they're mounted on tracks, so they are forced to have the same direction and speed. The electric force between them is repulsive, but how about the magnetic?

The magnetic field of q_1 points into the page (at the position of q_2), so the magnetic force on q_2 is toward the right.
Whereas the magnetic field of q_2 is out of the page (at the position of q_1), and the magnetic force on q_1 is upward.

The electromagnetic force of q_1 on q_2 is equal but not opposite to the force of q_2 on q_1, which violates Newton's third law.



The problem is that the loss of the third law would cancel the proof of conservation of momentum.
The solution to this problem is that actually the fields carry momentum.

Maxwell's Stress Tensor

Let's calculate the total electromagnetic force on the charges in volume V:

$$\vec{F} = \int_V (\vec{E} + \vec{v} \times \vec{B})\rho d\tau = \int_V (\rho\vec{E} + \vec{J} \times \vec{B}) d\tau$$

The force per unit volume is evidently:

$$\vec{f} = \rho\vec{E} + \vec{J} \times \vec{B}$$

As we have done, we can eliminate ρ with the Gauss law and $\vec{J}$ with Ampere Maxwell, so we get that:

$$\vec{f} = \epsilon_0(\nabla \cdot \vec{E})\vec{E} + \left(\frac{1}{\mu_0}\nabla \times \vec{B} - \epsilon_0\frac{\partial \vec{E}}{\partial t}\right) \times \vec{B}$$

But we know that $\dfrac{\partial}{\partial t}(\vec{E} \times \vec{B}) = \left(\dfrac{\partial \vec{E}}{\partial t} \times \vec{B}\right) + \left(\vec{E} \times \dfrac{\partial \vec{B}}{\partial t}\right)$

And Faraday says that $\dfrac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$

So that $\dfrac{\partial \vec{E}}{\partial t} \times \vec{B} = \dfrac{\partial}{\partial t}(\vec{E} \times \vec{B}) + \vec{E} \times (\nabla \times \vec{E})$
Therefore:

$$\vec{f} = \epsilon_0[(\nabla \cdot \vec{E})\vec{E} - \vec{E} \times (\nabla \times \vec{E})] - \frac{1}{\mu_0}[\vec{B} \times (\nabla \times \vec{B})] - \epsilon_0 \frac{\partial}{\partial t}(\vec{E} \times \vec{B})$$

We can add a term $(\nabla \vec{B})\vec{B}$ which is 0 to make things more symmetrical.
And we can use a product rule that says $\vec{E} \times (\nabla \times \vec{E}) = \dfrac{1}{2}\nabla(E^2) - (\vec{E} \cdot \nabla)\vec{E}$ (and the same for $\vec{B}$)

Therefore, we find that:

$$\vec{f} = \epsilon_0[(\nabla \cdot \vec{E})\vec{E} + (\vec{E} \cdot \nabla)\vec{E}] + \frac{1}{\mu_0}[(\nabla \cdot \vec{B})\vec{B} + (\vec{B} \cdot \nabla)\vec{B}] - \frac{1}{2}\nabla\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \epsilon_0 \frac{\partial}{\partial t}(\vec{E} \times \vec{B})$$

But we can introduce the **Maxwell Stress Tensor:**

$$\boxed{T_{ij} := \epsilon_0\left(E_i E_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij}B^2\right)}$$

Where i, j represent x, y or z.
Then, for example, we have:

$$T_{xx} = \frac{1}{2}\epsilon_0(E_x^2 - E_y^2 - E_z^2) + \frac{1}{2\mu_0}(B_x^2 - B_y^2 - B_z^2)$$
$$T_{xy} = \epsilon_0(E_x E_y) + \frac{1}{\mu_0}(B_x B_y)$$

It is a second order tensor, and when we apply it to a vector, we have for example:

$$(\vec{a} \cdot T)_j = a_i T_{ij}$$

And for example, the divergence of T is:

$$(\nabla \cdot T)_j = \epsilon_0\left[(\nabla \cdot \vec{E})E_j + (\vec{E} \cdot \nabla)E_j - \frac{1}{2}\nabla_j E^2\right] + \frac{1}{\mu_0}\left[(\nabla \cdot \vec{B})B_j + (\vec{B} \cdot \nabla)B_j - \frac{1}{2}\nabla_j B^2\right]$$

Therefore, we find that:

$$\boxed{\vec{f} = \nabla \cdot T - \epsilon_0\mu_0\frac{\partial \vec{S}}{\partial t}}$$

.

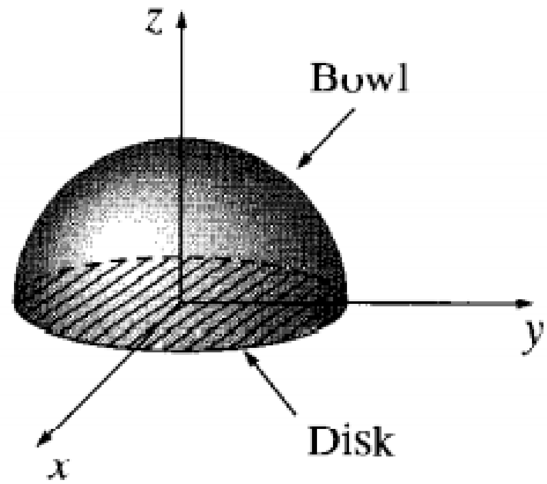
Then, the total force on the charges in V is as before $\vec{F} = \int_V \vec{f} d\tau$, so we have:

$$\boxed{\vec{F} = \oint_S T \cdot d\vec{a} - \epsilon_0\mu_0\frac{d}{dt}\int_V \vec{S} d\tau}$$

.

Physically, T is the force per unit area (**stress**) acting on the surface.
T_{ij} is the force per unit area in the ith direction acting on a surface aligned in the jth direction.
Therefore, diagonal elements (T_{xx}, T_{yy}, T_{zz}) represent pressures and off diagonal elements represent shears.

Example 8.2:
Determine the net force on the northern hemisphere of a uniformly charged solid sphere of radius R and charge Q



Solution:
The boundary consists of the bown and the disc.
For the bowl, $d\vec{a} = R^2 \sin\theta d\theta d\phi \hat{r}$
And the electric field there (caused by the solid sphere) is $\vec{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{R^2}\hat{r}$
In cartesian, $\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$
So:

$$T_{zx} = \epsilon_0 E_z E_x = \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 \sin\theta\cos\theta\cos\phi$$

$$T_{zy} = \epsilon_0 E_z E_y = \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 \sin\theta\cos\theta\sin\phi$$

$$T_{zz} = \frac{\epsilon_0}{2}(E_z^2 - E_x^2 - E_y^2) = \frac{\epsilon_0}{2} T_{zx} = \epsilon_0 E_z E_x = \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 (\cos^2\theta - \sin^2\theta)$$

The net force is obviuosly in the z-directoin, so it suffices to calculate:

$$(T \cdot d\vec{a})_z = T_{ij} da_j = T_{zx} da_x + T_{zt} da_y + T_{zz} da_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right)^2 \sin\theta\cos\theta d\theta d\phi$$

Then, the force on the z direction on this bowl is the integral of this thing:

$$F_{bowl} = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right)^2 2\pi\int_0^{\pi/2}\sin\theta\cos\theta d\theta = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{8R^2}$$

.

Meanwhile, for the equatorial disk, the differential of area is:

$$d\vec{a} = -rdrd\phi\hat{z}$$

And since we are now inside the sphere, the electric field $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\vec{r} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}r(\cos\phi\hat{x} + \sin\phi\hat{y})$

We will only care about the components T_{iz} because this is the only force that we will need. When taking the dot product with $d\vec{a}$, we will only care about T_{zz}:

$$T_{zz} = \frac{\epsilon_0}{2}(E_z^2 - E_x^2 - E_y^2) = -\frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 r^2$$

Hence:

$$(T\cdot d\vec{a})_z = T_{zj}\cdot d\vec{a}_j = T_{zz}da_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 r^3 drd\phi$$

Therefore, the force on the disk is:

$$F_{disk} = \oint_S T\cdot d\vec{a} = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 2\pi\int_0^R r^3 dr = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{16R^2}$$

Combining both forces, we sum them up and get:

$$\boxed{F = \frac{1}{4\pi\epsilon_0}\frac{3Q^2}{16R^2}}$$

.

Conservation of Momentum

According to Newton's second law, the force on an object is equal to the rate of change of its momentum:

$$\vec{F} = \frac{d\vec{p}_{mech}}{dt}$$

Therefore, we can write the equation for the force as:

$$\frac{d\vec{p}_{mech}}{dt} = -\epsilon_0\mu_0\frac{d}{dt}\int_V \vec{S}d\tau + \oint_S T\cdot d\vec{a}$$

Where $\vec{p}_{mech}$ is the total (mechanical) momentum of the particles contained in the volume V.
This is similar to Poyinting's theorem and deserves a similar interpretation
The first integral is the **momentum stored in the electromagnetic fields themselves:**

$$\vec{p}_{em} = \mu_0 \epsilon_0 \int_V \vec{S} d\tau$$

While the **second integral is the momentum per unit time flowing through the surface**.

So this is the general statement of **conservation of momentum in electrodynamics**.

Now, let $\mathcal{P}_{mech}$ be the density of **mechanical momentum** and $\mathcal{P}_{em}$ the density of momentum in the fields:

$$\mathcal{P}_{em} = \mu_0 \epsilon_0 \vec{S}$$

Then, the conservation of momentum in differential form is:

$$\boxed{\frac{\partial}{\partial t}(\mathcal{P}_{mech} + \mathcal{P}_{em}) = \nabla \cdot T}$$

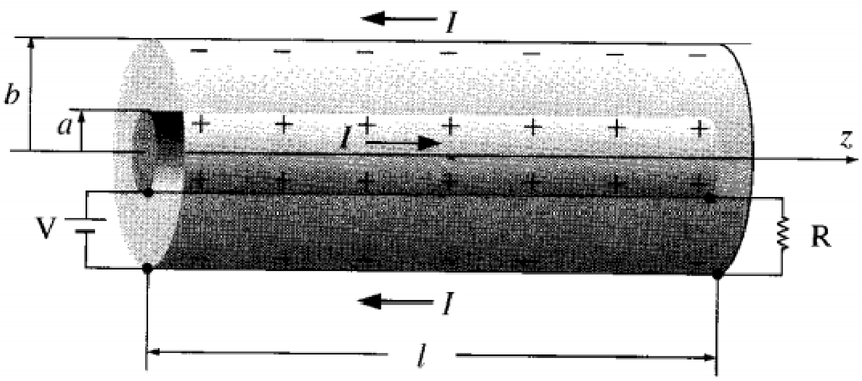
So $-T$ is the **momentum Flux density**, playing the role of $\vec{J}$ in the continuity equation or $\vec{S}$ in the in Poynting's theorem.
Specifically, $-T_{ij}$ is the momentum in the i direction crossing a surface oriented in the j direction, per unit area, per unit time.
Notice that $\vec{S}$ is the energy per unit area, per unit time transported by an EM field. While $\mu_0 \epsilon_0 \vec{S}$ is the momentum per unit volume stored in those fields.
Similarly, T plays a dual role: T is the EM stress acting on a surface and $-T$ is the flow of momentum

.

Example 8.3: A long coaxial cable, of length l, consists of an inner conductor (radius a) and an outer conductor (radius b). It is connected to a battery at one end and a resistor at the other. The inner conductor carries a uniform charge per unit length λ, and a steady current I to the right; the outer conductor has the opposite charge and current. Which is the electromagnetic momentum stored in the fields

Solution: the fields are:

$$\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{s} \quad , \quad \vec{B} = \frac{\mu_0}{2\pi}\frac{I}{s}\hat{\phi}$$



The Poynting vector is then:

$$\vec{S} = \frac{\lambda I}{4\pi^2\epsilon_0 s^2}\hat{z}$$

Evidently energy is flowing down the line, from the battery to the resistor. In fact, the power transported is:

$$P = \int \vec{S}\cdot d\vec{a} = \frac{\lambda I}{4\pi^2\epsilon_0}\int_a^b \frac{1}{s^2}2\pi s ds = \frac{\lambda I}{2\pi\epsilon_0}\log(b/a) = IV$$

The momentum in the fields is:

$$\vec{p}_{em} = \mu_0\epsilon_0\int \vec{S} d\tau = \frac{\mu_0\lambda I}{4\pi^2}\hat{z}\int_a^b \frac{1}{s^2} l 2\pi s ds = \frac{\mu_0\lambda I l}{2\pi}\log(b/a)\hat{z}$$

This is astonishing, the fields are static and the system too but there is some momentum. It turns out this isn't true, there is a hidden momentum, the mechanichal one, that makes the total momentum equal to 0.

Suppose now that we turn up the resistance, so the current decreases. The changing magnetic field will induce an electric field (Eq. 7.19):

$$\mathbf{E} = \left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln s + K\right]\hat{\mathbf{z}}.$$

This field exerts a force on $\pm\lambda$:

$$\mathbf{F} = \lambda l\left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln a + K\right]\hat{\mathbf{z}} - \lambda l\left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln b + K\right]\hat{\mathbf{z}} = -\frac{\mu_0\lambda l}{2\pi}\frac{dI}{dt}\ln(b/a)\,\hat{\mathbf{z}}.$$

The total momentum imparted to the cable, as the current drops from I to 0, is therefore

$$\mathbf{p}_{\text{mech}} = \int \mathbf{F}\,dt = \frac{\mu_0\lambda I l}{2\pi}\ln(b/a)\,\hat{\mathbf{z}},$$

which is precisely the momentum originally stored in the fields. (The cable will not recoil, however, because an equal and opposite impulse is delivered by the simultaneous disappearance of the hidden momentum.)

.

Angular Momentum

We have seen that EM fields carry energy:

$$u_{em} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

and momentum:

$$\mathcal{P}_{em} = \mu_0 \epsilon_0 \vec{S} = \epsilon_0 (\vec{E} \times \vec{B})$$

And for that matter, angular momentum:

$$\vec{l}_{em} = \vec{r} \times \surd^{em} = \epsilon_0 [\vec{r} \times (\vec{E} \times \vec{B})]$$

Example 8.4

Imagine a very long solenoid with radius R, n turns per unit length, and current I. Coaxial with the solenoid are two long cylindrical shells of length l—one, *inside* the solenoid at radius a, carries a charge $+Q$, uniformly distributed over its surface; the other, *outside* the solenoid at radius b, carries charge $-Q$ (see Fig. 8.7; l is supposed to be much greater than b). When the current in the solenoid is gradually reduced, the cylinders begin to rotate, as we found in Ex. 7.8. *Question*: Where does the angular momentum come from?[4]

Solution: It was initially stored in the fields. Before the current was switched off, there was an electric field,

$$\mathbf{E} = \frac{Q}{2\pi\epsilon_0 l}\frac{1}{s}\,\hat{\mathbf{s}} \quad (a < s < b).$$

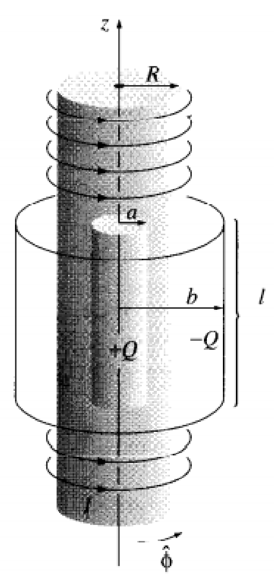


Figure 8.7

in the region between the cylinders, and a magnetic field,

$$\mathbf{B} = \mu_0 n I\,\hat{\mathbf{z}}(s < R),$$

inside the solenoid. The momentum density (Eq. 8.33) was therefore

$$\boldsymbol{\wp}_{\mathrm{em}} = -\frac{\mu_0 n I Q}{2\pi l s}\,\hat{\boldsymbol{\phi}},$$

in the region $a < s < R$. The *angular* momentum density was

$$\boldsymbol{\ell}_{\mathrm{em}} = \mathbf{r} \times \boldsymbol{\wp}_{\mathrm{em}} = -\frac{\mu_0 n I Q}{2\pi l}\,\hat{\mathbf{z}},$$

which is *constant*, as it turns out; to get the *total* angular momentum in the fields, we simply multiply by the volume, $\pi(R^2 - a^2)l$:

$$\mathbf{L}_{\mathrm{em}} = -\frac{1}{2}\mu_0 n I Q (R^2 - a^2)\,\hat{\mathbf{z}}. \tag{8.35}$$

When the current is turned off, the changing magnetic field induces a circumferential electric field, given by Faraday's law:

$$\mathbf{E} = \begin{cases} -\dfrac{1}{2}\mu_0 n \dfrac{dI}{dt}\dfrac{R^2}{s}\,\hat{\boldsymbol{\phi}}, & (s > R), \\[2ex] -\dfrac{1}{2}\mu_0 n \dfrac{dI}{dt} s\,\hat{\boldsymbol{\phi}}, & (s < R). \end{cases}$$

Thus the torque on the outer cylinder is

$$\mathbf{N}_b = \mathbf{r} \times (-Q\mathbf{E}) = \frac{1}{2}\mu_0 n Q R^2 \frac{dI}{dt}\,\hat{\mathbf{z}},$$

Thus the torque on the outer cylinder is

$$\mathbf{N}_b = \mathbf{r} \times (-Q\mathbf{E}) = \frac{1}{2}\mu_0 n Q R^2 \frac{dI}{dt}\hat{\mathbf{z}},$$

and it picks up an angular momentum

$$\mathbf{L}_b = \frac{1}{2}\mu_0 n Q R^2 \hat{\mathbf{z}} \int_I^0 \frac{dI}{dt} dt = -\frac{1}{2}\mu_0 n I Q R^2 \hat{\mathbf{z}}.$$

Similarly, the torque on the inner cylinder is

$$\mathbf{N}_a = -\frac{1}{2}\mu_0 n Q a^2 \frac{dI}{dt}\hat{\mathbf{z}},$$

and its angular momentum increase is

$$\mathbf{L}_a = \frac{1}{2}\mu_0 n I Q a^2 \hat{\mathbf{z}}.$$

So it all works out: $\mathbf{L}_{\text{em}} = \mathbf{L}_a + \mathbf{L}_b$. The angular momentum *lost* by the fields is precisely equal to the angular momentum *gained* by the cylinders, and the *total* angular momentum (fields plus matter) is conserved.

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Incidentally, the angular case is in some respects *cleaner* than the linear analog (Ex. 8.3), because there is no "hidden" angular momentum to compensate for the angular momentum in the fields, and the cylinders really *do* rotate when the magnetic field is turned off. If a localized system is not moving, its total *linear* momentum *has* to be zero,[5] but there is no corresponding theorem for angular momentum, and in Prob. 8.12 you will see a beautiful example in which nothing at *all* is moving—not even currents—and yet the angular momentum is nonzero.

Electromagnetic Waves

Waves in One dimension

A wave is a disturbance in a medium.
In the presence of absorption, the wave will diminish size, if the medium is dispersive,different frequencies move at different speeds, standing waves don't propagate, etc.

We begin with the simplest example of a wave, a fixed shape that propagates at constant speed. For this to happen, the wave function must be of the form:

$$f(x,t) = g(z - vt)$$

Wave equation:
In a taught string, we can get a differential equation that gives us the displacement of the wave (f) depending on the spatial coordinate z and time coordinate t, and that is:

$$\boxed{\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}}$$

This equation admits as solutions functions of the form $f(z,t) = g(z - vt) + h(z + vt)$ and that is why it is called the wave equation.
When we sum two functions like these g, h we no longer get a moving wave but we can get a standing wave or many other things.

Sinusoidal Wave

The sinusoidal wave is the simplest solution to the wave equation:

$$f(z,t) = A\cos[k(z - vt) + \delta]$$

- **Amplitude:** A, the maximum displacement it can have:
- **Phase:** What is inside the cosine $k(z - vt) + \delta$
- **Phase constant:** δ
- **Wave number:** k
- **Wavelength:** Length of a wave at an instant of time λ
- **Period:** At a fixed point z, the time it takes to vibrate up and down and complete a cycle, T
- **Frequency:** The number of oscillations per unit time ν
- **Angular frequency:** The number of radians swept per unit time

Clearly, we have the following relations:

- $\lambda = \dfrac{2\pi}{k}$
- $T = \dfrac{1}{\nu} = \dfrac{2\pi}{\omega}$
- $\omega = 2\pi\nu$

Then, we can write the general wave function as:

$$f(z,t) = A\cos(kz - \omega t + \delta)$$

.

Complex Notation
We can also write the wave equation as:

$$\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}$$

Where the **complex amplitude** $\tilde{A} = Ae^{i\delta}$ includes the phase constant. The actual wave function will be only the real part of this:

$$f(z,t) = Re[\tilde{f}(z,t)]$$

This sometimes simplifies the algebra.

Linear Combinations of sinusoidal waves

Any wave function can be written as a linear combination of sinusoidal ones as:

$$f(z,t) = \int_{\mathbb{R}} \tilde{A}(k)e^{i(kx-\omega t)}dk$$

Where ω is a function of k.
The formula for $\tilde{A}(k)$ can be obtained from knowing $f(z,0)$ and $\dot{f}(z,0)$ using Fourier, but we won't car much for that.

Boundary Conditions

For know we have assumed that the string is infinite.
Now we will add conditions, suppose for instance that the string is tied to a second string, such that maybe the speeds v at each string are different.
We have a **incident wave:**

$$\tilde{f}_I(z,t) = \tilde{A}_I e^{i(k_1 z - \omega t)} \qquad , \quad (z<0)$$

And a **reflected wave:**

$$\tilde{f}_R(z,t) = \tilde{A}_R e^{i(-k_1 z - \omega t)} \qquad , \quad (z<0)$$

traveling back along string 1.
And a **transmitted wave**:

$$\tilde{f}_T(z,t) = \tilde{A}_T e^{i(k_2 z - \omega t)} \qquad , \quad (z>0)$$

All parts of the system are oscillating at a frequency ω, but since the wave velocities are different in the two strings, so are the ks and:

$$\frac{\lambda_1}{\lambda_2} = \frac{k_2}{k_1} = \frac{v_1}{v_2}$$

Of course this is only true for waves of infinite extent having existed forever.

We know look for an incident wave of finite extent, like a pulse. The problem is that a finite pulse is not sinusoidal, it is a piece of a sinusoid, and it can be found using a linear combination of true sinusoidals, by putting together a whole range of frequencies and wavelengths. That is why working with finite pulses is more complicated.

Returning to infinite pulses, we have that:

$$\tilde{f}(z,t) = \begin{cases} \tilde{A}_I e^{i(k_1 z - \omega t)} + \tilde{A}_R e^{i(-k_1 z - \omega t)}, & \text{for } z<0, \\ \tilde{A}_T e^{i(k_2 z - \omega t)}, & \text{for } z>0. \end{cases}$$

The boundary conditions would now be:

$$\tilde{f}(0^-,t) = \tilde{f}(0^+,t)$$
$$\left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^-} = \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^+}$$

The actual conditions are for f (the real part) but it is the same for the complex parts.

when applied to the functions, we get:

$$\tilde{A}_I + \tilde{A}_r = \tilde{A}_T$$
$$k_1(\tilde{A}_I - \tilde{A}_R) = k_2 \tilde{A}_T$$

From which we get: $\tilde{A}_R = \dfrac{k_1 - k_2}{k_1 + k_2}\tilde{A}_I \;\; , \;\; \tilde{A}_T = \dfrac{2k_1}{k_1 + k_2}\tilde{A}_I$
Or because ω is the same in all, we can replace k with v.
If all the δs are the same, we get that:

$$A_R = \frac{v_2 - v_1}{v_2 + v_1} A_I \quad , \quad A_T = \frac{2v_2}{v_2 + v_1} A_I$$

.

We can see that if the second string is heavier ($v_2 < v_1$), then the reflected wave is out of phase by 180 degrees.
And if the second string is infinitely massive, then $A_R = A_I$ and $A_T = 0$

Polarization

The waves that travel down a string when you shake it are called **transverse**.
Now, there are two dimensions perpendicular to any given line of propagation, so transverse waves occur in two independent states of **polarization**. We can have:

$$\tilde{\mathbf{f}}_v(z,t) = \tilde{A}e^{i(kz-\omega t)}\widehat{x}$$
$$\tilde{\mathbf{f}}_h(z,t) = \tilde{A}e^{i(kz-\omega t)}\widehat{y}$$

Or along any other direction in the xy plane:

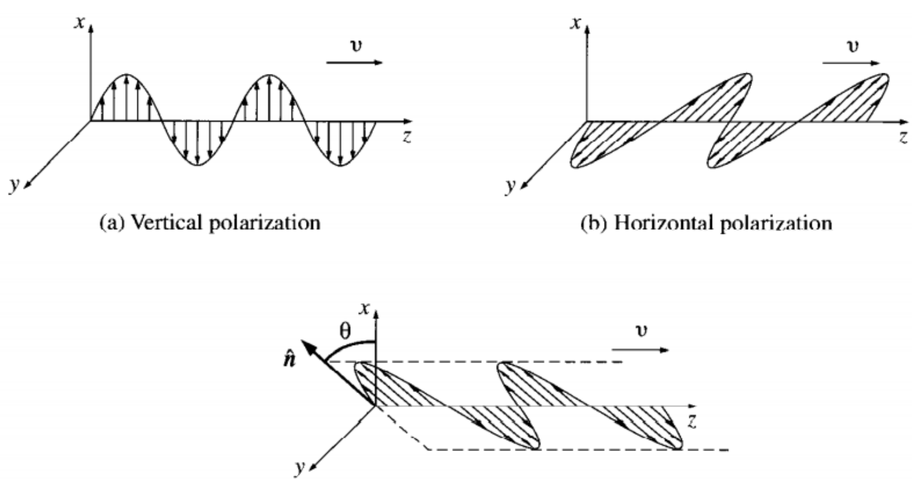
$$\tilde{\mathbf{f}}(z,t) = \tilde{A}e^{i(kz-\omega t)}\widehat{n}$$

Where $\widehat{n} \cdot \widehat{z} = 0$
We define the **polarization angle** as the angle of the propagation $\widehat{n}$ with respect to $\widehat{x}$, that is:

$$\widehat{n} = \cos\theta\widehat{x} + \sin\theta\widehat{y}$$

Thus:

$$\tilde{\mathbf{f}}(z,t) = (\tilde{A}\cos\theta)e^{i(kz-\omega t)}\widehat{x} + (\tilde{A}\sin\theta)e^{i(kz-\omega t)}\widehat{y}$$



(a) Vertical polarization

(b) Horizontal polarization

(c) Polarization vector

Figure 9.8

.

Electromagnetic Wave equation

If there is no charge of current, the Maxwell laws are:

- $\nabla \cdot \vec{E} = 0$
- $\nabla \cdot \vec{B} = 0$
- $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$
- $\nabla \times \vec{B} = \mu_0 \epsilon_0 \dfrac{\partial \vec{E}}{\partial t}$

We apply the curl to 3 and 4 and using the others, we get:

$$\nabla \times (\nabla \times \vec{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$
$$\nabla \times (\nabla \times \vec{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

Or, since $\nabla \cdot \vec{E} = 0$ and $\nabla \cdot \vec{B} = 0$, using the BAC CAB law we get:

$$\boxed{\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}}$$
$$\boxed{\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}}$$

These are the **Three dimensional wave equations** with a speed of $v = \dfrac{1}{\sqrt{\epsilon_0 \mu_0}} = 3 \times 10^8 m/s$

Monochromatic Plane Waves

We confine our attention to sinusoidal waves of frequency ω. This are called **monochromatic**.
Suppose that the wave travels in the z direction, with no x, y dependence, these are called **plane waves**.
The fields are independent of x, y for plane waves, so we have the form:

$$\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_0 e^{i(kz-\omega t)} \quad , \quad \tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_0 e^{i(kz-\omega t)}$$

Where $\tilde{\vec{E}}_0, \tilde{\vec{B}}_0$ are the complex amplitudes.
Now, whereas every solution to Maxwell's equations in empty space must obey the wave equation, the reverse is not true, since Maxwell's equations have extra constraints.

