Relatividad-Saúl

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Special Relativity Foundations

Introduction

The following principle were taken as correct in classic physics:

- Time is absolute, it is the same for every inertial observer
- Interactions occur instantaneously

These axioms where taken as correct. Nevertheless, the discovery of electrodynamics and the realization that they are not invariant under Galileo transformation led to the new theory.

Michelson Morley Experiment

It was thought that light moved in a substance called ether.
Michelson and Morley reported an experiment to try to measure the speed of light in the ether.

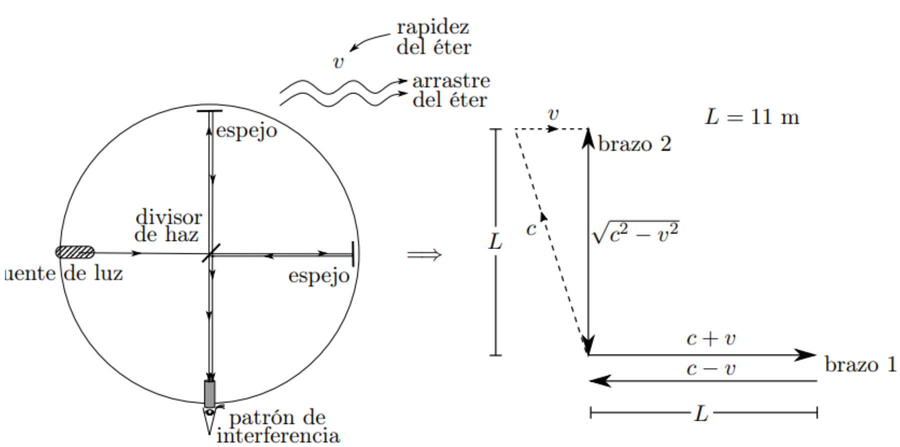


Figura 1.1: Interferómetro de Michelson y Morley. Un haz de luz es dividido a lo largo de dos direcciones perpendiculares (los *brazos* del interferómetro). Tras recorrer un distancia L, los haces son reflejados hacia el divisor de haz, que los recombina y dirige hacia un observador. En caso de que el éter exista y se desplace en la dirección de uno de los brazos, se debería observar un patrón de interferencia.

Because the Earth moves, it is expected that the ether exhibits a drag speed v in the opposite direction to the movement of earth. Due to this difference, we should observe an interference pattern.

If the horizontal arm is moving at speed v, then the horizontal time T_1 and vertical time T_2 are:

$$T_1 = \frac{L}{c+v} + \frac{L}{c-v} = \frac{2cL}{c^2 - v^2} = \frac{2L\gamma^2}{c} \quad , \quad T_2 = \frac{2L}{\sqrt{c^2 - v^2}} = \frac{2L\gamma}{c}$$

Where we defined:

$$\gamma := \frac{1}{\sqrt{1 - v^2/c^2}}$$

So there is a time difference of:

$$\Delta T = T_1 - T_2 = \frac{2L\gamma}{c}(\gamma - 1)$$

This means that there should be interference , but that is not what is seen, no matter the orientation.
A possible solution was that somehow the arm moving on the direction of the ether would somehow contract as:

$$L' = \frac{L}{\gamma} \quad \Rightarrow \quad T_1 = \frac{2L\gamma}{c} = T_2$$

Galilean Relativity

Galilean Relativity: Every physical law of nature should take the same form in all inertial reference systems.
We represent reference systems as $S, S', S'', etc.$
We use x^i for spatial coordinates and ct for temporal coordinate.

If there is an event, we represent it with coordinats ct, x_i.
In general, the coordinates in another system S' are not the same ones and there is a transformation rule between them:

$$x' = Gx$$

Where $x = (ct, x^1, x^2, x^3)^T$.

We will suppose two frames S and S' such that S' moves at speed $\vec{v} = (v, 0, 0)^T$ with respect to S in the direction x^1.
In Newtonian mechanics, the transformation rules are the **Galilean transformations**:

- $ct' = ct$
- $x^{1'} = x^1 - vt = x^1 - \frac{v}{c}(ct)$
- $x^{2'} = x^2$
- $x^{3'} = x^3$

Standard configuration: If $x^i = x^{i'} = 0$ at time $t' = t = 0$
The laws of nature should be conserved under this transformation. For example, it can be seen that acceleration is the same on both systems.
The laws of transformation can be written as:

$$x' = Gx \quad , \quad x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad , \quad G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This transformations $G(v)$ with the product define a group which satisfies the properties:

- **Closeness:** $G(v)G(u) = G(v + u)$
- **Inverse:** For all $G(v)$ there exists $(G(v))^{-1} = G(-v)$ such that $G(v)G(-v) = Id$
- **Neutral element:** There exists a neutral element $Id = G(v = 0)$

Because $\det G = 1$ for all G in the group, the group is called special.

Space Time Diagrams

A space time diagram is a diagram in which we put all the events from the point of buy of an observer. We represent the edges x^1 in horizontal and x^0 in the vertical.

Let's draw the diagram for S and draw in it the moving frame S' at speed v.

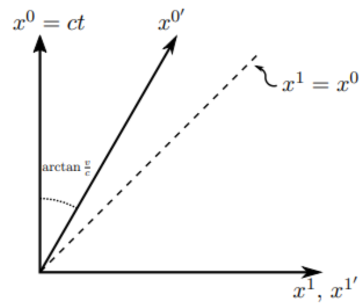


Figura 1.4: Diagrama de espacio–tiempo para dos marcos de referencia en configuración estándar. El marco de referencia S tiene coordenadas (x^0, x^1). La ecuación que satisface el eje vertical en S es $x^1 = 0$, mientras que en S' es $x^{1'} = 0$ o equivalentemente $x^1 - \frac{v}{c}x^0 = 0$.

Where the reference frames are in standard configuration, so their origins coincide.
The edge $x^{1'}$ can be drawn by asking $x^{0'} = 0$, which is equal to $x^0 = 0$, so the edge x^1.
On the other hand, demanding $x^{1'} = 0$ and applying the tranformation rule, we get $x^1 = \dfrac{v}{c}x^0$, so this is the edge $x^{0'}$

Galilean Interval

In Galilean transformations, the spatial interval is invariant, which we call simply distance. Consider a fixed bar in system S with edges at O_1, O_2. An observer in S sees the edges at coordinates:

$$O_1 = (x_1^0, x_1^1) \quad , \quad O_2 = (x_2^0, x_2^1)$$

Calculating the spatial interval, we find that (this is the defined interval in Galilean transformations):

$$\Delta x^1 = x_2^1 - x_1^1 = l$$

Now we can calculate the coordinates in the S' system:

$$O_1 : (x_1^{0'}, x_1^{1'}) = (x_1^0, x^1 - \frac{v}{c}x_1^0) \quad , \quad O_2 : (x_2^{0'}, x_2^{1'}) = (x_2^0, x^2 - \frac{v}{c}x_2^0) \quad ,$$

So:

$$\begin{aligned} \Delta x^{1'} &= x_2^{1'} - x_1^{1'} = \\ &= x_2^1 - \frac{v}{c}x_2^0 - (x_1^1 - \frac{v}{c}x_1^0) \\ &= \Delta x^1 - \frac{v}{c}\Delta x^0 = \Delta x^1 = l \end{aligned}$$

In general, the conserved interval is:

$$l^2 = (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2$$

Definition of an inertial observer

An observer is a huge information gathering system, that can record the location and time of any event. This coordinates must satisfy the following three properties to be called inertial:

- The distance between point $P_1 = (x_1, y_1, z_1)$ and point $P_2 = (x_2, y_2, z_2)$ is independent of time.
- The clocks that sit at every point ticking off the time coordinate t are synchronized and all run at the same rate.
- The geometry of space at any constant time t is Euclidean

New Units

We define $c = 1$, without any units whatsoever. Therefore, we measure time in meters, which is the time it takes light to travel one meter.

Space time diagrams

The plane t-x is the set of all events measured by an observer S.
A curve in space time gives a relation $x(t)$ that can represent the position of a particle in different times. This is called the particle's world line. Its slope is related to velocity:

$$\frac{dt}{dx} = \frac{1}{v}$$

We define the following:

- **Event:** Denoted in capitals A, B, P is a point in a space-time diagram.
- **Coordinates**: the coordinates of a event are numbers (t, x, y, z), the coordinates depend on the frame.
- We usually denote (t, x, y, z) as (x^0, x^1, x^2, x^3).
  Generically, x^α refers to any such coordinates, and x^i or x^j to any of the spatial ones.

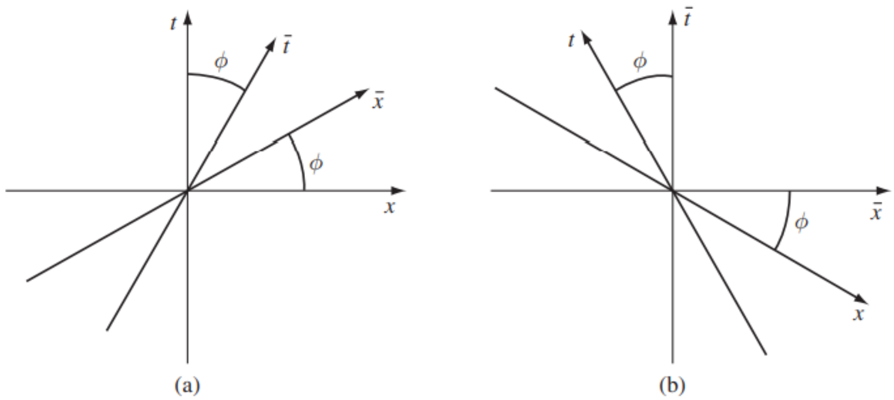
Coordinates of Other Observers

Since all observers look at the same events, it should be possible to draw the coordinate lines of O' inside the diagram of O.
Suppose an observer O uses coordinates t, x and another observer O' (who moves at speed v with respect to O) with coordinates t', x'. We want to draw the primed axis in the O diagram:

- t' axis: This is the locus of events at constant $x' = 0$, the system moves at speed v, these are the points $x = vt$, so a diagonal line with angle $\tan\theta = v$

- x' axis: Those are the locus of points with $t' = 0$

We can use a neat argument using light rays for the plan x'. Then we find:



This diagram are the embodiment that a light beam $c = 1$ has to be a diagonal for every observer.

Lorentz Transformations

Special relativity lies in 3 **postulates** inspired by the Michelson Morley experiment:

1) The laws of physics are the same in all inertial reference frames.
2) The speed of light is the same on all inertial references
3) Space is homogeneous, isotropic and continuous.

We look for the transformation that allows an observer in S to compare measurements of an event to those made by someone in S.

Consider S, S' at standard configuration and consider a clock moving in S freely and uniformly at the curve:

$$x^i = x^i(t) \quad \text{, such that} \quad \frac{dx^i}{dt} = cte$$

We define the **proper time** τ, which is the time measured in the perspective of the clock itself. As time is homogeneous, then:

$$\frac{dt}{d\tau} = cte \quad \Rightarrow \quad \frac{dx^0}{d\tau} = cte$$

Then, we have that:

$$\frac{dx^\mu}{d\tau} = cte \quad \Rightarrow \quad \frac{d^2x^\mu}{d\tau^2} = 0 \;\;,\;\; \mu = 0, 1, 2, 3, \cdots$$

The derivative of $x^{\mu'}$ can be expressed as:

$$\frac{dx^{\mu'}}{d\tau} = \sum_{\nu=0}^{3} \frac{\partial x^{\mu'}}{\partial x^\nu}\frac{dx^\nu}{d\tau} := \frac{\partial x^{\mu'}}{\partial x^\nu}\frac{dx^\nu}{d\tau}$$

If we differentiate again, we obtain:

$$\begin{aligned} \frac{d^2x^{\mu'}}{d\tau^2} &= \frac{\partial x^{\mu'}}{\partial x^\nu}\frac{d^2x^\nu}{d\tau^2} + \sum_{\sigma=0}^{3} \frac{\partial^2 x^{\mu'}}{\partial x^\sigma \partial x^\nu}\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau} \\ &= \frac{\partial^2 x^{\mu'}}{\partial x^\sigma \partial x^\nu}\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau} \quad =!0 \end{aligned}$$

Which leads us to:

$$\frac{\partial^2 x^{\mu'}}{\partial x^\sigma \partial x^\nu} = 0$$

Thus equation tells us that the relation between coordinates must be linear, that is:

$$x^{\mu'} = B^{\mu'}_{\nu} x^\nu$$

where $B := B^{\mu'}_{\nu}$ is a 4x4 matrix.

Now we want to find the rules of the transformations.

As with Galileo, we expect the $x^{0'}$ in S' to be described by:

$$x^1 = \frac{v}{c} x^0$$

that corresponds to $x^{1'} = 0$. We impose this restriction:

$$\begin{aligned} x^{1'} &= B^{1'}_{\nu} x^{\nu} = B^{1'}_0 x^0 + B^{1'}_1 x^1 + B^{1'}_2 x^2 + B^{1'}_3 x^3 \\ &= \left(B^{1'}_0 + \frac{v}{c} B^{1'}_1 \right) x^0 + B^{1'}_2 x^2 + B^{1'}_3 x^3 \doteq 0 \end{aligned}$$

This equality is satisfied only if:

$$B^{1'}_0 + \frac{v}{c} B^{1'}_1 = B^{1'}_2 = B^{1'}_3 = 0$$

Which implies that:

$$x^{1'} = (-\beta x^0 + x^1) B^{1'}_1$$

with $\beta = v/c$.
We notice that $B^{1'}_1$ should depend only on $|\vec{v}|$ (because of isotropy).
Now we imagine whe situation from the point of view of S', where S is moving in the opposite direction. In this case, the condition $x^1 = 0$ leads to:

$$x^1 = B^1_{1'} (\beta x^{0'} + x^{1'})$$

Where now the coefficient is of the inverse matrix.
But because this term only depends on $|\vec{v}|$, then:

$$B^1_{1'} = B^{1'}_1 := \gamma$$

Now we use the second postulate of relativity, of the form:

$$x^1 = x^0 \quad \Leftrightarrow \quad x^{1'} = x^{0'}$$

From which we obtain:

$$\begin{aligned} x^0 &= \gamma(\beta x^{0'} + x^{1'}) = \gamma(\beta + 1) x^{0'} \\ x^{0'} &= \gamma(\beta x^0 + x^1) = \gamma(-\beta + 1) x^0 \end{aligned}$$

When we take the product, we get:

$$x^0 x^{0'} = \gamma^2 (1 - \beta^2) x^0 x^{0'}$$

That is only satisfied if:

$$\gamma^2 = \frac{1}{1-\beta^2}$$

So we get:

$$\boxed{\gamma = \frac{1}{\sqrt{1-\beta^2}}}$$

Now, with the help of $x^{1'} = (-\beta x^0 + x^1)B_1^{1'}$ and $x^1 = B_{1'}^1(\beta x^{0'} + x^{1'})$, we replace the first one in the second and find:

$$x^1 = \gamma(\beta x^{0'} - \gamma\beta x^0 + \gamma x^1) = \gamma\beta x^{0'} - \gamma^2\beta x^0 + \gamma^2 x^1$$

Which can be rewritten as:

$$x^{0'} = \gamma x^0 + \frac{1-\gamma^2}{\gamma\beta} x^1 = \gamma(x^0 - \beta x^1)$$

So time is not absolute, and simultaneity is relative.

Finally, we find the **Lorenz transformation** for an observer S' moving at speed v with respect to S:

$$\begin{aligned}
x^{0'} &= \gamma(x^0 - \beta x^1) \\
x^{1'} &= \gamma(-\beta x^0 + x^1) \\
x^{2'} &= x^2 \\
x^{3'} &= x^3
\end{aligned}$$

Or we can write it as:

$$x' = Bx \quad , \quad B = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

.

We see that if $v > c$ then γ is imaginary, so we say that speed cannot be faster than light.
We notice that if $\beta^2 \simeq 0$ and so $\gamma \simeq 1$, then we get back the Galilean laws.

Inverse Lorentz transformation
If S' moves at speed v relative to S and we now the coordinates x'_i of an event measured in S', then the coordinates in S are:

$$\begin{aligned} x^0 &= \gamma(x^{0'} + \beta x^{1'}) \\ x^1 &= \gamma(\beta x^{0'} + x^{1'}) \\ x^2 &= x^{2'} \\ x^3 &= x^{3'} \end{aligned}$$

Or we can write it as:

$$x = B^{-1}x' \quad , \quad B^{-1} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The transformation rules are really for intervals, not for points themselves. But we write them like this now because we said that S, S' are in standard configuration.

Invariant Interval

Event: Is something that happens at a point in space time (a point in the space time diagram).

Types of separation between two events: Suppose we have two events, such that the difference in the four coordinates are Δx^0 , Δx^1. We consider three types of events:

- **Luminic:** When $\Delta x^0 = \pm \Delta x^1$, so the separation can only be achieved by light.
- **Temporal:** When $|\Delta x^0| > |\Delta x^1|$, so the speed to communicate from one event to the other is $c|\Delta x^1|/|\Delta x^0| < c$, so it is possible to communicate between them.
- **Spatial:** When $|\Delta x^1| > |\Delta x^0|$, so the speed needed to communicate between these two events is $c|\Delta x^1|/|\Delta x^0| > c$ which is impossible, and there is no possible causal relation between them.

Interval: We define the interval of two events $(x_1^0, x_1^1, x_1^2, x_1^3)$, $(x_2^0, x_2^1, x_2^2, x_2^3)$ using $\Delta x^i = x_2^i - x_1^i$ as:

$$\Delta s^2 := (\Delta x^0)^2 - (\Delta x^i)(\Delta x^i)$$

We can define it in a differential way too:

$$ds^2 := (dx^0)^2 - dx^i dx^i$$

Therefore, the type of interval between two events can be characterized as:

- Luminoid: $ds^2 = 0$
- Temporal: $ds^2 > 0$
- Spatial $ds^2 < 0$

Suppose we measured the two luminic events in two systems S, S' and we calculated their intervals in the two systems. Then, the universality of speed of light says that:

$$\Delta s^2 = (\Delta s')^2$$

Theorem: The interval is invariant in any reference system:
That is, for any pair of events in two systems:

$$(\Delta s')^2 = \Delta s^2$$

- **Solution**

  $(\Delta s')^2 = (\Delta x^{0'})^2 - (\Delta x^{1'})^2 - (\Delta x^{2'})^2 - (\Delta x^{3'})^2$
  $= (\gamma \Delta x^0 - \beta\gamma\Delta x^1)^2 - (-\beta\gamma\Delta x^0 + \gamma\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2$
  $= \gamma^2(\Delta x^0)^2 - 2\beta\gamma^2\Delta x^0\Delta x^1 + \beta^2\gamma^2(\Delta x^1)^2 - \beta^2\gamma^2(\Delta x^0)^2 + 2\beta\gamma^2\Delta x^0\Delta x^1 - \gamma^2(\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta$
  $= \gamma^2(1-\beta^2)(\Delta x^0)^2 + \gamma^2(\beta^2 - 1)(\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2$
  $= (\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2$
  $= \Delta s^2$

Invariant Hyperbolae

Consider a curve in O with equation:

$$-t^2 + x^2 = a^2$$

But by invariance of interval, these are the events at an interval a^2 from the origin at O, and so, these same point also fulfill:

$$-t'^2 + x'^2 = a^2$$

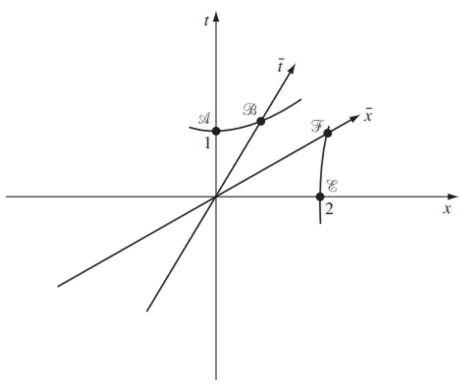
Similarly for any hiperbolae

We can now calibrate the axes of O' in the diagram of O. We draw the axes of O, O' from Os point of view and an invarainte hyperbola of timelike interval -1 from the origin, it has the equation:

$$-t^2 + x^2 = -1$$

We also draw a horizontal hyperbola given by:

$$-t^2 + x^2 = 4$$



Using the hyperbolae through events $\mathcal{A}$ and $\mathcal{E}$ to calibrate the $\bar{x}$ and $\bar{t}$ axes.