In particular, since $\nabla \cdot \vec{E} = \nabla \cdot \vec{B} = 0$, it follows that:

$$(\tilde{E}_0)_z = (\tilde{B}_0)_z = 0$$

That is **electromagnetic waves are transverse**.
The electric and magnetic fields are perpendicular to the direction of propagation.
Moreover, Faraday's law, $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ implies a relation between the amplitudes:

$$-k(\tilde{E}_0)_y = \omega(\tilde{B}_0)_x \;\; , \;\; k(\tilde{E}_0)_x = \omega(\tilde{B}_0)_y$$

Or, more compactly:

$$\tilde{\vec{B}}_0 = \frac{k}{\omega}(\tilde{z} \times \tilde{\vec{E}}_0)$$

Evidently, $\vec{E}, \vec{B}$ are **in phase and mutually perpendicular.** Their real amplitudes are related by:

$$B_0 = \frac{k}{\omega} E_0 = \frac{1}{c} E_0$$

Example 9.2

If $\mathbf{E}$ points in the x direction, then $\mathbf{B}$ points in the y direction (Eq. 9.46):

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}(z,t) = \frac{1}{c}\tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}\hat{\mathbf{y}},$$

or (taking the real part)

$$\boxed{\mathbf{E}(z,t) = E_0\cos(kz-\omega t+\delta)\,\hat{\mathbf{x}}, \quad \mathbf{B}(z,t) = \frac{1}{c}E_0\cos(kz-\omega t+\delta)\,\hat{\mathbf{y}}.} \tag{9.48}$$

[5]Because the real part of $\tilde{\mathbf{E}}$ differs from the imaginary part only in the replacement of sine by cosine, if the former obeys Maxwell's equations, so does the latter, and hence $\tilde{\mathbf{E}}$ as well.

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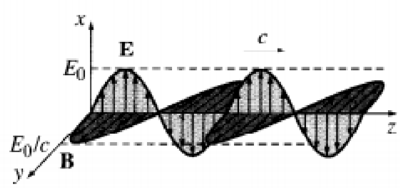


Figure 9.10

This is the paradigm for a monochromatic plane wave (see Fig. 9.10). The wave as a whole is said to be polarized in the x direction (by convention, we use the direction of $\mathbf{E}$ to specify the polarization of an electromagnetic wave).

Propagation vector $\vec{k}$ points in the direction of propagation and has the magnitude of the wave number k.
The scalar product $\vec{k}\cdot\vec{r}$ is the appropriate generalization of kz, so:

$$\boxed{\vec{E}(\vec{r},t) = \tilde{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega r)}\widehat{n}}$$

$$\boxed{\vec{B}(\vec{r},t) = \frac{1}{c}\tilde{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}(\widehat{k}\times\widehat{n}) = \frac{1}{c}\widehat{k}\times\vec{E}}$$

Where $\widehat{n}$ is the polarization vector. Because $\vec{E}$ is transverse, then:

$$\widehat{n}\cdot\widehat{k} = 0$$

The actual (real) electric and magnetic fields in a monochromatic plane wave with propagation vector $\vec{k}$ and polarization $\widehat{n}$ are:

$$\vec{E}(\vec{r},t) = E_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \delta)\widehat{n}$$
$$\vec{B}(\vec{r},t) = \frac{1}{c}E_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \delta)(\widehat{k}\times\widehat{n})$$

.

Energy and Momentum in Electromagnetic waves

According to what we saw before, the energy per unit volume in electromagnetic fields is:

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)$$

In the case of a monochromatic plane wave, $B^2 = \frac{1}{c^2}E^2 = \mu_0\epsilon_0 E^2$

so the electric and magnetic contributions are equal, therefore:

$$u = \epsilon_0 E^2 = \epsilon_0 E^2 \cos^2(kz - \omega t + \delta)$$

As the wave travels it carries energy along with it. The energy flux density (energy per unit area and per unit time) transported by the fields is given by the Poynting vector:

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})$$

For monochromatic plane waves propagating in the z direction:

$$\vec{S} = c\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\widehat{z} = cu\widehat{z}$$

So $\vec{S}$ is the energy density u times the speed $c\widehat{z}$ as it should. Since in a time Δt, a length $c\Delta t$ passes an area A, carrying an energy $uAc\Delta t$

Electromagnetic fields also carry momentum. As we saw before, the momentum density stored in the fields is:

$$\mathcal{P} = \frac{1}{c^2}\vec{S}$$

Then, for monochromatic plane waves, then:

$$\mathcal{P} = \frac{1}{c}\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\widehat{z} = \frac{1}{c}u\widehat{z}$$

We usually only care about the average of this quantities. The average of a cosine squared in a complete cycle is 1/2, so:

$$\langle u \rangle = \frac{1}{2}\epsilon_0 E_0^2$$
$$\langle \vec{S} \rangle = \frac{1}{2}c\epsilon_0 E_0^2 \widehat{z}$$
$$\langle \mathcal{P} \rangle = \frac{1}{2c}\epsilon_0 E_0^2 \widehat{z}$$

.

Intensity: Intensity is defined as the average power per unit area transported by an electromagnetic wave:

$$I := \langle S \rangle = \frac{1}{2}c\epsilon_0 E_0^2$$

.

Radiation Pressure:
When light falls on a perfect absorber, it delivers its momentum to the surface. In a time Δt the momentum transfer is $\Delta\vec{p} = \langle \mathcal{P} \rangle A c \Delta t$.
So the **radiation pressure** (average force per unit area) is:

$$P = \frac{1}{A}\frac{\Delta p}{\delta t} = \frac{1}{2}\epsilon_0 E_0^2 = \frac{I}{c}$$

On a perfect reflector, the pressure is twice as great, since the momentum switches direction, instead of being simply being absorbed)

Problem 9.12: Find all elements of the Maxwell stress tensor for a monochromatic plane wave traveling in the z direction and linearly polarized in the x direction. Does the answer make sense?

Problem 9.12

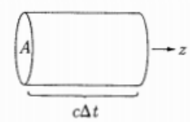
$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right).$$

With the fields in Eq. 9.48, **E** has only an x component, and **B** only a y component. So all the "off-diagonal" $(i \neq j)$ terms are zero. As for the "diagonal" elements:

$$\begin{aligned}
T_{xx} &= \epsilon_0 \left(E_x E_x - \frac{1}{2}E^2 \right) + \frac{1}{\mu_0}\left(-\frac{1}{2}B^2 \right) = \frac{1}{2}\left(\epsilon_0 E^2 - \frac{1}{\mu_0}B^2 \right) = 0. \\
T_{yy} &= \epsilon_0 \left(-\frac{1}{2}E^2 \right) + \frac{1}{\mu_0}\left(B_y B_y - \frac{1}{2}B^2 \right) = \frac{1}{2}\left(-\epsilon_0 E^2 + \frac{1}{\mu_0}B^2 \right) = 0. \\
T_{zz} &= \epsilon_0 \left(-\frac{1}{2}E^2 \right) + \frac{1}{\mu_0}\left(-\frac{1}{2}B^2 \right) = -u.
\end{aligned}$$

So $\boxed{T_{zz} = -\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)}$ (all other elements zero).

The momentum of these fields is in the z direction, and it is being *transported* in the z direction, so *yes*, it does make sense that T_{zz} should be the only nonzero element in T_{ij}. According to Sect. 8.2.3, $-\overleftrightarrow{T}\cdot d\mathbf{a}$ is the rate at which momentum crosses an area $d\mathbf{a}$. Here we have *no* momentum crossing areas oriented in the x or y direction; the momentum per unit time per unit area flowing across a surface oriented in the z direction is $-T_{zz} = u = \wp c$ (Eq. 9.59), so $\Delta p = \wp c A \Delta t$, and hence $\Delta p/\Delta t = \wp c A =$ momentum per unit time crossing area A. Evidently $\boxed{\text{momentum flux density} = \text{energy density.}}$ ✓



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Electromagnetic Waves in Matter

Propagation in Linear Media

Inside matter, but in regions where there is no free charge or free current, Maxwell's equations become:

- $\nabla \cdot \vec{D} = 0$
- $\nabla \cdot \vec{B} = 0$
- $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$
- $\nabla \times \vec{H} = \dfrac{\partial \vec{D}}{\partial t}$

Where if the medium is **linear:**

$$\vec{D} = \epsilon \vec{E} \quad , \quad \vec{H} = \frac{1}{\mu}\vec{B}$$

And if it is **homogeneous** (the whole space has the same type of matter, ϵ, μ don't vary from point to point). Maxwell's equations reduce to:

- $\nabla \cdot \vec{E} = 0$
- $\nabla \cdot \vec{B} = 0$

- $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$
- $\nabla \times \vec{B} = \mu\epsilon\dfrac{\partial \vec{E}}{\partial t}$

So we have again the same equations but now with a speed of:

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n}$$

Where

$$n := \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$$

is the **index of refraction** of the material.

For most materials, $\mu \simeq \mu_0$, so it reduces to:

$$n \simeq \sqrt{\epsilon_r}$$

Where ϵ_r is the dielectric constant.

All the previous results carry over.

- Energy density: $u = \dfrac{1}{2}\left(\epsilon E^2 + \dfrac{1}{\mu}B^2\right)$
- Poynting vector: $\vec{S} = \dfrac{1}{\mu}\vec{E} \times \vec{B}$
- Intensity $I = \dfrac{1}{2}\epsilon v E_0^2$
- Relation of amplitudes: $|B| = \dfrac{1}{v}|E|$
- $\omega = kv$

.

The interesting question is what happens when a wave passes from one transparent medium to another.
The conditions of the fields are:

- $\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}$
- $B_1^{\perp} = B_2^{\perp}$
- $\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$
- $\dfrac{1}{\mu_1}\vec{B}_1^{\parallel} = \dfrac{1}{\mu_2}\vec{B}_2^{\parallel}$

.

Reflection and Transmission at Normal incidence

Suppose the xy plane forms the boundary between two linear media. A plane wave of frequency ω, traveling in the z direction and polarized in the x direction approaches the interface from the left:

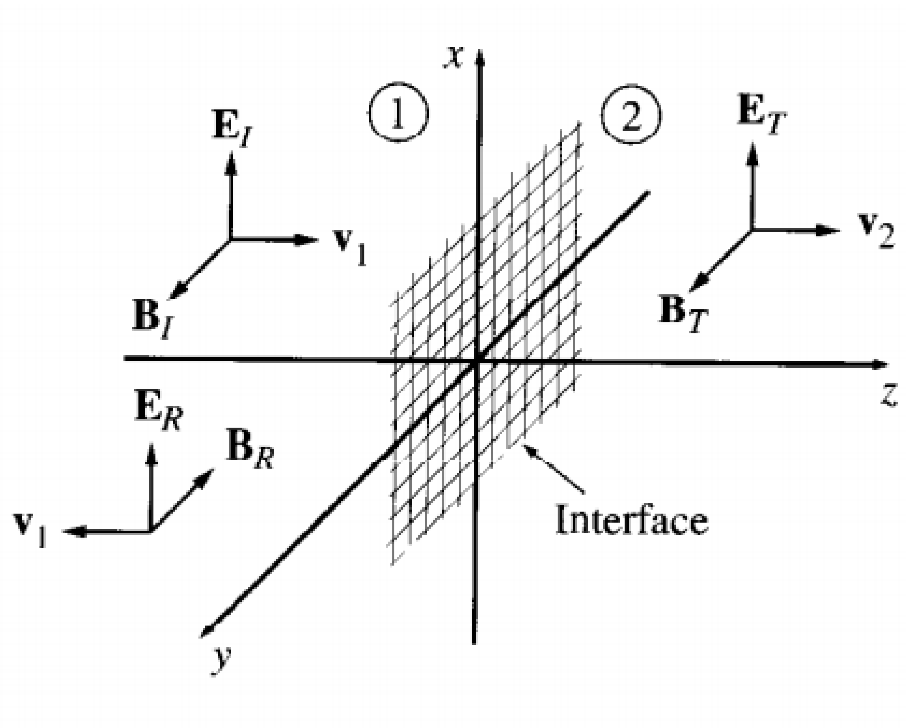
$$\vec{E}_I(z,t) = \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \widehat{x}$$
$$\vec{B}_I(z,t) = \frac{1}{v_1} \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \widehat{y}$$

It gives rise to a reflected wave that travels back:

$$\vec{E}_R(z,t) = \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \widehat{x}$$
$$\vec{B}_R(z,t) = -\frac{1}{v_1} \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \widehat{y}$$

And a transmitted wave:

$$\vec{E}_T(z,t) = \tilde{E}_{0T} e^{i(-k_2 z - \omega t)} \widehat{x}$$
$$\vec{B}_T(z,t) = -\frac{1}{v_2} \tilde{E}_{0T} e^{i(-k_2 z - \omega t)} \widehat{y}$$



Note that the minus sign in the reflected wave is required so that the reflected wave has an opposite Poyinting.

At $z = 0$, the combined fields on the left $\vec{E}_I + \vec{E}_R$ or $\vec{B}_I + \vec{B}_R$ must join the field on the right $\vec{E}_T, \vec{B}_T$ according to the boundary conditions.

There are no component perpendicular to the surface in this case, so we only care about the parallel continuity:

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$$

While the parallel continuity of B says that:

$$\frac{1}{\mu_1}\left(\frac{1}{v_1}\tilde{E}_{0I} - \frac{1}{v_1}\tilde{E}_{0R}\right) = \frac{1}{\mu_2}\left(\frac{1}{v_2}\tilde{E}_{0T}\right)$$

Or:

$$\tilde{E}_{0t} - \tilde{E}_{0R} = \beta\tilde{E}_{0T}$$

Where $\beta = \dfrac{\mu_1 v_1}{\mu_2 v_2} = \dfrac{\mu_1 n_2}{\mu_2 n_1}$.

We can solve these equations for outgoing amplitudes in terms of the incident amplitude:

$$\tilde{E}_{0R} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{0I} \quad , \quad \tilde{E}_{0T} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{0I}$$

These are remarkably similar to the ones for waves on a spring. If $\mu \simeq \mu_0$, then $\tilde{E}_{0R} = \left(\dfrac{v_2 - v_1}{v_2 + v_1}\right)\tilde{E}_{0I} \quad , \quad \tilde{E}_{0T} = \left(\dfrac{2v_2}{v_2 + v_1}\right)\tilde{E}_{0I}$
Which are exactly the equations for strings.
If $v_2 > v_1$, the reflected wave is in phase.
If $v_2 < v_1$, the reflected wave is completely out of phase.

Fraction of energies reflected and transmitted

The intensity of a wave is:

$$I = \frac{1}{2}\epsilon v E_0^2$$

So, if $\mu_1 = \mu_2 = \mu_0$, then:

$$R := \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

And the ratio to the transmitted one is:

$$T := \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0T}}{E_{0I}}\right)^2 = \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

.
R is the **reflection coefficient** and T is the **transmission coefficient**.
Notice that:

$$R + T = 1$$

As conservation of energy requires.

For example, when light passes from air ($n_1 = 1$) into glass $n_2 = 1{,}5$, then $R = 0{,}04$ and $T = 0{,}96$. Most of the light is transmitted.

Reflection and Transmission at Oblique incidence

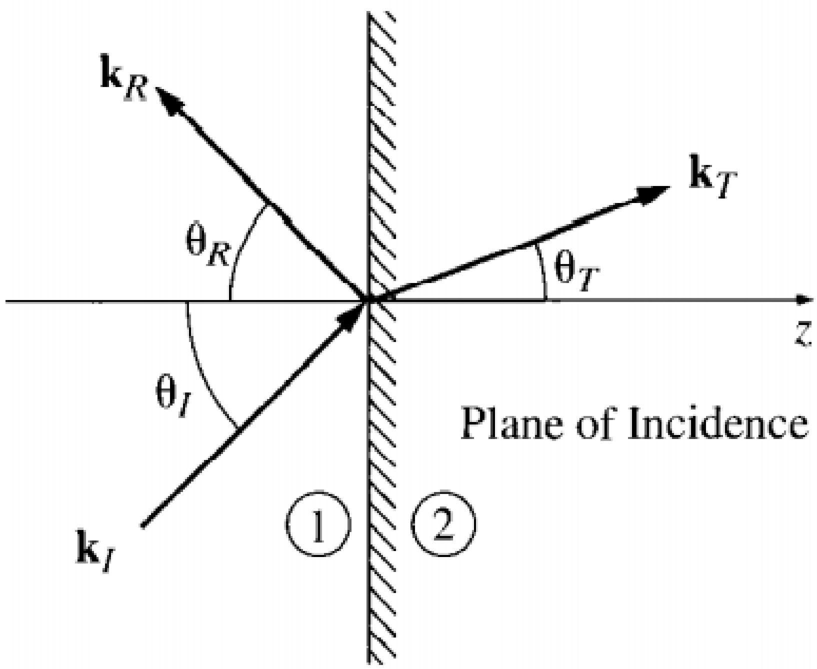


Figure 9.14

Suppose that a monochromatic wave:

$$\vec{E}_I(\vec{r}, t) = \vec{E}_{0I} e^{i(\vec{k}_I \cdot \vec{r} - \omega t)} \quad , \quad \vec{B}_I(\vec{r}, t) = \frac{1}{v_1}(\widehat{k}_1 \times \widehat{E}_I)$$

Approaches from the left, giving rise to a reflected and transmitted wave:

$$\vec{E}_R(\vec{r},t) = \vec{E}_{0R} e^{i(\vec{k}_R \cdot \vec{r} - \omega t)} \quad , \quad \vec{B}_R(\vec{r},t) = \frac{1}{v_1}(\widehat{k}_R \times \widehat{E}_R)$$

$$\vec{E}_T(\vec{r},t) = \vec{E}_{0T} e^{i(\vec{k}_T \cdot \vec{r} - \omega t)} \quad , \quad \vec{B}_T(\vec{r},t) = \frac{1}{v_2}(\widehat{k}_T \times \widehat{E}_T)$$

They all have the same frequency. The three wave numbers are related by:

$$k_I v_1 = k_R v_1 = k_T v_2 = \omega \quad \Rightarrow \quad k_l = k_R = \frac{v_2}{v_1} k_T = \frac{n_1}{n_2} k_T$$

Now we join the combined fields in medium 1 to the field in medium 2 using the boundary conditions

fields $\mathbf{E}_T$ and $\mathbf{B}_T$ in medium (2), using the boundary conditions 9.74. These all share the generic structure

$$(\ )e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)} + (\ )e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)} = (\ )e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)}, \quad \text{at } z = 0. \tag{9.93}$$

I'll fill in the parentheses in a moment; for now, the important thing to notice is that the x, y, and t dependence is confined to the exponents. *Because the boundary conditions must hold at all points on the plane, and for all times, these exponential factors must be equal.* Otherwise, a slight change in x, say, would destroy the equality (see Prob. 9.15). Of

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course, the time factors are *already* equal (in fact, you could regard this as an independent confirmation that the transmitted and reflected frequencies must match the incident one). As for the spatial terms, evidently

$$\mathbf{k}_I \cdot \mathbf{r} = \mathbf{k}_R \cdot \mathbf{r} = \mathbf{k}_T \cdot \mathbf{r}, \quad \text{when } z = 0, \tag{9.94}$$

Or more explicitly:

$$x(k_I)_x + y(k_I)_y = x(k_R)_x + y(k_R)_y = x(k_T)_x + y(k_T)_y$$

for all x, y

But when we take $x = 0$ or $y = 0$, we get that:

$$(k_I)_y = (k_R)_y = (k_T)_y$$
$$(k_I)_x = (k_R)_x = (k_T)_x$$

So:

First Law: The incident, reflected, and transmitted wave vectors for a plane (**plane of incidence**), which also includes the normal to the surface (here the z axis).

Meanwhile, the last equation we got says that:

$$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T$$

Where θ_I is the **angle of incidence**, θ_R the **angle of reflection** and θ_T the **angle or refraction**, all measured respect to the normal.

Second Law: The angle of incidence is equal to the angle of reflection:

$$\theta_I = \theta_R$$

This is the **law of reflection**

And the:

Third Law:

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2}$$

This is the **Snell Law**

.

We haven't even really used the specific boundary conditions.

The **Boundary conditions now are:**

- $\epsilon_1(\vec{E}_{0I} + \vec{E}_{0R})_z = \epsilon_2(\vec{E}_{0T})_z$
- $(\vec{B}_{0I} + \vec{B}_{0R}) = (\vec{B}_{0T})_z$
- $(\vec{E}_{0I} + \vec{E}_{0R})_{x,y} = (\vec{E}_{0T})_{x,y}$
- $\dfrac{1}{\mu_1}(\vec{B}_{0I} + \vec{B}_{0R})_{x,y} = \dfrac{1}{\mu_2}(\vec{B}_{0T})_{x,y}$

Where $\vec{B}_0 = (1/v)\widehat{k} \times \vec{E}_0$ in each case.

Suppose the polarization of the incident wave is parallel to the plane of incidence

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(the x-z plane in the image). So the reflected and transmitted waves are also polarized in this plane.

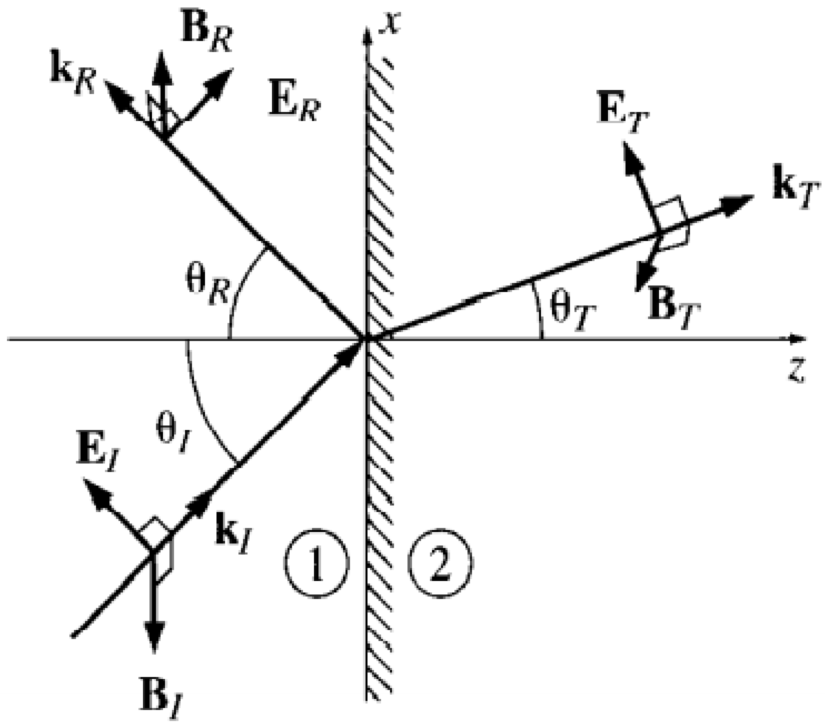

Figure 9.15

Then, the equation 1) reads:

$$\epsilon_1(-\tilde{E}_{0I}\sin\theta_I + \tilde{E}_{0R}\sin\theta_R) = \epsilon_2(-\tilde{E}_{0T}\sin\theta_T)$$

ii) adds nothing (0=0) and iii) becomes:

$$\tilde{E}_{0I}\cos\theta_I + \tilde{E}_{0R}\cos\theta_R = \tilde{E}_{0T}\cos\theta_T$$

And iv) says:

$$\frac{1}{\mu_1 v_1}(\tilde{E}_{0I} - \tilde{E}_{0R}) = \frac{1}{\mu_2 v_2}\tilde{E}_{0T}$$

Given the laws of reflection and refraction, the first and last equations reduce to:

$$\tilde{E}_{0I} - \tilde{E}_{0R} = \beta\tilde{E}_{0T}$$
$$\beta := \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$$

And the middle equations says that:

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \alpha \tilde{E}_{0T}$$
$$\alpha := \frac{\cos\theta_T}{\cos\theta_I}$$

.

Solving for the reflected and transmited we get the **Fresnel equations:**

$$\boxed{\tilde{E}_{0R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)\tilde{E}_{0I}}$$
$$\boxed{\tilde{E}_{0T} = \left(\frac{2}{\alpha + \beta}\right)\tilde{E}_{0I}}$$

Notice that the transmitted wave is always in phase with the incident one; the reflected wave is either in phase (if $\alpha < \beta$)or 180 degrees out (if $\alpha < \beta$)

We can write α in function of θ_I:

$$\alpha = \frac{\sqrt{1 - \sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1 - [(n_1/n_2)\sin\theta_I]^2}}{\cos\theta_I}$$

At grazing incidence ($\theta_I = 90deg$), α diverges, and the wave is totally reflected

There is an intermediate angle θ_B the **Brewster's angle**, at which the reflected wave is completely extinguished, this occurs when $\alpha = \beta$ or:

$$\tan\theta_B \simeq \frac{n_2}{n_1}$$

If $\mu_1 \simeq \mu_2$.

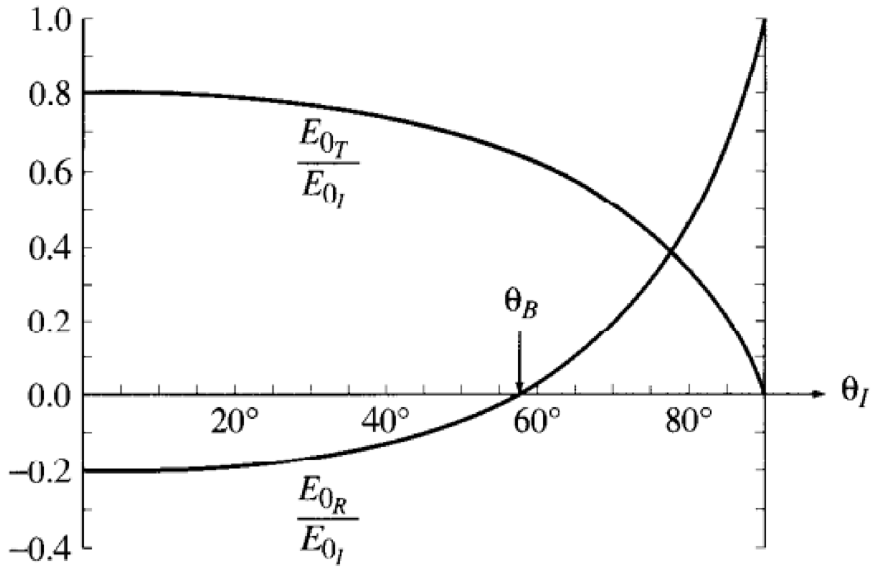

Figure 9.16

.

Now we see the powers:

The incident intensity is:

$$I_1 = \frac{1}{2}\epsilon_1 v_1 E_{0I}^2 \cos\theta_I$$

While the reflected and transmitted intensities are:

$$I_R = \frac{1}{2}\epsilon_1 v_1 E_{0R}^2 \cos\theta_R$$
$$I_T = \frac{1}{2}\epsilon_2 v_2 E_{0T}^2 \cos\theta_T$$

The cosines are there because the interface is at an angle to the wave front.
The reflection and transmission coefficients for waves polarized parallel to the plane of incidence are:

$$r_\parallel := \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2$$
$$t_\parallel := \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0T}}{E_{0I}}\right)^2 \frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2$$

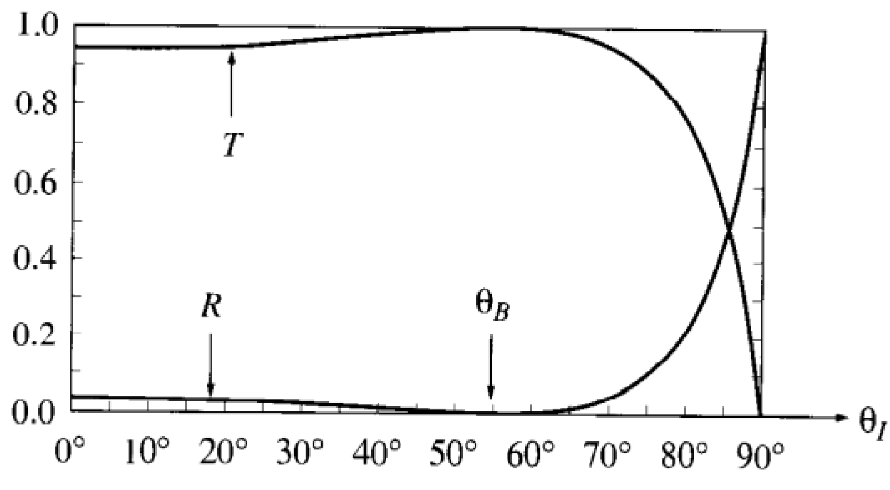

Figure 9.17

The electric field is polarized perpendicular to the plane of incidence

The relation between amplitudes is:

$$\tilde{E}_{0T} = \left(\frac{2}{1+\alpha\beta}\right)\tilde{E}_{0I}$$

$$\tilde{E}_{0R} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)\tilde{E}_{0I}$$

The reflected wave is in phase if $\alpha\beta < 1$ and 180 out of phase if $\alpha\beta < 1$.

Reflection and Transmission coefficients:

$$r_{\perp} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2$$

$$t_{\perp} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\alpha\left(\frac{E_{0T}}{E_{0I}}\right)^2 = \alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2$$

.

Absorption and Dispersion

Electromagnetic Waves in Conductors

In a general body, we will have free currents and charges. The free current is related to the electric field by:

$$\vec{J}_f = \sigma\vec{E}$$

And the equations for a linear media are:

- $\nabla \cdot \vec{E} = \frac{1}{\epsilon}\rho_f$
- $\nabla \cdot \vec{B} = 0$
- $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
- $\nabla \times \vec{B} = \mu\sigma\vec{E} + \mu\epsilon\frac{\partial \vec{E}}{\partial t}$

And the continuity equation for the free charge is:

$$\nabla \cdot \vec{J}_f = -\frac{\partial \rho_f}{\partial t}$$

That together with Ohm's law and Gauss's law (i), gives:

$$\frac{\partial \rho_f}{\partial t} = -\sigma(\nabla \cdot \vec{E}) = -\frac{\sigma}{\epsilon}\rho_f$$

So, for a homogeneous linear medium, it follows that:

$$\rho_f(t) = e^{-\sigma/\epsilon)t}\rho_f(0)$$

So if you put some free charge on a conductor, it will dissipate. The time constante $\tau := \epsilon/\sigma$ measures how good a conductor is (for a good one, τ is small).

At present, we are not interested in this transient behavior, we jump to the moment when the free charge $\rho_f = 0$ dissapears. We have:

- $\nabla \cdot \vec{E} = 0$
- $\nabla \cdot \vec{B} = 0$
- $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
- $\nabla \times \vec{B} = \mu\epsilon\frac{\partial \vec{E}}{\partial t} + \mu\sigma\vec{E}$

So the only difference is the addition of the term in iv).

Applying the curl to (iii) and (iv), as before, we obtain modified wave equations for **E** and **B**:

$$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{E}}{\partial t}, \quad \nabla^2 \mathbf{B} = \mu\epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{B}}{\partial t}. \tag{9.122}$$

These equations still admit plane-wave solutions,

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z-\omega t)}, \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(\tilde{k}z-\omega t)}, \tag{9.123}$$

but this time the "wave number" $\tilde{k}$ is complex:

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega, \tag{9.124}$$

as you can easily check by plugging Eq. 9.123 into Eq. 9.122. Taking the square root,

$$\tilde{k} = k + i\kappa, \tag{9.125}$$

where

$$k \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2}, \quad \kappa \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2}. \tag{9.126}$$

The imaginary part of $\tilde{k}$ results in an attenuation of the wave (decreasing amplitude with increasing z):

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)}, \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{-\kappa z} e^{i(kz-\omega t)}. \tag{9.127}$$

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)}, \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{-\kappa z} e^{i(kz-\omega t)}. \tag{9.127}$$

The distance it takes to reduce the amplitude by a factor of $1/e$ (about a third) is called the **skin depth**:

$$d \equiv \frac{1}{\kappa}; \tag{9.128}$$

it is a measure of how far the wave penetrates into the conductor. Meanwhile, the real part of $\tilde{k}$ determines the wavelength, the propagation speed, and the index of refraction, in the usual way:

$$\lambda = \frac{2\pi}{k}, \quad v = \frac{\omega}{k}, \quad n = \frac{ck}{\omega}. \tag{9.129}$$

The attenuated plane waves (Eq. 9.127) satisfy the modified wave equation (9.122) for *any* $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$. But Maxwell's equations (9.121) impose further constraints, which serve to determine the relative amplitudes, phases, and polarizations of **E** and **B**. As before, (i) and (ii) rule out any z components: the fields are *transverse*. We may as well orient our axes so that **E** is polarized along the x direction:

$$\tilde{\mathbf{E}}(z,t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}\,\hat{\mathbf{x}}. \tag{9.130}$$

Then (iii) gives

$$\tilde{\mathbf{B}}(z,t) = \frac{\tilde{k}}{\omega} \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}\,\hat{\mathbf{y}}. \tag{9.131}$$

(Equation (iv) says the same thing.) Once again, the electric and magnetic fields are mutually perpendicular.

Like any complex number, $\tilde{k}$ can be expressed in terms of its modulus and phase:

$$\tilde{k} = K e^{i\phi}, \tag{9.132}$$

where

$$K \equiv |\tilde{k}| = \sqrt{k^2+\kappa^2} = \omega\sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}} \tag{9.133}$$

and

$$\phi \equiv \tan^{-1}(\kappa/k). \tag{9.134}$$

According to Eq. 9.130 and 9.131, the complex amplitudes $\tilde{E}_0 = E_0 e^{i\delta_E}$ and $\tilde{B}_0 = B_0 e^{i\delta_B}$ are related by

$$B_0 e^{i\delta_B} = \frac{K e^{i\phi}}{\omega} E_0 e^{i\delta_E}. \tag{9.135}$$

Evidently the electric and magnetic fields are no longer in phase; in fact,

$$\delta_B - \delta_E = \phi; \tag{9.136}$$

the magnetic field *lags behind* the electric field. Meanwhile, the (real) amplitudes of **E** and **B** are related by

$$\frac{B_0}{E_0} = \frac{K}{\omega} = \sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}}. \tag{9.137}$$

The (real) electric and magnetic fields are, finally,

$$\left.\begin{aligned} \mathbf{E}(z,t) &= E_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E)\,\hat{\mathbf{x}}, \\ \mathbf{B}(z,t) &= B_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E+\phi)\,\hat{\mathbf{y}}. \end{aligned}\right\} \tag{9.138}$$

These fields are shown in Fig. 9.18.

.

Reflection on a Conducting Surface

In the presence of free charges and currents, the more general boundary laws are:

- $\epsilon_1 E_1^{\perp} - \epsilon_2 E_2 \perp = \sigma_g$
- $B_1^{\perp} - B_2^{\perp} = 0$
- $\vec{E}_1^{\|} - \vec{E}_2^{\|} = 0$
- $\dfrac{1}{\mu_1}\vec{B}_1^{\|} - \dfrac{1}{\mu_2}\vec{B}_2^{\|} = \vec{K}_f \times \widehat{n}$

Where σ_f is the free surface charge and $\vec{K}_f$ the free surface current. And $\widehat{n}$ the unit vector perpendicular to the surface.

For Ohmic conductors ($\vec{J} = \sigma \vec{E}$) there cannot be free surface current, since this would require infinite electric field at the boundary.

Suppose the xy plane forms the boundary between a nonconducting linear medium (1) and a conductor (2). A monochromatic wave, traveling in the z direction and polarized in the x direction approaches from the left:

$$\vec{E}_I(z,t) = \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \widehat{x} \quad , \quad \vec{B}_I(z,t) = \frac{1}{v_1} \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \widehat{y}$$

This incident wave gives rise to a reflected one:

$$\vec{E}_R(z,t) = \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \widehat{x} \ , \ \vec{B}_R(z,t) = -\frac{1}{v_1} \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \widehat{y}$$

And a transmitted one:

$$\vec{E}_T(z,t) = \tilde{E}_{0T} e^{i(\bar{k}_2 z - \omega t)} \widehat{x} \ , \ \vec{B}_T(z,t) = \frac{\bar{k}_2}{\omega} \tilde{E}_{0T} e^{i(\bar{k}_2 z - \omega t)} \widehat{y}$$

which is attenuated while it penetrates.

At $z = 0$, the combined wave in medium 1 must join the wave in medium 2 subject to the boundary conditions. Since $E^{\perp} = 0$ on both sides, the first boundary condition yields $\sigma_f = 0$. Since $B^{\perp} = 0$, ii) is satisfied.

Meanwhile, iii) gives:

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$$

And iv (with $K_f = 0$) gives:

$$\frac{1}{\mu_1 v_1}(\tilde{E}_{0I} - \tilde{E}_{0R}) - \frac{\bar{k}_2}{\mu_2 \omega} \tilde{E}_{0T} = 0$$

Or:

$$\tilde{E}_{0I} - \tilde{E}_{0T} = \tilde{\beta} \tilde{E}_{0T}$$

Where:

$$\tilde{\beta} := \frac{\mu_1 v_1}{\mu_2 \omega} \tilde{k}_2$$

It follows that:

$$\tilde{E}_{0R} = \frac{1 - \tilde{\beta}}{1 + \tilde{\beta}} \tilde{E}_{0I}$$
$$\tilde{E}_{0T} = \frac{2}{1 + \tilde{\beta}} \tilde{E}_{0I}$$

This is identical to the boundary between non conductors, but now $\tilde{\beta}$ is complex

For a perfect conductor $(\sigma = \infty), k_2 = \infty$, so $\tilde{\beta} = \infty$, we have:

$$\tilde{E}_{0R} = -\tilde{E}_{0I}$$
$$\tilde{E}_{0T} = 0$$

So the wave is totally reflected with a 180 degree phase shift. Good conductors make good mirrors.
In practice, you paint a thin silver coating onto the back of a pain of glass. The glass has nothing to do with the reflection, it is just there to support the silver.

The frequency Dependence of Permittivity

WE saw that the propagation of EM waves through matter is governed by three things that we took as constants:
Permittivity ϵ, permeability μ and conductivity σ

But actually, these three parameters depend on the frequency of the waves, and $n \simeq \sqrt{\epsilon_r}$ is also dependent on frequency.
The phenomenom of n depending on ω is called **dispersion** and the medium is called **dispersive**.

The **Phase Velocity** is the speed of a monochromatic wave:

$$v = \frac{\omega}{k}$$

But when a wave travels, it usually contains many different wavelengths in combination. The whole ."envelope."of waves travels at the **group velocity:**

$$v_g = \frac{d\omega}{dk}$$

The energy carried by a wave packet in a dispersive medium usually travels at the group velocity, not the phase velocity.

The electrons in a nonconductor are bound to specific molecules. We picture the binding forces as being of the type:

$$F_{bind} = -m\omega_0^2 x$$

Where ω_0 will be the angular frequency of the electrons and x is the displacement from equilibrium and ω_0 is the natural angular frequency $\sqrt{k/m}$
Practically any binding force can be approximated as this using Taylor series for the potential.

We put $U(x) = U(0) + xU'(0) + \frac{1}{2}x^2U''(0)$
But if we are in the equilibrium position, $U'(0) = 0$, we can take $U(0) = 0$ by moving the potential.

There will be presumably some damping force too:

$$F_{damp} = -m\gamma \frac{dx}{dt}$$

And, in the presence of an electromagnetic wave of frequency ω, polarized in the x direction, the electron is subject to a driving force of:

$$F_{driv} = qE = qE_0 \cos(\omega t)$$

Where q is the charge of the electron, E_0 the amplitude of the wave at the point of the electron. Therefore:

$$\begin{aligned} m\frac{d^2x}{dt^2} &= F_{tot} = F_{bind} + F_{damp} + F_{drive} \\ \Rightarrow \quad m\frac{d^2x}{dt^2} + m\gamma\frac{dx}{dt} &+ m\omega_0^2 x = qE_0\cos(\omega t) \end{aligned}$$

This is a damped and driven equation. We know the solution will be an oscillation of ω with a different amplitude and some phase.

This equation is easier to handle if we suppose the function is complex and write it as:

$$\frac{d^2\tilde{x}}{dt^2} + \gamma\frac{d\tilde{x}}{dt} + \omega_0^2\tilde{x} = \frac{q}{m}E_0 e^{-i\omega t}$$

We propose as a solution $\tilde{x}(t) = \tilde{x}_0 e^{-i\omega t}$
Inserting this into the equation, we find $\tilde{x}_0$ and the result is:

$$\tilde{x}(t) = \frac{q/m}{\omega_0^2 - \omega^2 - i\gamma\omega} E_0 e^{-i\omega t}$$

The actual solution is:

$$x_p(t) = \frac{qE_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} \cos(\omega t - \delta)$$

$$\delta = \arctan\left(\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right)$$

Then, the **dipole moment** is the real part of:

$$\tilde{p}(t) = q\tilde{x}(t) = \frac{q^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega} E_0 e^{-i\omega t}$$

The imaginary term means that p is out of phase with E, lagging by an angle of $\tan^{-1}[\gamma\omega/(\omega_0^2-\omega^2)]$

Resonance: The frequency $\omega = \omega_R$ that makes the amplitude maximum can be found by differentiating and equaling to 0, and it is:

$$\omega_R = \sqrt{\omega_0^2 - \gamma^2/2}$$

We usually express the quality of the damping with the **quality factor:**

$$Q := \frac{\omega_R}{\gamma}$$

For oscillators with a small damping, $Q >> 1$ and $\omega_R = \omega_0$ and we have:

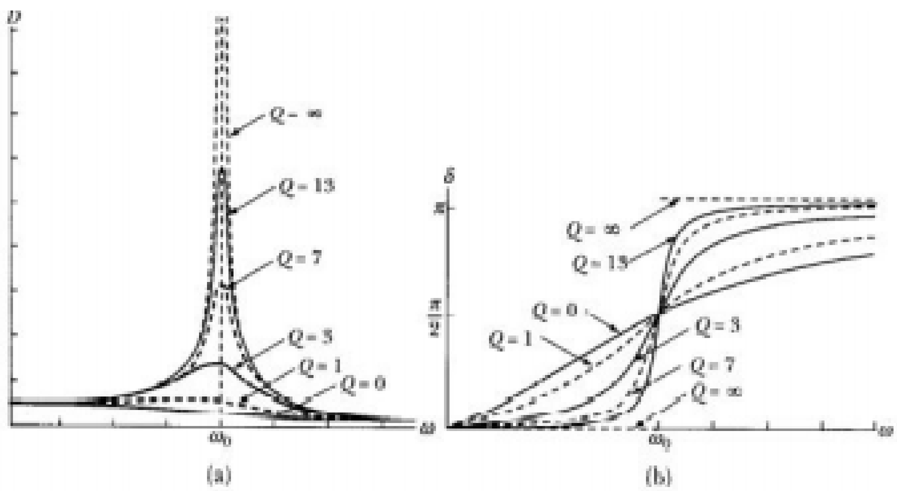

FIGURE 3-16 (a) The amplitude D is displayed as a function of the driving frequency ω for various values of the quality factor Q. Also shown is (b) the phase angle δ, which is the phase angle between the driving force and the resultant motion.

.

In general, differently situated electrons experience different natural frequencies and damping coefficients. Let's say there are f_j electrons with frequency ω_j and damping γ_j in each molecule. If there are N molecules per unit volume, the polarization vector $\vec{P}$ is given by the real part of $\tilde{\vec{P}} = N g_j \tilde{\vec{p}}$

$$\tilde{\vec{P}} = \frac{Nq^2}{m}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)\tilde{\vec{E}}$$

Now, the **electric susceptibility** is given by $\vec{P} = \epsilon_0\chi_e\vec{E}$.
In this case, $\vec{P}$ is not proportional to $\vec{E}$ because of the difference in phase (it is not a linear

medium).
However, the complex polarization $\tilde{\vec{P}}$ is proportional to the complex field $\tilde{\vec{E}}$ and we introduce a **Complex susceptibility** $\tilde{\chi}_e$:

$$\tilde{\vec{P}} = \epsilon_0 \tilde{\chi}_e \tilde{\vec{E}}$$

And, as before, the proportionality between $\tilde{\vec{D}}$ and $\tilde{\vec{E}}$ is the **complex permittivity** $\tilde{\epsilon} = \epsilon_0(1 + \tilde{\chi}_e)$
And the **Complex dielectric constant** is $\tilde{\epsilon}_r = 1 + \tilde{\chi}_e$

- **Complex Susceptibility:** $\tilde{\chi}_e = \dfrac{Nq^2}{m\epsilon_0}\left(\sum_j \dfrac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)$
- **Complex dielectric constant:** $\tilde{\epsilon}_r = 1 + \tilde{\chi}_e = 1 + \dfrac{Nq^2}{m\epsilon_0}\left(\sum_j \dfrac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)$
- **Complex permitivitty:** $\tilde{\epsilon} = \epsilon_0(\epsilon_0 + \tilde{\chi}_e) = 1 + \tilde{\chi}_e = \epsilon_0 + \dfrac{Nq^2}{m}\left(\sum_j \dfrac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)$

Ordinarily, the imaginary term is negligible, however, when ω is very close to a resonant frequency ω_j, it plays an important role.

In a dispersive medium, the wave equation for a given frequency is:

$$\nabla^2 \tilde{\vec{E}} = \tilde{\epsilon_0}\mu_0 \frac{\partial^2 \tilde{\vec{E}}}{\partial t^2}$$

It admits plane wave solutions as before:

$$\tilde{\vec{E}} = \tilde{\vec{E}_0} e^{i(\tilde{k}z - \omega t)}$$
$$\tilde{k} = \sqrt{\tilde{\epsilon}\mu_0}\omega$$

We can write $\tilde{k}$ as $\tilde{k} = k_1 + ik_2$, so, the solution is:

$$\tilde{\vec{E}}(z,t) = \tilde{E}_0 e^{-k_2 z} e^{i(k_1 z - \omega t)}$$

So, the wave is attenuated by the factor k_2. Because the intensity is proportional to $E^2(e^{-2k_2 z})$, we define:

$$\alpha := 2k_2$$

As the **absorption coefficient**
Meanwhile, the velocity is ω/k_1, and the index of refraction is:

$$n = \frac{ck_1}{\omega}$$

.

For gasses, the second term in $\tilde{\epsilon}_r = 1 + \dfrac{Nq^2}{m\epsilon_0}\sum_j \dfrac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}$.

So we can approximate $\tilde{k}$ as:

$$\tilde{k} = \frac{\omega}{c}\sqrt{\tilde{\epsilon}_r} \simeq \frac{\omega}{c}\left[1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right]$$

Therefore:

$$n = \frac{ck}{\omega} \simeq 1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j(\omega_j^2 - \omega_0^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}$$

And:

$$\alpha = 2k_1 \simeq \frac{Nq^2\omega^2}{m\epsilon_0 c}\sum_j \frac{f_j\gamma_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}$$

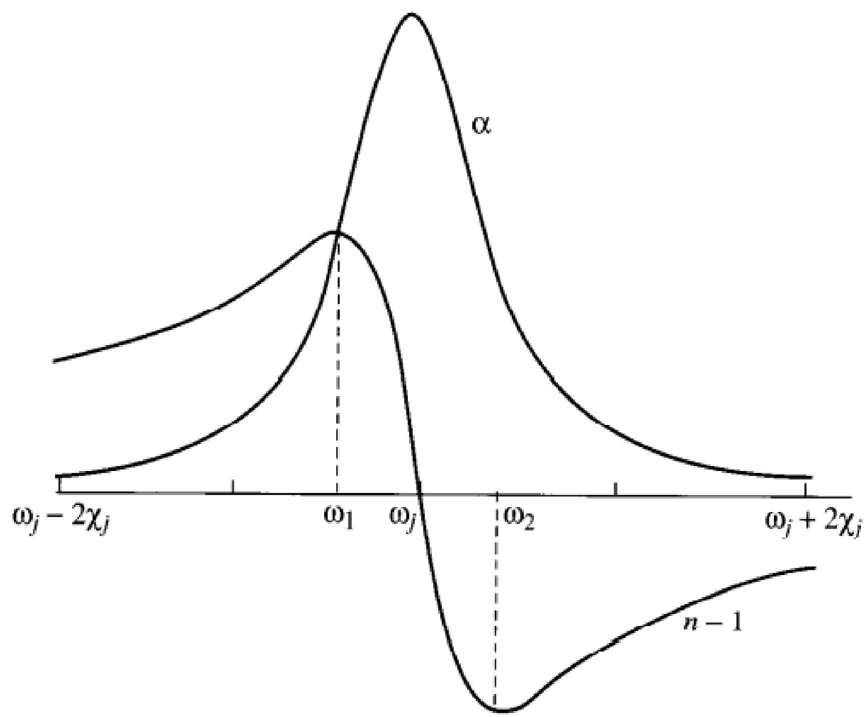

Figure 9.22

Here we have plotted the index of refraction and the absorption coefficient in the vicinity of one resonance.
Most of the time, the index rises gradually with increasing frequency.

However, in the immediate neighborhood of a resonance, the index drops sharply in what is called **anomalous dispersion**.
This part coincides with the part of maximum absorption, so actually, the material might be almost opaque here.
The reason is that we are close to one of the ' favorite ' frequencies of the electrons, so their amplitude of oscillation is relatively large and that dissipates energy.
Sometimes $n < 1$, suggesting that the speed is higher than c, but this is no cause for alarm, since energy travels at the group velocity.

If we stay away from the resonances, the damping can be ignored, and the formula is:

$$n = 1 + \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2}$$

Most substances have resonances scattered around the spectrum. But for transparent materials, the nearest significant resonances are typically uv, so that $\omega < \omega_j$. In that case:

$$\frac{1}{\omega_j^2 - \omega^2} = \frac{1}{\omega_j^2}\left(1 - \frac{\omega^2}{\omega_j^2}\right)^{-1} \simeq \frac{1}{\omega_j^2}\left(1 + \frac{\omega^2}{\omega_j^2}\right)$$

And then, the n is:

$$n = 1 + \left(\frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2}\right) + \omega^2 \left(\frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^4}\right)$$

Or, in terms of the wavelength in vacuum ($\lambda = 2\pi c/\omega$):

$$n = 1 + A\left(1 + \frac{B}{\lambda^2}\right)$$

Where A is the **coefficient of refraction** and B the **coefficient of dispersion**

.

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Guided Waves

Wave Guides

We consider EM waves confined to the interior of a hollow pipe, or **wave guide**
We will assume that the wave guide is a perfect conductor, so that $\vec{E} = 0 = \vec{B} = 0$ inside the material itself and hence, the boundary conditions at the inner wall are:

- $\vec{E}^{\parallel} = 0$
- $B^{\perp} = 0$

Free charges and currents will be induced on the surface in such a way as to enforce these constraints. We are interested in monochromatic waves that propagate down the tube, so $\vec{E}, \vec{B}$ have the form:

$$\tilde{\vec{E}}(x, y, z, t) = \tilde{\vec{E}}_0(x, y)e^{i(kz-\omega t)}$$
$$\tilde{\vec{B}}(x, y, z, t) = \tilde{\vec{B}}_0(x, y)e^{i(kz-\omega t)}$$

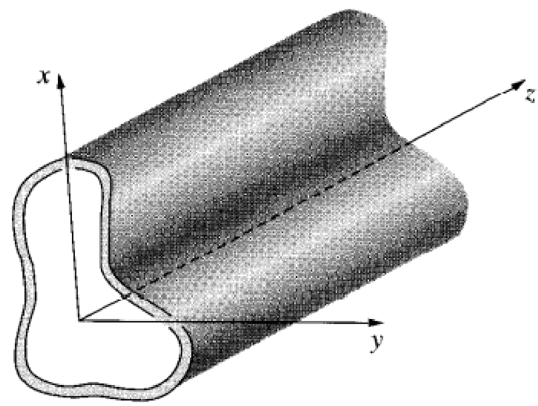


Figure 9.23

The electric and magnetic fields must satisfy Maxwell:

- $\nabla \cdot \vec{E} = 0$
- $\nabla \cdot \vec{B} = 0$
- $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$
- $\nabla \times \vec{B} = \dfrac{1}{c^2}\dfrac{\partial \vec{E}}{\partial t}$

As we shall see, confined waves are not (in general) transverse; in order to fulfill the boundary conditions, we shall have to include longitudinal components:

$$\tilde{\vec{E}}_0 = E_x\widehat{x} + E_y\widehat{y} + E_z\widehat{z}$$
$$\tilde{\vec{B}}_0 = B_x\widehat{x} + B_y\widehat{y} + B_z\widehat{z}$$

Where each component is a function of x, y.
Putting this into Maxwell's equations iii) and iv), we obtain:

$$
\left.\begin{array}{llll}
\text{(i)} & \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = i\omega B_z, & \text{(iv)} & \dfrac{\partial B_y}{\partial x} - \dfrac{\partial B_x}{\partial y} = -\dfrac{i\omega}{c^2} E_z, \\
\text{(ii)} & \dfrac{\partial E_z}{\partial y} - ikE_y = i\omega B_x, & \text{(v)} & \dfrac{\partial B_z}{\partial y} - ikB_y = -\dfrac{i\omega}{c^2} E_x, \\
\text{(iii)} & ikE_x - \dfrac{\partial E_z}{\partial x} = i\omega B_y, & \text{(vi)} & ikB_x - \dfrac{\partial B_z}{\partial x} = -\dfrac{i\omega}{c^2} E_y.
\end{array}\right\} \qquad (9.179)
$$

$$
\left.\begin{array}{ll}
\text{(i)} & E_x = \dfrac{i}{(\omega/c)^2 - k^2}\left(k\dfrac{\partial E_z}{\partial x} + \omega\dfrac{\partial B_z}{\partial y}\right), \\
\text{(ii)} & E_y = \dfrac{i}{(\omega/c)^2 - k^2}\left(k\dfrac{\partial E_z}{\partial y} - \omega\dfrac{\partial B_z}{\partial x}\right), \\
\text{(iii)} & B_x = \dfrac{i}{(\omega/c)^2 - k^2}\left(k\dfrac{\partial B_z}{\partial x} - \dfrac{\omega}{c^2}\dfrac{\partial E_z}{\partial y}\right), \\
\text{(iv)} & B_y = \dfrac{i}{(\omega/c)^2 - k^2}\left(k\dfrac{\partial B_z}{\partial y} + \dfrac{\omega}{c^2}\dfrac{\partial E_z}{\partial x}\right).
\end{array}\right\} \qquad (9.180)
$$

So, it suffices to determine E_z, B_z to get the others.
We now insert these equations into the Maxwell equations, to find the differential equations for the Z components.

$$
\left.\begin{array}{ll}
\text{(i)} & \left[\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right] E_z = 0, \\
\text{(ii)} & \left[\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right] B_z = 0.
\end{array}\right\} \qquad (9.181)
$$

If $E_z = 0$, we call these **TE (Transverse electric) waves**;
If $B_z = 0$, they are called **TM (transverse magnetic) waves**.
If both $E_z = B_z = -$, they are **TEM waves**

Theorem: In a hollow wave guide, TEM waves cannot occur.