The event A has $t = 1$. Similarly, event B lies in the t' axis, so has $x' = 0$. Since the same hyperbola also has the equation $-t'^2 + x'^2 = -1$.
It follows that event B has $t' = 1$, so we have calibrated the unit in t' axis.
We can do the same in the x' axis.

Velocity Composition Law

Suppose a particle has speed w' in the x' direction of O' i.i $\Delta x'/\Delta t' = w'$.
In another frame O, its velocity will be $w = \Delta x/\Delta t$.
If O' moves with velocity v with respect to O, then Lorentz transformations imply:

$$\Delta x = \gamma\beta\Delta x^{0'} + \gamma\Delta x^{1'}$$
$$\Delta t = \gamma\Delta x^{0'} + \gamma\beta\Delta x^{1'}$$

So the velocity in the O frame is:

$$\begin{aligned} w &= \frac{\gamma\beta\Delta x^{0'} + \gamma\Delta x^{1'}}{\gamma\Delta x^{0'} + \gamma\beta\Delta x^{1'}} = \frac{\beta\Delta x^{0'} + \Delta x^{1'}}{\Delta x^{0'} + \beta\Delta x^{1'}} \\ &= \frac{\beta + w'}{1 + \beta w'} \end{aligned}$$

Or, if $c = 1$, we can write it as:

$$w = \frac{v + w'}{1 + vw'}$$

We see that if $w' << 1$ and $v << 1$ then $w = v + w'$ as we should expect.

0.1 Los efectos Fundamentales

Los Efectos Fundamentales

- **Pérdida de Simultaneidad:** Dos eventos suceden simultáneamente en S'. Entonces $x' = x'$ y $t' = 0$. Luego, según la ecuación L2, tenemos que $t = \gamma v x'/c^2$. Por lo que hay una diferencia en los tiempos medidos, los eventos no son simultáneos.

- **Dilatación del Tiempo:** Dos eventos ocurren en el mismo lugar en S'. Entonces tenemos que $x' = 0$, $t' = t'$. Usado la ecuación L2, llegamos a que $t = \gamma t'$. Por lo que desde el punto de vista no primado, es mayor el tiempo de separación.

  Lo mismo si intercambiamos S y S'. En este caso, dos eventos en el mismo lugar de S tienen $x = 0$ y tienen $t = t$. Entonces la ec. L-2 nos dice que $t' = \gamma t$.

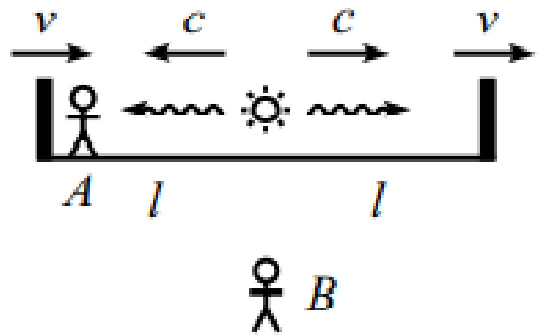
  Notar que los dos casos no son válidos a la vez.

- **Contracción de Longitud:** Consideramos un palo quieto en S', donde tiene longitud l'. Queremos medirlo en S. Para medirlo en S, necesitamos la distancia entre sus puntas al mismo tiempo, es decir $t = 0$ y $x = x$. Usando la ec L-1, llegamos a que: $x' = \gamma x$.

Entonces, tenemos que $l = l'/\gamma$

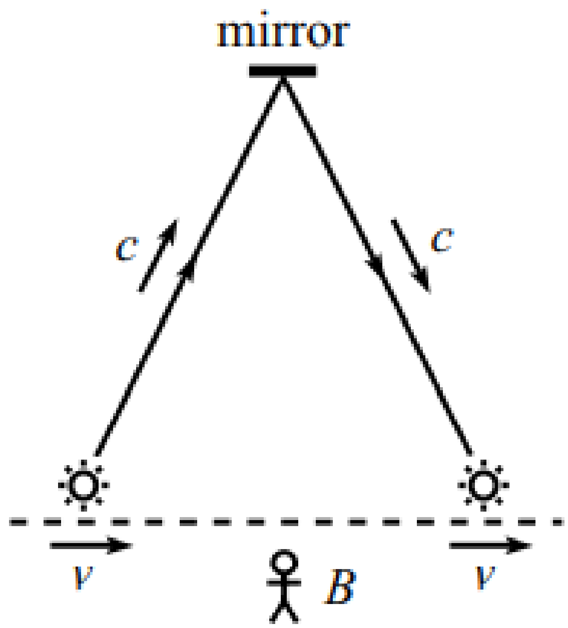
Al igual que en el tiempo, se puede hacer al revés pero nunca al derecho y al revés a la vez.

Pérdida de Simultaneidad



Consideramos la imagen de arriba, en la que A está en un tren de longitud $2l$ que se mueve a v respecto a B (que está en el piso)(la imagen es en el sistema de B).
Digamos que A tiene una fuente de luz en el centro y manda dos rayos de luz, entonces, desde el punto de vista de A, ambos rayos alcanzan las paredes al mismo tiempo en un tiempo l'/c (l' es la longitud en el punto de vista de A).
Desde el punto de vista de B, ambos rayos se mueven a la velocidad c por el postulado. Pero el primer rayo tiene una velocidad de $c - v$ respecto a la pared y el segundo $c + v$. Entonces toman un tiempo $t_l = \dfrac{l}{c+v}$ y $t_r = \dfrac{l}{c-v}$. Estos tiempos no son iguales, por lo que se pierde la simultaneidad.

Dilatación del Tiempo



A se encuentra en un tren y B lo ve desde el piso moverse a una velocidad v. A crea un reloj de luz.
En el punto de vista de A, el tren está en reposo y la luz toma un tiempo $t_A = \frac{2h}{c}$ en hacer una vuelta.
Desde el punto de vista de B, la luz se mueve en diagonal, pero la velocidad aún es c, por lo que $t_B = \frac{2h}{\sqrt{c^2 - v^2}}$.
Esto implica que:

$$t_B = \gamma t_A$$
$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Por lo tanto: **eventos que están en la misma coordenada espacial desde el punto de vista de** A y que toman un intervalo de tiempo t_A, desde el punto de vista de B toman un tiempo mayor de $t_B = \gamma t_A$.
Se vale también al revés, si dos eventos en la misma posición en el sistema B toman un tiempo t_B, entonces desde el punto de vista de A, toman un tiempo $t_A = \gamma t_B$.
Conste que $t_A = \gamma t_B$ y $t_B = \gamma t_A$ tienen distintas hipótesis y no son válidas simultáneamente.

Contracción de la longitud

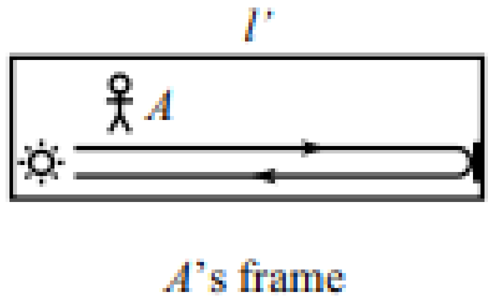
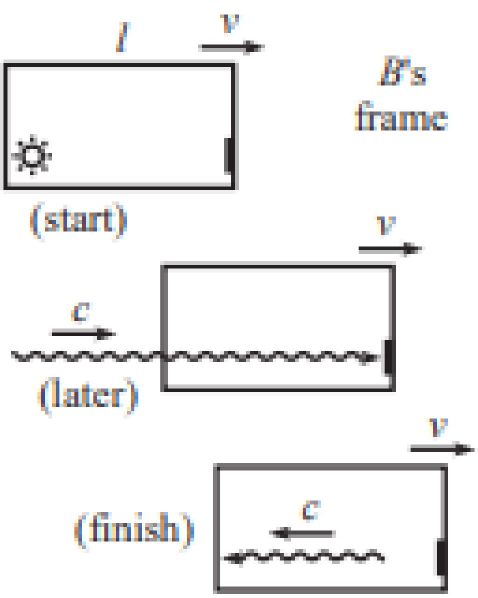


Fig. 11.12



A se mueve en un tren que él mide que tiene una longitud de l' y B está en el piso. El tren se mueve a v respecto con B.

Para medir el tren, A lanza un rayo de luz que toma un tiempo $t_A = \frac{2l'}{c}$.

Desde el punto de vista de B, el tren mide l (no sabemos cuánto es esto) y el tiempo es $t_B = \frac{l}{c-v} + \frac{l}{c+v} = \frac{2lc}{c^2-v^2} = \frac{2l}{c}\gamma^2$.

Pero como se cumplen las condiciones de $t_B = \gamma t_A$ (tomar en cuenta que la salida y llegada de la luz de la fuente en el frame de A son en la misma posición en este frame).

Usando esto, llegamos a:

$$l = \frac{l'}{\gamma}$$

Entonces, si alguien ve algo que se mueve a velocidad v, ve su longitud contraerse por γ.

Ejemplo Muones

En la atmósfera se crean muones con una vida media (desde su propio sistema de referencia) de $2 \times 10^{-6}s$ y tiene que recorrer una distancia de $50km$ hasta la tierra (medido desde nuestro punto de vista). Se mueve a 0.99998c (medido en nuestro punto de vista obvio), llega?

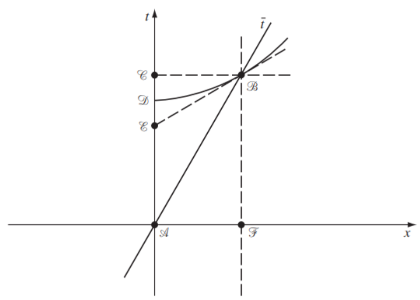
Desde el frame de la tierra: Desde este frame, el muón se mueve y entonces los $2 \cdot 10^{-6}s$ del muón en realidad son $t_T = \gamma * 2 \cdot 10^{-6}s$ que es 160 veces más. Entonces, la distancia que recorre desde nuestro frame es $d = vt_T$ que es mayor a 50km

Desde el frame del Muón: Desde este frame, parece que la tierra se mueve hacia el muón y las distancias se acortan a $50km/\gamma$. Entonces, en el tiempo $t_\mu = 2 \cdot 10^{-6}s$, se mueve lo suficiente como para recorrer la distancia $50km/\gamma$.

Failure of Relativity?

We have shown that if O' moves relative to O, the clocks of O' will be measured by O to be running more slowly than those of O and this seems to violate the first principle of relativity.

Different observers will agree in the outcome of certain kind of experiments, like flipping a coin for example.
Similarly, if two clocks are right next to each other, all observers will agree which is reading an earlier time than the other.
But to now the rate at which the clocks are running, we need to compare the same two clocks on two different occasions, and if the clocks are moving relative to one another, they can only be next to each other on one occasion.

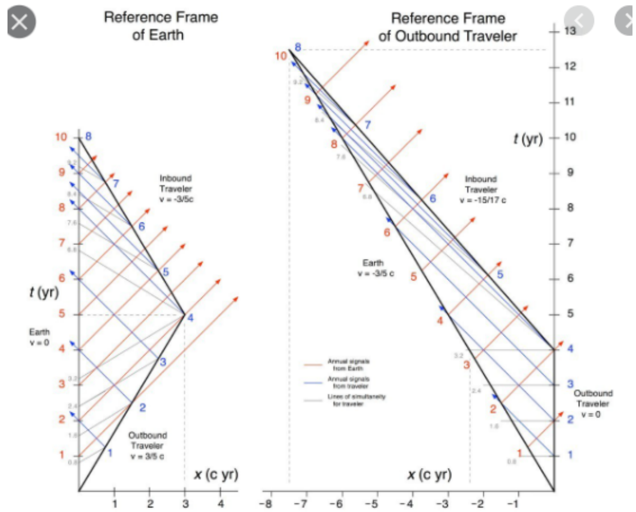


The proper length of $\mathcal{AB}$ is the time ticked by a clock at rest in $\bar{\mathcal{O}}$, while that of $\mathcal{AC}$ is the time it takes to do so as measured by $\mathcal{O}$.

We first analyze O measurement.
O' has a single clock which travels from A to B.
First, at A O and O' check their clocks to be at 0.
The second clock used by O is the one initially at F at $t = 0$, which coincides with O' clock at B.
This clock of O moves vertically in the dashed line.
What O says is that both clocks at A read $t = 0$, while at B the clock of O' reads $t' = 1$, while that of O reads $t = (1 - v^2)^{-1/2}$ (a later time).

Clearly O' agrees with this, since he is as capable as O of just looking at clocks. But for O to conclude that O' clock runs slowly he needs to know that both his clocks are synchronized, which they are.

But O' doesn't accept this, for him, O clocks are not synchronized.
The dotted line through B is the locus of events that O' regards as simultaneous to B. Event E is on this line, Event E is on this line, and is the tick of O first clock, which O' measures to be simultaneous with event B.
A simple calculation shows this to be at $t = (1 - v^2)^{-1/2}$. So O' can reject O measurement since the clocks involved are not synchronized.


Reference Frame
of Earth
Reference Frame
of Outbound Traveler
Inbound
Traveler
v = -3/5c
Earth
v = 0
Outbound
Traveler
v = 3/5 c
t (yr)
x (c yr)
Inbound
Traveler
v = -15/17 c
Earth
v = -3/5 c
Outbound
Traveler
v = 0
Annual signals
from Earth
Annual signals
from traveler
Lines of simultaneity
for traveler

Vector Analysis in SR

A vector is something whose components transform as do the coordinates under a coordinate transformation.
The **typical vector** is the displacement vector, which points from one event to another:

$$\Delta\vec{x} \rightarrow_O (\Delta t, \Delta x, \Delta y, \Delta z)$$

These are the components in the frame O.
the vector $\Delta\vec{x}$ is itslef an arrow, while the collection of components merely represents the vector on a given frame.
We can write it as:

$$\Delta\vec{x} \rightarrow_O \{\Delta x^\alpha\}$$

Where $\{\Delta x^\alpha\}$ means Δx^0 , Δx^1 , Δx^2 , Δx^3
If we want the components in a primed system, we write:

$$\Delta\vec{x} \rightarrow_{O'} \{\Delta x^{\alpha'}\}$$

The new components of the vector in the frame O' (which moves at speed v respect to O) are obtained by Lorentz transformation:

$$\Delta x^{\alpha'} = \Lambda^{\alpha'}_{\beta} \Delta x^\beta$$

Where $\Lambda^{\alpha'}_{\beta}$ are the transformation numbers from the normal system to the primed one.
So, in this case:

- $\Lambda^{0'}_{0} = \gamma$, $\Lambda^{1'}_{0} = -\beta\gamma$
- $\Lambda^{0'}_{1} = -\beta\gamma$, $\Lambda^{1'}_{1} = \gamma$
- $\Lambda^{2'}_{2} = 1$, $\Lambda^{3'}_{3} = 1$
- All the other numbers are 0

Genetal Vector:

Therefore, we define a **general vector (4-vector)** as any collection of numbers (in a frame O):

$$\vec{A} \rightarrow_O (A^0, A^1, A^2, A^3) = \{A^\alpha\}$$

Such that when changing to a frame O', the new coordinates are given by:

$$A^{\alpha'} = \lambda^{\alpha'}_{\beta} A^\beta$$

If $\vec{A}$ and $\vec{B}$ are vectors and μ is a number, then $\vec{A}+\vec{B}$ and $\mu\vec{A}$ are also vectors, defined by (their coordinate representation in O'):

$$\vec{A}+\vec{B} \to_O = (A^0+B^0, A^1+B^1, A^2+B^2, A^3+B^3)$$
$$\mu\vec{A} \to_O (\mu A^0, \mu A^1, \mu A^2, \mu A^3)$$

It is easy to prove this. For if $\vec{A}$ and $\vec{B}$ are vectors. Then $A^{\mu'} = \Lambda^{\alpha'}_{\beta} A^{\beta}$ and $B^{\mu'} = \Lambda^{\alpha'}_{\beta} B^{\beta}$. Therefore, if we define $\vec{A}+\vec{B}$ in O as before, by lineality of the coordinate transformation: $(\vec{A}+\vec{B})_{O'} = \Lambda\vec{A}_O + \Lambda\vec{B}_O = \Lambda(\vec{A}+\vec{B})_O$.

Vector Algebra

Basis Vectors

In any frame O there are four special vectors, defined by their components:

$$\vec{e}_0 \to_O (1,0,0,0)$$
$$\vec{e}_1 \to_O (0,1,0,0)$$
$$\vec{e}_2 \to_O (0,0,1,0)$$
$$\vec{e}_3 \to_O (0,0,0,1)$$

We can also define it in other frames O'.
Generally $\vec{e}_O \neq \vec{e}_{O'}$

If $\vec{A} \to_O (A^0, A^1, A^2, A^3)$ then:

$$\vec{A} = A^0\vec{e}_0 + A^1\vec{e}_1 + A^2\vec{e}_2 + A^3\vec{e}_3$$
$$\vec{A} = A^{\alpha}\vec{e}_{\alpha}$$

Transformation of Basis Vectors

The representation of $\vec{A} = A^{\alpha}\vec{e}_{\alpha}$ is equally true in another frame:

$$\vec{A} = A^{\alpha}\vec{e}_{\alpha}$$
$$= A^{\alpha'}\vec{e}_{\alpha'}$$

The expressions are not obtained merely by relabeling dummy indices, they have different meanings.
$\{A^{\alpha'}\}$ are different components than $\{A^{\alpha}\}$, just as the vectors $\{\vec{e}_{\alpha'}\}$ are different from $\{\vec{e}_{\alpha}\}$. Although they are different unit vectors and components, they sum to the same vector $\vec{A}$ in 4-space.

Therefore, we can find a relation between the unit vectors $\vec{e}_\alpha$ and $\vec{e}_{\alpha'}$:

$$\begin{aligned}
&A^\alpha \vec{e}_\alpha = A^{\alpha'} \vec{e}_{\alpha'} \\
\Rightarrow \quad &A^\alpha \vec{e}_\alpha = \Lambda^{\alpha'}_{\beta} A^\beta \vec{e}_{\alpha'} \\
\Rightarrow \quad &A^\alpha \vec{e}_\alpha = A^\beta \Lambda^{\alpha'}_{\beta} \vec{e}_{\alpha'}
\end{aligned}$$

So, we can find the transformation rules for the unit vectors:

$$\vec{e}_\alpha = \Lambda^{\beta'}_{\alpha} \vec{e}_{\beta'}$$

Therefore, if $A^\alpha \vec{e}_\alpha = A^{\alpha'} \vec{e}_{\alpha'}$, then the things transform as:

$$\begin{aligned}
\vec{A}^{\beta'} &= \Lambda^{\beta'}_{\alpha} A^\alpha \\
\vec{e}_\alpha &= \Lambda^{\beta'}_{\alpha} \vec{e}_{\beta'}
\end{aligned}$$

Therefore, to know the transformation rules, we only need to match up the indexes such that they cancel up and down. Note that the transformation is the inverse and transpose.

Example
Let O' move with velocity v in the x direction relative to O. Then the matrix for $[\Lambda^{\beta'}_{\alpha}]$ is:

$$(\Lambda^{\beta'}_{\alpha}) = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then, if for example $\vec{A} \rightarrow_O (5, 0, 0, 2)$, we find its components in O' by:

$$\begin{aligned}
A^{0'} &= \Lambda^{0'}_{\alpha} A^\alpha = \gamma * 5 + (-v\gamma) * 0 = 5\gamma \\
A^{1'} &= -5v\gamma \\
A^{2'} &= 0 \\
A^{3'} &= 2
\end{aligned}$$

Therefore:

$$\vec{A} \rightarrow_{O'} \rightarrow (5\gamma, -5v\gamma, 0, 2)$$

The basis vectors are expressible as:

$$\vec{e}_\alpha = \Lambda^{\beta'}_{\alpha} \vec{e}_{\beta'}$$

Then:

$$\begin{aligned}
\vec{e}_0 &= \Lambda^{\beta'}_{0} \vec{e}_{\beta'} = \Lambda^{0'}_{0} \vec{e}_{0'} + \Lambda^{1'}_{0} \vec{e}_{1'} + \Lambda^{2'}_{0} \vec{e}_{2'} + \Lambda^{3'}_{0} \vec{e}_{3'} = \gamma \vec{e}_{0'} - v\gamma \vec{e}_{1'} \\
\vec{e}_1 &= \cdots = -v\gamma \vec{e}_{0'} + \gamma \vec{e}_{1'} \\
\vec{e}_2 &= \vec{e}_{2'} \\
\vec{e}_3 &= \vec{e}_{3'}
\end{aligned}$$

Inverse Transformation

We know that $\Lambda_{\alpha}^{\beta'}$ is a transformation that depends only on speed $\vec{v}$.
Then:

$$\vec{e}_{\alpha} = \Lambda_{\alpha}^{\beta'}(\vec{v})\vec{e}_{\beta'}$$

And the inverse transformation says:

$$\vec{e}_{\mu'} = \Lambda_{\mu'}^{\nu}(-\vec{v})\vec{e}_{\nu}$$

Where the transformation $\Lambda_{\mu'}^{\nu}$ is the inverse of the matrix $\Lambda_{\alpha}^{\beta'}$.
And it is exactly the same matrix but with $-v$ instead v.

Matrix $\Lambda_{\alpha}^{\beta'}$ goes from the normal components to the primed ones (or from the primed unit vectors to the unprimed ones). It is obtained by using the velocity of O' with respect to O.
Matrix $\Lambda_{\mu'}^{\nu}$ goes from the primed components to the unprimed ones (or from the unprimed unit vectors to the primed ones). It is obtained by using the velocity of $'$ with respect to O'.

The matrices are inverses, in the sense that:

$$\Lambda_{\alpha}^{\beta'}(\vec{v})\Lambda_{\beta'}^{\nu}(-\vec{v}) = \delta_{\alpha}^{\nu}$$
$$\Lambda_{\beta'}^{\nu}(-\vec{v})\Lambda_{\alpha}^{\beta'}(\vec{v}) = \delta_{\alpha}^{\nu}$$

With this inverse, we can also rewrite the formulas for change of components:

$$\Lambda_{\beta'}^{\nu}(-\vec{v})A^{\beta'} = A^{\nu}$$

Finally, all the formulas together are:

If O' is a system moving at speed v to the right with respect to O, then we have the transformation matrix is $(\Lambda_{\alpha}^{\beta'}) = \begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix}$
And $\Lambda_{\mu'}^{\nu}$ is the inverse matrix which only has $-v$ instead of v in every place.

Then, the components of a vector $\vec{A}$ follow the rule of transformation:

$$A^{\beta'} = \Lambda_{\alpha}^{\beta'}A^{\alpha}$$

And the vectors transform as:

$$\vec{e}_{\mu'} = \Lambda_{\mu'}^{\nu}\vec{e}_{\nu}$$

The transformations rules for the inverse sense are:

$$A^{\nu} = \Lambda_{\beta'}^{\nu}A^{\beta'}$$
$$\vec{e}_{\alpha} = \Lambda_{\alpha}^{\beta'}\vec{e}_{\beta'}$$

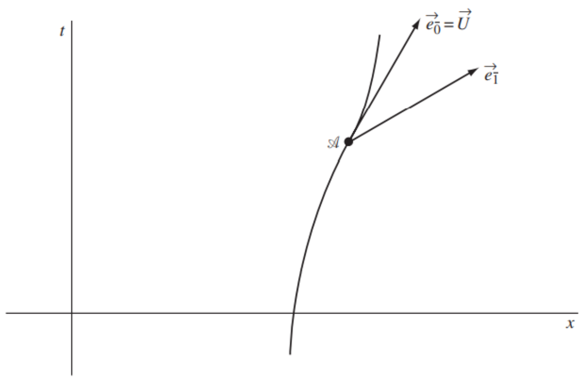
The Four Velocity

In 4-space we define the four velocity $\vec{U}$ to be a vector tangent to the world line of the particle and of such length that it stretches one unit of time in that particle's frame.

If we are in the inertial frame of the object, the four velocity will be simply the vector $\vec{e}_0$

An accelerated particle has no inertial frame in which it is always at rest. However, there is an inertial frame which momentarily has the same velocity of the particle. This is the **Momentarily comoving reference frame (MCFR)**.

The 4-velocity of an accelerated particle at one event is defined as the $\vec{e}_0$ basis vector of its MCRF at that event. (because it point in the direction the vector would be in 1s). This vector is tangent to the curve line of the particle in any frame.



The four-velocity and MCRF basis vectors of the world line at $\mathcal{A}$

The 4-momentum

The four momentum $\vec{p}$ is defined as:

$$\vec{p} = m\vec{U}$$

Where m is the **rest mass** of the particle (the mass measured in its rest frame). In some frame O it has components denoted by:

$$\vec{p} \rightarrow_O (E, p^1, p^2, p^3)$$

We call p^0 the **energy** E of the particle in the frame O (which in the rest frame of the particle is equal to $m(1) = m$)

Example:
A particle of rest mass m moves with velocity $\vec{v}$ in the x direction of frame O. What are the components of the 4-momentum and velocity? Its rest frame O' has time basis vector $\vec{e}_{0'}$ so, by definition of $\vec{p}$ and $\vec{U}$, we have:

$$\vec{U} = \vec{e}_{0'} \quad , \quad \vec{p} = m\vec{U} = m\vec{e}_{0'}$$

Now, due to the transformations laws, we have that $\vec{e}_{\mu'} = \Lambda^{\nu}_{\mu'}\vec{e}_{\nu}$, so in particular $\vec{e}_{0'} = \Lambda^{\nu}_{0'}\vec{e}_{\nu}$
So the velocity in the lab frame O is:

$$\begin{aligned}\vec{e}_{0'} &= \Lambda^{\nu}_{0'}\vec{e}_{\nu}\\ &= \Lambda^{0}_{0'}\vec{e}_{0} + \Lambda^{1}_{0'}\vec{e}_{1} + \cdots\\ &= \gamma\vec{e}_{0} + v\gamma\vec{e}_{1}\end{aligned}$$

Other form of doing it: We know that the vector is $\vec{U} \rightarrow_{O'} (1,0,0,0)$ in the rest frame of the particle, it has coordinates $U^{0'} = 1, U^{1'} = 0, U^{2'} = 0, U^{3'} = 0$. So the vector in the lab frame O is:

$$U^{\nu} = \Lambda^{\nu}_{\beta'}U^{\beta'}$$

Then, we have:

$$\begin{aligned}U^0 &= \Lambda^0_{\beta'}U^{\beta'} = \Lambda^0_{0'}U^{0'} + \Lambda^0_{1'}U^{1'} + \Lambda^0_{2'}U^{2'} + \Lambda^0_{3'}U^{3'} = \gamma(1) = \gamma\\ U^1 &= \Lambda^1_{\beta'}U^{\beta'} = \Lambda^1_{0'}U^{0'} + \Lambda^1_{1'}U^{1'} + \Lambda^1_{2'}U^{2'} + \Lambda^1_{3'}U^{3'} = v\gamma(1) = v\gamma\\ U^2 &= \Lambda^2_{\beta'}U^{\beta'} = 0\\ U^3 &= \Lambda^3_{\beta'}U^{\beta'} = 0\end{aligned}$$

So the vector $\vec{U}$ is:

$$\begin{aligned}\vec{U} &\rightarrow_{O'} (1,0,0,0)\\ \vec{U} &\rightarrow_{O} (\gamma, \gamma v, 0, 0)\end{aligned}$$

And the momentum is:

$$\begin{aligned}\vec{p} &\rightarrow_{O'} (m,0,0,0)\\ \vec{p} &\rightarrow_{O} (\gamma m, \gamma m v, 0, 0)\end{aligned}$$

For small v, the spatial components of $\vec{U}$ are $(v,0,0)$ and the spatial of $\vec{p}$ are $(mv,0,0)$. And for small v, the energy is:

$$E := p^0 = m\gamma = m(1-v^2)^{-1/2} \simeq m + \frac{1}{2}mv^2$$

.

Therefore, in general:

- **The Rest Energy** is m

- **The energy of a particle seen to move at speed v is:** $m\gamma$.
  Which is the sum of rest energy plus kinetic
  (γmc^2 in normal units)

- **The kinetic energy of a particle seen to move at speed v is:** $(\gamma - 1)m$

- **The momentum of a particle seen to move at speed $\vec{v}$ is:** $\vec{p} = \gamma m\vec{v}$

By invariance of inner product, if a general 4-momentum is (E, p^x, p^y, p^z), then, in the rest-frame it is $(m, 0, 0, 0)$, so we have that:

$$E^2 - p^2 = m^2$$

Conservation of 4-momentum

The correct relativistic law for conservation of momentum is that 4-momentum is conserved. We define the total momentum of a set of particles as:

$$\vec{p} = \sum_{allparticles} \vec{p}_{(i)}$$

Where $\vec{p}_{(i)}$ is the ith particle's momentum.

This law is an extra postulate, inspired by the fact that the law reduces to a correct one for small v

The vector $\vec{p}$ is the same in all frames (just different coordinates).

Scalar Product

Magnitude of a vector

By analogy of the interval, we define the **magnitude** of A as:

$$\vec{A}^2 = -(A^0)^2 + (A^1)^2 + (A^2)^2 + (A^3)^2$$

Because the components transform as Lorentz transformations, we have a guarantee that $\vec{A}^2$ **doesn't depend on frame**.
The magnitude of $\vec{A}^2$ doesn't have to be positive

Scalar Product of two vectors

We define it as:

$$\vec{A} \cdot \vec{B} = -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3$$

This number is **also conserved in all frames of reference**h. This can be proven by seeing that $(\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B}) = \vec{A}^2 + \vec{B}^2 + 2\vec{A} \cdot \vec{B}$
But because $(\vec{A} + \vec{B})^2$, $\vec{A}, \vec{B}$ are conserved, then $\vec{A} \cdot \vec{B}$

Two vectors are **orthogonal** if $\vec{A} \cdot \vec{B} = 0$

Example
The basis vectors of a frame O satisfies:

$$\begin{aligned} \vec{e}_0 \cdot \vec{e}_0 &= -1 \\ \vec{e}_1 \cdot \vec{e}_1 &= \vec{e}_2 \cdot \vec{e}_2 = \vec{e}_3 \cdot \vec{e}_3 = 1 \\ \vec{e}_\alpha \cdot \vec{e}_\beta = 0 \quad , \quad \alpha &\neq \beta \end{aligned}$$

We can write all this as:

$$\vec{e}_\alpha \cdot \vec{e}_\beta = \eta_{\alpha\beta}$$

Where $\eta_{00} = -1$ and otherwise it is just as a Kronecker delta.
Remember that this is true for any frame, it doesn't have to be the frame the unit vector are on.

Example: The four velocity $\vec{U}$ of a particle is just the time basis vector of its MCRF, so we have:

$$\vec{U} \cdot \vec{U} = -1$$

Applications

Four - velocity and acceleration as derivatives

Suppose a particle makes an infinitesimal displacement $d\vec{x}$, whose components in O are (dt, dx, dy, dz). The magnitude squared of this displacement is just:

$$ds^2 = d\vec{x} \cdot d\vec{x} = -dt^2 + dx^2 + dy^2 + dz^2$$

Since the worl line is timelike, this is negative, so we define the proper time $d\tau$ by:

$$(d\tau)^2 = -d\vec{x} \cdot d\vec{x}$$

Now we consider the vector $d\vec{x}/d\tau$, where $d\tau$ is the square root of what we wrote before. This vector is tangent ti the world line since it is a multiple of $d\vec{x}$. Its magnitude is:

$$\frac{d\vec{x}}{d\tau} \cdot \frac{d\vec{x}}{d\tau} = \frac{d\vec{x} \cdot d\vec{x}}{(d\tau)^2} = -1$$

It is therefore a timelike vector of unit magnitude tangent to the world line. In an MCRF,

$$d\vec{x} \rightarrow_{MCDR d\tau = dt} \rightarrow (dt, 0, 0, 0)$$

So that:

$$\frac{d\vec{x}}{d\tau} = \rightarrow_{MCRF} (1, 0, 0, 0)$$

Or:

$$\frac{d\vec{x}}{d\tau} = (\vec{e}_0)_{MCRF}$$

This is just the definition of the 4-velocity, so:

$$\vec{U} = d\vec{x}/d\tau$$

Moreover, we can examine the vector:

$$\frac{d\vec{U}}{d\tau} = \frac{d^2\vec{x}}{d\tau^2}$$

First we consider

$$\frac{d}{d\tau}(\vec{U} \cdot \vec{U}) = 2\vec{U} \cdot \frac{d\vec{U}}{d\tau}$$

But since $\vec{U} \cdot \vec{U} = -1$ is constant, we have $\vec{U} \cdot \dfrac{d\vec{U}}{d\tau} = 0$
Since in the MCRF, $\vec{U}$ has only a zero component, this orthogonality means that:

$$\frac{d\vec{U}}{d\tau} \rightarrow_{MCRF} (0, a^1, a^2, a^3)$$

Which is defined as the **Acceleration 4-vector** $\vec{a}$
As we said it is defined by:

$$\vec{a} = \frac{d\vec{U}}{d\tau}$$

And fulfills:

$$\vec{U} \cdot \vec{a} = 0$$

Energy and Momentum

Consider a particle with momentum $\vec{p}$. Then:

$$\vec{p} \cdot \vec{p} = m^2 \vec{U} \cdot \vec{U} = -m^2$$

But $\vec{p} \cdot \vec{p} = -E^2 + (p^1)^2 + (p^2)^2 + (p^3)^2$, so:

$$E^2 = m^2 + \sum_{i=1}^{3} (p^i)^2$$

This is the familiar expression of the total energy of a particle.

Now suppose an observer O' moves with four velocity $\vec{U}_{obs}$ not necessarily equal to the particle's four-velocity. Then:

$$\vec{p} \cdot \vec{U}_{obs} = \vec{p} \cdot \vec{e}_{0'}$$

Where $\vec{e}_{0'}$ is the basis vector of the frame of the observer. In that frame, the 4-momentum has components:

$$\vec{p} \rightarrow_{O'} (E', p^{1'}, p^{2'}, p^{3'})$$

Therefore, we obtain:

$$-\vec{p} \cdot \vec{U}_{obs} = E'$$

It says that the energy of the particle relative to the observer E' can be computed by anyone in any frame by taking $\vec{p} \cdot \vec{U}_{obs}$.
This is called a ' frame invariant ' expression for the energy relative to the observer.

Photons

No four velocity

Photons move on null lines, so, for a photon path:

$$d\vec{x} \cdot d\vec{x} = 0$$

Therefore $d\tau$ is 0 and therefore, the four velocity $\vec{U} = d\vec{x}/d\tau$ cannot be defined.
Note that we can find a vector tangent to the path (which is a line) (for example $d\vec{x}$ itself) but they all have 0 magnitude.

Four Momentum

The four momentum is not a unit vector. Instead it is a vector whose components in some frame give the energy and momentum relative to the frame.
If a photon carries energy E in some frame, then $p^0 = E$ in that frame. If it moves in the x direction, then $p^y = p^z = 0$. And in order for the four momentum to be parallel to its world line (hence be null) we must have $p^x = E$. This ensures that:

$$\vec{p} \cdot \vec{p} = -E^2 + E^2 = 0$$

So photons have spatial momentum equal to their energy.
We know from QM that a photon has energy:

$$E = hv$$

Therefore, the **Momentum** of a photon is also $p = hv$ (and when correcting for coordinates, we divide by c and get $p = h/\lambda$ as De Broglie says).

This equation and Lorentz gives the **Doppler Effect**
Suppose in frame O a photon has freq ν and moves in the x direction. Then, in O', which has velocity v in the x direction relative to O, the energy is:

$$\begin{aligned} E' &= \frac{E}{\sqrt{1-v^2}} - \frac{p^x v}{\sqrt{1-v^2}} \\ &= \frac{h\nu}{\sqrt{1-v^2}} - \frac{h\nu v}{\sqrt{1-v^2}} \end{aligned}$$

Setting this equal to $h\nu'$ (the new freq), then:

$$\frac{\nu'}{\nu} = \frac{1-v}{\sqrt{1-v^2}} = \sqrt{\frac{1-v}{1+v}}$$

Zero Rest-Mass Particles

The rest mass of a photon is 0, since:

$$m^2 = -\vec{p} \cdot \vec{p} = 0$$

Any particle with null mass must have null 4-momentum and conversely.

Excercises

12 **Given $\vec{A} \rightarrow_O (0, -2, 3, 5)$, find**
The components of $\vec{A}$ in O' which moves at speed 0.8 relative to O in the direction x

We have that:

$$\begin{aligned} A^{\mu'} &= \Lambda^{\mu'}_{\alpha} A^{\alpha} \\ &= \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ 3 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 1.666 & -1.333 & 0 & 0 \\ -1.3333 & 1.6666 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 2.666 \\ -3.333 \\ 3 \\ 5 \end{pmatrix} \end{aligned}$$

b) Find the components of $\vec{A}$ in O'' which moves at speed 0.6 relative to O' in the positive x direction
We can transform the components of A in O'

$$\begin{aligned} A^{\mu''} &= \Lambda^{\mu''}_{\alpha'} A^{\alpha'} \\ &== \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 8/3 \\ -10/3 \\ 3 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 5/4 & -3/4 & 0 & 0 \\ -3/4 & 5/4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 8/3 \\ -10/3 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 35/6 \\ -37/6 \\ 3 \\ 5 \end{pmatrix} \end{aligned}$$

We could also calculate it as directly from the components of A in O using the speed $v'' = \dfrac{v + v'}{1 + vv'} = 0.9459$

14) **The following is a Lorentz Transformation from O to O':**

$$\begin{pmatrix} 1.25 & 0 & 0 & 0.75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.75 & 0 & 0 & 1.25 \end{pmatrix}$$

What is the velocity of O' relative to O
We need to find, after a second in Os frame, the position of O' (that is the position with $x' = y' = z' = 0$)

$$\begin{pmatrix} k \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.25 & 0 & 0 & 0.75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.75 & 0 & 0 & 1.25 \end{pmatrix} \begin{pmatrix} 1 \\ v_x \\ v_y \\ v_z \end{pmatrix}$$

We see that $0.75 + 1.25 v_z = 0$, so $v_z = -0.6$

- **Find the general Lorentz transformation for going from coordinates in O to coordinates in O' where O' moves with velocity (v_x, v_y, v_z) respect to O (obviously $|v| < 1$**
  We can compose transformations corresponding first to a frame moving at v_x, then one at v_y and then one at v_z:

$$\begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = \begin{pmatrix} \gamma_z & 0 & 0 & -\gamma v_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma v_z & 0 & 0 & \gamma_z \end{pmatrix} \begin{pmatrix} \gamma_y & 0 & -\gamma_y v_y & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma_y v_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma_x v_x & 0 & 0 \\ -\gamma_x v_x & \gamma_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} \gamma & -\gamma v_x & -\gamma v_y & -\gamma v_z \\ -\gamma v_z & 1 + (\gamma - 1) v_x^2/v^2 & (\gamma - 1)\dfrac{v_x v_y}{v^2} & (\gamma - 1)\dfrac{v_x v_z}{v^2} \\ -\gamma v_y & (\gamma - 1)\dfrac{v_x v_y}{v^2} & 1 + (\gamma - 1)\dfrac{v_y^2}{v^2} & (\gamma - 1)\dfrac{v_y v_z}{v^2} \\ -\gamma v_z & (\gamma - 1)\dfrac{v_x v_z}{v^2} & (\gamma - 1)\dfrac{v_y v_z}{v^2} & 1 + (\gamma - 1)\dfrac{v_z^2}{v^2} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

15) **a) Compute the four velocity components in O of a particle whose speed in O is v in the x direction, by using the Lorentz transformation from the rest frame of the particle**
The answer will obviously be $(\gamma, \gamma v, 0, 0)$
To actually calculate it, we see that the four velocity in O' (rest frame of particle) is as always $(1, 0, 0, 0)$ and the transformation from O' to O is $\Lambda^{\mu}_{\beta'}$, so:

$$\begin{pmatrix} U^0 \\ U^x \\ U^y \\ U^z \end{pmatrix}_O = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So we see that $\vec{U}_O = (\gamma, \gamma v, 0, 0)$ as expected.

b) **Generalize this result to find the four velocity components when the particle has an arbitrary velocity $\vec{v}$ with $|v| < 1$**

We use the general Lorenz transformation as obtained earlier (but inverting the direction):

$$\begin{pmatrix} U^0 \\ U^1 \\ U^2 \\ U^3 \end{pmatrix}_O = \begin{pmatrix} \gamma & \gamma v_x & \gamma v_y & \gamma v_z \\ \gamma v_z & 1+(\gamma-1)v_x^2/v^2 & (\gamma-1)\frac{v_x v_y}{v^2} & (\gamma-1)\frac{v_x v_z}{v^2} \\ \gamma v_y & (\gamma-1)\frac{v_x v_y}{v^2} & 1+(\gamma-1)\frac{v_y^2}{v^2} & (\gamma-1)\frac{v_y v_z}{v^2} \\ \gamma v_z & (\gamma-1)\frac{v_x v_z}{v^2} & (\gamma-1)\frac{v_y v_z}{v^2} & 1+(\gamma-1)\frac{v_z^2}{v^2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So, as fucking expected:

$$\vec{U}_O = \begin{pmatrix} \gamma \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix}$$

16) **Derive the Einstein velocity addition formula by performing a lorentz transformation with velocity v on the four velocity of a particle whose speed in the original frame was W**

The speed in the frame O' is W, so the 4-velocity is $\vec{U}_{O'} = (\gamma_W, \gamma_W W, 0, 0)$

We now transform to a O from which O' seems to be moving at v, so we use inverse Lorentz.

$$\begin{pmatrix} U^{0'} \\ U^{1'} \\ U^{2'} \\ U^{3'} \end{pmatrix} = \begin{pmatrix} \gamma_v & v\gamma_v & 0 & 0 \\ \gamma_v v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_W \\ \gamma_W W \\ 0 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} \gamma_v\gamma_W + vW\gamma_v\gamma_W \\ v\gamma_v\gamma_W + W\gamma_v\gamma_W \\ 0 \\ 0 \end{pmatrix}$$

The first term should be $\gamma_{v\oplus W}$ and the second $\gamma_{v\oplus W}(v\oplus w)$

So we divide the second by the first: $v\oplus w = \dfrac{\gamma_v\gamma_W(v+W)}{\gamma_v\gamma_W(1+vW)} = \dfrac{v+W}{1+vW}$

Example Morin 1: A particle with mass m and energy E_1 approaches an identical particle at rest. They collide elastically (no mass changes) in such a way that they both scatter at an angle θ relative to the incident direction. What is θ?