Proof: If $E_z = 0$, Gauss's law (Eq. 9.177i) says

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0,$$

and if $B_z = 0$, Faraday's law (Eq. 9.177iii) says

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0.$$

Indeed, the vector $\tilde{\mathbf{E}}_0$ in Eq. 9.178 has zero divergence and zero curl. It can therefore be written as the gradient of a scalar potential that satisfies Laplace's equation. But the boundary condition on $\mathbf{E}$ (Eq. 9.175) requires that the surface be an equipotential, and since Laplace's equation admits no local maxima or minima (Sect. 3.1.4), this means that the potential is constant throughout, and hence the electric field is *zero*—no wave at all. qed

This is valid only for a completely empty pipe.

TE waves in a rectangular Wave Guide

Suppose we have a wave guide of rectangular shape, with height a and width b, and we are interested in the Propagation of TE waves. We solve the equation for B_z:

$$B_z(x, y) = X(x)Y(y)$$

So that:

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} + [(\omega/c)^2 - k^2]XY = 0$$

We separate into:

- $\dfrac{1}{X}\dfrac{d^2X}{dx^2} = -k_x^2$
- $\dfrac{1}{Y}\dfrac{d^2Y}{dy^2} = -k_y^2$

With $-k_x^2 - k_y^2 + (\omega/c)^2 - k^2 = 0$

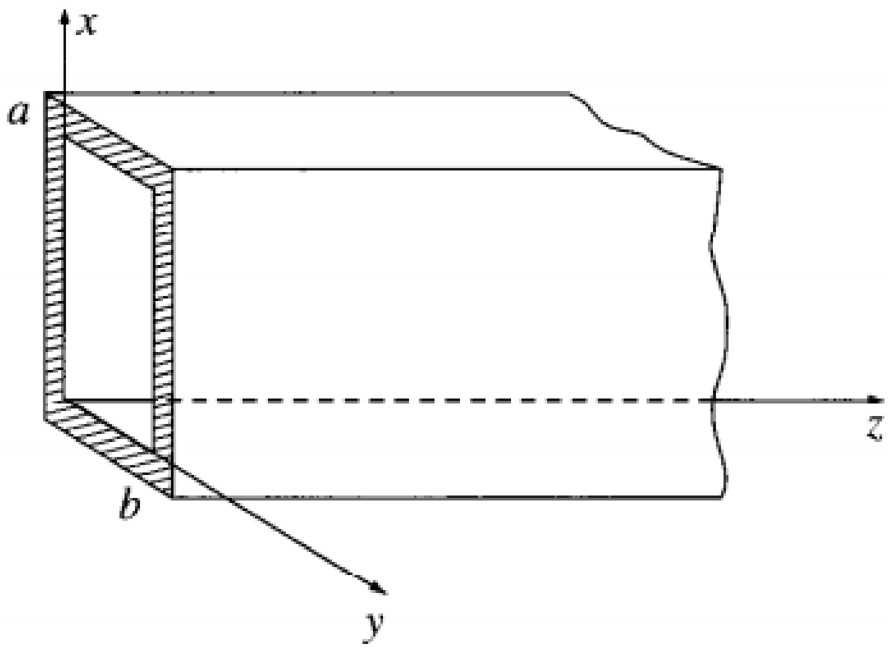


Figure 9.24

The general solution is $X(x) = A\sin(k_x x) + B\cos(k_x x)$
But the boundary conditions require that B_x (and hence, dX/x by the relation of B_z to B_x) vanishes at $x = 0, x = a$. So $A = 0$ and $k_x = m\pi/a$

Same goes for Y, so that $k_y = n\pi/b$

And we conclude that:

$$B_z = B_0 \cos(m\pi x/a)\cos(n\pi y/b)$$

This is called the TE_{mn} mode.
The wave number is obtained as:

$$k = \sqrt{(\omega/c)^2 - \pi^2[(m/a)^2 + (n/b)^2]}$$

If:

$$\omega < c\pi\sqrt{(m/a)^2 + (n/b)^2} := \omega_{mn}$$

ω_{mn} is called the **cutoff frequency** for the mode in question. The lowest cutoff freq for a given wave guide occurs for TE_{10}:

$$\omega_{10} = c\pi/a$$

The wave number can be written more simply in terms of the cutoff freq:

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}$$

The wave velocity is:

$$v = \frac{\omega}{k} = \frac{c}{\sqrt{1-(\omega_{mn}/\omega)^2}}$$

However, the energy carried by the wave travels at the group velocity:

$$v_g = \frac{1}{dk/d\omega} = c\sqrt{1-(\omega_{mn}/\omega)^2} < c$$

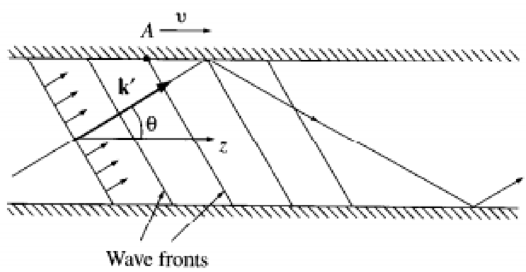


Figure 9.25

There's another way to visualize the propagation of an electromagnetic wave in a rectangular pipe, and it serves to illuminate many of these results. Consider an ordinary *plane* wave, traveling at an angle θ to the z axis, and reflecting perfectly off each conducting surface (Fig. 9.25). In the x and y directions the (multiply reflected) waves interfere to form standing wave patterns, of wavelength $\lambda_x = 2a/m$ and $\lambda_y = 2b/n$ (hence wave number $k_x = 2\pi/\lambda_x = \pi m/a$ and $k_y = \pi n/b$), respectively. Meanwhile, in the z direction there remains a traveling wave, with wave number $k_z = k$. The propagation vector for the "original" plane wave is therefore

$$\mathbf{k}' = \frac{\pi m}{a}\,\hat{\mathbf{x}} + \frac{\pi n}{b}\,\hat{\mathbf{y}} + k\,\hat{\mathbf{z}},$$

and the frequency is

$$\omega = c|\mathbf{k}'| = c\sqrt{k^2 + \pi^2[(m/a)^2 + (n/b)^2]} = \sqrt{(ck)^2 + (\omega_{mn})^2}.$$

Only certain angles will lead to one of the allowed standing wave patterns:

$$\cos\theta = \frac{k}{|\mathbf{k}'|} = \sqrt{1-(\omega_{mn}/\omega)^2}.$$

The plane wave travels at speed c, but because it is going at an angle θ to the z axis, its net velocity down the wave guide is

$$v_g = c\cos\theta = c\sqrt{1-(\omega_{mn}/\omega)^2}.$$

The *wave* velocity, on the other hand, is the speed of the wave fronts (A, say, in Fig. 9.25) down the pipe. Like the intersection of a line of breakers with the beach, they can move much faster than the waves themselves—in fact

$$v = \frac{c}{\cos\theta} = \frac{c}{\sqrt{1-(\omega_{mn}/\omega)^2)}}.$$

The Potential Formulation

Scalar And vector Potentials

We seek general solutions to Maxwells equations:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Given $\rho(\vec{r}, t)$ and $\vec{J}(\vec{r}, t)$, what are the fields?
In the static case, the solution is given by Coulomb's law and the Biot Savart law. We are looking for a generalization.

The problem is that we cannot write $\vec{E} = -\nabla V$ because it is not curless anymore. Neverthe-less, $\vec{B}$ is divergengenceless, so we may write:

$$\vec{B} = \nabla \times \vec{A}$$

Then, Faraday's law yields:

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{A})$$
$$\Rightarrow \quad \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

So this quantity is really divergenceless. Therefore, we can write it as a gradient of a scalar quantity V as:

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V$$

Therefore:

$$\boxed{\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}}$$

This reduces to the old form when A is constant.

This representation automatically satisfies Maxwell equations ii) and iii).
When we put it into equation i) we find that:

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = -\frac{1}{\epsilon_0}\rho$$

This replaces **Poisson's** equation (to which it reduces in the static case).
Putting it into Maxwll iv) we find that:

$$\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J} - \mu_0 \epsilon_0 \nabla \left(\frac{\partial V}{\partial t} \right) - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$$

Or, using BAC CAB, we get:

$$\left(\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} \right) - \nabla \left(\nabla \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} \right) = -\mu_0 \vec{J}$$

These equations are equivalent to **Maxwell**:

$$\boxed{\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = -\frac{1}{\epsilon_0}\rho}$$

$$\boxed{\left(\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} \right) - \nabla \left(\nabla \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} \right) = -\mu_0 \vec{J}}$$

.
Then we can get $\vec{E}$ and $\vec{B}$ with:

$$\boxed{\vec{B} = \nabla \times \vec{A}}$$

$$\boxed{\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}}$$

Example: Find the charge and current distributions that would give rise to the potentials:

$$V = 0$$

$$\vec{A} = \begin{cases} \frac{\mu_0 k}{4c}(ct - |x|)^2 \hat{z} \quad , \quad |x| < ct \\ 0 \quad , \quad |x| > ct \end{cases}$$

Where k is a constant and $c = 1/\sqrt{\mu_0 \epsilon_0}$

- **Solution:** First we'll determine the electric and magnetic fields:

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 k}{2}(ct - |x|)\hat{z}$$

$$\vec{B} = \nabla \times \vec{A} = \pm \frac{\mu_0 k}{2c}(ct - |x|)\hat{y}$$

For $|x| < ct$. If $|x| > ct$ then $\vec{E} = \vec{B} = 0$
We can calculate every derivative:

- $\nabla \cdot \vec{E} = 0$
- $\nabla \cdot \vec{B} = 0$
- $\nabla \times \vec{E} = \mp \frac{\mu_0 k}{2} \widehat{y}$
- $\nabla \times \vec{B} = -\frac{\mu_0 k}{2c} \widehat{z}$
- $\frac{\partial \vec{E}}{\partial t} = -\frac{\mu_0 k c}{2} \widehat{z}$
- $\frac{\partial \vec{B}}{\partial t} = \pm \frac{\mu_0 k}{2} \widehat{y}$

We can check that all Maxwell equations are satisfied.
We notice that $\vec{B}$ has a discontinuity at $x = 0$, which implies the presence of $\vec{K}$ in the yz plane.

Gauge Transformations

We are free to impose extra restrictions on V and $\vec{A}$ as long as nothing happens to $\vec{E}, \vec{B}$
This is called the **gauge freedom**.
Suppose we have to sets of potentials $(V, \vec{A})$ and $(V', \vec{A}')$ which correspond to the same fields, how can they differ?
We write:

$$\vec{A}' = \vec{A} + \vec{\alpha} \quad , \quad V' = V + \beta$$

Since the two $\vec{A}$ give the same $\vec{B} = \nabla \times \vec{A}$, we must have:

$$\nabla \times \vec{\alpha} = 0$$

Which implies that:

$$\vec{\alpha} = \nabla \lambda$$

On the other hand, both potentials give the same $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$ which must mean that:

$$\nabla \beta + \frac{\partial \vec{\alpha}}{\partial t} = 0$$

$$\Rightarrow \quad \nabla \left(\beta + \frac{\partial \lambda}{\partial t} \right) = 0$$

So, the term in parenthesis is independent of position, call it $k(t)$:

$$\beta = -\frac{\partial \lambda}{\partial t} + k(t)$$

Actually, we might as well absorb $k(t)$ into λ, defining a new λ by adding $\int_0^t k(t')dt'$ to the old one. This will not affect the gradient of λ, it will only add $k(t)$ to $\partial\lambda/\partial t$. It follows that:

$$\vec{A'} = \vec{A} + \nabla\lambda$$
$$V' = V - \frac{\partial\lambda}{\partial t}$$

Conclusion: For any old scalar function λ, we can with impunity add $\nabla\lambda$ to $\vec{A}$, provided we simultaneously substract $\partial\lambda/\partial t$ from V.
None of this will affect the physical quantities $\vec{E}$ and $\vec{B}$.
Such changes in $V, \vec{A}$ are called **Gauge transformations**.
We can use them to simplify some problems depending on how we choose them.

Coulumb Gauge

As in magnetostatics, we use our liberty to choose that:

$$\nabla \cdot \vec{A} = 0$$

With this, the Potential Maxwell equation 1 turns into:

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho$$

so that:

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r'}, t)}{\mathfrak{r}} d\tau'$$

However, V by itself doesn't tell us $\vec{E}$, we need $\vec{A}$ as well.

There is a peculiar thing about the scalar potential in the Coulomb gauge: it is determined by the distribution of charge *right now*. If I move an electron in my laboratory, the potential V on the moon immediately records this change. That sounds particularly odd in the light of special relativity, which allows no message to travel faster than the speed of light. The point is that V *by itself* is not a physically measurable quantity—all the man in the moon can measure is **E**, and that involves **A** as well. Somehow it is built into the vector potential, in the Coulomb gauge, that whereas V instantaneously reflects all changes in ρ, the combination $-\nabla V - (\partial\mathbf{A}/\partial t)$ does *not*; **E** will change only after sufficient time has elapsed for the "news" to arrive.[1]

The *advantage* of the Coulomb gauge is that the *scalar* potential is particularly simple to calculate; the *disadvantage* (apart from the acausal appearance of V) is that **A** is particularly *difficult* to calculate. The differential equation for **A** (10.5) in the Coulomb gauge reads

$$\nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} + \mu_0\epsilon_0\nabla\left(\frac{\partial V}{\partial t}\right). \quad (10.11)$$

Lorentz Gauge

In the Lorentz Gauge, we pick:

$$\boxed{\nabla \cdot \vec{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t}}$$

This is designed to eliminate the middle term in the second potential Maxwell equation. With this, the second Maxwell potential equation is:

$$\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

Meanwhile, the differential equation for V, becomes:

$$\nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0} \rho$$

So, it is very symmertrical. If we define the **d' Alambertian** as:

$$\boxed{\Box^2 = \nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}}$$

Then, the equations are:

$$\boxed{\Box^2 V = -\frac{\rho}{\epsilon_0}}$$

$$\boxed{\Box^2 \vec{A} = -\mu_0 \vec{J}}$$

They are some kind of 4-d Poisson equation, or an **Inhomogeneous wave equation** with a source term.

We will only use this Gauge and solve different types of problems.

Continuous Distributions

Retarded Potentials

In the static case, the d'alembertian equations reduce to Poisson equations:

$$\nabla^2 V = -\frac{1}{\epsilon_0} \rho$$

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

With the familiar solutions:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r'})}{\mathfrak{r}}d\tau'$$
$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r'})}{\mathfrak{r}}d\tau'$$

Where $\mathfrak{r} = |\vec{r} - \vec{r'}|$
Now, electromagnetic 'news' travel at the speed of light. Therefore, in the nonstatic case, the status of the source right now doesn't matter, what matters is the condition it had some earlier time t_r (called the **retarded time**) when the 'message' left. Since the message must travel a distance $\mathfrak{r}$, the delay is $\mathfrak{r}/c$:

$$\boxed{t_r := t - \frac{\mathfrak{r}}{c}}$$

So, the natural generalization for nonstatic sources is:

$$\boxed{\vec{V}(\vec{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r'}, t_r)}{\mathfrak{r}}d\tau'}$$
$$\boxed{\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r'}, t_r)}{\mathfrak{r}}d\tau'}$$

Where $\rho(\vec{r'}, t_r)$ is the charge density at $\vec{r'}$ at the retarded time t_r. These are called **retarded potentials**

We need to prove this formulas obey the d'ambertianian equations and the Lorentz condition for the gauge. The reasoning since plausible, but that is not enough. For example, if we took the same reasoning for $\vec{E}$, $\vec{B}$ using their normal formulas but with a retarded time, the answer would be **wrong**.

So let's prove these equations for the potentials:

- **Lorentz Gauge:** $\nabla \cdot \vec{A} = -\mu_0\epsilon_0\dfrac{\partial V}{\partial t}$

Problem 10.8

¿From the product rule:

$$\nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)=\frac{1}{\mathscr{r}}(\nabla\cdot\mathbf{J})+\mathbf{J}\cdot\left(\nabla\frac{1}{\mathscr{r}}\right),\quad \nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)=\frac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{J})+\mathbf{J}\cdot\left(\nabla'\frac{1}{\mathscr{r}}\right).$$

But $\nabla\frac{1}{\mathscr{r}}=-\nabla'\frac{1}{\mathscr{r}}$, since $\boldsymbol{\mathscr{r}}=\mathbf{r}-\mathbf{r}'$. So

$$\nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)=\frac{1}{\mathscr{r}}(\nabla\cdot\mathbf{J})-\mathbf{J}\cdot\left(\nabla'\frac{1}{\mathscr{r}}\right)=\frac{1}{\mathscr{r}}(\nabla\cdot\mathbf{J})+\frac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{J})-\nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right).$$

But

$$\nabla\cdot\mathbf{J}=\frac{\partial J_x}{\partial x}+\frac{\partial J_y}{\partial y}+\frac{\partial J_z}{\partial z}=\frac{\partial J_x}{\partial t_r}\frac{\partial t_r}{\partial x}+\frac{\partial J_y}{\partial t_r}\frac{\partial t_r}{\partial y}+\frac{\partial J_z}{\partial t_r}\frac{\partial t_r}{\partial z},$$

and

$$\frac{\partial t_r}{\partial x}=-\frac{1}{c}\frac{\partial\mathscr{r}}{\partial x},\quad \frac{\partial t_r}{\partial y}=-\frac{1}{c}\frac{\partial\mathscr{r}}{\partial y},\quad \frac{\partial t_r}{\partial z}=-\frac{1}{c}\frac{\partial\mathscr{r}}{\partial z},$$

so

$$\nabla\cdot\mathbf{J}=-\frac{1}{c}\left[\frac{\partial J_x}{\partial t_r}\frac{\partial\mathscr{r}}{\partial x}+\frac{\partial J_y}{\partial t_r}\frac{\partial\mathscr{r}}{\partial y}+\frac{\partial J_z}{\partial t_r}\frac{\partial\mathscr{r}}{\partial z}\right]=-\frac{1}{c}\frac{\partial\mathbf{J}}{\partial t_r}\cdot(\nabla\mathscr{r}).$$

Similarly,

$$\nabla'\cdot\mathbf{J}=-\frac{\partial\rho}{\partial t}-\frac{1}{c}\frac{\partial\mathbf{J}}{\partial t_r}\cdot(\nabla'\mathscr{r}).$$

[The first term arises when we differentiate with respect to the *explicit* $\mathbf{r}'$, and use the continuity equation.] thus

$$\nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)=\frac{1}{\mathscr{r}}\left[-\frac{1}{c}\frac{\partial\mathbf{J}}{\partial t_r}\cdot(\nabla'\mathscr{r})\right]+\frac{1}{\mathscr{r}}\left[-\frac{\partial\rho}{\partial t}-\frac{1}{c}\frac{\partial\mathbf{J}}{\partial t_r}\cdot(\nabla'\mathscr{r})\right]-\nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)=-\frac{1}{\mathscr{r}}\frac{\partial\rho}{\partial t}-\nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)$$

(the other two terms cancel, since $\nabla\mathscr{r}=-\nabla'\mathscr{r}$). Therefore:

$$\nabla\cdot\mathbf{A}=\frac{\mu_0}{4\pi}\left[-\frac{\partial}{\partial t}\int\frac{\rho}{\mathscr{r}}d\tau-\int\nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)d\tau\right]=-\mu_0\epsilon_0\frac{\partial}{\partial t}\left[\frac{1}{4\pi\epsilon_0}\int\frac{\rho}{\mathscr{r}}d\tau\right]-\frac{\mu_0}{4\pi}\oint\frac{\mathbf{J}}{\mathscr{r}}\cdot d\mathbf{a}.$$

The last term is over the suface at "infinity", where $\mathbf{J}=0$, so it's zero. Therefore $\nabla\cdot\mathbf{A}=-\mu_0\epsilon_0\frac{\partial V}{\partial t}$. ✓

- **First D'Alambertian** $\Box^2 V=-\dfrac{\rho}{\epsilon_0}$

In calculating the Laplacian of $V(\mathbf{r}, t)$, the crucial point to notice is that the integrand (in Eq. 10.19) depends on $\mathbf{r}$ in *two* places: *explicitly*, in the denominator ($\mathscr{r} = |\mathbf{r} - \mathbf{r}'|$), and *implicitly*, through $t_r = t - \mathscr{r}/c$, in the numerator. Thus

$$\nabla V = \frac{1}{4\pi\epsilon_0}\int\left[(\nabla\rho)\frac{1}{\mathscr{r}} + \rho\nabla\left(\frac{1}{\mathscr{r}}\right)\right]d\tau', \tag{10.20}$$

and

$$\nabla\rho = \dot{\rho}\nabla t_r = -\frac{1}{c}\dot{\rho}\nabla\mathscr{r} \tag{10.21}$$

(the dot denotes differentiation with respect to time).[3] Now $\nabla\mathscr{r} = \hat{\boldsymbol{\mathscr{r}}}$ and $\nabla(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$ (Prob. 1.13), so

$$\nabla V = \frac{1}{4\pi\epsilon_0}\int\left[-\frac{\dot{\rho}}{c}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}} - \rho\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right]d\tau'. \tag{10.22}$$

Taking the divergence,

$$\nabla^2 V = \frac{1}{4\pi\epsilon_0}\int\left\{-\frac{1}{c}\left[\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}}\cdot(\nabla\dot{\rho}) + \dot{\rho}\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}}\right)\right] - \left[\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\cdot(\nabla\rho) + \rho\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)\right]\right\}d\tau'.$$

But

$$\nabla\dot{\rho} = -\frac{1}{c}\ddot{\rho}\nabla\mathscr{r} = -\frac{1}{c}\ddot{\rho}\,\hat{\boldsymbol{\mathscr{r}}},$$

as in Eq. 10.21, and

$$\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}^2}$$

(Prob. 1.62), whereas

$$\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

(Eq. 1.100). So

$$\nabla^2 V = \frac{1}{4\pi\epsilon_0}\int\left[\frac{1}{c^2}\frac{\ddot{\rho}}{\mathscr{r}} - 4\pi\rho\delta^3(\boldsymbol{\mathscr{r}})\right]d\tau' = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \frac{1}{\epsilon_0}\rho(\mathbf{r}, t),$$

confirming that the retarded potential (10.19) satisfies the inhomogeneous wave equation (10.16). qed

[2] I'll give you the straightforward but cumbersome proof; for a clever indirect argument see M. A. Heald and J. B. Marion, *Classical Electromagnetic Radiation*, 3d ed., Sect. 8.1 (Orlando, FL: Saunders (1995)).

[3] Note that $\partial/\partial t_r = \partial/\partial t$, since $t_r = t - \mathscr{r}/c$ and $\mathscr{r}$ is independent of t.

- similar proof for $\Box^2\vec{A} = -\mu_0\vec{J}$

.

Example 10.2: An infinite straight wire carries the current:

$$I(t) = \begin{cases} 0 & , \; t \le 0 \\ I_0 & , t > 0 \end{cases}$$

That is, a constant current I_0 that is turned on abruptly.

- **Solution:** The wire is presumably electrically neutral, so the scalar potential is zero. Let the wire lie along the z-axis. The retarded vector potential at point P is:

$$\vec{A}(s,t) = \frac{\mu_0}{4\pi}\hat{z}\int_{-\infty}^{\infty}\frac{I(t_r)}{\mathfrak{r}}dz$$

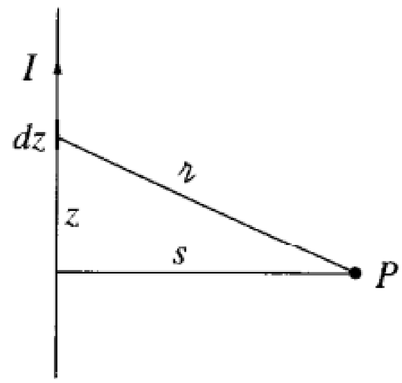


Figure 10.4

For $t < s/c$, the 'news' has not yet reached P, and the potential is zero. For $t > s/c$, only the segment with $|z| \leq \sqrt{(ct)^2 - s^2}$
Contributes (outside this range, t_r is negative, so $I(t_r) = 0$);

$$\begin{aligned}
\vec{A}(s,t) &= \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r'},t_r)}{\mathfrak{r}}d\tau' \\
&= \frac{\mu_0}{4\pi}\hat{z}\int_{-\infty}^{\infty}\frac{I(t_r)}{\mathfrak{r}}dz \\
&= \frac{\mu_0}{4\pi}\hat{z}\int_{-\infty}^{\infty}\frac{I(t-\frac{\mathfrak{r}}{c})}{\mathfrak{r}}dz \\
&= \frac{\mu_0}{4\pi}\hat{z}\int_{-\infty}^{\infty}\frac{I(t-\frac{\sqrt{s^2+z^2}}{c})}{\sqrt{s^2+z^2}}dz \\
&= \frac{\mu_0}{4\pi}\hat{z}\int_{-\sqrt{(ct)^2-s^2}}^{\sqrt{(ct)^2-s^2}}\frac{I_0 dz}{\sqrt{s^2+z^2}}
\end{aligned}$$

This is because, $I = I_0 \neq 0$ only when its argument is greater than 0, so $t > \dfrac{\sqrt{s^2+z^2}}{c}$ and that's how we get the boundaries.

Then, we solve the integral to get:

$$\vec{A}(s,t) = \frac{\mu_0 I_0}{2\pi} \log\left(\frac{ct + \sqrt{(ct)^2 - s^2}}{s}\right)\hat{z}$$

Now we can calculate the electric and magnetic fields:

$$\begin{aligned}\vec{E}(s,t) &= -\nabla V - \frac{\partial \vec{A}}{\partial t} \\ &= -\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 I_0 c}{2\pi\sqrt{(ct)^2 - s^2}}\hat{z}\end{aligned}$$

And the magnetic field is:

$$\begin{aligned}\vec{B}(s,t) &= \nabla \times \vec{A} \\ &= -\frac{\partial A_z}{\partial s}\hat{\phi} \\ &= \frac{\mu_0 I_0}{2\pi s}\frac{ct}{\sqrt{(ct)^2 - s^2}}\hat{\phi}\end{aligned}$$

Notice that as $t \to \infty$, we recover the static case: $\vec{E} = 0$, $\vec{B} = (\mu_0 I_0 / 2\pi s)\hat{\phi}$

.

Jefimenko's Equations

Given the retarded potentials:

$$\begin{aligned}V(\vec{r},t) &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r'}, t_r)}{\mathfrak{r}}d\tau' \\ \vec{A}(\vec{r},t) &= \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r'}, t_r)}{\mathfrak{r}}d\tau'\end{aligned}$$

And it is in principle straightforward to determine the fields:

$$\begin{aligned}\vec{E} &= -\nabla V - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \nabla \times \vec{A}\end{aligned}$$

But it can be difficult, because the integrands depend on $\vec{r}$ both through $\mathfrak{r} = |\vec{r} - \vec{r'}|$ and through $t_r = t - \mathfrak{r}/c$.

We have already calculated the gradient of V in a proof before, and the time derivative of $\vec{A}$ is easy:

$$\nabla V = \frac{1}{4\pi\epsilon_0}\int\left[-\frac{\dot{\rho}}{c}\frac{\widehat{\mathfrak{r}}}{\mathfrak{r}} - \rho\frac{\widehat{\mathfrak{r}}}{\mathfrak{r}^2}\right]d\tau'$$

$$\frac{\partial\vec{A}}{\partial t} = \frac{\mu_0}{4\pi}\int\frac{\dot{\vec{J}}}{\mathfrak{r}}d\tau'$$

Putting them together, we get:

$$\boxed{\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(\vec{r'},t_r)}{\mathfrak{r}^2}\widehat{\mathfrak{r}} + \frac{\dot{\rho}(\vec{r'},t_r)}{c\mathfrak{r}}\widehat{\mathfrak{r}} - \frac{\dot{\vec{J}}(\vec{r'},t_r)}{c^2\mathfrak{r}}\right]d\tau'}$$

This is the **time dependent generalization** of Coulomb's law, to which it reduces in the static case.