In the frame in which the problem is described, the momenta are:

$$P_1 = (E_1, p, 0, 0) \quad , \quad P_2 = (m, 0, 0, 0)$$

The final 4-momenta are:

$$P_1' = (E', p'\cos\theta, p'\sin\theta, 0) \quad P_2' = (E', p'\cos\theta, -p\sin\theta, 0)$$

Conservation of y-momentum is what allows us to say that their final momentum p' is the same.
Conservaition of momentum $P_1+P_2 = P_1'+P_2'$ says that: $E+m = 2E'$ and $p = 2p'\cos\theta$
So $\cos^2\theta = \dfrac{p^2}{4p'^2}$
But we also know that $p'^2 = E'^2 - m^2$ and that $p^2 = E^2 - m^2$ so:

$$\begin{aligned}\cos^2\theta = \frac{p^2}{4p'^2} &= \frac{E^2 - m^2}{4(E'^2 - m^2)} \\ &= \frac{E^2 - m^2}{4(E^2/4 + mE/2 + m^2/4 - m^2)} \\ &= \frac{E^2 - m^2}{E^2 + 2mE - 3m^2} \\ &= \frac{E+m}{E+3m}\end{aligned}$$

- **Decay at an Angle: A particle with mass M and energy E decays into two identical particles. In the lab frame, one is emitted at a 90 degree angle, what are the energies of the created particles?**

The 4 momentum before decay is $P = (E, p, 0, 0)$
Where $p^2 = E^2 - M^2$ 77 The 4 momenta after decay are: $P_1 = (E_1, 0, p_1, 0)$ and $P_2 = (E_2, p_2\cos\theta, -p_2\sin\theta, 0)$
Now, the total momentum is P and by conservation we have:

$$\begin{aligned}&P - P_1 = P_2 \\ \Rightarrow\ &(P - P_1)\cdot(P - P_1) = P_2 \cdot P_2 \\ \Rightarrow\ &P^2 - 2P\cdot P_1 + P_1^2 = P_2^2 \\ \Rightarrow\ &-M^2 + 2EE_1 - m^2 = -m^2 \\ \Rightarrow\ &E_1 = \frac{M^2}{2E}\end{aligned}$$

And then, $E_2 = E - E_1 = \dfrac{2E^2 - M^2}{2E}$

.

La 4-Velocidad de Nuevo

Motivation: A possible 4-velocity for a particle moving at speed $\vec{v}$ would be $u^\mu = (-, v^x, v^y, v^z)$ So that u^m is the speed of the particle in the mth direction. We are still missing the first components, which would be the speed of the particle through time.

Proper Time

The 4-speed must be the change in spacetime of the particle with respect to some parameter. But what parameter?

Consider a particle moving at constant speed $v < 1$ respect to a reference system O with coordinates x^μ.
Because the particle moves at constant speed, we can define it an inertial reference frame, its **repose system**, which we denote O'.

The particle moves through space time from an event A to an event B. According to time dilation:

$$\Delta t = \gamma \Delta t'$$

Note that $\Delta t'$ is the time that the clock on the particle measures while it moves from A to B. What this clock in the particle measures is the **proper time** of the particle.
In O', the interval is equal to the time (because the origin stays with the particle) so:

$$(\Delta s')^2 = -(\Delta t')^2 = -(\Delta t)^2 + (\Delta r)^2$$

So, the **proper time** $\Delta\tau$ of the particle is given by:

$$\Delta\tau = \sqrt{-(\Delta s)^2}$$

Any observer can measue what the proper time of the particle would be, because he can easily measure Δs.
So proper time is invariant (because the interval is).

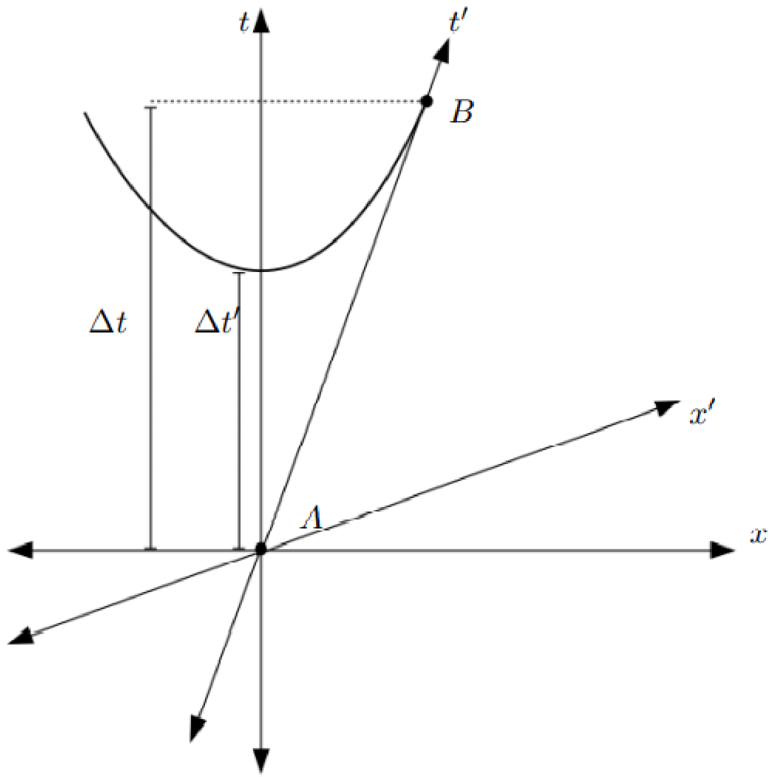


Figura 1: El tiempo propio y el tiempo coordenado.

If a system O measures a time Δt between two events of a particle moving at v, then we have the relation:

$$\Delta t = \gamma \Delta\tau$$

Where Δt is called the **coordinate time**.

Constant 4-velocity

We will use the proper time $\Delta\tau$ as the parameter because it is independent of reference frame.
If the particle moves at speed $v < 1$ with respect to a reference system O with coordinates x^μ from an event A to B, the expression is:

$$u^\mu = \frac{\Delta x^\mu}{\Delta\tau}$$

.

Speed in the particle's frame: Suppose O' is the repose system of the particle (with respect to which this particle doesn't move). Then we have that $\Delta x'^i = 0$ and that $\Delta t' = \Delta\tau$, so that:

$$u'^{\mu} = (1, 0, 0, 0)$$

So the particle only moves through time.

Now we pass to a system O with respect to which the particle moves at constant speed $\vec{v}$. In this case, the 4-speed says that:

$$u^{\mu} = \frac{\Delta x^{\mu}}{\Delta \tau} = \gamma \frac{\Delta x^{\mu}}{\Delta t}$$

So that, since the coordinates of speed are $v^i = \Delta x^i / \Delta t$, we have:

$$u^{\mu} = \gamma(1, v^x, v^y, v^z)$$

The spatial vector $\gamma\vec{v}$ is called the **proper speed** of the particle.
The 4-velocity is a tensor, which we can see by seeing that it transforms as such.

We also see that its **norm**:

$$u^{\mu} u_{\mu} = -1$$

Therefore, the **4-velocity** is a time like vector.

Four Velocity

Consider a particle that moves through space time at a non constant speed with respect to a reference frame O. We describe its trajectory parametrically as $x^{\mu}(\lambda)$ for some parameter λ.
The tangent vector has components:

$$T^{\mu} = \frac{dx^{\mu}}{d\lambda}$$

The proper time is the parameter such that speed has norm 1, that is $u^{\mu} u_{\mu} = -1$

Therefore, the **proper time** is the **affine parameter** τ that satisfies:

$$\eta_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} = -1$$

From integrating, we obtain the proper time in general:

$$\tau = \int^{\tau} \sqrt{-\eta_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda}} d\lambda$$

And

$$d\tau = \sqrt{-\eta_{\mu\nu} dx^{\mu} dx^{\nu}}$$

Then, the definition of proper time is analogous to the definition of arc length in $\mathbb{R}^n$. While in $\mathbb{R}^n$, the parameter is the path length, in spacetime it is the proper time.

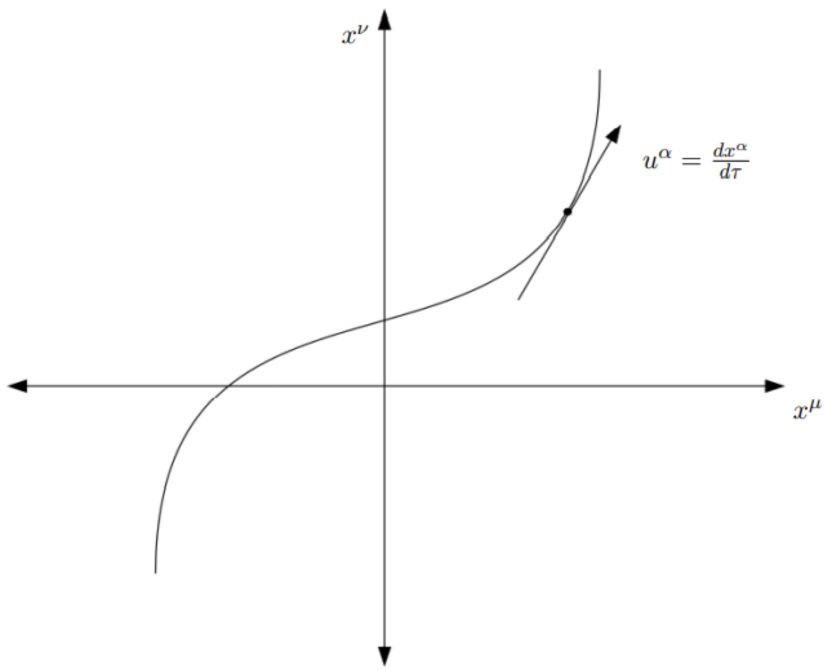


Figura 2: La 4-velocidad es el vector tangente unitario a la curva.

The four speed is the normalized tangent to the curve, given by:

$$u^\mu = \frac{dx^\mu}{d\tau}$$

That is, not any tangent vector to the curve is a 4-velocity, only that with square norm equal to -1.

4-acceleration (Rindler Coordinates)

The definition is natural:

$$a^\mu = \frac{du^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2}$$

Suppose we have a particle with proper acceleration α constant with respect to a reference system O in the x direction. That means that:

$$a^\mu a_\mu = \alpha^2$$

doesn't depend on the proper time.
So α is a Lorentz invariant.
If the particle is at the origin of O at time $\tau = 0$, its position is parametrized by:

$$t(\tau) = \frac{1}{\alpha}\sinh(\alpha\tau) \ \ , \ \ x(\tau) = \frac{1}{\alpha}(\cosh(\alpha\tau) - 1) \ \ , \ \ y(\tau) = z(\tau) = 0$$

Because this parametric curve satisfies that:

- Initial condition $x^{\mu}(0) = 0$
- 4-velocity is normalized $u^{\mu}u_{\mu} = -1$
- Constant proper acceleration $a^{\mu}a_{\mu} = \alpha^2$

Furthermore, we can revert $t(\tau)$ to get $\tau(t)$ and therefore $x(t)$ as:

$$x(t) = \frac{\sqrt{1+\alpha^2 t^2} - 1}{\alpha}$$

This is the position relative to the time (as seen by O).
So we can calculate the speed and acceleration:

$$v(t) = \frac{t\alpha}{\sqrt{1+t^2\alpha^2}}$$
$$a(t) = \frac{\alpha}{(1+t^2\alpha^2)^{3/2}}$$

Una buena forma de visualizar todo esto es pintar el diagrama espacio-temporal: las trayectorias son hipérbolas como en la Fig. (1). Observen cómo éstas tienden asintóticamente al cono de luz cuando α tiende a infinito. Esto significa que no importa cuánto se acelere la partícula, nunca va a poder superar la velocidad de la luz.

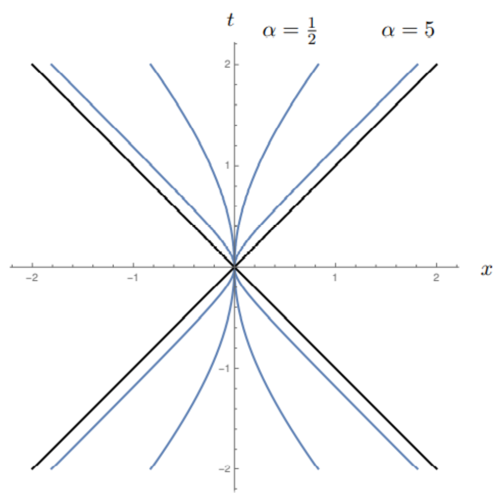


Figura 1: Trayectorias de partículas con aceleración propia constante.

To go from a inertial system x^{μ} to a x'^{μ} we use a Lorentz transformation $x'^{\mu} = \Lambda^{\mu}_{\nu}x^{\nu}$
This transformation leaves the metric invariant $\eta_{\mu\nu} = \Lambda^{\alpha}_{\mu}\Lambda^{\beta}_{\nu}\eta_{\alpha\beta}$

When going to a non inertial frame, the transformation will not be Lorentz.
We need a system of coordinates in which the particle is always at the origin and the time is equal to its proper time. The transformation that fulfills this is:

$$t = \left(\frac{1}{\alpha} + x'\right)\sinh(\alpha t') \ \ , \ \ x = \left(\frac{1}{\alpha} + x'\right)\cosh(\alpha t') - \frac{1}{\alpha} \ \ , \ \ y = y' \ , \ z = z'$$

Because when taking $x' = y' = z' = 0$ and $t' = \tau$ we recover the initial stuff.
These are called the **Rindler coordinates**.
We can calculate how the components of the metric are:

$$\eta'_{\mu\nu} = \begin{pmatrix} -(1+x'\alpha)^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The important thing to notice is that on a non inertial system, the metric components change, but the metric itself is the same.

1 Tensor analysis in special Relativity

The metric tensor

Consider the representation of two vector $\vec{A}, \vec{B}$ on the basis $\{\vec{e}_\alpha\}$ of some frame O:

$$\vec{A} = A^\alpha \vec{e}_\alpha \quad , \quad \vec{B} = B^\beta \vec{e}_\beta$$

Their scalar product is:

$$\begin{aligned} \vec{A} \cdot \vec{B} &= (A^\alpha \vec{e}_\alpha) \cdot (B^\beta \vec{e}_\beta) \\ &= A^\alpha B^\beta (\vec{e}_\alpha \cdot \vec{e}_\beta) \end{aligned}$$

Which is equal to:

$$\vec{A} \cdot \vec{B} = A^\alpha B^\beta \eta_{\alpha\beta}$$

This is a Frame invariant way of writing $-A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3$

Definition of Tensors

Tensor of type (0, N)

We define a tensor of type (0 N) as a function of N vectors into the real numbers, which is linear in each of its N arguments.

For example, the dot product is a (0,2) tensor because it is bilinear. We denote the metric tensor by g:

$$g(\vec{A}, \vec{B}) := \vec{A} \cdot \vec{B}$$

The definition of tensor does not mention the components of the vectors. A tensor must be a rule which gives the same real number independently of reference frame in which the components are calculated. We showed already that the dot product $A^\alpha B^\beta \eta_{\alpha\beta}$ satisfies this property.

Components of a Tensor

A tensor has components. In a reference frame O, a tensor of type $(0, N)$ are the values of the function when its arguments are the basis vector $\{\vec{e}_\alpha\}$ of the frame O
That is, the components are:

$$g(\vec{e}_\alpha, \vec{e}_\beta) = \vec{e}_\alpha \cdot \vec{e}_\beta = \eta_{\alpha\beta}$$

So $\eta_{\alpha\beta}$ were always the components of the tensor. In this case, it doesn't depend on the frame.

The (0,1) tensors (One-Forms)

A tensor of type $(0,1)$ is called a covector or a one form

General properties
We denote them with a tilde, like $\tilde{p}$.
Then, in general $\tilde{p}(\vec{A})$ is a real number.

If $\tilde{p}, \tilde{q}$ are covectors, then $\tilde{s} = \tilde{p} + \tilde{q}$ and $\tilde{r} = \alpha\tilde{p}$ also are, defined by:
$\tilde{s}(\vec{A}) = \tilde{p}(\vec{A}) + \tilde{q}(\vec{A})$
And $\tilde{r}(\vec{A}) = \alpha\tilde{p}(\vec{A})$

Therefore, the set of all one forms forms a vector space, called the **dual vector space**.

Component:
The components of a covector $\tilde{p}$ are:

$$p_\alpha := \tilde{p}(\vec{e}_\alpha)$$

By convention, we write the index down.