For $\vec{B}$, we need to calculate the curl of $\vec{A}$:

As for $\mathbf{B}$, the curl of $\mathbf{A}$ contains two terms:

$$\nabla\times\mathbf{A} = \frac{\mu_0}{4\pi}\int\left[\frac{1}{\mathfrak{r}}(\nabla\times\mathbf{J}) - \mathbf{J}\times\nabla\left(\frac{1}{\mathfrak{r}}\right)\right]d\tau'.$$

Now

$$(\nabla\times\mathbf{J})_x = \frac{\partial J_z}{\partial y} - \frac{\partial J_y}{\partial z},$$

and

$$\frac{\partial J_z}{\partial y} = \dot{J}_z\frac{\partial t_r}{\partial y} = -\frac{1}{c}\dot{J}_z\frac{\partial\mathfrak{r}}{\partial y},$$

so

$$(\nabla\times\mathbf{J})_x = -\frac{1}{c}\left(\dot{J}_z\frac{\partial\mathfrak{r}}{\partial y} - \dot{J}_y\frac{\partial\mathfrak{r}}{\partial z}\right) = \frac{1}{c}\left[\dot{\mathbf{J}}\times(\nabla\mathfrak{r})\right]_x.$$

But $\nabla\mathfrak{r} = \hat{\boldsymbol{\mathfrak{r}}}$ (Prob. 1.13), so

$$\nabla\times\mathbf{J} = \frac{1}{c}\dot{\mathbf{J}}\times\hat{\boldsymbol{\mathfrak{r}}}. \tag{10.30}$$

Meanwhile $\nabla(1/\mathfrak{r}) = -\hat{\boldsymbol{\mathfrak{r}}}/\mathfrak{r}^2$ (again, Prob. 1.13), and hence

So:

$$\boxed{\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\int\left[\frac{\vec{J}(\vec{r'},t_r)}{\mathfrak{r}^2} + \frac{\dot{\vec{J}}(\vec{r'},t_r}{c\mathfrak{r}}\right]\times\widehat{\mathfrak{r}}d\tau'}$$

This is the time-dependent generalization of Biot Savart

These are the **Jefimenko's Equations**.
They are the (causal) solutions to Maxwell's equations.
They are not very practical, since it is often easier to calculate the potentials and then differentiate them. Still, they bring a closure to the theory, and we can see that the retarded fields aren't just the normal fields with $t \to t_r$ (as with the potentials)

Point Charges

Lienard Wiechert Potentials

We want the retarded potentials $V, \vec{A}$ of a point charge q moving in the trajectory:

$$\vec{w}(t) = \text{position of q at time t}$$

The retarded time is determined implicitly by the equation:

$$|\vec{r} - \vec{W}(t_r)| = c(t - t_r)$$

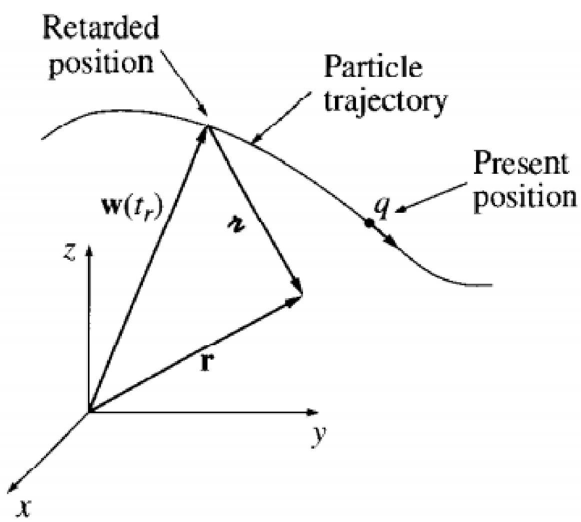
Where $\vec{r}$ is the point of measurement, and $\vec{w}(t_r)$ is the position that the particle has at time t_r. So that $|\vec{r} - \vec{w}(t_r)|$ is the distance between the particle at time t_r and the measuring point. We call $\vec{w}(t)$ the **retarded position** of the charge.

We also define:

$$\mathfrak{r} = \vec{r} - \vec{w}(t_r)$$

It is important to note that at most one point on the trajectory is 'in communication' with $\vec{r}$ at any particular time t. For suppose there are two such points, with retarded times t_1, t_2:

$$\mathfrak{r} = c(t - t_1)$$
$$\mathfrak{r} = c(t - t_1)$$



Then $\mathfrak{r}_1 - \mathfrak{r}_2 = c(t_2 - t_1)$, so the average velocity of the particle in the direction of $\vec{r}$ would have to be c -and that is not counting the velocity in other directions- So only one retarded point contributes to the potentials at any given moment.

Now, a naive reading of the formula $V(\vec{r}, t) = \dfrac{1}{4\pi\epsilon_0} \int \dfrac{\rho(\vec{r'}, t_r)}{\mathfrak{r}} d\tau'$

Might suggest to you that the retarded potential is simply $\dfrac{1}{4\pi\epsilon_0}\dfrac{q}{\mathfrak{r}}$

But this is wrong for a very subtle reason. It is true that for a point source, $\mathfrak{r}$ comes out of the integral for V, but what remains: $\int \rho(\vec{r'}, t_r)d\tau'$
Is not equal to the charge of the particle.
To calculate the total charge of a configuration, you need to integrate ρ at an instant, but the retardation $t_r = t - \mathfrak{r}/c$ obliges us to evaluate ρ at different times for different parts of the configuration, so we get a distorted picture of the total charge.

Theorem: For an extended particle (no matter how small), the retardation throws in a factor of $(1 - \hat{\mathfrak{r}} \cdot \vec{v}/c)^{-1}$ when calculating the integral of ρ. Where $\vec{v}$ is the velocity of the particle at the retarded time:

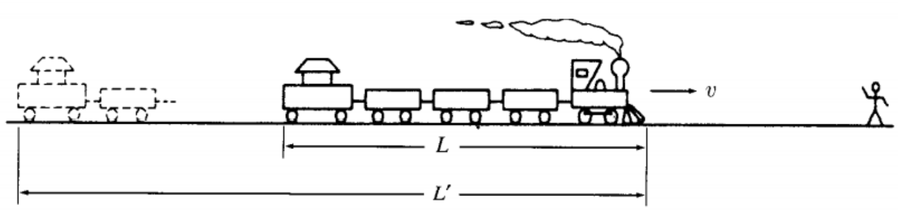
$$\int \rho(\vec{r'}, t_r)d\tau' = \frac{q}{1 - \hat{\mathfrak{r}} \cdot \vec{v}/c}$$

Proof: This is a purely *geometrical* effect, and it may help to tell the story in a less abstract context. You will not have noticed it, for obvious reasons, but the fact is that a train coming towards you looks a little longer than it really is, because the light you receive from the caboose left earlier than the light you receive simultaneously from the engine, and at that earlier time the train was farther away (Fig. 10.7). In the interval it takes light from the caboose to travel the extra distance L', the train itself moves a distance $L' - L$:

$$\frac{L'}{c} = \frac{L' - L}{v}, \quad \text{or} \quad L' = \frac{L}{1 - v/c}.$$

So approaching trains appear *longer*, by a factor $(1 - v/c)^{-1}$. By contrast, a train going *away* from you looks *shorter*,[8] by a factor $(1 + v/c)^{-1}$. In general, if the train's velocity makes an angle θ with your line of sight,[9] the extra distance light from the caboose must cover is $L' \cos\theta$ (Fig. 10.8). In the time $L'\cos\theta/c$, then, the train moves a distance $(L' - L)$:

$$\frac{L'\cos\theta}{c} = \frac{L' - L}{v}, \quad \text{or} \quad L' = \frac{L}{1 - v\cos\theta/c}.$$



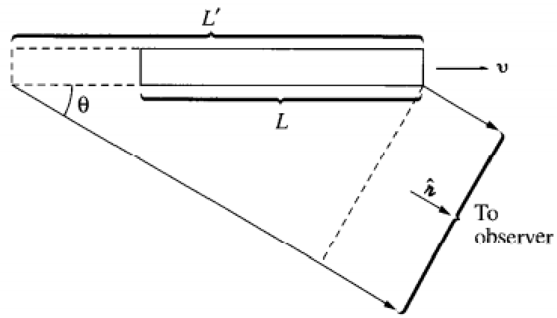


Figure 10.8

Notice that this effect does *not* distort the dimensions perpendicular to the motion (the height and width of the train). Never mind that the light from the far side is delayed in reaching you (relative to light from the near side)—since there's no *motion* in that direction, they'll still look the same distance apart. The apparent *volume* τ' of the train, then, is related to the *actual* volume τ by

$$\tau' = \frac{\tau}{1 - \hat{\mathfrak{r}} \cdot \mathbf{v}/c}, \tag{10.38}$$

where $\hat{\mathfrak{r}}$ is a unit vector from the train to the observer.

In case the connection between moving trains and retarded potentials escapes you, the point is this: Whenever you do an integral of the type 10.37, in which the integrand is evaluated at the retarded time, the effective volume is modified by the factor in Eq. 10.38, just as the apparent volume of the train was—and for the same reason. Because this correction factor makes no reference to the size of the particle, it is every bit as significant for a point charge as for an extended charge. qed

It follows then, that for a point charge, we have that:

$$\boxed{V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathfrak{r}c - \vec{\mathfrak{r}} \cdot \vec{v})}}$$

Where $\vec{v}$ is the velocity of the charge at the retarded time, $\vec{\mathfrak{r}}$ is the vector from the retarded position to the field point $\vec{r}$.

Meanwhile, we have that:

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\rho(\vec{r'}, t_r)\vec{v}(t_r)}{\mathfrak{r}} d\tau' = \frac{\mu_0}{4\pi} \frac{\vec{v}}{\mathfrak{r}} \int \rho(\vec{r'}, t_r) d\tau'$$

Therefore:

$$\boxed{\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{qc\vec{v}}{(\mathfrak{r}c - \vec{\mathfrak{r}} \cdot \vec{v})} = \frac{\vec{v}}{c^2} V(\vec{r}, t)}$$

These are the **Lienard Wiecheret Potentials.**

Example 10.3: Find the potentials of a point charge moving at constant velocity:

- **Solution:** For convenience, let's say that the particle passes through the origin at time 0, so that:

  $$\vec{w}(t) = \vec{v}t$$

  We need to compute the retarded time:

  $$|\vec{r} - \vec{v}t_r| = c(t - t_r)$$

  Or squaring, $r^2 - 2\vec{r}\cdot\vec{v}t_r + v^2t_r^2 = c^2(t^2 - 2tt_r + t_r^2)$
  Solving for t_r, we get that:

  $$t_r = \frac{(c^2t - \vec{r}\cdot\vec{v}) - \sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}{c^2 - v^2}$$

  The minus sign was fixed by considering the case for $v = 0$, for which we would get $t_r = t \pm \frac{r}{c}$ and clearly the correct sign would be the minus one.

  Now, from the definition of $\vec{\mathfrak{r}} = \vec{r} - \vec{w}(t_r)$ we get that:

  $$\mathfrak{r} = c(t - t_r)$$
  $$\hat{\mathfrak{r}} = \frac{\vec{r} - \vec{v}t_r}{c(t - t_r)}$$

  So that:

  so

  $$\begin{aligned} \mathfrak{r}(1 - \hat{\mathfrak{r}}\cdot\mathbf{v}/c) &= c(t - t_r)\left[1 - \frac{\mathbf{v}}{c}\cdot\frac{(\mathbf{r} - \mathbf{v}t_r)}{c(t - t_r)}\right] = c(t - t_r) - \frac{\mathbf{v}\cdot\mathbf{r}}{c} - \frac{v^2}{c}t_r \\ &= \frac{1}{c}[(c^2t - \mathbf{r}\cdot\mathbf{v}) - (c^2 - v^2)t_r] \\ &= \frac{1}{c}\sqrt{(c^2t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)} \end{aligned}$$

  Therefore, we get that:

  $$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{\sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}$$
  $$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi}\frac{qc\vec{v}}{\sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}$$

The fields of a Moving Point Charge

For a point charge in arbitrary motion, according to the Lienard Wiechert potential, we have:

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{\mathfrak{r}c - \vec{\mathfrak{r}}\cdot\vec{v}}$$
$$\vec{A}(\vec{r},t) = \frac{\vec{v}}{c^2}V(\vec{r},t)$$

And the equations for $\vec{E},\vec{B}$, they are:

$$\vec{E} = -\nabla V - \frac{\partial\vec{A}}{\partial t}$$
$$\vec{B} = \nabla\times\vec{A}$$

The differentiation is tricky, because:

$$\vec{\mathfrak{r}} = \vec{r} - \vec{w}(t_r)$$
$$\vec{v} = \dot{\vec{w}}(t_r)$$

Are both evaluated at the retarded time, and t_r is defined implicitly as:

$$|\vec{r} - \vec{w}(t_r)| = c(t - t_r)$$

And so t_r is a function itself of $\vec{r}, t$.
So the derivatives are rough:

Let's begin with the gradient of V:

$$\nabla V = \frac{qc}{4\pi\epsilon_0}\frac{-1}{(\mathfrak{r}c - \boldsymbol{\mathfrak{r}}\cdot\mathbf{v})^2}\nabla(\mathfrak{r}c - \boldsymbol{\mathfrak{r}}\cdot\mathbf{v}). \tag{10.49}$$

Since $\mathscr{r} = c(t - t_r)$,

$$\nabla \mathscr{r} = -c\nabla t_r. \tag{10.50}$$

As for the second term, product rule 4 gives

$$\nabla(\boldsymbol{\mathscr{r}} \cdot \mathbf{v}) = (\boldsymbol{\mathscr{r}} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\boldsymbol{\mathscr{r}} + \boldsymbol{\mathscr{r}} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \boldsymbol{\mathscr{r}}). \tag{10.51}$$

Evaluating these terms one at a time:

$$\begin{aligned}(\boldsymbol{\mathscr{r}} \cdot \nabla)\mathbf{v} &= \left(\mathscr{r}_x \frac{\partial}{\partial x} + \mathscr{r}_y \frac{\partial}{\partial y} + \mathscr{r}_z \frac{\partial}{\partial z}\right)\mathbf{v}(t_r) \\ &= \mathscr{r}_x \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial x} + \mathscr{r}_y \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial y} + \mathscr{r}_z \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial z} \\ &= \mathbf{a}(\boldsymbol{\mathscr{r}} \cdot \nabla t_r),\end{aligned} \tag{10.52}$$

where $\mathbf{a} \equiv \dot{\mathbf{v}}$ is the *acceleration* of the particle at the retarded time. Now

$$(\mathbf{v} \cdot \nabla)\boldsymbol{\mathscr{r}} = (\mathbf{v} \cdot \nabla)\mathbf{r} - (\mathbf{v} \cdot \nabla)\mathbf{w}, \tag{10.53}$$

and

$$\begin{aligned}(\mathbf{v} \cdot \nabla)\mathbf{r} &= \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}\right)(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) \\ &= v_x\,\hat{\mathbf{x}} + v_y\,\hat{\mathbf{y}} + v_z\,\hat{\mathbf{z}} = \mathbf{v},\end{aligned} \tag{10.54}$$

while

$$(\mathbf{v} \cdot \nabla)\mathbf{w} = \mathbf{v}(\mathbf{v} \cdot \nabla t_r)$$

(same reasoning as Eq. 10.52). Moving on to the third term in Eq. 10.51,

$$\begin{aligned}\nabla \times \mathbf{v} &= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}} \\ &= \left(\frac{dv_z}{dt_r}\frac{\partial t_r}{\partial y} - \frac{dv_y}{dt_r}\frac{\partial t_r}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{dv_x}{dt_r}\frac{\partial t_r}{\partial z} - \frac{dv_z}{dt_r}\frac{\partial t_r}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{dv_y}{dt_r}\frac{\partial t_r}{\partial x} - \frac{dv_x}{dt_r}\frac{\partial t_r}{\partial y}\right)\hat{\mathbf{z}} \\ &= -\mathbf{a} \times \nabla t_r.\end{aligned} \tag{10.55}$$

Finally,

$$\nabla \times \boldsymbol{\mathscr{r}} = \nabla \times \mathbf{r} - \nabla \times \mathbf{w}, \tag{10.56}$$

but $\mathbf{V} \times \mathbf{r} = 0$, while, by the same argument as Eq. 10.55,

$$\nabla \times \mathbf{w} = -\mathbf{v} \times \mathbf{V} t_r. \tag{10.57}$$

Putting all this back into Eq. 10.51, and using the "BAC-CAB" rule to reduce the triple cross products,

$$\begin{aligned}\nabla(\boldsymbol{\mathscr{r}}\cdot\mathbf{v}) &= \mathbf{a}(\boldsymbol{\mathscr{r}}\cdot\nabla t_r)+\mathbf{v}-\mathbf{v}(\mathbf{v}\cdot\nabla t_r)-\boldsymbol{\mathscr{r}}\times(\mathbf{a}\times\nabla t_r)+\mathbf{v}\times(\mathbf{v}\times\nabla t_r)\\ &= \mathbf{v}+(\boldsymbol{\mathscr{r}}\cdot\mathbf{a}-v^2)\nabla t_r. \end{aligned} \tag{10.58}$$

Collecting Eqs. 10.50 and 10.58 together, we have

$$\nabla V=\frac{qc}{4\pi\epsilon_0}\frac{1}{(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})^2}\left[\mathbf{v}+(c^2-v^2+\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\nabla t_r\right]. \tag{10.59}$$

To complete the calculation, we need to know ∇t_r. This can be found by taking the gradient of the defining equation (10.48)—which we have already done in Eq. 10.50—and expanding out $\nabla\mathscr{r}$:

$$\begin{aligned}-c\nabla t_r &= \nabla\mathscr{r}=\nabla\sqrt{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}}=\frac{1}{2\sqrt{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}}}\nabla(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}})\\ &= \frac{1}{\mathscr{r}}[(\boldsymbol{\mathscr{r}}\cdot\nabla)\boldsymbol{\mathscr{r}}+\boldsymbol{\mathscr{r}}\times(\nabla\times\boldsymbol{\mathscr{r}})]. \end{aligned} \tag{10.60}$$

But

$$(\boldsymbol{\mathscr{r}}\cdot\nabla)\boldsymbol{\mathscr{r}}=\boldsymbol{\mathscr{r}}-\mathbf{v}(\boldsymbol{\mathscr{r}}\cdot\nabla t_r)$$

(same idea as Eq. 10.53), while (from Eq. 10.56 and 10.57)

$$\nabla\times\boldsymbol{\mathscr{r}}=(\mathbf{v}\times\nabla t_r).$$

Thus

$$-c\nabla t_r=\frac{1}{\mathscr{r}}[\boldsymbol{\mathscr{r}}-\mathbf{v}(\boldsymbol{\mathscr{r}}\cdot\nabla t_r)+\boldsymbol{\mathscr{r}}\times(\mathbf{v}\times\nabla t_r)]=\frac{1}{\mathscr{r}}[\boldsymbol{\mathscr{r}}-(\boldsymbol{\mathscr{r}}\cdot\mathbf{v})\nabla t_r],$$

and hence

$$\nabla t_r=\frac{-\boldsymbol{\mathscr{r}}}{\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v}}. \tag{10.61}$$

Incorporating this result into Eq. 10.59, I conclude that

$$\nabla V=\frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})^3}\left[(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})\mathbf{v}-(c^2-v^2+\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\boldsymbol{\mathscr{r}}\right]. \tag{10.62}$$

A similar calculation, which I shall leave for you (Prob. 10.17), yields

$$\begin{aligned}\frac{\partial\mathbf{A}}{\partial t} &= \frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})^3}\Big[(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})(-\mathbf{v}+\mathscr{r}\mathbf{a}/c)\\ &+ \frac{\mathscr{r}}{c}(c^2-v^2+\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v}\Big]. \end{aligned} \tag{10.63}$$

Combining these results, and introducing the vector

$$\mathbf{u} \equiv c\hat{\boldsymbol{\mathfrak{r}}} - \mathbf{v}, \tag{10.64}$$

I find

$$\boxed{\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\mathfrak{r}}{(\boldsymbol{\mathfrak{r}} \cdot \mathbf{u})^3}[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathfrak{r}} \times (\mathbf{u} \times \mathbf{a})].} \tag{10.65}$$

Meanwhile,

$$\nabla \times \mathbf{A} = \frac{1}{c^2}\nabla \times (V\mathbf{v}) = \frac{1}{c^2}[V(\nabla \times \mathbf{v}) - \mathbf{v} \times (\nabla V)].$$

We have already calculated $\nabla \times \mathbf{v}$ (Eq. 10.55) and ∇V (Eq. 10.62). Putting these together,

$$\nabla \times \mathbf{A} = -\frac{1}{c}\frac{q}{4\pi\epsilon_0}\frac{1}{(\mathbf{u} \cdot \boldsymbol{\mathfrak{r}})^3}\boldsymbol{\mathfrak{r}} \times [(c^2 - v^2)\mathbf{v} + (\boldsymbol{\mathfrak{r}} \cdot \mathbf{a})\mathbf{v} + (\boldsymbol{\mathfrak{r}} \cdot \mathbf{u})\mathbf{a}].$$

The quantity in brackets is strikingly similar to the one in Eq. 10.65, which can be written, using the BAC-CAB rule, as $[(c^2 - v^2)\mathbf{u} + (\boldsymbol{\mathfrak{r}} \cdot \mathbf{a})\mathbf{u} - (\boldsymbol{\mathfrak{r}} \cdot \mathbf{u})\mathbf{a}]$; the main difference is that we have $\mathbf{v}$'s instead of $\mathbf{u}$'s in the first two terms. In fact, since it's all crossed into $\boldsymbol{\mathfrak{r}}$ *anyway*, we can with impunity *change* these $\mathbf{v}$'s into $-\mathbf{u}$'s; the extra term proportional to $\hat{\boldsymbol{\mathfrak{r}}}$ disappears in the cross product. It follows that

$$\boxed{\mathbf{B}(\mathbf{r}, t) = \frac{1}{c}\hat{\boldsymbol{\mathfrak{r}}} \times \mathbf{E}(\mathbf{r}, t).} \tag{10.66}$$

Again:

$$\vec{u} := c\hat{\mathfrak{r}} - \vec{v}$$

And:

$$\boxed{\vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0}\frac{\mathfrak{r}}{(\vec{\mathfrak{r}} \cdot \vec{u})^3}[(c^2 - v^2)\vec{u} + \vec{\mathfrak{r}} \times (\vec{u} \times \vec{a})]}$$

$$\boxed{\vec{B}(\vec{r}, t) = \frac{1}{c}\hat{\mathfrak{r}} \times \vec{E}(\vec{r}, t)}$$

If the particle has a speed 0 and acceleration 0, then the first term only survives and becomes $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathfrak{r}^2}\hat{\mathfrak{r}}$

The second term falls off as the inverse of $\mathfrak{r}$ and is therefore dominant at large distances. If is called the **radiation field** or Acceleration field

Now, we can write the force felt by a particle Q at speed $\vec{V}$ due to a particle q with speed $\vec{v}$, acceleration $\vec{a}$. Such that $\vec{\mathfrak{r}}, \vec{u}, \vec{v}, \vec{a}$ is evaluated at the retarded time.

$$\vec{F} = \frac{qQ}{4\pi\epsilon_0}\frac{\mathfrak{r}}{(\vec{\mathfrak{r}} \cdot \vec{u})^3}\left[[(c^2 - v^2)\vec{u} + \vec{\mathfrak{r}} \times (\vec{u} \times \vec{a})] + \frac{\vec{V}}{c} \times \left[\hat{\mathfrak{r}} \times [(c^2 - v^2)\vec{u} + \vec{\mathfrak{r}} \times (\vec{u} \times \vec{a})]\right]\right]$$

.

Example 10.4: Calculate the electric and magnetic fields of a point charge moving with constant velocity

Example 10.4

Calculate the electric and magnetic fields of a point charge moving with constant velocity.

Solution: Putting $\mathbf{a} = 0$ in Eq. 10.65,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \frac{(c^2 - v^2)\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3} \mathbf{u}.$$

In this case, using $\mathbf{w} = \mathbf{v}t$,

$$\mathscr{r}\mathbf{u} = c\boldsymbol{\mathscr{r}} - \mathscr{r}\mathbf{v} = c(\mathbf{r} - \mathbf{v}t_r) - c(t - t_r)\mathbf{v} = c(\mathbf{r} - \mathbf{v}t).$$

In Ex. 10.3 we found that

$$\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v} = \boldsymbol{\mathscr{r}} \cdot \mathbf{u} = \sqrt{(c^2 t - \mathbf{r} \cdot \mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}.$$

In Prob. 10.14, you showed that this radical could be written as

$$Rc\sqrt{1 - v^2 \sin^2\theta / c^2},$$

where

$$\mathbf{R} \equiv \mathbf{r} - \mathbf{v}t$$

is the vector from the *present* location of the particle to $\mathbf{r}$, and θ is the angle between $\mathbf{R}$ and $\mathbf{v}$ (Fig. 10.9). Thus

$$\boxed{\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{\left(1 - v^2 \sin^2\theta/c^2\right)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2}.} \qquad (10.68)$$

Notice that $\mathbf{E}$ points along the line from the *present* position of the particle. This is an *extraordinary* coincidence, since the "message" came from the *retarded* position. Because of the $\sin^2\theta$ in the denominator, the field of a fast-moving charge is flattened out like a pancake in the direction perpendicular to the motion (Fig. 10.10). In the forward and backward directions $\mathbf{E}$ is *reduced* by a factor $(1 - v^2/c^2)$ relative to the field of a charge at rest; in the perpendicular direction it is *enhanced* by a factor $1/\sqrt{1 - v^2/c^2}$.

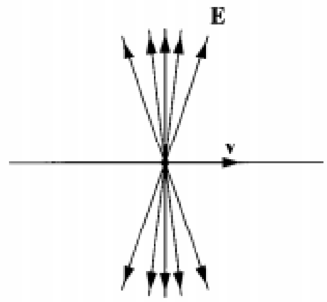


Figure 10.10

As for **B**, we have

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\mathbf{r} - \mathbf{v}t_r}{\mathscr{r}} = \frac{(\mathbf{r} - \mathbf{v}t) + (t - t_r)\mathbf{v}}{\mathscr{r}} = \frac{\mathbf{R}}{\mathscr{r}} + \frac{\mathbf{v}}{c},$$

and therefore

$$\boxed{\mathbf{B} = \frac{1}{c}(\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}) = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E}).} \tag{10.69}$$

Lines of **B** *circle around* the charge, as shown in Fig. 10.11.

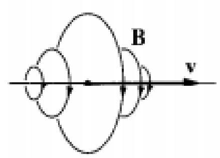


Figure 10.11

The fields of a point charge moving at constant velocity (Eqs. 10.68 and 10.69) were first obtained by Oliver Heaviside in 1888.[13] When $v^2 \ll c^2$ they reduce to

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\hat{\mathbf{R}}; \qquad \mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\frac{q}{R^2}(\mathbf{v} \times \hat{\mathbf{R}}). \tag{10.70}$$

The first is essentially Coulomb's law, and the latter is the "Biot-Savart law for a point charge" I warned you about in Chapter 5 (Eq. 5.40).

Radiation

What Is Radiation

Radiation is the transmission of energy in form of EM fields. We already saw the description of EM waves in space, radiation is the source of them.

EM waves propagate to infinity, carrying energy with them. We shall assume the source is localized near the origin. The total power passing out through a sphere of radius r is the integral of the Poynting vector:

$$P(r) = \oint \vec{S} \cdot d\vec{a} = \frac{1}{\mu_0} \oint (\vec{E} \times \vec{B}) \cdot d\vec{a}$$

The **power radiated** is the limit of this quantity as $r \to \infty$:

$$P_{rad} := \lim_{r \to \infty} P(r)$$

This is the energy transported to infinity that never comes back.
Now, the area of the sphere is $4\pi r^2$, so for radiation to occur, the Poynting vector must decrease no faster than $1/r^2$ (e.g. if it went as $1/r^3$, then $P(r)$ would go like $1/r$, and P_{rad} would be 0).