Therefore, a covector acting on a vector can be written as:

$$\begin{aligned}\tilde{p}(\vec{A}) &= \tilde{p}(A^\alpha \vec{e}_\alpha)\\ &= A^\alpha \tilde{p}(\vec{e}_\alpha)\\ &= A^\alpha p_\alpha\end{aligned}$$

Notice that ll terms have plus signs, and it is called a contraction of $\vec{A}$ and $\tilde{p}$

Transformation
If we have $\tilde{p}$ in a basis $\{\vec{e}_{\beta'}\}$, then the components of $\tilde{p}$ are:

$$\begin{aligned}p_{\beta'} &:= \tilde{p}(\vec{e}_{\beta'}) = \tilde{p}(\Lambda^\alpha_{\beta'}\vec{e}_\alpha)\\ &= \Lambda^\alpha_{\beta'}\tilde{p}(\vec{e}_\alpha) = \Lambda^\alpha_{\beta'} p_\alpha\end{aligned}$$

So, the components transform as:

$$p_{\beta'} = \Lambda^\alpha_{\beta'} p_\alpha$$

That is, they transform just as the vectors $\vec{e}_{\beta'} = \Lambda^\alpha_{\beta'}\vec{e}_\alpha$

Therefore, we can prove that $\tilde{p}(\vec{A})$ is frame independent:

$$\begin{aligned}A^{\alpha'} p_{\alpha'} &= (\Lambda^{\alpha'}_\beta A^\beta)(\Lambda^\mu_{\alpha'} p_\mu)\\ &= \Lambda^\mu_{\alpha'}\Lambda^{\alpha'}_\beta A^\beta p_\mu\\ &= \delta^\mu_\beta A^\beta p_\mu\\ &= A^\beta p_\beta\end{aligned}$$

This is the same way as the vector $A^\alpha \vec{e}_\alpha$ remains frame independent.

Remember: The transformation of a basis is the expression of new vectors in terms of old ones.
The transformation of components is the expression of the same object in terms of the new basis.

Basis One-Forms

The set of all one-forms is a vector space, so we could find it a basis
To do this, given a basis $\{\vec{e}_\alpha\}$, we define its dual basis as the forms that fulfill:

$$\tilde{\omega}^\alpha(\vec{e}_\beta) = \delta^\alpha_\beta$$

In this frame, any covector $\tilde{p}$ can be written as:

$$\tilde{p} = p_\alpha \tilde{w}^\alpha$$

The components of unit covectors are:

$$\begin{aligned}\tilde{\omega}^0 &\to_O (1,0,0,0)\\ \tilde{\omega}^1 &\to_O (0,1,0,0)\\ \tilde{\omega}^2 &\to_O (0,0,1,0)\\ \tilde{\omega}^3 &\to_O (0,0,0,1)\end{aligned}$$

The cobasis vectors transform as:

$$\tilde{\omega}^{\alpha'} = \Lambda^{\alpha'}_\beta \tilde{\omega}^\beta$$

Vectors and Covectors

We have a vector basis $\vec{e}_\alpha$ with corresponding covector basis $\tilde{\omega}^\alpha$.
And another basis $\vec{e}_{\alpha'}$ with corresponding covector basis $\tilde{\omega}^{\alpha'}$.
Then, the vectors transform as:

$$\begin{aligned}\vec{e}_{\mu'} &= \Lambda^\nu_{\mu'} \vec{e}_\nu\\ \vec{e}_\alpha &= \Lambda^{\beta'}_\alpha \vec{e}_{\beta'}\\ \tilde{\omega}^{\alpha'} &= \Lambda^{\alpha'}_\beta \tilde{\omega}^\beta\\ \tilde{\omega}^\beta &= \Lambda^\beta_{\alpha'} \tilde{\omega}^{\alpha'}\end{aligned}$$

Suppose we have a vector $\vec{A} = A^\alpha \vec{e}_\alpha = A^{\alpha'} \vec{e}_{\alpha'}$ and a covector $\tilde{p} = p_\alpha \tilde{\omega}^\alpha = p_{\alpha'} \tilde{\omega}^{\alpha'}$. Then, the components transform as:

$$
\begin{aligned}
A^\nu &= \Lambda^\nu_{\beta'} A^{\beta'} \\
A^{\beta'} &= \Lambda^{\beta'}_\alpha A^\alpha \\
p_{\beta'} &= \Lambda^\alpha_{\beta'} p_\alpha \\
p_\nu &= \Lambda^{\mu'}_\nu p_{\mu'}
\end{aligned}
$$

Gradient of a Function as a one-form

Consider a scalar field $\phi(\vec{x})$ defined at every event $\vec{x}$.

If we parametrize each point on a world line by the value of the proper time τ along it (reading the clock moving on the line), then we can express the coordinates of events on the curve as functions of τ:

$$[t = t(\tau)\ ,\ x = x(\tau)\ ,\ y = y(\tau)\ ,\ z = z(\tau)]$$

The four velocity has components $\vec{U} \to \left(\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right)$

Since ϕ is a function of t, x, y, z, it is implicitly a function of τ along the curve:

$$\phi(\tau) = \phi[t(\tau), x(\tau), y(\tau), z(\tau)]$$

And the rate of change is:

$$
\begin{aligned}
\frac{d\phi}{d\tau} &= \frac{\partial\phi}{\partial t}\frac{dt}{d\tau} + \frac{\partial\phi}{\partial x}\frac{dx}{d\tau} + \frac{\partial\phi}{\partial y}\frac{dy}{d\tau} + \frac{\partial\phi}{\partial z}\frac{dz}{d\tau} \\
&= \frac{\partial\phi}{\partial t}U^t + \frac{\partial\phi}{\partial x}U^x + \frac{\partial\phi}{\partial y}U^y + \frac{\partial\phi}{\partial z}U^z
\end{aligned}
$$

The number $d\phi/d\tau$ is clearly a linear function of $\vec{U}$, so we have defined a one form.

This one form is called the gradient of ϕ and denoted by $\tilde{d}\phi$:

$$\tilde{d}\phi \to_O \left(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

We can see this is consistent with definition. The components should transform as:

$$(\tilde{d}\phi)_{\alpha'} = \Lambda^\beta_{\alpha'} (\tilde{d}\phi)_\beta$$

But we know how to transform partial derivatives:

$$\frac{\partial\phi}{\partial x^{\alpha'}} = \frac{\partial\phi}{\partial x^\beta}\frac{\partial x^\beta}{\partial x^{\alpha'}}$$

Which means that

$$(\tilde{d}\phi)_{\alpha'} = \frac{\partial x^\beta}{\partial x^{\alpha'}} (\tilde{d}\phi)_\beta$$

So, clearly:

$$\frac{\partial x^\beta}{\partial x^{\alpha'}} = \Lambda^\beta_{\alpha'}$$

Notation of Derivatives

We shall write:

$$\frac{\partial \phi}{\partial x^\alpha} := \phi_{,\alpha}$$

In particular, we have:

$$x^\alpha_{,\beta} := \delta^\alpha_\beta$$

Therefore, remembering the definition of the cobasis associated with a basis, we have that:

$$\tilde{d}x^\alpha := \tilde{\omega}^\alpha$$

Therefore, we can write for any function:

$$\tilde{d}f = \frac{\partial f}{\partial x^\alpha}\tilde{d}x^\alpha$$

The $(0,2)$ Tensors

A tensor (0 2) takes two vector arguments.
The simplest type is constructed by taking the product of two one forms.
If $\tilde{p}, \tilde{q}$ are 1 forms, we define the $\tilde{p} \otimes \tilde{q}$ tensor as a (0 2) tensor such that:

$$(\tilde{p} \otimes \tilde{q})(\vec{A}, \vec{B}) = \tilde{p}(\vec{A})\tilde{p}(\vec{B})$$

Not all (0,2) tensors have this form.

Components

The most general $(0,2)$ tensor is not a simple outer product, but it can be written as a sum of simple outer products.

For a tensor (0,2) f, we define its components as:

$$f_{\alpha\beta} := f(\vec{e}_\alpha, \vec{e}_\beta)$$

There are 16 components. The value of f on an arbitrary pair of vectors is:

$$\begin{aligned} f(\vec{A}, \vec{B}) &= f(A^\alpha \vec{e}_\alpha, B^\beta \vec{e}_\beta) \\ &= A^\alpha B^\beta f(\vec{e}_\alpha, \vec{e}_\beta) \\ &= A^\alpha B^\beta f_{\alpha\beta} \end{aligned}$$

Can we form a basis of this tenors??

Yes. the basis is of all the tensors of the form $\tilde{w}^\alpha \otimes \tilde{w}^\beta$ (it is a space of dimension 16).
It is easy to see this is true by considering:

$$f = f_{\alpha\beta}\tilde{w}^\alpha \otimes \tilde{w}^\beta$$

Symmetries

A tensor $(0, 2)$ is symmetric if:

$$f(\vec{A}, \vec{B}) = f(\vec{B}, \vec{A})$$

And that means that:

$$f_{\alpha\beta} = f_{\beta\alpha}$$

For any tensor h we can define a new symmetric tensor $h_{(s)}$ as:

$$h_{(s)}(\vec{A}, \vec{B}) = \frac{1}{2}h(\vec{A}, \vec{B}) + \frac{1}{2}h(\vec{B}, \vec{A})$$

For the components, this means that:

$$h_{(s)\alpha\beta} = \frac{1}{2}(h_{\alpha\beta} + h_{\beta\alpha}) := h_{(\alpha\beta)}$$

A tensor f is antisymmetric if:

$$f(\vec{A}, \vec{B}) = -f(\vec{B}, \vec{A})$$
$$f_{\alpha\beta} = -f_{\beta\alpha}$$

For any (0,2) tensor h, we can construct a antisymmetric one:

$$h_{(A)}(\vec{A}, \vec{B}) = \frac{1}{2}h(\vec{A}, \vec{B}) - \frac{1}{2}h(\vec{B}, \vec{A})$$

For the components, this means that:

$$h_{(A)\alpha\beta} = \frac{1}{2}(h_{\alpha\beta} - h_{\beta\alpha}) := h_{[\alpha\beta]}$$

In general:

$$h_{\alpha\beta} = h_{(\alpha\beta)} + h_{[\alpha\beta]}$$

Metric As a Mapping of vectors into one-forms

Consider g (metric tensor) and a vector $\vec{V}$.
We can consider $g(\vec{V}, \quad)$ as a function that requires one vector and gives out a real number

(a covector!)
Therefore, we define:

$$g(\vec{V},\ \ ) := \tilde{V}(\ \ )$$

For example:

$$\tilde{V}(\vec{A}) := g(\vec{V}, \vec{A}) = \vec{V} \cdot \vec{A}$$

Let's find the components of $\tilde{V}$. We see that:

$$\begin{aligned} V_\alpha &:= \tilde{V}(\vec{e}_\alpha) = \vec{V} \cdot \vec{e}_\alpha = \vec{e}_\alpha \cdot V \\ &= \vec{e}_\alpha \cdot (V^\beta \vec{e}_\beta) \\ &= (\vec{e}_\alpha \cdot \vec{e}_\beta) V^\beta \end{aligned}$$

Therefore (in the special case in which g has components $\eta_{\alpha\beta}$:

$$V_\alpha = \eta_{\alpha\beta} V^\beta$$

So V^α are the components of $\vec{V}$ and V_β are the components of $\tilde{V}$

In this special case:

$$\begin{aligned} \vec{V} &\to (a, b, c, d) \\ \tilde{V} &\to (-a, b, c, d) \end{aligned}$$

The inverse, going from $\tilde{A}$ to $\vec{A}$

If $\eta_{\alpha\beta}$ has an inverse, we call the components of this **inverse** $\eta^{\alpha\beta}$ and we have:

$$A^\alpha := \eta^{\alpha\beta} A_\beta$$

In particular, with $\tilde{d}\phi$ we can associate a vector $\vec{d}\phi$, which is the one associated with the gradient.
We can see that this vector is normal to surfaces of constant ϕ, for its inner product with a vector $\vec{V}$ in a surface of constant ϕ is identical to $\tilde{d}\phi(\vec{V})$
This is the rate of change of ϕ along $\vec{V}$, which is 0.

Magnitudes and Scalar products of One-Forms

A one form $\tilde{p}$ is defined to have the same magnitude as its associated vector $\vec{p}$:

$$\tilde{p}^2 := \vec{p}^2 = \eta_{\alpha\beta} p^\alpha p^\beta$$

But this is equal to $\eta_{\alpha\beta}(\eta^{\alpha\mu}p_\mu)(\eta^{\beta\nu}p_\nu)$, so:

$$\tilde{p}^2 = \eta^{\alpha\mu}p_\mu p_\alpha$$

So, it can be written more explicitly as: $\tilde{p}^2 = -(p_0)^2 + (p_1)^2 + (p_2)^2 + (p_3)^2$

And in general:

$$\tilde{p} \cdot \tilde{q} = -p_0 q_0 + p_1 q_1 + p_2 q_2 + p_3 q_3$$

Finally (M, N) Tensors

We can define vectors as functions of one-forms as:

$$\vec{V}(\tilde{p}) := \tilde{p}(\vec{V}) := p_\alpha V^\alpha := \langle \tilde{p}, \vec{V} \rangle$$

(M,0) Tensors

It is a linear function of M one forms into the real numbers.

A simple $(2,0)$ tensor is $\vec{V} \otimes \vec{W}$ which acts as
$(\vec{V} \otimes \vec{W})(\tilde{p}, \tilde{q}) = \vec{V}(\tilde{p})\vec{W}(\tilde{q}) = \tilde{p}(\vec{V})\tilde{q}(\vec{W}) = V^\alpha p_\alpha W^\beta q_\beta$

(M,N) Tensors

An (M, N) tensor is a linear function of M one forms and N vectors into the real numbers.

For example, a (1,1) tensor R Such a tensor has components given by:

$$R^\alpha_\beta := R(\tilde{\omega}^\alpha, \vec{e}_\beta)$$

And it transforms as:

$$\begin{aligned} R^{\alpha'}_{\beta'} &= R(\tilde{\omega}^{\alpha'}, \vec{e}_{\beta'}) \\ &= R(\Lambda^{\alpha'}_\mu \tilde{\omega}^\mu, \Lambda^\nu_{\beta'} \vec{e}_\nu) \\ &= \Lambda^{\alpha'}_\mu \Lambda^\nu_{\beta'} R^\mu_\nu \end{aligned}$$

Index Raising and Lowering

The metric maps a vector $\vec{V}$ into a one form $\tilde{V}$
Similarly, it maps a (N, M) tensor into a $(N-1, M+1)$ tensor.
And the inverse metric maps an (N, M) tensor into a $(N+1, M-1)$ one.

Suppose $T\alpha\beta\ _{\gamma}$ are the components of a (2,1) tensor, then:

$$T^{\alpha}_{\beta\gamma} := \eta_{\beta\mu}T^{\alpha\mu}_{\ \ \ \gamma}$$

Are the components of a (1,2) tensor.

And also $T_{\alpha\ \gamma}^{\ \beta} = \eta_{\alpha\mu}T^{\mu\beta}_{\ \ \ \gamma}$ are the components of another (1,2) tensor.

Or $T^{\alpha\beta\gamma} := \eta^{\gamma\mu}T^{\alpha\beta}_{\ \ \ \gamma}$ are the components of a (3,0) tensor

We can define the **Mixed components of a metric as**:

$$\eta^{\alpha}_{\beta} := \eta^{\alpha\mu}\eta_{\mu\beta}$$

Which we get by raising the first index of the $\eta_{\mu\beta}$. We see that in this case this is simply the identity.

Derivative of Tensor

Suppose we have $T = T^{\alpha}_{\beta}\tilde{\omega}^{\beta} \otimes \vec{e}_{\alpha}$
Then the derivarive is defined as:

$$\frac{d\vec{T}}{d\tau} = \lim \frac{T(\tau + \Delta\tau) - T(\tau)}{\Delta\tau}$$

But in flat space the covectors and vectors are constant, so:

$$\frac{dT}{d\tau} = \frac{dT^{\alpha}_{\beta}}{d\tau}\tilde{\omega}^{\beta} \otimes \vec{e}_{\alpha}$$

Where $dT^{\alpha}_{\beta}/d\tau$ is the ordinary derivative of the function T^{α}_{β} along the world line:

$$\frac{dT^{\alpha}_{\beta}}{d\tau} = T^{\alpha}_{\beta\ ,\gamma}U^{\gamma}$$

Therefore:

$$\frac{dT}{d\tau} = (T^{\alpha}_{\beta,\gamma}\tilde{\omega}^{\beta} \otimes \vec{e}_{\alpha})U^{\gamma}$$

From which we can deduce that:

$$\nabla T := (T^{\alpha}_{\beta,\gamma}\tilde{\omega}^{\beta} \otimes \tilde{\omega}^{\gamma} \otimes \vec{e}_{\alpha})$$

is a $(1,2)$ tensor. This tensor is the gradient of T. We also have a convenient notation:

$$\frac{dT}{d\tau} = \nabla_{\vec{U}}T$$

$$\nabla_{\vec{U}}T \rightarrow \{T^{\alpha}_{\beta,\gamma}U^{\gamma}\}$$

.

Where for example, if $D \to (xt, 5tx + y, t + x, 0) = D^\alpha$, then the derivative is:

$$D^\alpha_{,\gamma} = \frac{dD^\alpha}{dx^\gamma}$$

That is, we need to derive each term with respect to each variable. In this case we obtain:

$$D^\alpha_{,\gamma} = \begin{pmatrix} x & t & 0 & 0 \\ 5x & 5t & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, $\frac{dD^\alpha_{,\gamma}}{d\tau} = D^\alpha_{,\gamma}U^\gamma = D^\alpha_{,0}U^0 + D^\alpha_{,1}U^1 + D^\alpha_{,2}U^2 + D^\alpha_{,3}U^3$
$= -D^\alpha_{,0} = (x, 5x, 1, 0)$
In which we used U as the speed in the particle's frame.

Exercise

30: In some frame O, the vector fields $\vec{U}$, $\vec{D}$ have the components:

$$\vec{U} \to (1 + t^2, t^2, \sqrt{2}t, 0)$$
$$\vec{D} \to (x, 5tx, \sqrt{2}t, 0)$$

And the scalar $\rho = x^2 + t^2 - y^2$

- **Find $\vec{U} \cdot \vec{U}$, $\vec{U} \cdot \vec{D}$, $\vec{D} \cdot \vec{D}$ Is $\vec{U}$ suitable for a 4-velocity? is $\vec{D}$?**

  $\vec{U} \cdot \vec{U} = -(1 + t^2)^2 + (t^2)^2 + (\sqrt{2}t)^2 + (0)^2 = -1 - 2t^2 - t^4 + t^4 + 2t^2 = -1$
  $\vec{U} \cdot \vec{D} = -(1 + t^2)(x) + t^2(5tx) + \sqrt{2}t(\sqrt{2}t) + (0)(0) = -x - t^2x + 5t^3x + 2t^2$
  $\vec{U} \cdot \vec{U} = -(x)^2 + (5tx)^2 + (\sqrt{2}t)^2 + (0)^2 = -x^2 + 25t^2x^2 + 2t^2$
  $\vec{U}$ is a velocity field and $\vec{D}$ isn't.

- **Find the spatial velocity v of a particle whose four velocity is $\vec{U}$, for arbitrary t**

  We need $U^0 = 1$, so:

$$v^i = \frac{U^i}{U^0} = \left(\frac{t^2}{1 + t^2}, \frac{\sqrt{2}t}{1 + t^2}, 0\right)$$

- **Find** U_α
  For the covector in flat space, we simply change de sign of the first term:
$$U_\alpha = (-(1+t)^2, t^2, \sqrt{2}t, 0)$$

- **Find** $U^\alpha_{,\beta}$
  We derive each term by each variable:
$$U^\alpha_{,\beta} = \frac{\partial U^\alpha}{dx^\beta} = \begin{pmatrix} 2t & 0 & 0 & 0 \\ 2t & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- **Show that** $U_\alpha U^\alpha_{,\beta} = 0$**, show that** $U^\alpha U_{\alpha,\beta} = 0$ **for all** β
  for various values of β:
$$\begin{aligned}
\beta = 0 \quad &\Rightarrow \quad U_\alpha U^\alpha_{,0} = \frac{\partial}{\partial t}(-(1+t^2)^2 + t^4 + 2t) = -2(1+t^2)\cdot 2t + 4t^3 + 4t = 0 \\
\beta = 1 \quad &\Rightarrow \quad U_\alpha U^\alpha_{,1} = \frac{\partial}{\partial x}(-(1+t^2)^2 + t^4 + 2t) = 0 \\
\beta = 2 \quad &\Rightarrow \quad U_\alpha U^\alpha_{,2} = \frac{\partial}{\partial y}(-(1+t^2)^2 + t^4 + 2t) = 0 \\
\beta = 3 \quad &\Rightarrow \quad U_\alpha U^\alpha_{,3} = \frac{\partial}{\partial z}(-(1+t^2)^2 + t^4 + 2t) = 0
\end{aligned}$$
  And $U^\alpha_{\alpha,\beta} = (U_\alpha U^\alpha)_{,\beta} = 0$

- **Find** $D^\beta_{,\beta}$
  This is the divergence of the vector:
$$\begin{aligned}
D^\beta_{,\beta} &= \frac{\partial D^\beta}{\partial \beta} = \frac{\partial D^0}{\partial t} + \frac{\partial D^0}{\partial x} + \frac{\partial D^0}{\partial y} + \frac{\partial D^0}{\partial z} \\
&= 0 + 5t + 0 + 0 = 5t
\end{aligned}$$

- **Find** $(U^\alpha D^\beta)_{,\beta}$
  The components of tensor $U^\alpha D^\beta$ are:
$$U^\alpha D^\beta = \begin{pmatrix} (1+t^2)x & 5tx(1+t^2) & \sqrt{2}t(1+t^2) & 0 \\ t^2 x & 5t^3 x & \sqrt{2}t^3 & 0 \\ \sqrt{2}tx & 5\sqrt{2}t^2 x & 2t^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
  So the derivatives are:
$$(U^\alpha D^\beta)_{,\beta} = \frac{\partial U^\alpha D^\beta}{\partial x^\beta}$$

We can find it for every α:

$$\begin{aligned}
\alpha = 0 &\Rightarrow \; 2tx + 5t(1+t^2) + 0 + 0 = 2tx + 5t(1+t^2) \\
\alpha = 1 &\Rightarrow \; 2tx + 5t^3 + 0 + 0 = 2tx + 5t^3 \\
\alpha = 2 &\Rightarrow \; \sqrt{2}x + 5\sqrt{2}t^2 + 0 + 0 = \sqrt{2}x + 5\sqrt{2}t^2 \\
\alpha = 3 &\Rightarrow \; 0
\end{aligned}$$

- **Find** $U_\alpha(U^\alpha D^\beta)_{,\beta}$
  We have that $M^\alpha := (U^\alpha D^\beta)_{,\beta} = (2tx + 5t(1+t^2), 2tx + 5t^3, \sqrt{2}x + 5\sqrt{2}t^2, 0)$
  So $U_\alpha M^\alpha = -(1+t^2)(2tx + 5t(1+t^2)) + t^2(2tx + 5t^3) + \sqrt{2}t(\sqrt{2}xt + 5\sqrt{2}t^2)) = -5t$
  We see that $U_\alpha(U^\alpha D^\beta)_{,\beta} = U_\alpha U^\alpha D^\beta_{,\beta} = -D^\beta_{,\beta}$

- Find $\rho_{,\alpha}$ and $\rho^{,\alpha}$

$$\rho_{,\alpha} = \frac{\partial \rho}{\partial x^\alpha} = (2t, 2x, -2y, 0)$$

And $\rho^{,\beta} = (-2t, 2x, -2y, 0)$

- Find $\nabla_{\vec{U}}\vec{D}$
  That is defined as $D^\alpha_{,\gamma}U^\gamma$
  But:

$$D^\alpha_{,\gamma} = \frac{\partial D^\alpha}{\partial x^\gamma} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 5x & 5t & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

And therefore:

$$D^\alpha_{,\gamma}U^\gamma = D^\alpha_{,0}U^0 + D^\alpha_{,1}U^1 + D^\alpha_{,2}U^2 + D^\alpha_{,3}U^3$$

$$\begin{aligned}
\alpha = 0 &\Rightarrow \; D^0_{,0}U^0 + D^0_{,1}U^1 + D^0_{,2}U^2 + D^0_{,3}U^3 = 1(t^2) = t^2 \\
\alpha = 1 &\Rightarrow \; D^1_{,0}U^0 + D^1_{,1}U^1 + D^1_{,2}U^2 + D^1_{,3}U^3 = 1(t^2) = 5x(1+t^2) + 5t(t^2) = 5t^3 + 5x(1+t^2) \\
\alpha = 2 &\Rightarrow \; D^2_{,0}U^0 + D^2_{,1}U^1 + D^2_{,2}U^2 + D^2_{,3}U^3 = \sqrt{2}(1+t^2) \\
\alpha = 3 &\Rightarrow \; D^3_{,0}U^0 + D^3_{,1}U^1 + D^3_{,2}U^2 + D^3_{,3}U^3 = 0
\end{aligned}$$

Perfect Fluids in Special Relativity

A fluid is a special kind of continuum. A continuum is a collection of particles so numerous that the dynamics of individual particles cannot be followed, leaving only a description of the collection in terms in bulk quantities.

Dust: The number flux vector $\vec{N}$

Dust is defined to be a collection of particles, all of which are at rest in some Lorentz frame

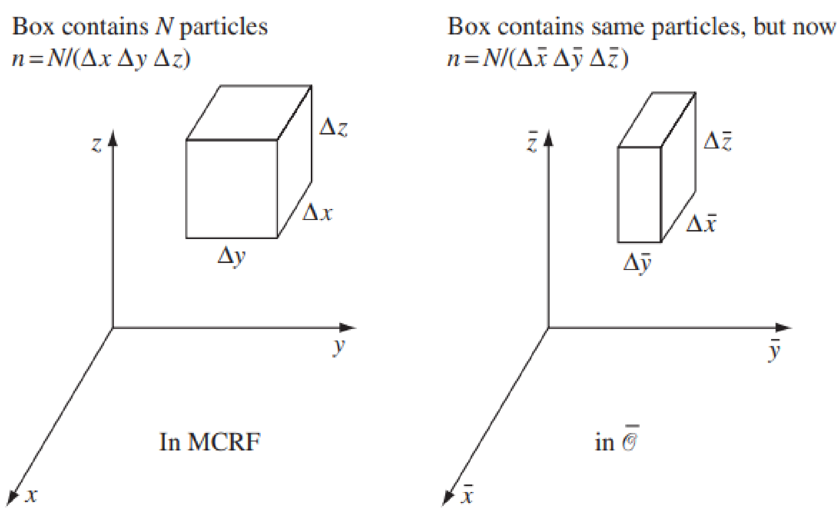
The number density n

In the rest frame we can calculate the amount of particles per unit volume.

$$\text{n} := \text{number density in the MCRF of the element}$$

But in a system O' the particles are not at rest and the density is different. The same particles now occupy a different volume.
If they where originally in a rectangular solid of volume $\Delta x \Delta y \Delta z$, then the volume in the other system O' reduces by γ, that is, the new density is bigger by γ

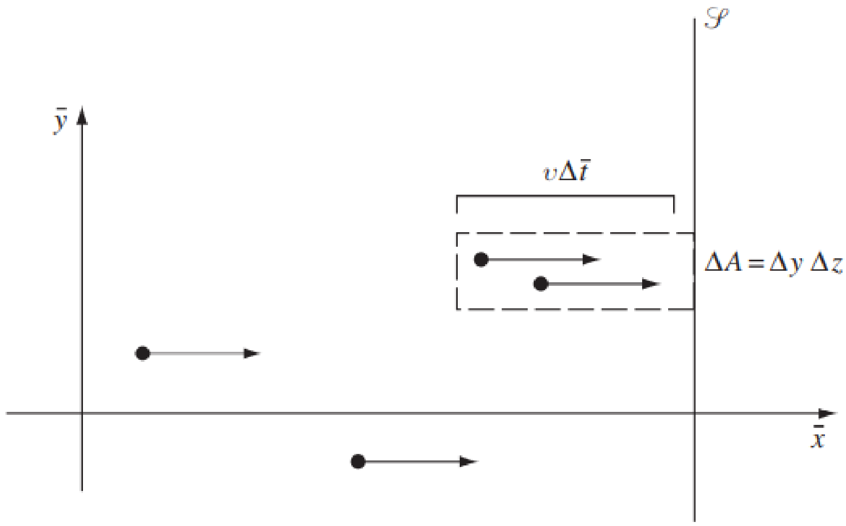
$$\frac{n}{\sqrt{1-v^2}} = \text{number density in frame in which particles move with v}$$



Flux Across a surface

The flux of particles across a surface is the number crossing a unit area of that surface in a unit time.
This clearly depends on the reference frame

In the rest frame, the flux is zero, since all particles are at rest.
In the frame O', all particles move with velocity v in the x' direction and we consider a surface S perpendicular to x'.



The rectangular volume outlined contains all particles that will pass through the surface ΔA of S in a time $\Delta t'$.
It has a volume $v\Delta t'\Delta A$ and therefore contains $[n/\sqrt{(1-v^2)}]v\Delta t'\Delta A$ particles.
The number crossing per unit time and per unit area is the flux across surfaces of constant x':

$$(flux)^{x'} = \frac{nv}{\sqrt{1-v^2}}$$

If, more generally, the particles had a y component of velocity in O' as well, then the flux considers only the velocity in direction x:

$$(flux)^{x'} = \frac{nv^{x'}}{\sqrt{1-v^2}}$$

The number flux 4-vector $\vec{N}$

We define:

$$\vec{N} = n\vec{U}$$

Where $\vec{U}$ is the 4-velocity of the particles. In a frame O' in which the particles have a velocity (v^x, v^y, v^z), we have:

$$\vec{U} \rightarrow_{O'} (\gamma, \gamma v^x, \gamma v^y, \gamma v^z)$$

It follows that:

$$\vec{N} \rightarrow_{O'} (n\gamma, nv^x\gamma, nv^y\gamma, nv^z\gamma)$$

Therefore, the time component of $\vec{N}$ is the number density
The spatial components are the fluxes across surfaces of the various coordinates

$$\vec{N} \cdot \vec{N} = -n^2 \quad , \quad n = (-\vec{N} \cdot \vec{N})^{1/2}$$

So n (rest density) is a scalar (independent of frame)

One forms and surfaces

Let us look at the flux across x' surfaces again, this time in a space time diagram in which we plot only t', x'
The surface S perpendicular to x has the world lines shown.
At any time t' it just one point, since we are suppressing y', z'
The world lines of particles that go through S in time $\Delta t'$ are shown.
The flux is the number of world lines that cross S in $\Delta t' = 1$

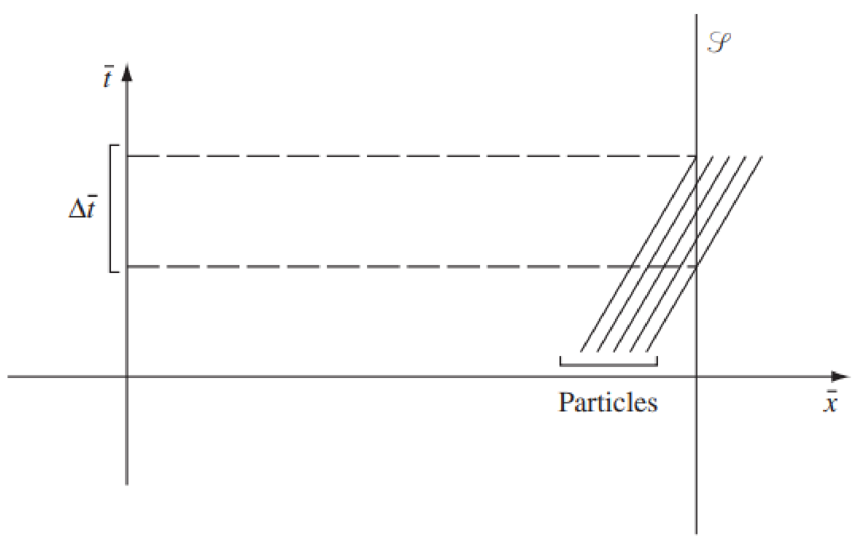


Fig. 4.2 in a spacetime diagram, with the $\bar{y}$ direction suppressed.

Also: The term $N^{0'}$ is just the density of particles, so it is the flux of particles across a surface of constant t', for as time passes, the particles go through this time.

One forms determine a surface

In general a surface is:

$$\phi(t, x, y, z) = cte$$

The gradient of the function ϕ, $\tilde{d}\phi$ is a normal one form (in the sense that applied to any vector parallel to the surface, it results in 0).
In some sense, $\tilde{d}\phi$ defines the surface $\phi = cte$. Or all its multiples, therefore we normalize the one form:

$$\tilde{n} := \tilde{d}\phi/|\tilde{d}\phi|$$

.

$$|\tilde{d}\phi| = |\eta^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}|^{1/2}$$

We define the surface element as the unit normal times an area element:

$$\tilde{n}dx^{\alpha}dx^{\beta}dx^{\gamma}$$

Flux across a surface

The flux of particles across a surface of constant ϕ is:

$$\langle\widehat{n}, \vec{N}\rangle$$

(just as in electro the flux is $\vec{E}\cdot\vec{n}$)

To see this, let $\phi = x'$ for example. Then the normal is $\tilde{d}x'$ which is already normalized. Then $\langle\tilde{d}x', \vec{N}\rangle = N^{\alpha}(\tilde{d}x')_{\alpha} = N^{x'}$ which is the flux across x' surfaces.

So, given the vector $\vec{N}$, we can calculate the flux across any surface by finding the unit normal one form and then contracting with $\vec{N}$

Representation of a frame by a one-form

Before continuing, we will see a useful fact.
An inertial frame, can be define by a one form instead of by its 4-velocity
Namely, by the one form $g(\vec{U},\ )$ that has components:

$$U_{\alpha} = \eta_{\alpha\beta}U^{\beta}$$

In its rest frame, this is just $\tilde{d}t'$.
So $\tilde{d}t$ can be pictured as a set of surfaces of constant t, and this clearly does define the frame.

The stress energy tensor

The particles also have momentum and energy.

For now, we suppose all the particles have rest energy m.

Energy density

In the MCRF, the energy of each particle is m, and the number per unit volume is n. Therefore, the energy per unit volume is mn. We denote it by ρ:

$$\rho = \text{Energy density in the MCRF}$$

Thus ρ is just a scalar (as n and m). In the case of dust:

$$\rho = nm \quad \text{(dust)}$$

In more general fluids, there is kinetic energy always, since there is no rest frame for all particles at the same time, so the expression is more complicated.

Now, in frame O' the number density is $n/\sqrt{1-v^2} = n\gamma$, but now the energy of each particle is $m/\sqrt{1-v^2} = m\gamma$, since it is moving. Therefore, the energy density is $mn\gamma\gamma$

$$\frac{\rho}{1-v^2} = \rho\gamma^2 = \quad \text{energy density in frame in which particles move at v}$$

We needed two one forms, the 4-momentum and the density one, so we will need a tensor of order 2.

Stress Energy Tensor

We define the components of the stress-energy tensor (which is type (2,0), it takes in 2 one forms) in terms of the components in some arbitrary frame:

$$T(\tilde{d}x^\alpha, \tilde{d}x^\beta) = T^{\alpha\beta} = \text{Flux of } \alpha \text{ momentum across a surface of constant } x^\beta$$

α momentum means $p^\alpha = \langle \tilde{d}x^\alpha, \vec{p} \rangle$
It can be proven that this is really a tensor.

Consider T^{00}.
This is the flux of p^0 (energy) across a surface of $t = cte$, that is just the energy density:

$$T^{00} = \text{energy density}$$

Similarly, T^{0i} is the flux of energy across the $x^i = cte$ surface:

$$T^{0i} = \text{energy flux across the } x^i \text{ surface}$$

Then, T^{i0} is the flux of i momentum across $t = cte$ surface, that is the density of i momentum:

$$T^{i0} = \text{ i momentum density}$$

And T^{ij} is:

$$T^{ij} = \text{ flux of i momentum across j surface}$$

For any system, Giving the components of T in some frame defines it completely (then it is just a matter of transforming to other frame).

For Dust, The components of T in the MCRF are super easy. There is no motion so all directions, therefore:

$$\begin{aligned}(T^{00})_{MCRF} &= \rho = mn \\ (T^{ij})_{MCRF} = (T^{0i})_{MCRF} = (T^{i0})_{MCRF} &= 0\end{aligned}$$

It is easy to see that the tensor $\vec{p} \otimes \vec{N}$ has exactly this components in the $MCRF$, where $\vec{p} = m\vec{U}$ is the 4-momentum of a particle. THerefore:

$$\text{Dust: } T = \vec{p} \otimes \vec{N} = mn\vec{U} \otimes \vec{U} = \rho\vec{U} \otimes \vec{U}$$

So, we can conclude that in any frame:

$$\begin{aligned}T^{\alpha\beta} &= T(\tilde{\omega}^\alpha, \tilde{\omega}^\beta) \\ &= \rho\vec{U}(\tilde{\omega}^\alpha)\vec{U}(\tilde{\omega}^\beta) \\ &= \rho U^\alpha U^\beta\end{aligned}$$

Therefore, in the frame O' where $\vec{U} \to_{O'} \left(\frac{1}{\sqrt{1-v^2}}, \frac{v^x}{\sqrt{1-v^2}}, \cdots\right)$ we have:

$$\begin{aligned}T^{0'0'} &= \rho U^{0'} U^{0'} = \rho/(1-v^2) \\ T^{0'i'} &= \rho U^{0'} U^{i'} = \rho v^i/(1-v^2) \\ T^{i'0'} &= \rho U^{i'} U^{0'} = \rho v^i/(1-v^2) \\ T^{i'j'} &= \rho U^{i'} U^{j'} = \rho v^i v^j/(1-v^2)\end{aligned}$$

Notice that T is symmetric.

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General Fluids

In general fluids things are much more complicated.
We have to take into account that:

- Besides the bulk motions of the fluid, each particle has some random velocity
- There may be various forces between particles that contribute potential energies to the total.

Table 4.1 Macroscopic quantities for single-component fluids

Symbol	Name	Definition
$\vec{U}$	Four-velocity of fluid element	Four-velocity of MCRF
n	Number density	Number of particles per unit volume in MCRF
$\vec{N}$	Flux vector	$\vec{N} := n\vec{U}$
ρ	energy density	Density of *total* mass energy (rest mass, random kinetic, chemical, ...)
Π	Internal energy per particle	$\Pi := (\rho/n) - m \Rightarrow \rho = n(m + \Pi)$ Thus Π is a general name for all energies other than the rest mass.
ρ_0	Rest-mass density	$\rho_0 := mn$. Since m is a constant, this is the 'energy' associated with the rest mass only. Thus, $\rho = \rho_0 + n\Pi$.
T	Temperature	Usual thermodynamic definition in MCRF (see below).
p	Pressure	Usual fluid-dynamical notion in MCRF. More about this later.
S	Specific entropy	Entropy per particle (see below).

Fluid Element: A small enough piece of fluid to have defined local variables as pressure and temperature, but big enough to contain many particles.

Definition of Macroscopic quantities:
For each fluid element, we go to the frame in which it is at rest (its MCRF).
This frame is momentarily comoving, since fluid elements can be accelerated.
The MCRF is specific to a single fluid element, since other fluid elements may be moving with respect to this one.
All scalar quantities related with a fluid element are defined to be their values in the MCRF.

First Law of Thermo

In the MCRF we imagine that the fluid element is able to exchange energy with its surroundings by heat conduction (ΔQ) and by work ($p\Delta V$).

If we let E be the total energy of the element, then we can write:

$$\Delta E = \Delta Q - p\Delta V$$

If the element contains a total of N particles, and if this number doesn't change, we can write:

$$V = \frac{N}{n} \quad , \quad \Delta V = -\frac{N}{n^2}\Delta n$$

And from the definition of ρ:

$$E = \rho V = \rho N/n$$
$$\Delta E = \rho\Delta V + V\Delta\rho$$

These two results then imply:

$$\Delta Q = \frac{N}{n}\Delta\rho - N(\rho + p)\frac{\Delta n}{n^2}$$

If we define $q = Q/N$, which is the heat absorbed by particle.
Then we obtain:

$$n\Delta q = \Delta\rho - \frac{\rho + p}{n}\Delta n$$

If we suppose that the changes are infinitesimal. It can be shown that a fluid's state can be given by two parameters out or ρ, T, n.