According to Coulumb and Biot Savart, static fields go like $1/r^2$, so Poynting vector goes like $1/r^4$.
For radiation to occur, we need non static fields, so that they go like $1/r$ and then there is radiation.

We care about the **Radiation zone**, this is the field in the faraway zone.

Dipole Radiation

Electric Dipole radiation

We will model a dipole like two charges in the z axis. The upper one with charge $q(t)$ and the lower one with $-q(t)$. Where:

$$q(t) = q_0 \cos(\omega t)$$

This is a simpler model than a real dipole with one charge moving, and it is equivalent.

The electric dipole is:

$$\vec{p}(t) = p_0 \cos(\omega t)\hat{z}$$

Where:

$$p_0 = q_0 d$$

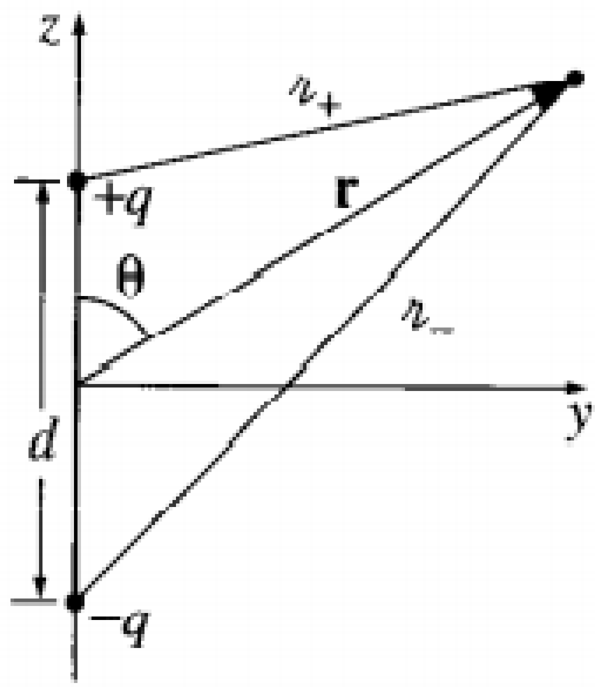


Figure 11.2

Now, the retarded potential is:

$$\begin{aligned} V(\vec{r},t) &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r'},t_r)}{\mathfrak{r}}d\tau' \\ &= \frac{1}{4\pi\epsilon_0}\left[\frac{q_0\cos[\omega(t-\mathfrak{r}_+/c)]}{\mathfrak{r}_+} - \frac{q_0\cos[\omega(t-\mathfrak{r}_-/c)]}{\mathfrak{r}_-}\right] \end{aligned}$$

Where, by the laws of cosines:

$$\mathfrak{r}_\pm = \sqrt{r^2 \mp rd\cos\theta + (d/2)^2}$$

Now we make some approximations:

- **Approximation 1:** $d << r$

$$\begin{aligned} \mathfrak{r}_\pm &\simeq r\left(1 \mp \frac{d}{2r}\cos\theta\right) \\ \Rightarrow \quad \frac{1}{\mathfrak{r}_\pm} &\simeq \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right) \end{aligned}$$

And also:

$$\cos[\omega(t - \mathfrak{r}_{\pm}c)] \simeq \cos\left[\omega(t - r/c) \pm \frac{\omega d}{2c}\cos\theta\right]$$
$$= \cos[\omega(t - r/c)]\cos\left(\frac{\omega d}{2c}\cos\theta\right) \mp \sin[\omega(t - r/c)]\sin\left(\frac{\omega d}{2c}\cos\theta\right)$$

- **Approximation 2:** $d << \dfrac{c}{\omega}$

  Since $\lambda = 2\pi c/\omega$, this is basically $d << \lambda$. Under this conditions:

$$\cos[\omega(t - \mathfrak{r}_{\pm}/c)] \simeq \cos[\omega(t - r/c)] \mp \frac{\omega d}{2c}\cos\theta\sin[\omega(t - r/c)]$$

  Therefore, we can get the potential:

$$V(r,\theta,t) = \frac{p_0\cos\theta}{4\pi\epsilon_0 r}\left[-\frac{\omega}{c}\sin[\omega(t - r/c)] + \frac{1}{r}\cos[\omega(t - r/c)]\right]$$

  In the static limit ($\omega \to 0$), this reduces to the well known normal potential $V = \dfrac{p_0\cos\theta}{4\pi\epsilon_0 r^2}$

- **Approximation 3:** $r >> \dfrac{c}{\omega}$
  That is, $r >> \lambda$, it corresponds to the **radiation zone.**

  In this region, the potential simplifies to:

$$\boxed{V(r,\theta,t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin[\omega(t - r/c)]}$$

  .

The vector potential: The vector potential is determined by the current flowing in the wire:

$$\vec{I}(t) = \frac{dq}{dt}\widehat{z} = -q_0\omega\sin(\omega t)\widehat{z}$$

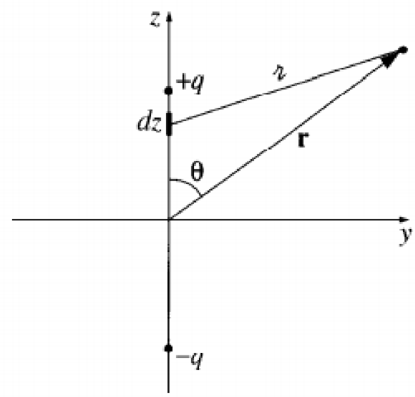


Figure 11.3

Referring to the figure, we have that:

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{I(\vec{r},t_r)}{\mathfrak{r}}dr$$
$$= \frac{\mu_0}{4\pi}\int_{-d/2}^{d/2} \frac{-q_0\omega\sin[\omega(t-\mathfrak{r}/c)]\widehat{z}}{\mathfrak{r}}dz$$

After integrating an using d to first order, we get:

$$\boxed{\vec{A}(r,\theta,t) = -\frac{\mu_0 p_0\omega}{4\pi r}\sin[\omega(t-r/c)]\widehat{z}}$$

.

Fields:

$$\nabla V = \frac{\partial V}{\partial r}\widehat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\widehat{\theta}$$
$$\simeq \frac{p_0\omega^2}{4\pi\epsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos[\omega(t-r/c)]\widehat{r}$$

Likewise:

$$\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 p_0\omega^2}{4\pi r}\cos[\omega(t-r/c)](\cos\theta\widehat{r} - \sin\theta\widehat{\theta})$$

.
Also:

$$\nabla\times\vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\widehat{\phi}$$
$$= -\frac{\mu_0 p_0\omega}{4\pi r}\left[\frac{\omega}{c}\sin\theta\cos[\omega(t-r/c)] + \frac{\sin\theta}{r}\sin[\omega(t-r/c)]\right]\widehat{\phi}$$

.

Therefore:

$$\boxed{\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\widehat{\theta}}$$

And:

$$\boxed{\vec{B} = \nabla \times \vec{A} = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\widehat{\phi}}$$

.

We can see that $\vec{E}$ and $\vec{B}$ are in phase, mutually perpendicular and $E_0/B_0 = c$. They are **spherical waves**, that is why their amplitudes decrease as $1/r$.

The energy radiated; is given by **Poynting vector:**

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{\mu_0}{c}\left[\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\right]^2\widehat{r}$$

The intensity is obtained by averaging over a complete cycle:

$$\langle\vec{S}\rangle = \left(\frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\widehat{r}$$

Notice that there is no radiation along the axis of the dipole ($\sin\theta = 0$); The **intensity profile** takes the form of a donut with a maximum in the equatorial plane.

The **Total power irradiated** over a sphere of radius r:

$$\langle P\rangle = \int\langle\vec{S}\rangle\cdot d\vec{a} = \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\int\frac{\sin^2\theta}{r^2}r^2\sin\theta d\theta d\phi = \frac{\mu_0 p_0^2\omega^4}{12\pi c}$$

We see that the energy irradiated in a sphere of radius r is independent of r.

- **Example:**
  Incident solar radiation covers a broad range of frequencies. But the energy absorbed and reradiated by the atmospheric dipoles is stronger at the higher frequencies because of the ω^4. It is more intense in the blue than in the red. It is this reradiated light that you see when you look up in the sky.

Magnetic Dipole radiation

Suppose we have a wire loop of radius b, around which we drive an alternating current:

$$I(t) = I_0 \cos(\omega t)$$

Then, the dipole moment is:

$$\vec{m}(t) = \pi b^2 I(t)\widehat{z} = m_0 \cos(\omega t)\widehat{z}$$

Where $m_0 := \pi b^2 I_0$

The loop is uncharged, so the scalar potential is zero. The retarded vector potential is:

$$\begin{aligned}\vec{A}(\vec{r},t) &= \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r'},t_r)}{\mathfrak{r}}d\tau' = \frac{\mu_0}{4\pi}\int \frac{I(t'_r)d\vec{l'}}{\mathfrak{r}} \\ &= \frac{\mu_0}{4\pi}\int \frac{I_0\cos[\omega(t-\mathfrak{r}/c)]}{\mathfrak{r}}d\vec{l'}\end{aligned}$$

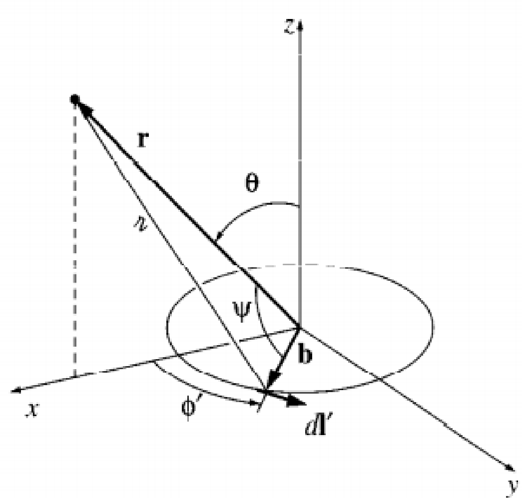


Figure 11.8

For a point $\vec{r}$ directly above the x axis, $\vec{A}$ must point in the y direction. Thus:

$$\vec{A}(\vec{r},t) = \frac{\mu_0 I_0 b}{4\pi}\widehat{y}\int_0^{2\pi} \frac{\cos[\omega(t-\mathfrak{r}/c)]}{\mathfrak{r}}\cos\phi' d\phi'$$

$\cos\phi'$ serves to pick out the y component of $d\vec{l'}$. By the law of cosines:

$$\mathfrak{r} = \sqrt{r^2 + b^2 - 2rb\cos\phi}$$

Where ψ is the angel between $\vec{r}, \vec{b}$:

$$\vec{r} = r\sin\theta\widehat{x} + r\cos\theta\widehat{z}$$
$$\vec{b} = b\cos\phi'\widehat{x} + b\sin\phi'\widehat{y}$$

So $rb\cos\psi = \vec{r}\cdot\vec{b} = rb\sin\theta\cos\phi'$ and therefore:

$$\mathfrak{r} = \sqrt{r^2 + b^2 - 2rb\sin\theta\cos\phi'}$$

.

- **Approx 1:** $b << r$

  Then:

  $$\mathfrak{r} \simeq r\left(1 - \frac{b}{r}\sin\theta\cos\phi'\right)$$
  $$\frac{1}{\mathfrak{r}} \simeq \frac{1}{r}\left(1 + \frac{b}{r}\sin\theta\cos\phi'\right)$$

  And therefore:

  $$\cos[\omega(t - \mathfrak{r}/c)] \simeq \cos\left[\omega(t - r/c) + \frac{\omega b}{c}\sin\theta\cos\phi'\right]$$
  $$= \cos[\omega(t - r/c)]\cos\left(\frac{\omega b}{c}\sin\theta\cos\phi'\right) - \sin[\omega(t - r/c)]\sin\left(\frac{\omega b}{c}\sin\theta\cos\phi'\right)$$

- **Approx 2:** $b << \frac{c}{\omega}$

  In this case:

  $$\cos[\omega(t - \mathfrak{r}/c)] \simeq \cos[\omega(t - r/c)] - \frac{\omega b}{c}\sin\theta\cos\phi'\sin[\omega(t - r/c)]$$

  Then, we insert this into the equation into $\vec{A}$, to get:

  $$\vec{A}(\vec{r}, t) = \frac{\mu_0 I_0 b}{4\pi r}\widehat{y}\int_0^{2\pi}\left[\cos[\omega(t - r/c)] + b\sin\theta\cos\phi'\left(\frac{1}{r}\cos[\omega(t - r/c)] - \frac{\omega}{c}\sin[\omega(t - r/c)]\right)\right]\cos\phi' d$$

  The first term integrates to 0, $\int_0^{2\pi}\cos\phi' d\phi' = 0$
  The second involves a cosine squared.

  Putting this in, we get:

  $$\vec{A}(r, \theta, t) = \frac{\mu_0 m_0}{4\pi}\left(\frac{\sin\theta}{r}\right)\left[\frac{1}{r}\cos[\omega(t - r/c)] - \frac{\omega}{c}\sin[\omega(t - r/c)]\right]\widehat{\phi}$$

  In the static limit ($\omega = 0$), we recover the familiar formula $\vec{A}(r, \theta) = \frac{\mu_0}{4\pi}\frac{m_0\sin\theta}{r^2}\widehat{\phi}$

- **Aprox 3:** $r >> \dfrac{c}{\omega}$

  Then, the firs term in $\vec{A}$ is negligible, so that:

  $$\boxed{\vec{A}(r,\theta,t) = -\frac{\mu_0 m_0 \omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin[\omega(t-r/c)]\widehat{\phi}}$$

  .

  From this, we get the $\vec{B}$ and $\vec{E}$ for large r:

  $$\boxed{\vec{E} = -\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\widehat{\phi}}$$
  $$\boxed{\vec{B} = \nabla\times\vec{A} = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\widehat{\theta}}$$

The fields are perpendicular and $E_0/B_0 = c$.
The **Poynting vector is:**

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{\mu_0}{c}\left[\frac{m_0\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\right]^2\widehat{r}$$

The **intensity** is:

$$\langle\vec{S}\rangle = \left(\frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3}\right)\frac{\sin^2\theta}{r^2}\widehat{r}$$

The total **radiated power** is:

$$\langle P\rangle = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}$$

We see that it has the same shape as that of an electric dipole. But the powers are related by:

$$\frac{P_{mag}}{P_{ele}} = \left(\frac{m_0}{p_0 c}\right)^2 = \left(\frac{\omega b}{c}\right)^2$$

.

Radiation from arbitrary source

The radiation from any source is:

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r'}, t-\mathfrak{r}/c)}{\mathfrak{r}}d\tau'$$

Where $\mathfrak{r} = \sqrt{r^2 + r'^2 - 2\vec{r}\cdot\vec{r'}}$.

- **Approximation 1:** $r' << r$

On this assumption:

$$\mathfrak{r} \simeq r\left(1 - \frac{\vec{r}\cdot\vec{r'}}{r^2}\right)$$
$$\frac{1}{\mathfrak{r}} \simeq \frac{1}{r}\left(1 + \frac{\vec{r}\cdot\vec{r'}}{r^2}\right)$$

Therefore:

$$\rho(\vec{r'}, t - \mathfrak{r}/c) \simeq \rho\left(\vec{r'}, t - \frac{r}{c} + \frac{\widehat{r}\cdot\vec{r'}}{c}\right)$$

We expand ρ as a Taylor series in t about the retarded time at the origin $t_0 := t - \dfrac{r}{c}$
We have:

$$\rho(\vec{r'}, t - \mathfrak{r}/c) \simeq \rho(\vec{r'}, t_0) + \dot{\rho}(\vec{r'}, t_0)\left(\frac{\widehat{r}\cdot\vec{r'}}{c}\right) + \cdots$$

The next terms are $\dfrac{1}{2}\ddot{\rho}\left(\dfrac{\widehat{r}\cdot\vec{r'}}{c}\right)^2$, $\dfrac{1}{3!}\ddot{\rho}\left(\dfrac{\widehat{r}\cdot\vec{r'}}{c}\right)^3, \cdots$

- **Approx 2:** $r' << \dfrac{c}{|\ddot{\rho}/\dot{\rho}|}, \dfrac{c}{|\dddot{\rho}/\dot{\rho}|^{1/2}}, \cdots$

For an oscillating system, these ratios are all c/ω.
We put this into the formula for V and discard the second order term:

$$V(\vec{r}, t) \simeq \frac{1}{4\pi\epsilon_0 r}\left[\int \rho(\vec{r'}, t_0)d\tau' + \frac{\widehat{r}}{r}\cdot\int \vec{r'}\rho(\vec{r'}, t_0)d\tau' + \frac{\widehat{\mathfrak{r}}}{c}\cdot\frac{d}{dt}\int \vec{r}\rho(\vec{r'}, t_0)d\tau'\right]$$

The two integral $\int \vec{r'}\rho(\vec{r'}, t_0)d\tau'$ is the dipole moment $\vec{\rho}$, so:

Scalar potential

$$\boxed{V(\vec{r}, t) \simeq \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\widehat{r}\cdot\vec{p}(t_0)}{r^2} + \frac{\widehat{r}\cdot\dot{\vec{p}}(t_0)}{rc}\right]}$$

In the static case, the first two terms are the monopole and dipole contributions.

Meanwhile the **Vector Potential** is:

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r'}, t - \mathfrak{r}/c)}{\mathfrak{r}}d\tau'$$

To first order in r', it suffices to replace $\mathfrak{r}$ by r in the integrand:

$$\vec{A}(\vec{r},t) \simeq \frac{\mu_0}{4\pi r}\int \vec{J}(\vec{r'},t_0)d\tau'$$

The integral of $\vec{J}$ is the time derivative of the dipole moment (proven as exercise), so that:

$$\boxed{\vec{A}(\vec{r},t) \simeq \frac{\mu_0}{4\pi}\frac{\dot{\vec{p}}(t_0)}{r}}$$

- **Approx 3:** Discard $1/r^2$ terms in $\vec{E}$ and $\vec{B}$.

With some work, we get:

$$\nabla V \simeq -\frac{1}{4\pi\epsilon_0 c^2}\frac{[\widehat{r}\cdot\ddot{\vec{p}}(t_0)]}{r}\widehat{r}$$

And similarly:

$$\nabla\times\vec{A} \simeq -\frac{\mu_0}{4\pi rc}[\widehat{r}\times\ddot{\vec{p}}(t_0)]$$
$$\frac{\partial\vec{A}}{\partial t} \simeq \frac{\mu_0}{4\pi}\frac{\ddot{\vec{p}}(t_0)}{r}$$

So:

$$\boxed{\vec{E}(\vec{r},t) \simeq \frac{\mu_0}{4\pi r}[(\widehat{r}\cdot\ddot{\vec{p}})\widehat{r}-\ddot{\vec{p}}] = \frac{\mu_0}{4\pi r}[\widehat{r}\times(\widehat{r}\times\ddot{\vec{p}})]}$$

Where $\ddot{\vec{p}}$ is evaluated at $t_0 = t - r/c$.
And:

$$\boxed{\vec{B}(\vec{r},t) \simeq -\frac{\mu_0}{4\pi rc}[\widehat{r}\times\ddot{\vec{p}}]}$$

.

If we choose coordinates, $\ddot{\vec{p}}(t_0)$ point in the z axis, then:

$$\vec{E}(r,\theta,t) \simeq \frac{\mu_0\ddot{p}(t_0)}{4\pi}\left(\frac{\sin\theta}{r}\right)\widehat{\theta}$$
$$\vec{B}(r,\theta,t) \simeq \frac{\mu_0\ddot{p}(t_0)}{4\pi c}\left(\frac{\sin\theta}{r}\right)\widehat{\phi}$$

.

The Poynting vector is:

$$\vec{S} \simeq \frac{1}{\mu_0}\vec{E} \times \vec{B} = \frac{\mu_0}{16\pi^2 c}[\ddot{p}(t_0)]^2 \left(\frac{\sin^2\theta}{r^2}\right)\widehat{r}$$

The total **Radiated power is:**

$$P \simeq \int \vec{S} \cdot d\vec{a} = \frac{\mu_0 \ddot{p}^2}{6\pi c}$$

.

Example 11.2:

- In the case of an oscillating electric dipole, $p(t) = p_0 \cos(\omega t)$, and we recover the equation we had.

- For a single point charge q, the dipole moment is $\vec{p}(t) = q\vec{d}(t)$
  Where $\vec{d}$ is the position of q with respect to the origin.
  Accordingly, $\ddot{\vec{p}}(t) = q\vec{a}(t)$

  In this case, the power irradiated is given by **Larmor Formula**:

  $$P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

  .

.

.

Point Charges

In chapter 10 we derived the fields of a charge q in arbitrary motion:

$$\vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0}\frac{\mathfrak{r}}{(\vec{\mathfrak{r}} \cdot \vec{u})^3}[(c^2 - v^2)\vec{u} + \vec{\mathfrak{r}} \times (\vec{u} \times \vec{a})]$$

Where $\vec{u} = c\widehat{\mathfrak{r}} - \vec{v}$, and:

$$\vec{B}(\vec{r}, t) = \frac{1}{c}\widehat{\mathfrak{r}} \times \vec{E}(\vec{r}, t)$$

In the first equation, the first therm is the **velocity field** and the second the **acceleration field**.

The **Poynting vector is:**

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) = \frac{1}{\mu_0 c}[\vec{E} \times (\hat{\mathfrak{r}} \times \vec{E})] = \frac{1}{\mu_0 c}[E^2\hat{\mathfrak{r}} - (\hat{\mathfrak{r}} \cdot \vec{E})\vec{E}]$$

However, not all this energy is radiation. The radiation energy is in some sense, the stuff that detaches from the charge and propagates to infinite.
To calculated the total power radiated by the particle at time t_r, we draw a huge sphere of radius $\mathfrak{r}$ centered at the position of the particle at time t_r, wait the appropriate interval:

$$t - t_r = \frac{\mathfrak{r}}{c}$$

For the radiation to reach the sphere and at that moment integrate the Poynting vector over the surface.
Because the radius goes as $1/\mathfrak{r}^2$, the terms with $1/\mathfrak{r}^3$, etc. wont matter.

For this reason, only the acceleration field represent true radiation (**acceleration fields**):

$$\vec{E}_{rad} = \frac{q}{4\pi\epsilon_0}\frac{\mathfrak{r}}{(\vec{\mathfrak{r}} \cdot \vec{u})^3}[\hat{\mathfrak{r}} \times (\vec{u} \times \vec{a})]$$

The velocity fields carry energy, but this energy is carried along with the particle, it is not radiation.
Now $\vec{E}_{rad}$ is perpendicular to $\vec{\mathfrak{r}}$, so a term cancels in the expression of $\vec{S}$. And we have:

$$\boxed{\vec{S}_{rad} = \frac{1}{\mu_0 c}E^2_{rad}\hat{\mathfrak{r}}}$$

If The charge is instantaneously at rest (at time t_r), then $\vec{u} = c\hat{\mathfrak{r}}$ and:

$$\vec{E}_{rad} = \frac{q}{4\pi\epsilon_0 c^2\mathfrak{r}}[\hat{\mathfrak{r}} \times (\hat{\mathfrak{r}} \times \vec{a})] = \frac{\mu_0 q}{4\pi\mathfrak{r}}[(\hat{\mathfrak{r}} \cdot \vec{a})\hat{\mathfrak{r}} - \vec{a}]$$

.

In that case:

$$\vec{S}_{rad} = \frac{1}{\mu_0 c}\left(\frac{\mu_0 q}{4\pi\mathfrak{r}}\right)^2[a^2 - (\hat{\mathfrak{r}} \cdot \vec{a})^2]\hat{\mathfrak{r}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\theta}{\mathfrak{r}^2}\right)\hat{\mathfrak{r}}$$

.

Where θ is the angle between $\hat{\mathfrak{r}}$ and $\vec{a}$. No power is radiated in the forward or backward direction, it is emitted in a donut about the direction of instantaneous acceleration

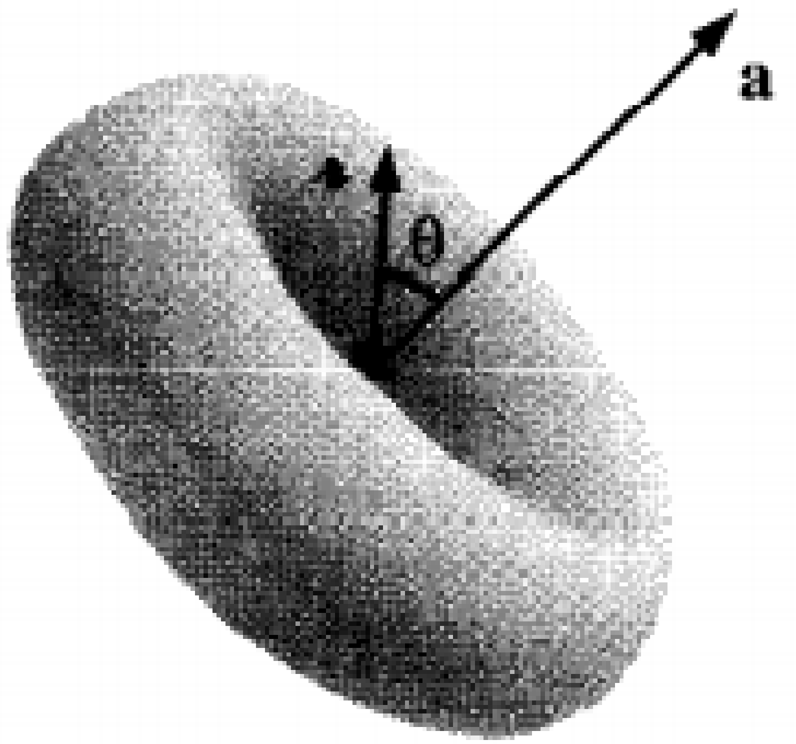


Figure 11.12

The **total power radiated** is:

$$P = \oint \vec{S}_{rad} \cdot d\vec{a} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \int \frac{\sin^2\theta}{\mathfrak{r}^2} \mathfrak{r}^2 \sin\theta d\theta d\phi$$

Or:

$$\boxed{P = \frac{\mu_0 q^2 a^2}{6\pi c}}$$

This is again the **Larmor Formula** we derived in an excercise.

Although we assumed that $v = 0$, this formula holds nicely for $v << c$.

We assumed $v = 0$ at the moment we measured the power (or at least $v << c$). If we don't make this assumption, the result is much more difficult and is given by **Lienard's generalization**:

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\vec{v} \times \vec{a}}{c}\right|^2\right)$$

.

Radiation Reaction

According to the laws of EM, an accelerating charge radiates. This radiation carries off energy at the expense of the particle's KE. Under the influence of a given force, therefore, a charged particle accelerates less than a neutral one of the same mass.
The radiation exerts a force $\vec{F}_{rad}$ back on the charge (recoil force).
We will derive this **radiation reaction**.

For a non relativistic particle:

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

Conservation of energy suggests that this is the rate at which the particle loses energy under the influence of $\vec{F}_{rad}$: $\quad \vec{F}_{rad} \cdot \vec{v} = -\frac{\mu_0 q^2 a^2}{6\pi c}$
This is wrong, because when we calculated P, we did it over a sphere very big, ignoring velocity fields (that is not radiation, but nevertheless carries energy and produces a radiation reaction).

The energy lost by the particle in any given time interval must then equal the energy carried away by the radiation *plus* whatever extra energy has been pumped into the velocity fields.