But from thermodynamics, we know that $dq = TdS$
So the last equation (written infinitesimally) is:

$$\boxed{d\rho - (\rho + p)\frac{dn}{n} = nTdS}$$

The general stress-energy tensor

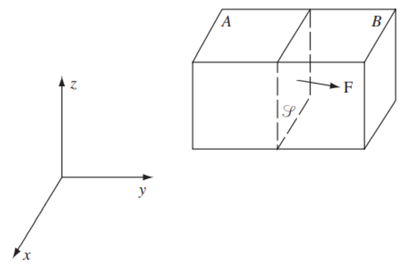
The definition of $T^{\alpha\beta}$ is perfectly general. Let us look at it in the MCRF, where there is no bulk flow of the fluid element, and no spatial momentum of the particles. Then, in the MCRF:

- T^{00} = energy density = ρ
- T^{0i} = energy flux. Although there is no motion in the MCRF, energy might be transmitted by heat, so T^{0i} in the MCRF is basically heat conduction.
- T^{i0} = Momentum density. Again, particles themselves have no net momentum in the MCRF, buy heat is conducted, then the moving energy will have an associated momentum. We will see later that $T^{i0} = T^{0i}$
- T^{ij} = momentum flux. We call it stress.

The Spatial components of T

By definition, T^{ij} is the flux of i momentum across the j surface.

Consider two adjacent fluid elements, represented as cubes, with the common interface S. In general, they exert forces on each other.
Shown in the diagram is the force $\vec{F}$ exerted by A on B (B exerts an equal and opposite force on A)



The force **F** exerted by element *A* on its neighbor *B* may be in any direction depending on properties of the medium and any external forces.

Since force equals the rate of change of momentum (Newton is valid here in the MCRF), A is pouring momentum into B at the rate $\vec{F}$ per unit time.
So there is a flow of momentum across S form A to B at rate $\vec{F}$.
If the area is a, then the flux of momentum across S is $\vec{F}/a$

T^{ij} represents forces between adjacent fluid elements. If the forces are always perpendicular to the elements then $T^{ij} = 0$ for $i \neq j$.

Symmetry of $T^{\alpha\beta}$ in MCRF

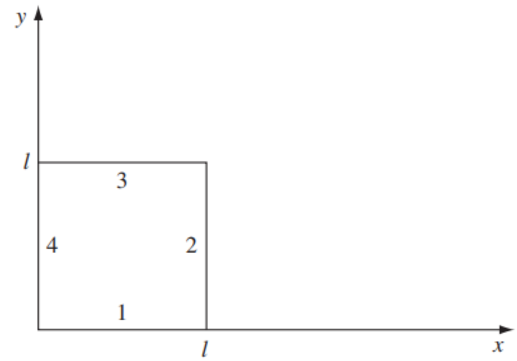
It can be proven, with some work, that:

$$T^{ij} = T^{ji}$$

This is true in the MCRF frame, so it must be frame in every frame.

Conservation of Energy-Momentum

Since T represents energy and momentum, there must be some way of using it to express the law of conservation of energy.



A section z = const. of a cubical fluid element.

We see a cross section of a cubical fluid element. Energy can flow in across all sides.
The rate of flow across face (4) is $l^2T^{0x}(x=0)$, and across (2) is $-l^2T^{0x}(x=a)$ (it has a minus sign since T^{0x} represents energy in the x direction).

Similarly, the total energy flowing in the y direction is $l^2T^{0y}(y=0) - l^2T^{0y}(y=l)$.

Then, by conservation of energy, the total rate of increase in energy inside, $\dfrac{\partial(T^{00}l^3}{\partial t}$ is equal to the sum of these changes:

$$\frac{\partial}{\partial t}l^3T^{00} = l^2\Big[T^{0x}(x=0) - T^{0x}(x=l) + T^{0y}(y=0) \\ - T^{0y}(y=l) + T^{0z}(z=0) - T^{0z}(z=l)\Big]$$

Then, we divide by l^3 and take the limit $l \to 0$ to give:

$$\frac{\partial}{\partial t}T^{00} = -\frac{\partial}{\partial x}T^{0x} - \frac{\partial}{\partial y}T^{0y} - \frac{\partial}{\partial z}T^{0z}$$

Where we used: $\lim_{l\to 0}\dfrac{T^{0x}(x=0) - T^{0x}(x=l)}{l} := -\dfrac{\partial}{\partial x}T^{0x}$

We can write the equation as:

$$T^{00}{}_{,0} + T^{0x}{}_{,x} + T^{0y}{}_{,y} + T^{0z}{}_{,z} = 0$$

Or, more succinctly:

$$T^{0\alpha}{}_{,\alpha} = 0$$

The same mathematics applies for the conservation of momentum.
Then, in general, **conservation of energy and momentum is:**

$$\boxed{T^{\alpha\beta}{}_{,\beta} = 0}$$

Conservation of Particles

During the flow of fluid , the number of particles in a fluid element will change, but of course the total number of particles in the fluid will not change.

This conservation law can be obtained in the same way as the one for energy was (using that the rate of change of the number of particles in a fluid element will be due only to loss or gain across the boundaries) , and we get:

$$\frac{\partial}{\partial t}N^0 = -\frac{\partial}{\partial x}N^x - \frac{\partial}{\partial y}N^y - \frac{\partial}{\partial z}N^z$$

Therefore, the **conservation of particles says:**

$$\boxed{N^\alpha{}_{,\alpha} = (nU^\alpha)_{,\alpha} = 0}$$

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Perfect Fluids

A perfect fluid in relativity is defined: A fluid that has no viscosity and no heat conduction in the MCRF.
It is, after dust, the simplest fluid to deal with.

No heat Conduction

From the definition of T, the no heat conduction supposition implies that

$$T^{0i} = T^{i0} = 0$$

No viscosity

Viscosity is a force parallel to the interface between particles. Its absence means that the forces should always be perpendicular to the surface.
This implies that $T^{ij} = 0$ if $i \neq j$
So T^{ij} must be a diagonal matrix, moreover it must be diagonal in all MCRF frames (since no viscosity is a statement independent of spatial axes).

The only matrix that is diagonal in all frames is a multiple of the identity.

Thus, an x surface will have across it only a force in the x direction, and similarly for y and z. These force-per-unit-area are all equal , and we call the **pressure** p.
So we have $T^{ij} = p\delta^{ij}$
So the 0 viscosity assumption reduces a lot the 3x3 matrix T^{ij}

Form of T

In the MCRF, T has the components we have deduced:

$$(T^{\alpha\beta}) = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0p & 0 & 0 & \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

It is not hard to show that in the MCRF:

$$T^{\alpha\beta} = (\rho + p)U^{\alpha}U^{\beta} + p\eta^{\alpha\beta}$$

For instance, for $\alpha = \beta = 0$, $U^0 = 1$ and $\eta^{00} = -1$ so we get ρ, as we should, we can prove similarly that we get every value as we should with this expression.

Therefore, the **stress energy tensor of a perfect fluid** is:

$$T = (\rho + p)\vec{U} \otimes \vec{U} + pg^{-1}$$

A comparison with dust shows that dust is a pressure free liquid, which is because it has no random velocities of the particles.

The conservation Laws

The conservation law leads us to:

$$T^{\alpha\beta}_{\ \ ,\beta} = [(\rho + p)U^{\alpha}U^{\beta} + \rho\eta^{\alpha\beta}]_{,\beta} = 0 \quad 4.39$$

There are four equations here,one for every α

Theorem: $U^{\alpha}_{\ ,\beta}U_{\alpha} = 0$

- To prove this, we begin with:

$$U^{\alpha}U_{\alpha} = -1 \ \Rightarrow \ (U^{\alpha}U_{\alpha})_{,\beta} = 0$$

Or:

$$0 = (U^{\alpha}U^{\gamma}\eta_{\alpha\gamma})_{,\beta} = (U^{\alpha}U^{\gamma})_{,\beta}\eta_{\alpha\gamma} = 2U^{\alpha}_{\ ,\beta}U^{\gamma}\eta_{\alpha\gamma}$$

Finally, because of the symmetry of $\eta_{\alpha\beta}$, this means $U^\alpha{}_{,\beta}U^\gamma\eta_{\alpha\gamma} = U^\alpha U^\gamma{}_{,\beta}\eta_{\alpha\gamma}$.
So, we have:

$$(U^\alpha U^\gamma \eta_{\alpha\gamma})_{,\beta} = 2U^\alpha_{,\beta}U_\alpha$$

Thus, we prove the theorem.
Importantly, we needed to use that η is constant.

On the other hand, we assume that $(nU^\beta)_{,\beta} = 0$, and with that, we can write the first term of 4.39 as:

$$\begin{aligned} [(\rho+p)U^\alpha U^\beta]_{,\beta} &= \left[\frac{\rho+p}{n}U^\alpha nU^\beta\right]_{,\beta} \\ &= nU^\beta\left(\frac{\rho+p}{n}U^\alpha\right)_{,\beta} \end{aligned}$$

Therefore, the original equation 4.39 now reads:

$$nU^\beta\left(\frac{\rho+p}{n}U^\alpha\right)_{,\beta} + p_{,\beta}\eta^{\alpha\beta}U_\alpha = 0$$

We now multiply by U_α and sum on α:

$$nU^\beta U_\alpha\left(\frac{\rho+p}{n}U^\alpha\right)_{,\beta} + p_{,\beta}\eta^{\alpha\beta}U_\alpha = 0$$

The last term is just $p_{,\beta}U^\beta$, which is the derivative of p along the world line of the fluid element $dp/d\tau$. So, we obtain:

$$U^\beta\left[-n\left(\frac{\rho+p}{n}\right)_{,\beta} + p_{,\beta}\right] = 0$$

A little algebra converts this in:

$$-U^\beta\left[\rho_{,\beta} - \frac{\rho+p}{n}n_{,\beta}\right] = 0$$

Or, written another way:

$$\frac{d\rho}{d\tau} - \frac{\rho+p}{n}\frac{dn}{d\tau} = 0$$

This, compared to equation 4.25 means that:

$$\boxed{U^\alpha S_{,\alpha} = \frac{dS}{d\tau} = 0}$$

Thus, the flow of a particle-conserving perfect fluid conserves entropy.

From the remaining components and some algebra, we get:

$$\boxed{(\rho+p)a_i + p_{,i} = 0}$$

Importance of general relativity

T is very important in GR. In classical mechanics, the source of forces and stuff is the density ρ.
but this makes no sense in GR, since rest mass and energy are interconvertible.
So the source in GR must be all energies, which means we need T

Gauss' law

Gauss' law is:

$$\int V^{\alpha}_{\ ,\alpha} d^4x = \oint V^{\alpha} n_{\alpha} d^3S$$

Where $\tilde{n}$ is the unit-normal one form outward pointing.

Exercises

4) Show that the number density of dust measured by an arbitrary observer whose 4-velocity is $\vec{U}_{obs}$ is $-\vec{N}\cdot\vec{U}_{obs}$

The density is by definition N^0 in the frame in which $\vec{U} \to (1,0,0,0)$. In this frame $\vec{N}\cdot\vec{U} = -N^0$

5) Complete the proof that

$$T(\tilde{d}x^\alpha, \tilde{d}x^\beta) = T^{\alpha\beta} = \text{Flux of } \alpha \text{ momentum in surface of constant } x^\beta$$

defines a vector by arguing that it is linear in both arguments.

The α component of 4-momentum is $p^\alpha = \langle \tilde{d}x^\alpha, \vec{p}\rangle = \vec{p}(\tilde{d}x^\alpha)$, which is linear on $\tilde{d}x^\alpha$

4.7: Derive the expressions for the components of the stress-energy tensor of dust:

$$\begin{aligned}
T^{0'0'} &= pU^{0'}U^{0'} = \rho/(1-v^2)\\
T^{0'i'} &= \rho U^{0'}U^{i'} = \rho v^i/(1-v^2)\\
T^{i'j'} &= \rho U^{i'}U^{j'} = \rho v^i v^j/(1-v^2)\\
T^{i'0'} &= \rho U^{0'}U^{i'} = \rho v^i/(1-v^2)
\end{aligned}$$

Solution: The terms follow directly from the expression $T^{\alpha'\beta'} = \rho U^{\alpha'}U^{\beta'}$
Using that $\vec{U} \to_{O'} (\gamma, v^x\gamma, v^y\gamma, v^z\gamma)$

25: Electromagnetism: Maxwells equations are:

$$\begin{aligned}
\nabla\times\vec{B} - \frac{\partial\vec{E}}{\partial t} &= 4\pi\vec{J}\\
\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} &= 0\\
\nabla\cdot\vec{E} &= 4\pi\rho\\
\nabla\cdot\vec{B} &= 0
\end{aligned}$$

In units $\mu_0 = \epsilon_0 = c = 1$.

- **An antisymmetric $(2,0)$ tensor F can be defined by $F^{0i} = E^i$ $(i = 1,2,3)$, $F^{x,y} = B^z, F^{yz} = B^x, F^{zx} = B^y$. Represent it as a matrix in this frame:**

Because it is antisymmetric, with the terms given is enough

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{pmatrix}$$

- **A rotation by an angle θ about the z axis is one kind of Lorentz transformation, with matrix**

$$(\Lambda^{\beta'}_{\alpha}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Show that the new components of F

$$F^{\alpha'\beta'} = \Lambda^{\alpha'}_{\mu}\Lambda^{\beta'}_{\nu}F^{\mu\nu}$$

Define new electric and magnetic three vector components (by the rule given in a) that are just the same as the components of the old E and B in the rotated three space. This shows that a spatial rotation of F makes a spatial rotation of E and B

We define $E^{i'}$ as:

$$\begin{aligned} E^{i'} &= F^{0'i'} = \Lambda^{0'}_{\mu}\Lambda^{i'}_{\nu}F^{\mu\nu} \\ &= \Lambda^{i'}_{\nu}F^{0\nu} \quad \text{sumamos sobre mu} \\ &= \Lambda^{i'}_{i}F^{0i} \quad \text{porque } F^{00} = 0 \\ &= \Lambda^{i'}_{i}E^{i} \end{aligned}$$

This definition of $E^{i'}$ what we would get after rotating, as can be seen.

Summary

- **Dust:** A big set of equal particles of mass m such that there is a frame in which they are at rest.

- **Number density n:** Amount of particles per unit volume in the MCRF frame. In a frame moving at speed v, tha volume reduces by γ, so:

$$\text{Number density in frame } O'\text{: } \frac{n}{\sqrt{1-v^2}}$$

- **Flux:** The amount of particles passing through a surface of constant x' is: $(flux)^{x'} = \frac{nv^{x'}}{\sqrt{1-v^2}}$
- **Number-Flux vector**
Each component N^α is the number flux (amount of particles per unit time and area) passing through a surface of constant x^α.
The component N^0 is just the number density (because it passes a surface of constant x^0)
In the rest frame, the vector is:
$$\vec{U} \to_O (n, 0, 0, 0)$$
In a frame O' moving at v, it is:
$$\vec{N} \to \left(\frac{n}{\sqrt{1-v^2}}, \frac{nv^x}{\sqrt{1-v^2}}, \frac{nv^y}{\sqrt{1-v^2}}, \frac{nv^z}{\sqrt{1-v^2}}\right) = (n\gamma, n\gamma v^x, n\gamma v^y, n\gamma v^z)$$
In general:
$$\vec{N} = n\vec{U}$$
And $\vec{N} \cdot \vec{N} = -n^2$

- **Energy density:** It is the amount of energy per unit volume.
For dust, in the rest field it is:
$$\rho = mn$$
In the O' frame, the number density is $n/\sqrt{1-v^2}$ and the energy is $m/\sqrt{1-v^2}$, so the density is:
$$\frac{\rho}{1-v^2}$$
- **Stress Energy Tensor:**
$$T(\tilde{d}x^\alpha, \tilde{d}x^\beta) = T^{\alpha\beta} = \text{flux of } \alpha \text{ momentum across a surface of constant } x^\beta$$
So T^{00} is the energy density, T^{i0} is the i momentum density, T^{0i} is the flux of energy across the x^i surface, T^{ij} flux of i momentum across j surface.
In the MCRF frame:
$$(T^{00})_{MCRF} = \rho = mn$$
And all the others are 0.
We can see that in this frame $T^{\alpha\beta} = \rho U^\alpha U^\beta$, since they are tensors, this is true in general:
$$T^{\alpha\beta} = \rho U^\alpha U^\beta$$

In the O' frame: Using the change of U, we get:

$$T^{0'0'} = \rho/(1-v^2)$$
$$T^{0'i'} = \rho v^i/(1-v^2)$$
$$T^{i'0'} = \rho v^i/(1-v^2)$$
$$T^{i'j'} = \rho v^i v^j/(1-v^2)$$

General Fluids: We cannot find a rest frame, and things are defined locally.
Thermo leads to $d\rho - (\rho + p)dn/n = nTdS$
We can still define a stress energy tensor.

- T^{00} = energy density
- T^{0i} = energy flux. Although there is no motion in the MCRF, energy may be transmitted by heat. So it is heat conduction in the MCRF frame
- T^{i0} = momentum density.
- T^{ij} = momentum flux, it is called stress

It can be shown that $T^{ij} = T^{ji}$

Conservation of energy and momentum
The conservation laws are summarized in:

$$T^{\alpha\beta}_{,\beta} = 0$$

The conservation of particles is summarized in:

$$N^{\alpha}_{,\alpha} = 0$$

- **Perfect fluid:** There is no heat conduction and no viscosity (no cross elements in the stress part).
  After applying this conditions, it can be found that in the MCRF, the tensor is:

$$T^{\alpha\beta} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

  And so, if the metric is g, it can be shown that:

$$T = (\rho + p)\vec{U} \otimes \vec{U} + pg^{-1}$$

  Conservation Laws: They are $T^{\alpha\beta}_{,\beta} = 0$
  And in this case, it implies that:

$$(\rho + p)a_i + p_{,i} = 0$$

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Energy moment EM tensor

If we have fields with a Faraday tensor F, the EM energy moment tensor is:

$$T^{\mu\nu} = F^{\mu\sigma}F^{\nu}_{\sigma} - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}$$

The interpretation is the same as before. In particular, the T^{tt} component is the energy density.
Using the definition of F as:

$$F^{\mu\nu} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{pmatrix}$$

Then, we have that:

$$T^{tt} = \frac{1}{2}(|\vec{B}|^2 + |\vec{E}|^2)$$

Which is the energy in the fields.

In completeness, and using normal units, the EM Stress energy tensor is:

$$T^{\mu\nu} = \begin{pmatrix} \frac{1}{2}(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2) & \frac{1}{c}S_x & \frac{1}{c}S_y & \frac{1}{c}S_z \\ \frac{1}{c}S_x & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ \frac{1}{c}S_y & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ \frac{1}{c}S_z & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{pmatrix}$$

Where $\vec{S} = \dfrac{1}{\mu_0}\vec{E} \times \vec{B}$ is the Poynting vector
And $\sigma_{ij} = \epsilon_0 E_i E_j + \dfrac{1}{\mu_0}B_i B_j - \dfrac{1}{2}\left(\epsilon_0 E^2 + \dfrac{1}{\mu_0}B^2\right)\delta_{ij}$ is the Maxwell stress tensor

Caroll

Special relativity and Flat Spacetime

We will work in flat spacetime, and therefore our framework will always be orthonormal.

Event: A single moment in space and time characterized by coordinates (t, x, y, z)

The Spacetime interval between two events is:

$$\Delta s^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

Beginning from the 2 Relativity postulates, it can be shown that the spacetime interval is conserved.