The correct result is:

$$\boxed{\vec{F}_{rad} = \frac{\mu_0 q^2}{6\pi c}\dot{\vec{a}}}$$

the **Abraham Lorentz formula**

$$\boxed{\mathbf{F}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}\dot{\mathbf{a}}.} \tag{11.80}$$

This is the **Abraham-Lorentz formula** for the radiation reaction force.

Of course, Eq. 11.79 doesn't *prove* Eq. 11.80. It tells you nothing whatever about the component of $\mathbf{F}_{\text{rad}}$ *perpendicular* to $\mathbf{v}$; and it only tells you the *time average* of the parallel component—the average, moreover, over very special time intervals. As we'll see in the next section, there are other reasons for believing in the Abraham-Lorentz formula, but for now the best that can be said is that it represents the *simplest* form the radiation reaction force could take, consistent with conservation of energy.

The Abraham-Lorentz formula has disturbing implications, which are not entirely understood nearly a century after the law was first proposed. For suppose a particle is subject to no *external* forces; then Newton's second law says

$$F_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}\dot{a} = ma,$$

from which it follows that

$$a(t) = a_0 e^{t/\tau}, \tag{11.81}$$

where

$$\tau \equiv \frac{\mu_0 q^2}{6\pi m c}. \tag{11.82}$$

(In the case of the electron, $\tau = 6 \times 10^{-24}$s.) The acceleration spontaneously *increases* exponentially with time! This absurd conclusion can be avoided if we insist that $a_0 = 0$, but it turns out that the systematic exclusion of such **runaway solutions** has an even more unpleasant consequence: If you *do* apply an external force, the particle starts to respond *before the force acts!* (See Prob. 11.19.) This **acausal preacceleration** jumps the gun by only a short time τ; nevertheless, it is (to my mind) philosophically repugnant that the theory should countenance it at all.[10]

Example 11.4

Calculate the **radiation damping** of a charged particle attached to a spring of natural frequency ω_0, driven at frequency ω.

Solution: The equation of motion is

$$m\ddot{x} = F_{\text{spring}} + F_{\text{rad}} + F_{\text{driving}} = -m\omega_0^2 x + m\tau\dddot{x} + F_{\text{driving}}.$$

With the system oscillating at frequency ω,

$$x(t) = x_0\cos(\omega t + \delta),$$

so

$$\dddot{x} = -\omega^2\dot{x}.$$

Therefore

$$m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = F_{\text{driving}}, \tag{11.83}$$

and the damping factor γ is given by

$$\gamma = \omega^2\tau. \tag{11.84}$$

[When I wrote $F_{\text{damping}} = -\gamma m v$, back in Chap. 9 (Eq. 9.152), I assumed for simplicity that the damping was proportional to the velocity. We now know that *radiation* damping, at least, is proportional to $\ddot{v}$. But it hardly matters: for sinusoidal oscillations *any* even number of derivatives of v would do, since they're all proportional to v.]

Physical basis of radiation reaction

To check that Eq. 12.132 is equivalent to Eq. 12.130, let's evaluate a few terms explicitly. For $\mu = 0$, $\nu = 1$,

$$\begin{aligned} F^{01} &= \frac{\partial A^1}{\partial x_0} - \frac{\partial A^0}{\partial x_1} = -\frac{\partial A_x}{\partial (ct)} - \frac{1}{c}\frac{\partial V}{\partial x} \\ &= -\frac{1}{c}\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla V\right)_x = \frac{E_x}{c}. \end{aligned}$$

That (and its companions with $\nu = 2$ and $\nu = 3$) is the first equation in 12.130. For $\mu = 1$, $\nu = 2$, we get

$$F^{12} = \frac{\partial A^2}{\partial x_1} - \frac{\partial A^1}{\partial x_2} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = (\nabla \times \mathbf{A})_z = B_z,$$

which (together with the corresponding results for F^{13} and F^{23}) is the second equation in 12.130.

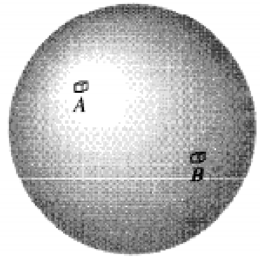


Figure 11.17

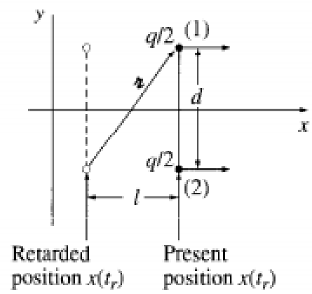


Figure 11.18

a fixed distance d (Fig. 11.18). This is the simplest possible arrangement of the charge that permits the essential mechanism (imbalance of internal electromagnetic forces) to function. Never mind that it's an unlikely model for an elementary particle: in the point limit ($d \to 0$) *any* model must yield the Abraham-Lorentz formula, to the extent that conservation of energy alone dictates that answer.

Let's assume the dumbbell moves in the x direction, and is (instantaneously) at rest at the retarded time. The electric field at (1) due to (2) is

$$\mathbf{E}_1 = \frac{(q/2)}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}[(c^2 + \boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{u} - (\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a}] \tag{11.85}$$

(Eq. 10.65), where

$$\mathbf{u} = c\,\hat{\boldsymbol{\mathscr{r}}} \quad \text{and} \quad \boldsymbol{\mathscr{r}} = l\,\hat{\mathbf{x}} + d\,\hat{\mathbf{y}}, \tag{11.86}$$

so that

$$\boldsymbol{\mathscr{r}}\cdot\mathbf{u} = c\mathscr{r}, \quad \boldsymbol{\mathscr{r}}\cdot\mathbf{a} = la, \quad \text{and} \quad \mathscr{r} = \sqrt{l^2+d^2}. \tag{11.87}$$

Actually, we're only interested in the x component of $\mathbf{E}_1$, since the y components will cancel when we add the forces on the two ends (for the same reason, we don't need to worry about the magnetic forces). Now

$$u_x = \frac{cl}{\mathscr{r}}, \tag{11.88}$$

and hence

$$E_{1_x} = \frac{q}{8\pi\epsilon_0 c^2}\frac{(lc^2 - ad^2)}{(l^2+d^2)^{3/2}}. \tag{11.89}$$

By symmetry, $E_{2_x} = E_{1_x}$, so the net force on the dumbbell is

$$\mathbf{F}_{\text{self}} = \frac{q}{2}(\mathbf{E}_1 + \mathbf{E}_2) = \frac{q^2}{8\pi\epsilon_0 c^2}\frac{(lc^2 - ad^2)}{(l^2+d^2)^{3/2}}\,\hat{\mathbf{x}}. \tag{11.90}$$

So far everything is exact. The idea now is to expand in powers of d; when the size of the particle goes to zero, all *positive* powers will disappear. Using Taylor's theorem

$$x(t) = x(t_r) + \dot{x}(t_r)(t - t_r) + \frac{1}{2}\ddot{x}(t_r)(t - t_r)^2 + \frac{1}{3!}\dddot{x}(t_r)(t - t_r)^3 + \cdots ,$$

we have,

$$l = x(t) - x(t_r) = \frac{1}{2}aT^2 + \frac{1}{6}\dot{a}T^3 + \cdots , \tag{11.91}$$

where $T \equiv t - t_r$, for short. Now T is determined by the retarded time condition

$$(cT)^2 = l^2 + d^2, \tag{11.92}$$

so

$$d = \sqrt{(cT)^2 - l^2} = cT\sqrt{1 - \left(\frac{aT}{2c} + \frac{\dot{a}T^2}{6c} + \cdots\right)^2} = cT - \frac{a^2}{8c}T^3 + (\ )T^4 + \cdots .$$

This equation tells us d, in terms of T; we need to "solve" it for T as a function of d. There's a systematic procedure for doing this, known as **reversion of series**,[13] but we can get the first couple of terms more informally as follows: Ignoring all higher powers of T,

$$d \cong cT \quad \Rightarrow \quad T \cong \frac{d}{c};$$

using this as an approximation for the cubic term,

$$d \cong cT - \frac{a^2}{8c}\frac{d^3}{c^3} \quad \Rightarrow \quad T \cong \frac{d}{c} + \frac{a^2 d^3}{8c^5},$$

and so on. Evidently

$$T = \frac{1}{c}d + \frac{a^2}{8c^5}d^3 + (\ )d^4 + \cdots . \tag{11.93}$$

Returning to Eq. 11.91, we construct the power series for l in terms of d:

$$l = \frac{a}{2c^2}d^2 + \frac{\dot{a}}{6c^3}d^3 + (\ )d^4 + \cdots . \tag{11.94}$$

Putting this into Eq. 11.90, I conclude that

$$\mathbf{F}_{\text{self}} = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{a}{4c^2 d} + \frac{\dot{a}}{12c^3} + (\ )d + \cdots\right]\hat{\mathbf{x}}. \tag{11.95}$$

Here a and $\dot{a}$ are evaluated at the *retarded* time (t_r), but it's easy to rewrite the result in terms of the *present* time t:

$$a(t_r) = a(t) + \dot{a}(t)(t - t_r) + \cdots = a(t) - \dot{a}(t)T + \cdots = a(t) - \dot{a}(t)\frac{d}{c} + \cdots ,$$

[13] See, for example, the *CRC Standard Mathematical Tables* (Cleveland: CRC Press).

and it follows that

$$\mathbf{F}_{\text{self}} = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{a(t)}{4c^2 d} + \frac{\dot{a}(t)}{3c^3} + (\ )d + \cdots\right]\hat{\mathbf{x}}. \tag{11.96}$$

The first term on the right is proportional to the acceleration of the charge; if we pull it over to the other side of Newton's second law, it simply adds to the dumbbell's mass. In effect, the total inertia of the charged dumbbell is

$$m = 2m_0 + \frac{1}{4\pi\epsilon_0}\frac{q^2}{4dc^2}, \tag{11.97}$$

where m_0 is the mass of either end alone. In the context of special relativity it is not surprising that the electrical repulsion of the charges should enhance the mass of the dumbbell. For the potential energy of this configuration (in the static case) is

$$\frac{1}{4\pi\epsilon_0}\frac{(q/2)^2}{d}, \tag{11.98}$$

and according to Einstein's formula $E = mc^2$, this energy contributes to the inertia of the object.[14]

The second term in Eq. 11.96 is the radiation reaction:

$$F_{\text{rad}}^{\text{int}} = \frac{\mu_0 q^2 \dot{a}}{12\pi c}. \tag{11.99}$$

It alone (apart from the mass correction[15]) survives in the "point dumbbell" limit $d \to 0$. Unfortunately, it differs from the Abraham-Lorentz formula by a factor of 2. But then, this is only the self-force associated with the *interaction* between 1 and 2—hence, the superscript "int." There remains the force of *each end on itself.* When the latter is included (see Prob. 11.20) the result is

$$F_{\text{rad}} = \frac{\mu_0 q^2 \dot{a}}{6\pi c}, \tag{11.100}$$

reproducing the Abraham-Lorentz formula exactly. *Conclusion: The radiation reaction is due to the force of the charge on itself*—or, more elaborately, the net force exerted by the fields generated by different parts of the charge distribution acting on one another.

Electro And relativity

Special Theory

Principle of Relativity: The same laws apply to any inertial reference frame.

This doesn't seem to be true for EM. First of all, in EM the laws depend on velocity (with respect to what?) And second, one observer might see a M field where another one sees an E field.

Still, there was some hope. Consider a train carrying a wire loop and going into a place with a M field perpendicular to the tracks.
From the view of someone outside, the Lorentz force causes the particles to go around the loop. From the view of someone inside, the changing flux causes an electric field by Faradays law and therefore a current.
The mechanism seems different but the result is completely the same.

At the time, people believed this was due to a weird coincidence, they believed everything was inside a medium called **ether** and absolute speeds were measured form this point of reference.
It was of crucial importance to find this rest frame.

Scientists try to find it doing experiments like the one of the train above, but weird coincidences seemed to hide it away.

Therefore came the Michelson Morley experiment. They tried to compare the speed of light in different directions and they in fact discovered that this speed is *exactly the same in all directions.*

Einstein saw all this and realized the best logical reason was that there is no ether. Any inertial system is suitable for the application of Maxwell's equations and the velocities of the charged particles can be measured with respect to this system with no problem. He gave his postulates:

- **Principle of relativity:** The laws of physics apply in all inertial frames
- **Universal speed of light**: The speed of light in vacuum is the same for all inertial observers

Geometry of Relativity

- **Simultaneity:** Two events that are simultaneous in one inertial system are not in general simultaneous in another
- **Time dilation:** Moving clocks run slow: If two events happen at the same position in frame S' with time t' (frame S' moving at speed v with respect to to S), then the time in S is $t = \gamma t'$

- **Contraction:** moving objects are shortened $\Delta x' = \gamma \Delta x$

Lorentz Transformations

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t - \frac{v}{c^2}x) \end{aligned}$$

$$\begin{aligned} x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \gamma(t' + \frac{v}{c^2}x') \end{aligned}$$

Structure of Spacetime

- **Four vectors:** A set of numbers x^ν that when changing basis transforms as:

$$x^{\mu'} = \Lambda^{\mu'}_{\nu} x^\nu$$

Where $\Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & & 1 \end{pmatrix}$

- **Four dimensional Scalar product:**

$$\vec{A} \cdot \vec{B} = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$

This is **invariant** (that is why we choose it).

- **Covariant:** Given a vector a_μ, its covariant covector is a^μ and has components:

$$a_\mu = (a_0, a_1, a_2, a_3) = (-a^0, a^1, a^2, a^3)$$

This way, $\vec{a} \cdot \vec{b} = a_\mu b^\mu$

- **Invariant Interval:** Given two events A, B in some frame, we define the displacement vector as:

$$\Delta x^\mu := x^\mu_A - x^\mu_B$$

The **Interval** between the events is:

$$I := (\Delta x)_\mu (\Delta x)^\mu = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2$$

If $I < 0$, the events are **timelike**, $I > 0$ for **spacelike** and $I = 0$ for **lightlike**.

- **Space time diagrams**

Relativistic mechanics

Proper time and proper velocity

As you progress along your world line, your watch runs slaw; while the clock on the wall ticks of an interval dt, your clock advances by:

$$d\tau = \sqrt{1 - u^2/c^2}dt$$

This is the **proper time**.

If you are moving at speed $\vec{u}$ with respect to the ground, this speed is $\vec{u} = \dfrac{d\vec{l}}{dt}$ with $\vec{l}$ the displacement vector.

On the other hand, the **proper velocity** would be:

$$\vec{\eta} = \frac{d\vec{l}}{d\tau}$$

The two velocities are related by:

$$\vec{\eta} = \frac{1}{\sqrt{1 - u^2/c^2}}\vec{u}$$

.

In fact, η is a 4-vector, since:

$$\eta^0 := \frac{dx^\mu}{d\tau}$$

Whose zeroth component is:

$$\eta^0 = \frac{dx^0}{d\tau} = c\frac{dt}{d\tau} = \frac{c}{\sqrt{1 - u^2/c^2}}$$

(from the point of view of an observer on Earth).
From the point of view of myself, $\eta_{S'} = (c, 0, 0, 0)$

It is a 4-vector since dx^μ is and $d\tau$ is invariant.
If we change system of reference, then as always:

$$\eta^{\mu'} = \Lambda^\mu_\nu \eta^\nu$$

.

Relativistic energy and momentum

We define the relativistic momentum as:

$$\vec{p} := m\vec{\eta} = m\gamma\vec{u}$$

This is a 4-vector.

From the point of view the particle

$$\vec{p}_{S'} = (mc, 0, 0, 0)$$

And in the reference frame from earth:

$$\vec{p}_S = (\gamma mc, \gamma mu, 0, 0)$$

.
Relativistic energy:

$$E := cp^0 = \gamma mc^2$$

Rest energy:

$$E_{rest} := mc^2$$

Kinetic energy:

$$EK = E - mc^2 = mc^2(\gamma - 1)$$

We can see that $KE = \frac{1}{2}mu^2 + \frac{3}{8}\frac{mu^4}{c^2} + \cdots$

Postulate: in every closed system, the total energy and momentum are conserved This is what justifies the definition for energy and momentum.

We define:

$$\mathbf{p} = \gamma m\mathbf{u}$$

That is, the last three coordinates of the four vector $\vec{p}$.
Then, the norm of $\vec{p}$ is:

$$p^\mu p_\mu = -(p^0)^2 + (\mathbf{p} \cdot \mathbf{p}) = -m^2c^2$$

Therefore:

$$E^2 - p^2c^2 - m^2c^4$$

.

Momentum and energy of light: they are related by

$$E = pc$$

Where $E = h\nu$

Kinematics

Example 12.7

Two lumps of clay, each of (rest) mass m, collide head-on at $\frac{3}{5}c$ (Fig. 12.26). They stick together. *Question*: what is the mass (M) of the composite lump?

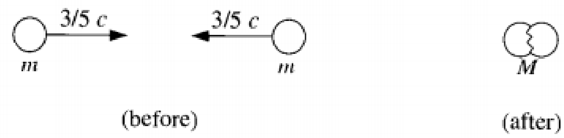


Figure 12.26

Solution: In this case conservation of momentum is trivial: zero before, zero after. The energy of each lump prior to the collision is

$$\frac{mc^2}{\sqrt{1-(3/5)^2}} = \tfrac{5}{4}mc^2,$$

and the energy of the composite lump after the collision is Mc^2 (since it's at rest). So conservation of energy says

$$\tfrac{5}{4}mc^2 + \tfrac{5}{4}mc^2 = Mc^2,$$

and hence

$$M = \tfrac{5}{2}m.$$

Notice that this is *greater* than the sum of the initial masses! Mass was not conserved in this collision; kinetic energy was converted into rest energy, so the mass increased.

In the *classical* analysis of such a collision, we say that kinetic energy was converted into *thermal* energy—the composite lump is *hotter* than the two colliding pieces. This is, of course, true in the relativistic picture too. But what *is* thermal energy? It's the sum total of the random kinetic and potential energies of all the atoms and molecules in the substance. Relativity tells us that these microscopic energies are represented in the *mass* of the object: a hot potato is *heavier* than a cold potato, and a compressed spring is *heavier* than a relaxed spring. Not by *much*, it's true—internal energy (U) contributes an amount U/c^2 to the mass, and c^2 is a very large number by everyday standards. You could never get two lumps of clay going anywhere *near* fast enough to detect the nonconservation of mass in their collision. But in the realm of elementary particles, the effect can be very striking. For example, when the neutral pi meson (mass 2.4×10^{-28} kg) decays into an electron and a positron (each of mass 9.11×10^{-31} kg), the rest energy is converted *almost entirely* into kinetic energy—less than 1% of the original mass remains.

Example 12.8

A pion at rest decays into a muon and a neutrino (Fig. 12.27). Find the energy of the outgoing muon, in terms of the two masses, m_π and m_μ (assume $m_\nu = 0$).

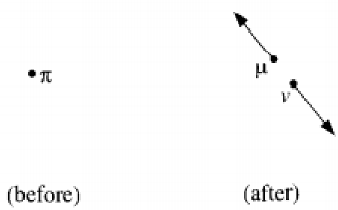


Figure 12.27

Solution: In this case

$$E_{\text{before}} = m_\pi c^2, \qquad \mathbf{p}_{\text{before}} = 0.$$
$$E_{\text{after}} = E_\mu + E_\nu, \qquad \mathbf{p}_{\text{after}} = \mathbf{p}_\mu + \mathbf{p}_\nu.$$

Conservation of momentum requires that $\mathbf{p}_\nu = -\mathbf{p}_\mu$. Conservation of energy says that

$$E_\mu + E_\nu = m_\pi c^2.$$

Now, $E_\nu = |\mathbf{p}_\nu| c$, by Eq. 12.56, whereas $|\mathbf{p}_\mu| = \sqrt{E_\mu^2 - m_\mu^2 c^4}/c$, by Eq. 12.55, so

$$E_\mu + \sqrt{E_\mu^2 - m_\mu^2 c^4} = m_\pi c^2,$$

from which it follows that

$$E_\mu = \frac{(m_\pi^2 + m_\mu^2)c^2}{2m_\pi}.$$

Dynamics

The force is still:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Example 12.10

Motion under a constant force. A particle of mass m is subject to a constant force F. If it starts from rest at the origin, at time $t = 0$, find its position (x), as a function of time.

Solution:

$$\frac{dp}{dt} = F \quad \Rightarrow \quad p = Ft + \text{constant},$$

but since $p = 0$ at $t = 0$, the constant must be zero, and hence

$$p = \frac{mu}{\sqrt{1 - u^2/c^2}} = Ft.$$

Solving for u, we obtain

$$u = \frac{(F/m)t}{\sqrt{1 + (Ft/mc)^2}}. \tag{12.61}$$

The numerator, of course, is the classical answer—it's approximately right, if $(F/m)t \ll c$. But the relativistic denominator ensures that u never exceeds c; in fact, as $t \to \infty$, $u \to c$.

To complete the problem we must integrate again:

$$\begin{aligned} x(t) &= \frac{F}{m} \int_0^t \frac{t'}{\sqrt{1 + (Ft'/mc)^2}} dt' \\ &= \frac{mc^2}{F} \sqrt{1 + (Ft'/mc)^2} \Big|_0^t = \frac{mc^2}{F} \left[\sqrt{1 + (Ft/mc)^2} - 1 \right]. \end{aligned} \tag{12.62}$$

In place of the classical parabola, $x(t) = (F/2m)t^2$, the graph is a *hyperbola* (Fig. 12.30); for this reason, motion under a constant force is often called **hyperbolic motion.** It occurs, for example, when a charged particle is placed in a uniform electric field.

Work as always is:

$$W := \int \vec{F} \cdot d\vec{l}$$

The **Work energy theorem** still holds:

$$W = \int \frac{d\vec{p}}{dt} \cdot d\vec{l} = \int \frac{d\vec{p}}{dt} \cdot \frac{d\vec{l}}{dt} dt = \int \frac{d\vec{p}}{dt} \cdot \vec{u} dt$$

Where $\frac{d\vec{p}}{dt} \cdot \vec{u} = \frac{d}{dt}\left(\frac{m\vec{u}}{\sqrt{1 - u^2/c^2}}\right) \cdot \vec{u} = \cdots = \frac{d}{dt}\left(\frac{mc^2}{\sqrt{1 - u^2/c^2}}\right) = \frac{dE}{dt}$

Therefore:

$$W = \int \frac{dE}{dt} dt = E_f - E_i$$

.

Transformation of $\vec{F}$

Because $\vec{F}$ is the derivative of momentum with respect to ordinary time, to transform it to a reference frame S' moving at speed u_x in the x directoin with respect to S, we must transform the numerator and denominator:

Because **F** is the derivative of momentum with respect to *ordinary* time, it shares the ugly behavior of (ordinary) velocity, when you go from one inertial system to another: both the numerator *and the denominator* must be transformed. Thus,[12]

$$\bar{F}_y = \frac{d\bar{p}_y}{d\bar{t}} = \frac{dp_y}{\gamma\, dt - \frac{\gamma\beta}{c}dx} = \frac{dp_y/dt}{\gamma\left(1 - \frac{\beta}{c}\frac{dx}{dt}\right)} = \frac{F_y}{\gamma(1-\beta u_x/c)}, \tag{12.66}$$

and similarly for the z component:

$$\bar{F}_z = \frac{F_z}{\gamma(1-\beta u_x/c)}.$$

The x component is even worse:

$$\bar{F}_x = \frac{d\bar{p}_x}{d\bar{t}} = \frac{\gamma\, dp_x - \gamma\beta\, dp^0}{\gamma\, dt - \frac{\gamma\beta}{c}dx} = \frac{\frac{dp_x}{dt} - \beta\frac{dp^0}{dt}}{1 - \frac{\beta}{c}\frac{dx}{dt}} = \frac{F_x - \frac{\beta}{c}\left(\frac{dE}{dt}\right)}{1-\beta u_x/c}.$$

We calculated dE/dt in Eq. 12.64; putting that in,

$$\bar{F}_x = \frac{F_x - \beta(\mathbf{u}\cdot\mathbf{F})/c}{1-\beta u_x/c}. \tag{12.67}$$

Only in one special case are these equations reasonably tractable: *If the particle is (instantaneously) at rest in $\mathcal{S}$*, so that $\mathbf{u} = 0$, then

$$\bar{\mathbf{F}}_\perp = \frac{1}{\gamma}\mathbf{F}_\perp, \quad \bar{F}_\parallel = F_\parallel. \tag{12.68}$$

That is, the component of **F** *parallel* to the motion of $\bar{\mathcal{S}}$ is unchanged, whereas components perpendicular are divided by γ.

It has perhaps occurred to you that we could avoid the bad transformation behavior of **F** by introducing a "proper" force, analogous to proper velocity, which would be the derivative of momentum with respect to *proper* time:

$$K^\mu \equiv \frac{dp^\mu}{d\tau}. \tag{12.69}$$

This is called the **Minkowski force**; it is plainly a 4-vector, since p^μ is a 4-vector and proper time is invariant. The spatial components of K^μ are related to the "ordinary" force by

$$\mathbf{K} = \left(\frac{dt}{d\tau}\right)\frac{d\mathbf{p}}{dt} = \frac{1}{\sqrt{1-u^2/c^2}}\mathbf{F}, \tag{12.70}$$

while the zeroth component,

$$K^0 = \frac{dp^0}{d\tau} = \frac{1}{c}\frac{dE}{d\tau}, \tag{12.71}$$

is, apart from the $1/c$, the (proper) rate at which the energy of the particle increases—in other words, the (proper) *power* delivered to the particle.

Relativistic dynamics can be formulated in terms of the ordinary force *or* in terms of the Minkowski force. The latter is generally much *neater,* but since in the long run we are interested in the particle's trajectory as a function of *ordinary* time, the former is often more useful. When we wish to generalize some classical force law, such as Lorentz's, to the relativistic domain, the question arises: Does the classical formula correspond to the *ordinary* force or to the Minkowski force? In other words, should we write

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}),$$

or should it rather be

$$\mathbf{K} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})?$$

Since proper time and ordinary time are identical in classical physics, there is no way at this stage to decide the issue. The Lorentz force law, as it turns out, is an *ordinary* force—later on I'll explain why this is so, and show you how to construct the electromagnetic Minkowski force.

Relativistic Electrodynamics

Magnetism as a relativistic phenomenon

What one observer interprets as an electric phenomenon, another sees as magnetic, but the actual phenomenon is the same.

Why there had to be Magnetism:
We can calculate the force of a moving particle and a current carrying wire without using magnetism.
Suppose you have a string of positive charges moving along tho the right at speed v. We will see the line of charge as a charge density λ superimposed with a negative $-\lambda$ so that the wire is neutral (moving to the left at the same speed v).
We have then, a net current moving to the right of magnitude:

$$I = 2\lambda v$$

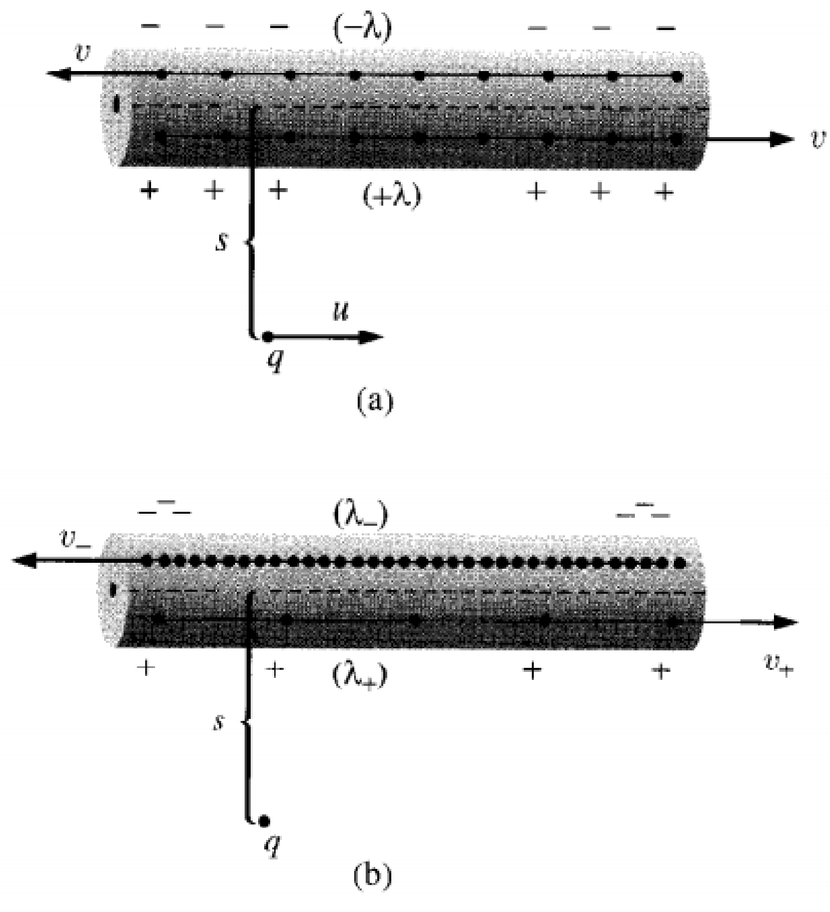


Figure 12.34

Meanwhile, a distance s away there is a point charge q traveling to the right at spedd $u < v$. There is no electrical force on q (the wire is neutral) in this system S.

Now we see the situation from system S' moving along the particle (at speed u).
In this reference frame, q is at rest, and the velocities of the positive and negative lines are:

$$v_{\pm} = \frac{v \mp u}{1 \mp vu/c^2}$$

Because v_- is greater than v_+, the Lorentz contraction of the spacing between negative charges is more severe than that between positive charges. In this frame, therefore, the wire carries a negative charge!
In fact:

$$\lambda_{\pm} = \pm(\gamma_{\pm})\lambda_0$$

Where

$$\gamma_{\pm} = \frac{1}{\sqrt{1 - v_{\pm}^2/c^2}}$$

Where λ_0 is the charge density of the positive line in its own rest system (that is not the same as λ), because in S they are already moving at v, so:

$$\lambda = \gamma\lambda_0$$

Where $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$

We can put $\gamma_{\pm}$ in a simpler form:

$$\begin{aligned}\gamma_{\pm} &= \frac{1}{\sqrt{1 - \frac{1}{c^2}(v \mp u)^2(1 \mp vu/c^2)^{-2}}} = \frac{c^2 \mp uv}{\sqrt{(c^2 \mp uv)^2 - c^2(v \mp u)^2}} \\ &= \frac{c^2 \mp uv}{\sqrt{(c^2 - v^2)(c^2 - u^2)}} = \gamma \frac{1 \mp uv/c^2}{\sqrt{1 - u^2/c^2}}. \qquad (12.\end{aligned}$$

Evidently, then, the net line charge in S' is:

$$\lambda_{tot} = \lambda_+ + \lambda_- = \lambda_0(\gamma_+ - \gamma_-) = \frac{-2\lambda uv}{c^2\sqrt{1 - u^2/c^2}}$$

.

Conclusion: As a result of unequal Lorentz contraction of the positive and negative lines, a current carrying wire that is electrically neutral in one system, is charged in another.

In the frame of the particle S' there is therefore an electric force given by a charged line:

$$E = \frac{\lambda_{tot}}{2\pi\epsilon_0 s}$$

$$F' = qE = -\frac{\lambda v}{\pi\epsilon_0 c^2 s}\frac{qu}{\sqrt{1-u^2/c^2}}$$

.

But id there is a force on q in S', there must be one in S. We can calculate it by using the transformation of forces. Since q is at rest in S' and F' is perpendicular to u, the force in S is:

$$F = \sqrt{1-u^2/c^2}F' = -\frac{\lambda v}{\pi\epsilon_0 c^2}\frac{qu}{s}$$

.

So, although in S there is no electrical force, something attracts the particle towards the wire.
So electrostatics + relativity imply the existence of some other force due to movement of charges.

This force in S is of course the magnetic one and using that $c^2 = (\epsilon_0\mu_0)^{-1}$ and $I = 2\lambda v$ we actually get:

$$F = -qu\left(\frac{\mu_0 I}{2\pi s}\right)$$

Something we already know!

How fields transform

Given fields in S, what are the fields in S'?

Charge is invariant: The charge of an object is the same no matter how it moves.
Assumption: The transformation laws are the same no matter how the fields are produced.

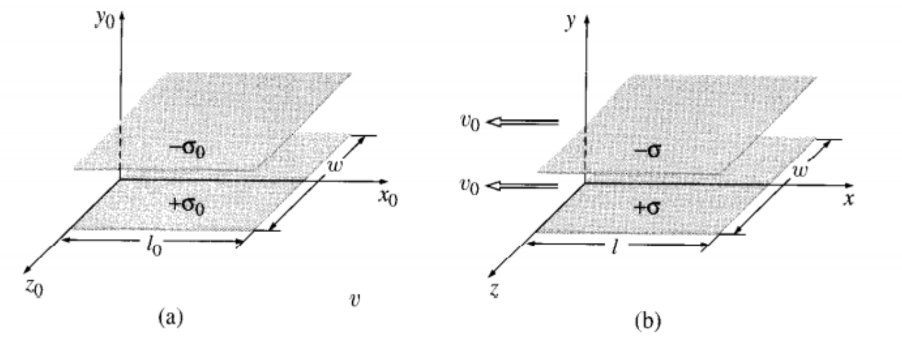
We consider the simplest Electric field, that of a parallel plate capacitor at rest in S_0 with surface charges $\pm\sigma_0$:

$$\vec{E}_0 = \frac{\sigma_0}{\epsilon_0}\widehat{y}$$

.

We know examine the field in a system S moving to the right at speed v_0, it is :

$$\vec{E} = \frac{\sigma}{\epsilon_0}\widehat{y}$$



The charge in the capacitors is the same, but know the length of one dimension is contracted by $\dfrac{1}{\gamma_0} = \sqrt{1 - v_0^2/c^2}$,
So the charge per unit area is increased to:

$$\sigma = \gamma_0 \gamma_0$$

Accordingly:

$$\vec{E}^{\perp} = \gamma_0 \vec{E}_0^{\perp}$$

($\perp$ means the fields are perependicular to the motion of the reference frame S w/ respect to S_0).

For parallel components, we consider the capacitor to be lined up with the yz plane. Now the distance d is contracted, but that doesn't change the field, so:

$$E^{\perp} = E_0^{\perp}$$

Example 12.13

Electric field of a point charge in uniform motion. A point charge q is at rest at the origin in system $\mathcal{S}_0$. *Question*: What is the electric field of this same charge in system $\mathcal{S}$, which moves to the right at speed v_0 relative to $\mathcal{S}_0$?

Solution: In $\mathcal{S}_0$ the field is

$$\mathbf{E}_0 = \frac{1}{4\pi\epsilon_0}\frac{q}{r_0^2}\hat{\mathbf{r}}_0,$$

or

$$\begin{cases} E_{x0} = \dfrac{1}{4\pi\epsilon_0}\dfrac{qx_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}, \\[2ex] E_{y0} = \dfrac{1}{4\pi\epsilon_0}\dfrac{qy_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}, \\[2ex] E_{z0} = \dfrac{1}{4\pi\epsilon_0}\dfrac{qz_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}. \end{cases}$$

From the transformation rules (Eqs. 12.90 and 12.91), we have

$$\begin{cases} E_x = E_{x0} = \dfrac{1}{4\pi\epsilon_0}\dfrac{qx_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}, \\[2ex] E_y = \gamma_0 E_{y0} = \dfrac{1}{4\pi\epsilon_0}\dfrac{\gamma_0 q y_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}, \\[2ex] E_z = \gamma_0 E_{z0} = \dfrac{1}{4\pi\epsilon_0}\dfrac{\gamma_0 q z_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}. \end{cases}$$

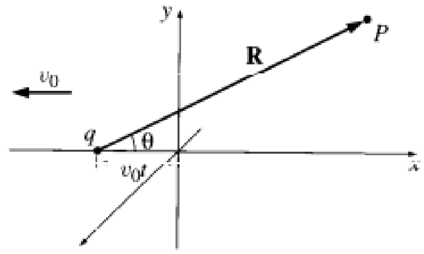


Figure 12.37

These are still expressed in terms of the $\mathcal{S}_0$ coordinates (x_0, y_0, z_0) of the field point (P); I'd prefer to write them in terms of the $\mathcal{S}$ coordinates of P. From the Lorentz transformations (or, actually, the inverse transformations),

$$\begin{cases} x_0 = \gamma_0(x + v_0 t) = \gamma_0 R_x, \\ y_0 = y = R_y, \\ z_0 = z = R_z, \end{cases}$$

where $\mathbf{R}$ is the vector from q to P (Fig. 12.37). Thus

$$\begin{aligned} \mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{\gamma_0 q \mathbf{R}}{(\gamma_0^2 R^2 \cos^2\theta + R^2 \sin^2\theta)^{3/2}} \\ &= \frac{1}{4\pi\epsilon_0}\frac{q(1 - v_0^2/c^2)}{[1 - (v_0^2/c^2)\sin^2\theta]^{3/2}}\frac{\hat{\mathbf{R}}}{R^2}. \end{aligned} \tag{12.92}$$

This, then, is the field of a charge in uniform motion; we got the same result in Chapter 10 using the retarded potentials (Eq. 10.68). The present derivation is far more efficient, and sheds some light on the remarkable fact that the field points away from the instantaneous (as opposed to the retarded) position of the charge: E_x gets a factor of γ_0 from the Lorentz transformation of the *coordinates;* E_y and E_z pick up theirs from the transformation of the *field.* It's the balancing of these two γ_0's that leaves $\mathbf{E}$ parallel to $\mathbf{R}$.

.

But these transformation laws are not the most general, since we began in a system S_0 where charges are at rest (there is no magnetic field).
For a **general rule** we must start with a system with magentic and electric fields.

The reference frame S will serve nicely.
Here we have an electric field:

$$E_y = \frac{\sigma}{\epsilon_0}$$

And a magnetic field due to surface currents of:

$$\vec{K}_{\pm} = \mp\sigma v_0 \widehat{x}$$

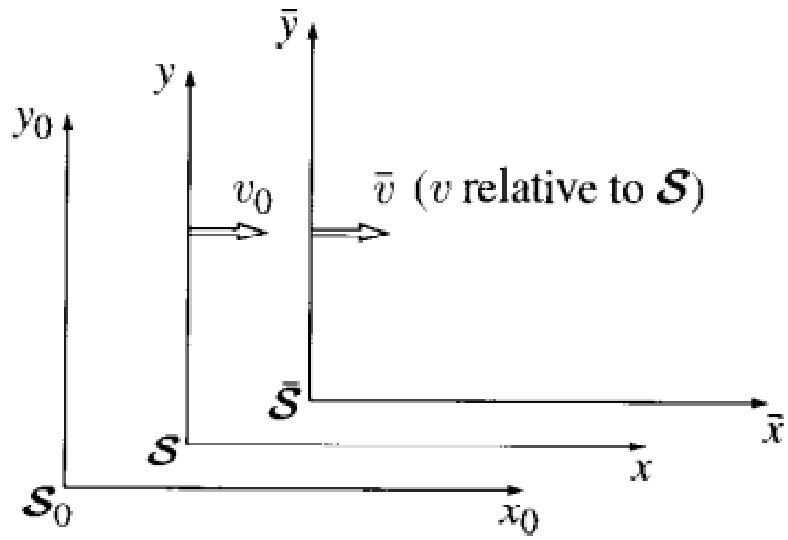


Figure 12.38

By the RHL, this field points in the negative z direction, its magnitude is given by Ampere to be:

$$B_z = -\mu_0 \sigma v_0$$

In a third system S' traveling to the right with speed v relative to S, the fields would be:

$$E'_y = \frac{\sigma'}{\epsilon_0}$$
$$B'_z = -\mu_0 \sigma' v'$$

Where v' is the velocity of S' relative to S_0, that is:

$$v' = \frac{v + v_0}{1 + v v_0 / c^2}$$
$$\gamma' = \frac{1}{\sqrt{1 - v'^2/c^2}}$$

And :

$$\sigma' = \gamma' \sigma_0$$

.

In remains only to express $\vec{E'}$ and $\vec{B'}$ in terms of $\vec{E}$ and $\vec{B}$.
Since $\sigma' = \gamma'\sigma_0$ and $\gamma = \gamma_0\gamma_0$, we have:

$$E'_y = \frac{\sigma'}{\epsilon_0} = \left(\frac{\gamma'}{\gamma_0}\right)\frac{\sigma}{\epsilon_0}$$
$$B'_z = -\mu_0\sigma' v' = -\left(\frac{\gamma'}{\gamma_0}\right)\mu_0\sigma v'$$

With a little algebra, we can find that $\dfrac{\gamma'}{\gamma_0} = \dfrac{\sqrt{1-v_0^2/c^2}}{\sqrt{1-v'^2/c^2}} = \cdots = \gamma(1+\dfrac{vv_0}{c^2})$

Where $\gamma = \dfrac{1}{\sqrt{1-v^2/c^2}}$

Thus:

$$E'_y = \gamma\left(1+\frac{vv_0}{c^2}\right)\frac{\sigma}{\epsilon_0} = \gamma\left(E_y - \frac{v}{c^2\epsilon_0\mu_0}B_z\right)$$
$$B'_z = -\gamma(1+\frac{vv_0}{c^2})\mu_0\sigma\left(\frac{v+v_0}{1+vv_0/c^2}\right) = \gamma(B_z - \mu_0\epsilon_0 vE_y)$$

Or, since $\mu_0\epsilon_0 = 1/c^2$, we have:

$$E'_y = \gamma(E_y - vB_z)$$
$$B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

This tells us how E_y and B_z transform.

To transform E_Z and B_y, we align the capacitor to the xy plane instead of xz. The fields in S are then:

$$E_z = \frac{\sigma}{\epsilon_0}$$
$$B_y = \mu_0\sigma v_0$$

The rest of the argument is identical and we get:

$$E'_z = \gamma(E_z + vB_y)$$
$$B'_y = \gamma(B_y + \frac{v}{c^2}E_z)$$

For the x components, we align the capacitor to the yz plane and get:

$$E'_x = E_x$$

.
.

Since in this case there is no acompanying magnetic field, we deduce it in another way.
We imagine a long solenoid parallel to the x axis at rest in I. It has a field:

$$B_x = \mu_0 n I$$

In system S', the length contracts, so n increases $n' = \gamma n$.
On the other hand, time dilates (The S clock, which rides with the solenoid, runs slow, so the current in S' is given by) $I' = \frac{1}{\gamma} I$
The two factors γ cancel, so:

$$B'_x = B_x$$

.

Transformation Laws

For frame S' moving at speed v in the x direction with respect to S

- $E'_x = E_x$
- $E'_y = \gamma(E_y - vB_z)$
- $E'_z = \gamma(E_z + vB_y)$

- $B'_x = B_x$
- $B'_y = \gamma(B_y + \frac{v}{c^2} E_z)$
- $B'_z = \gamma(B_z - \frac{v}{c^2} E_y)$

.

Special Cases:

- **If** $\vec{B} = 0$ **in** S:

$$\vec{B'} = \gamma \frac{v}{c^2}(E_z \widehat{y} - E_y \widehat{z}) = \frac{v}{c^2}(E'_z \widehat{y} - E'_y \widehat{z})$$

  Or since $\vec{v} = v\widehat{x}$, :

$$\vec{B'} = -\frac{1}{c^2}(\vec{v} \times \vec{E'})$$

- **If** $\vec{E} = 0$ **in** S, then:

$$\vec{E'} = -\gamma v(B_z \widehat{y} - B_y \widehat{z}) = -v(B'_z \widehat{y} - B'_y \widehat{z})$$

  Or

$$\vec{E'} = \vec{v} \times \vec{B}$$

Therefore, if $\vec{E}$ or $\vec{B}$ are 0 in one system, the fields are related simply by the corresponding rule in any other system

Example 12.14

Magnetic field of a point charge in uniform motion. Find the *magnetic* field of a point charge q moving at constant velocity $\mathbf{v}$.

Solution: In the particle's *rest* frame ($\mathcal{S}_0$) the magnetic field is zero (everywhere), so in a system $\mathcal{S}$ moving to the right at speed v,

$$\mathbf{B} = -\frac{1}{c^2}(\mathbf{v} \times \mathbf{E}).$$

We calculated the *electric* field in Ex. 12.13. The magnetic field, then, is

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{qv(1 - v^2/c^2)\sin\theta}{[1 - (v^2/c^2)\sin^2\theta]^{3/2}} \frac{\hat{\boldsymbol{\phi}}}{R^2}, \tag{12.111}$$

where $\hat{\boldsymbol{\phi}}$ aims counterclockwise as you face the oncoming charge. Incidentally, in the nonrelativistic limit ($v^2 \ll c^2$), Eq. 12.111 reduces to

$$\mathbf{B} = \frac{\mu_0}{4\pi} q \frac{\mathbf{v} \times \mathbf{R}}{R^2},$$

which is exactly what you would get by naïve application of the Biot-Savart law to a point charge (Eq. 5.40).

The field Tensor

We can see that $\vec{E}$ and $\vec{B}$ do not transform like the spacial form of 4-vectors as we might have expected.
But they do form a **antisymmetric second rank tensor**

Recall that a 4*vector transforms as:

$$a'^{\mu} = \Lambda^{\mu'}_{\nu} a^{\nu}$$

If S' is moving in the x direction at speed v, $\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

A second order tensor $t^{\mu\nu}$ has 16 components and transforms as:

$$t^{\mu'\nu'} = \Lambda^{\mu'}_{\lambda} \Lambda^{\nu'}_{\sigma} t^{\lambda\sigma}$$

If it is antisymmetric, then $t^{\mu\nu} = -t^{\nu\mu}$

Using this and some algebra, we can find that:

$$\left.\begin{array}{lll} \bar{t}^{01} = t^{01}, & \bar{t}^{02} = \gamma(t^{02} - \beta t^{12}), & \bar{t}^{03} = \gamma(t^{03} + \beta t^{31}), \\ \bar{t}^{23} = t^{23}, & \bar{t}^{31} = \gamma(t^{31} + \beta t^{03}), & \bar{t}^{12} = \gamma(t^{12} - \beta t^{02}). \end{array}\right\} \tag{12.117}$$

Therefore, we can see that the six components of E, B form a antisymmetric tensor.

We define the **Field tensor** $F^{\mu\nu}$ by direct comparison with the transformation laws, and we get:

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

But we could have also arrange the 6 terms in another way to get:

This leads to **dual tensor,** $G^{\mu\nu}$:

$$G^{\mu\nu} = \left\{ \begin{matrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{matrix} \right\}. \tag{12.119}$$

$G^{\mu\nu}$ can be obtained directly from $F^{\mu\nu}$ by the substitution $\mathbf{E}/c \to \mathbf{B}$, $\mathbf{B} \to -\mathbf{E}/c$. Notice that this operation leaves Eq. 12.108 unchanged—that's why both tensors generate the correct transformation rules for **E** and **B.**

EM in tensor notation

We must first determine the sources of the fields and how they transform.
Imagine a cloud of charge drifting by, we concentrate on an infinitesimal volume V, which contains charge Q moving at velocity $\vec{u}$. The charge density is:

$$\rho = \frac{Q}{V}$$

And the current density is:

$$\vec{J} = \rho\vec{u}$$

.

We define the **Proper charge density** ρ_0 as the density in the rest system of the charge:

$$\rho_0 = \frac{Q}{V_0}$$

Where V_0 is the rest volume of the chunk. Because one dimension is contracted, we have $V = \sqrt{1 - u^2/c^2}V_0$.
And hence:

$$\rho = \rho\frac{1}{\sqrt{1 - u^2/c^2}}$$

$$\vec{J} = \rho_0\frac{\vec{u}}{\sqrt{1 - u^2/c^2}}$$

We recognize here the components of **proper velocity** multiplied by the invariant ρ_0. Evidently, charge density and current density make together a 4-vector:

$$J^\mu = \rho_0 \eta^\mu$$

Whose components are:

$$J^\mu = (c\rho, J_x, J_y, J_z)$$

We call it the **Current density 4-vector**

The continuity equation $\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$
Takes a nice compact form, since:

$$\nabla \cdot \vec{J} = \sum_{i=1} \frac{\partial J^i}{\partial x^i}$$
$$\frac{\partial \rho}{\partial t} = \frac{1}{c}\frac{\partial J^0}{\partial t} = \frac{\partial J^0}{\partial x^0}$$

Thus:

$$\boxed{\partial_\mu J^\mu = 0}$$

As for **Maxwell's equations**, they can be written as:

$$\boxed{\partial_\nu F^{\mu\nu} = \mu_0 J^\mu}$$
$$\boxed{\frac{\partial G^{\mu\nu}}{\partial x^\nu}}$$

This is a different equation for each μ:

- $\frac{\partial F^{0\nu}}{\partial x^\nu} = \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3}$
  $= \frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = \frac{1}{c}\nabla \cdot \vec{E}$
  $= \mu_0 J^0 = \mu_0 c\rho$
  So $\nabla \cdot \vec{E} = \frac{1}{\epsilon_0}\rho$

$$\begin{aligned}\frac{\partial F^{1\nu}}{\partial x^\nu} &= \frac{\partial F^{10}}{\partial x^0}+\frac{\partial F^{11}}{\partial x^1}+\frac{\partial F^{12}}{\partial x^2}+\frac{\partial F^{13}}{\partial x^3}\\ &= -\frac{1}{c^2}\frac{\partial E_x}{\partial t}+\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}=\left(-\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}+\nabla\times\mathbf{B}\right)_x\\ &= \mu_0 J^1=\mu_0 J_x.\end{aligned}$$

Combining this with the corresponding results for $\mu = 2$ and $\mu = 3$ gives

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t},$$

which is Ampère's law with Maxwell's correction. ∎

Meanwhile, the second equation in 12.126, with $\mu = 0$, becomes

$$\begin{aligned}\frac{\partial G^{0\nu}}{\partial x^\nu} &= \frac{\partial G^{00}}{\partial x^0}+\frac{\partial G^{01}}{\partial x^1}+\frac{\partial G^{02}}{\partial x^2}+\frac{\partial G^{03}}{\partial x^3}\\ &= \frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=\nabla\cdot\mathbf{B}=0\end{aligned}$$

∎

(the third of Maxwell's equations), whereas $\mu = 1$ yields

$$\begin{aligned}\frac{\partial G^{1\nu}}{\partial x^\nu} &= \frac{\partial G^{10}}{\partial x^0}+\frac{\partial G^{11}}{\partial x^1}+\frac{\partial G^{12}}{\partial x^2}+\frac{\partial G^{13}}{\partial x^3}\\ &= -\frac{1}{c}\frac{\partial B_x}{\partial t}-\frac{1}{c}\frac{\partial E_z}{\partial y}+\frac{1}{c}\frac{\partial E_y}{\partial z}=-\frac{1}{c}\left(\frac{\partial \mathbf{B}}{\partial t}+\nabla\times\mathbf{E}\right)_x=0.\end{aligned}$$

So, combining this with the corresponding results for $\mu = 2$ and $\mu = 3$,

$$\nabla\times\mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t},$$

which is Faraday's law. In relativistic notation, then, Maxwell's four rather cumbersome equations reduce to two delightfully simple ones. ∎

which is Faraday's law. In relativistic notation, then, Maxwell's four rather cumbersome equations reduce to two delightfully simple ones.

In terms of $F^{\mu\nu}$ and the proper velocity η^μ, the *Minkowski* force on a charge q is given by

$$\boxed{K^\mu = q\eta_\nu F^{\mu\nu}.} \tag{12.127}$$

For if $\mu = 1$, we have

$$\begin{aligned} K^1 &= q\eta_\nu F^{1\nu} = q(-\eta^0 F^{10} + \eta^1 F^{11} + \eta^2 F^{12} + \eta^3 F^{13}) \\ &= q\left[\frac{-c}{\sqrt{1-u^2/c^2}}\left(\frac{-E_x}{c}\right) + \frac{u_y}{\sqrt{1-u^2/c^2}}(B_z) + \frac{u_z}{\sqrt{1-u^2/c^2}}(-B_y)\right] \\ &= \frac{q}{\sqrt{1-u^2/c^2}}[\mathbf{E} + (\mathbf{u}\times\mathbf{B})]_x, \end{aligned}$$

with a similar formula for $\mu = 2$ and $\mu = 3$. Thus,

$$\mathbf{K} = \frac{q}{\sqrt{1-u^2/c^2}}[\mathbf{E} + (\mathbf{u}\times\mathbf{B})], \tag{12.128}$$

and therefore, referring back to Eq. 12.70,

$$\mathbf{F} = q[\mathbf{E} + (\mathbf{u}\times\mathbf{B})],$$

which is the Lorentz force law. Equation 12.127, then, represents the Lorentz force law in relativistic notation. I'll leave for you the interpretation of the zeroth component (Prob. 12.54).

■

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Relativistic Potentials

WE know that:

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$
$$\vec{B} = \nabla \times \vec{A}$$

We can **Define the 4-potential**:

$$\boxed{A^\mu = (V/c, A_x, A_y, A_z)}$$

.

In terms of this 4 potential, the EM tensor is:

$$\boxed{F^{\mu\nu} = \frac{\partial A^\nu}{\partial x^\mu} - \frac{\partial A^\mu}{\partial x_\nu}}$$

Observe that differentiation is with respect to covariant vectors x_μ, x_ν (we need to add a - to the 0th component)

To check that Eq. 12.132 is equivalent to Eq. 12.130, let's evaluate a few terms explicitly. For $\mu = 0$, $\nu = 1$,

$$F^{01} = \frac{\partial A^1}{\partial x_0} - \frac{\partial A^0}{\partial x_1} = -\frac{\partial A_x}{\partial(ct)} - \frac{1}{c}\frac{\partial V}{\partial x}$$
$$= -\frac{1}{c}\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla V\right)_x = \frac{E_x}{c}.$$

That (and its companions with $\nu = 2$ and $\nu = 3$) is the first equation in 12.130. For $\mu = 1$, $\nu = 2$, we get

$$F^{12} = \frac{\partial A^2}{\partial x_1} - \frac{\partial A^1}{\partial x_2} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = (\nabla \times \mathbf{A})_z = B_z,$$

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which (together with the corresponding results for F^{13} and F^{23}) is the second equation in 12.130.

The potential formulation automatically takes care of the homogeneous Maxwell equation ($\partial G^{\mu\nu}/\partial x^\nu = 0$). As for the inhomogeneous equaiton ($\partial F^{\mu\nu}/\partial x^\nu = \mu_0 J^\mu$).
Seeing how we related $F^{\mu\nu}$ to A^λ, this equation is:

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial A^\nu}{\partial x^\nu}\right) - \frac{\partial}{\partial x_\nu}\frac{\partial A^\mu}{\partial x^\nu}\mu_0 J^\mu$$

This is a hard equation, however, because of **Gauge invariance**, we can agregate a scalar function λ as:

$$A^\mu \to A^{\mu'} = A^\mu + \frac{\partial \lambda}{\partial x_\mu}$$

without changing $F^{\mu\nu}$ or any physical content.

Specifically, we use the Lorentz Gauge condition:

$$\nabla \cdot \vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}$$

Or, in relativistic notation:

$$\frac{\partial A^\mu}{\partial x^\nu} = 0$$

In relativistic notation:

$$\frac{\partial A^\mu}{\partial x^\nu} = 0$$

Therefore, the hard equation reduces to:

$$\boxed{\Box^2 A^\mu = -\mu_0 J^\mu}$$

Where $\Box^2 := \partial_\nu \partial^\nu = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$
This is the simles representation of all of **Maxwell's equations**