Theorem: Given the 2 postulates of SR (1. Light speed is same in all inertial frames, 2. All inertial frames are equivalent), then the spacetime interval is invariant under change of inertial frame

- **Solution:** Linearity can be proved as Saul did. Then, in the O' frame, $(\Delta s')^2 = -(\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2$
  And by linearity, $\Delta s'$ should be a quadratic function of the unbarred coordinate increments:

  $$(\Delta s')^2 = M_{\alpha\beta}\Delta x^\alpha \Delta x^\beta \qquad 1.2$$

  Changing $\Delta x^\alpha \Delta x^\beta$ to $\Delta x^\beta \Delta x^\alpha$ does nothing, so we may assume that $M_{\alpha\beta} = M_{\alpha\beta}$

  Now, suppose in the frame O that $\Delta s^2 = 0$, so $\Delta t = \Delta r$ where $\Delta r = [(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2]^{1/2}$
  Putting this into 1.2 we get:

  $$(\Delta s')^2 = M_{00}(\Delta r)^2 + 2M_{0i}\Delta x^i \Delta r + M_{ij}\Delta x^i \Delta x^j$$

  But by the first postulate, if $\Delta s^2 = 0$ then $(\Delta s')^2 = 0$. So this last thing must be 0 for arbitrary Δx^i (given that $\Delta t = \Delta r$). It is easy to show that this implies:

  $$M_{0i} = 0$$
  $$M_{ij} = -M_{00}\delta_{ij}$$

  Then, we conclude that $(\Delta s')^2 = M_{00}[(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2]$
  We define a function $\phi(v) = -M_{00}$ since this term should only depend on v.
  Therefore, $\Delta s'^2 = \phi(v)\Delta s^2$
  To prove that it is the same, we use homogeneity of space to conclude that the relation is valid if v moves in the other direction. Therefore, we can set a system O'' moving at speed -v from O' and $\Delta s'' = \phi(v)\Delta s' = \phi(v)^2\Delta s$
  But $O'' = O$, so $\Delta s'' = \Delta s$, therefore $\phi(v) = 1$.
  So we have proven that:

  $$\Delta s'^2 = \Delta s^2$$

.

We will always suppose that the two systems are in standard configuration (they cross the origin when both clocks read 0).

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We write the coordinates of an event as:

$$x^\mu : \begin{aligned} x^0 &= t \\ x^1 &= x \\ x^2 &= y \\ x^3 &= z \end{aligned}$$

We define the **metric** in flat space as:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then, the interval can be written in a simple way as:

$$s^2 = \eta_{\mu\nu} = \Delta x^\mu \Delta x^\nu$$

Transformation

We look for a transformation that leaves the spacetime interval invariant. Such transformation can be of the form:

$$x^{\mu'} = \Lambda^{\mu'}_{\ \nu} x^\nu$$

Or in vector form:

$$x' = \Lambda x$$

We would like the interval to remain invariant, that is:

$$\begin{aligned} s^2 = (\Delta x)^T \eta (\Delta x) &\doteq (\Delta x')^T \eta (\Delta x') \\ &= (\Delta x)^T \Lambda^T \eta \Lambda (\Delta x) \end{aligned}$$

Therefore, to conserve the interval, we want:

$$\begin{aligned} \eta &= \Lambda^T \eta \Lambda \\ \eta_{\rho\sigma} &= \Lambda^{\mu'}_{\rho} \Lambda^{\eta'}_{\sigma} \eta_{\mu'\nu'} \end{aligned}$$

The matrices that satisfy this are the **Lorentz Transformations**
They are similar to Euclidian rotations, which satisfy $I = R^T I R$

Types of Lorentz Transformations

- **Rotations:** Conventional rotation, such as one in the x-y plane:

$$\Lambda^{\mu'}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- **Boost:** ' rotation between space and time directions ':

$$\Lambda^{\mu'}_{\nu} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  Where $\phi \in (-\infty, \infty)$

Poincare Group: The set of both translations and Lorentz transformations. It is a 10 parameter non abelian group.

Lorentz Group: The sets of matrices that satisfy $\eta = \Lambda^T \eta \Lambda$
They are similar to the set of orthogonal matrices.
In a Lorentz matrix, all the columns form an orthonormal set (with the dot product defined by η), and that is why it is a 6 parameter group.
3 parameters correspond to the speed of system O' (and correspond to boosts) and the other 3 to the rotation (euler angles) of O' axis respect to O axis.

We will focus only on boosts, since we will suppose that the coordinate systems are aligned.

Boost in the x direction:
If O' moves at speed v in the positive x direction with respect to O, then the coordinates of a event (t, x, y, z) are now (t', x', y', z') in O'. The transformation is the boost we already showed, so the components are given by:

$$t' = t\cosh\phi - x\sinh\phi$$
$$x' = -t\sinh\phi + x\cosh\phi$$

We see that the point $x' = 0$ (where the person in the O' frame is) is moving at speed:

$$v = \frac{x}{t} = \frac{\sinh\phi}{\cosh\phi} = \tanh\phi$$

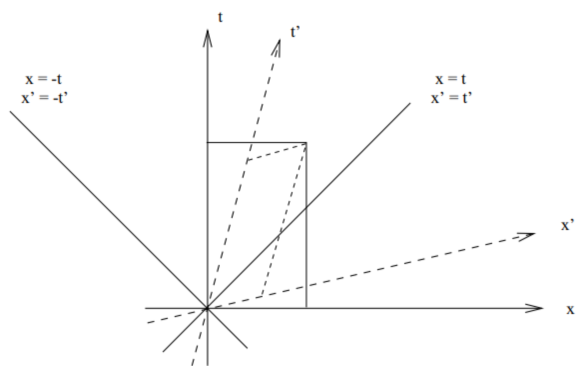
We can replace $\phi = \tanh -1v$ and get the more known transformation:

$$\boxed{t' = \gamma(t - vx)}$$
$$\boxed{x' = \gamma(x - vt)}$$

Where $\gamma = 1/\sqrt{1 - v^2}$

Spacetime Diagram

We can draw the coordinate system for the frame O and the coordinate system for the O' frame in the same diagram.



For a point p we can draw a **light cone** of all the points that are connected from p by a light wave.

Tensors

Given a point p we define the **Tangent space** T_p as all the vectors located at the point p.

A **Vector Space** is a collection of objects that is closed under linear operations and with all the other properties.

A **Basis** is any set of vectors that spans the space and is L.I

At each tangent space we set up a basis of four vectors $\widehat{e}_\mu$
The basis vector $\widehat{e}_\mu$ points in the direction of the x^μ axis and has norm 1.

Then, any abstract vector A can be written as:

$$A = A^\mu \tilde{e}_\mu$$

Where A^μ are the **components** of the vector A in this frame.

The real vector is an abstract geometrical entity, while the components are just the coefficients of the vector in a given basis.

Curve: A curve is a path through spacetime specified by a parameter λ, such as $x^\mu(\lambda)$. The tangent vector $V(\lambda)$ has components:

$$V^\mu = \frac{dx^\mu}{d\lambda}$$

Thus, the entire vector is $V = V^\mu \widehat{e}_\mu$
Under a Lorentz transformation, the coordinates x^μ change, therefore, so do the V^μ, to give:

$$V^\mu \to V^{\mu'} = \Lambda^{\mu'}_{\nu} V^\nu$$

But the vector itself is invariant.

We use this fact to derive the change of components of the basis vectors:

$$V = V^\mu \widehat{e}_\mu = V^{\nu'} \widehat{e}_{\nu'} = \Lambda^{\nu'}_{\mu} V^\mu \widehat{e}_{\nu'}$$

Therefore:

$$\widehat{e}_\mu = \Lambda^{\nu'}_{\mu} \widehat{e}_{\nu'}$$

Inverse: We define the inverse of the Lorentz transformation as:

$$\Lambda^{\mu}_{\nu'} \Lambda^{\sigma'}_{\mu} = \delta^{\sigma'}_{\nu'}$$

It is a very subtle notation.

Therefore, we obtain the transformation of basis vectors:

$$\widehat{e}_{\nu'} = \Lambda^{\mu}_{\nu'} \widehat{e}_\mu$$

Dual vector space: It is the space of all linear maps from the vector space to the real numbers.
If T_p is the **tangent space** to some point p, the set of covectors form the **cotangent space** T_p^*
If $\omega \in T_p^*$ is a dual vector, then it acts on vectors like:

$$\omega(aV + bW) = a\omega(V) + b\omega(W) \in \mathbb{R}$$

We can sum and scale covectors defining:

$$(a\omega + b\eta)(V) = a\omega(V) + b\eta(V)$$

So they form a vector space.

We can create a **dual basis** given an original one by defining:

$$\widehat{\theta}^{\nu}(\widehat{e}_{\nu}) = \delta^{\nu}_{\mu}$$

And every dual vector can be written as:

$$\omega = \omega_{\mu}\widehat{\theta}^{\mu}$$

Applying a one form (covector) to a vector:
We can see that:

$$\begin{aligned} \omega(V) &= \omega_{\mu}V^{\mu}\theta^{\mu}(e_{\nu}) = \omega_{\mu}V^{\nu}\delta^{\mu}_{\nu} \\ &= \omega_{\mu}V^{\mu} \end{aligned}$$

this also allows us to think of vectors as acting on covectors:

$$V(\omega) := \omega(V) = \omega_{\mu}V^{\mu}$$

This quantity is **invariant** under Lorentz transforms (which is clear because it is defined in vector and not components, or can be proven with components also).

We can get the rules of transformations for co-things

$$\begin{aligned} \omega_{\mu'} &= \Lambda^{\nu}_{\mu'}\omega_{\nu} \\ \theta^{\rho'} &= \Lambda^{\rho'}_{\sigma}\theta^{\sigma} \end{aligned}$$

In space time we are interested in **vector fields** (a vector in each point of space) in the **tangent bundle** (the set of all cotangent spaces of a manifold M, written as $T(M)$) or the **cotangent bundle**. In that case, the action of a covector in a vector is a **scalar** (a function of spacetime that assigns a number to each point)

A simple example of a dual vector is a horizontal vector, which acts on a vector by matrix product.

Gradient: Given a scalar function ϕ, the set of partial derivatives with respect to spacetime coordinates is denoted by d:

$$d\phi = \frac{\partial\phi}{\partial x^{\mu}}\widehat{\theta}^{\mu}$$

It is a dual vector because it takes the vector whose direction we want to differentiate in and outputs a number.
Using the chain rule, we can see that it transforms correctly for the components of a covector:

$$\frac{\partial\phi}{\partial x^{\mu'}} = \Lambda^{\mu}_{\mu'}\frac{\partial\phi}{\partial x^{\mu}}$$

Notation:
We write derivatives as:

$$\frac{\partial \phi}{\partial x^\mu} = \partial_\mu \phi = \phi_{,\mu}$$

Note that it is a lower index (component of a covector should)

Tensor: A tensor of type (k,l) is a multilinear map that takes k covectors and l vectors and gives a real number.

$$T: \quad T_p^* \times \cdots (k \; times) \times \cdots \times T_p^* \times \cdots (l \; times) \times \cdot \times T_p \to \mathbb{R}$$

Tensor product: If T is a (k, l) tensor and S is a (m, n) tensor, we define the $(k+m, l+n)$ tensor $T \otimes S$ as:

$$\begin{aligned} T \otimes S(\omega^1, \cdots, \omega^k, \cdots, \omega^{k+m}, V^1, \cdots, V^l, \cdots, V^{l+n}) \\ = T(\omega^1, \cdots, \omega^k, V^1, \cdots, V^l) S(\omega^{k+1}, \cdots, \omega^{k+m}, V^{l+1}, \cdots, V^{l+n}) \end{aligned}$$

Note that it is not commutative.

Basis: A basis for the set of (k, l) tensors is now straightforward to get, it consists of all tensors of the form:

$$\widehat{e}_{\mu_1} \otimes \cdots \otimes \widehat{e}_{\mu_k}) \otimes \widehat{\theta}^{\nu_1} \otimes \cdots \otimes \widehat{\theta}^{\nu_1}$$

And a general tensor can be written (in this basis) as:

$$T = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \widehat{e}_{\mu_1} \otimes \cdots \otimes \widehat{e}_{\mu_k} \otimes \widehat{\theta}^{\nu_1} \otimes \cdots \otimes \widehat{\theta}^{\nu_l}$$

Where the components can be obtained by orthonormality and are:

$$T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} = T(\widehat{\theta}^{\mu_1}, \cdots, \widehat{\theta}^{\mu_k}, \widehat{e}_{\nu_1}, \cdots, \widehat{e}_{\nu_l})$$

Then, the action of a tensor in covectors ω and vectors V is:

$$T(\omega^1, \cdots, \omega^k, V^1, \cdots, V^l) = T^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_l} \omega^1_{\mu_1} \cdots \omega^k_{\mu_k} V^{1\,\nu_1} \cdots V^{l\,\nu_l}$$

Finally, if we change basis, the components change as:

$$T^{\mu'_1 \cdots \mu'_k}{}_{\nu'_1 \cdots \nu'_l} = \Lambda^{\mu'_1}_{\mu_1} \cdots \Lambda^{\mu'_k}_{\mu_k} \Lambda^{\nu_1}_{\nu'_1} \cdots \Lambda^{\nu_l}_{\nu'_l} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}$$

We don't have to act on all components. For example, a (1,1) tensor acts on a vector transforming it in another vector (still hungry for a covector):

$$T^\mu_\nu : V^\nu \to T^\mu_\nu V^\nu$$

Generally, we will have a **tensor field** (a tensor in each point of space time).

Tensor can be written as matrices if we like to write their components like that.

Inner product

The most familiar example of a (0,2) tensor is the metric $\eta_{\mu\nu}$. The action of this tensor in two vectors is called the **inner product**:

$$\eta(V,W) = \eta_{\mu\nu}V^\mu W^\nu = V \cdot W$$

This result is invariant under Lorentz transforms (as is any tensor acting on vectors and covectors).

Norm of a vector is:

$$||V||^2 = V \cdot V = \nu_{\mu\nu}V^\mu V^\nu$$

If $|V|^2 < 0$, then V^μ is timelike.

Inverse metric: The inverse metric is $\nu^{\mu\eta}$ and is a type (2,0) tensor defined as the inverse of the metric:

$$\eta^{\mu\nu}\eta_{\nu\rho} = \eta_{\rho\nu}\eta^{\nu\mu} = \delta^\rho_\mu$$

Levi Civita tensor: Is a $(0,4)$ tensor with:

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} 1 \quad , \; \mu\nu\rho\sigma \quad \text{, is an even permutation of } 0123 \\ -1 \quad , \; \mu\nu\rho\sigma \quad \text{, is an odd permutation of } 0123 \\ 0 \quad , cc \end{cases}$$

It is remarkable that for the metric, the inverse metric, the kroeneker delta and levi civita, under a Lorentz transformation, their components remain unchanged (which can be proven using the transformation of components).

Contraction: Contraction turns a (k,l) tensor into a $(k-1,l-1)$ one. Contraction proceeds by summing over one upper and one lower index.

$$S^{\mu\rho}_{\ \ \sigma} = T^{\mu\nu\rho}_{\ \ \ \ \sigma\nu}$$

Raise and Lower indices: Given a tensor $T^{\alpha\beta}_{\ \ \gamma\delta}$ we can use the metric to define new tensors which we choose to denote with the same letter T:

$$\begin{aligned} T^{\alpha\beta\mu}_{\ \ \ \ \delta} &= \eta^{\mu\gamma}T^{\alpha\beta}_{\ \ \gamma\delta} \\ T_\mu{}^\beta{}_{\gamma\delta} &= \eta_{\mu\alpha}T^{\alpha\beta}_{\ \ \gamma\delta} \\ T_{\mu\nu}{}^{\rho\sigma} &= \eta_{\mu\alpha}\eta_{\nu\beta}\eta^{\rho\gamma}\eta^{\sigma\delta}T^{\alpha\beta}_{\ \ \gamma\delta} \end{aligned}$$

As an example, we can turn a vector into a dual vector by lowering indices:

$$\begin{aligned} V_\mu &= \eta_{\mu\nu}V^\nu \\ \omega^\mu &= \eta^{\mu\nu}\omega_\nu \end{aligned}$$

Explanation: The **metric** given by $\eta(\vec{V}, \vec{W}) = \vec{V} \cdot \vec{W}$
So, given a vector $\vec{V}$, we can define a covector as:

$$\tilde{V} = \eta(\vec{V}, \cdot)$$

And we can see that the components of this covector are $\tilde{V}^\alpha = \eta(\vec{V}, \vec{e}_\alpha)$, therefore, the **covector related to vector** V has components:

$$V_\mu = (-V^0, V^1, V^2, V^3)$$

Therefore, the dot product of two vectors is:

$$V \cdot W = \eta(V, W) = \eta_{\mu\nu} V^\mu W^\nu = V_\nu W^\nu = \tilde{V}(\vec{W})$$

Since δ is the metric of 3-space, we can rise and lower indices with no care.

Symmetry

A tensor is **Symmetric** if we can change any two of the indices without changing the tensor:

$$S_{\mu\nu\rho} = S_{\nu\mu\rho}$$

It is **antisymmetric** is changing two indices causes a change in sign:

$$A_{\mu\nu\rho} = -A_{\rho\nu\mu}$$

Symmetrize: We can symmetrized under a set of indices (the ones under parenthesis) by summing over them:

$$T_{(\mu_1 \cdots \mu_n)\rho}{}^{\sigma} = \frac{1}{n!}(T_{\mu_1\mu_2\cdots\mu_n\rho}{}^{\sigma} + \text{sum over permutation of indices } \mu_1 \cdots \mu)$$

While antisymmetrization comes from:

$$T_{[\mu_1 \cdots \mu_n]\rho}{}^{\sigma} = \frac{1}{n!}(T_{\mu_1\mu_2\cdots\mu_n\rho}{}^{\sigma} + \text{alternating sum over permutation of indices } \mu_1 \cdots \mu)$$

A symmetric tensor satisfies $S_{\mu_1 \cdots \mu_n} = S_{(\mu_1 \cdots \mu_n)}$ and an antysimmetric satisfies $S_{\mu_1 \cdots \mu_n} = S_{[\mu_1 \cdots \mu_n]}$

Derivatives

This is only valid In Flat Space

If we are working in flat spacetime, then the partial derivative of a (k,l) tensor is a $(k,l+1)$ tensor (just as the derivative of a scalar is a covector) for example:

$$T_{\alpha}{}^{\mu}{}_{\nu} = \partial_a R^{\mu}{}_{\nu}$$

This new tensor is really a tensor since it follows the transformation rules (this is not true in non flat spacetimes, except for $\partial_\alpha \phi$, which is a tensor always).

Electromagnetic Strngth Tensor

We know that the electromagnetic field has an electric vector E_i and a magnetic one B_i.
This are only ' vectors ' under rotations in space, not under Lorentz.
Either way, we can define the **Electromagnetic strength tensor** as:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} = -F_{\nu\mu}$$

And we can transform the components under Lorentz transformation as before to get the components in a moving frame. That way, seeing how electric fields transform in magnetic and vice versa.

Maxwell's equations:

$$\begin{aligned} \nabla \times \vec{B} - \partial_t \vec{E} &= 4\pi \vec{J} \\ \nabla \cdot \vec{E} &= 4\pi\rho \\ \nabla \times \vec{E} + \partial_t \vec{B} &= 0 \\ \nabla \cdot \vec{B} &= 0 \end{aligned}$$

These equations are invariant under Lorentz transformations. To see this, we first write them with tensor notation:

$$\begin{aligned} \epsilon^{ijk}\partial_j B_k - \partial_0 E^i &= 4\pi J^i \\ \partial_i E^i &= 4\pi J^0 \\ \epsilon^{ijk}\partial_j E_k + \partial_0 B^i &= 0 \\ \partial_i B^i &= 0 \end{aligned}$$

Since δ is the metric of 3-space, we have risen and lowered indices without abandon.
We used the **Current 4-vector:** $J^\mu = (\rho, J^1, J^2, J^3)$

To write Maxwell's equations in a tensor form, we first realize that:

$$F^{0i} = E^i$$
$$F^{ij} = \epsilon^{ijk} B_k$$

(Note that $F^{01} = \eta^{00}\eta^{11}F_{01}$ and $F^{12} = \epsilon^{123}B_3$). Then, the first two equations become:

$$\partial_j F^{ij} - \partial_0 F^{0i} = 4\pi J^i$$
$$\partial_i F^{0i} = 4\pi J^0$$

Using the antisymmetry of F, we can write it in just one equation:

$$\boxed{\partial_\mu F^{\nu\mu} = 4\pi J^\nu}$$

A similar reasoning reveals that the third and fourth equation can be written as:

$$\boxed{\partial_{[\mu} F_{\nu\lambda]} = 0}$$

And these are now tensor equations that transform as tensors. The are called the **covariant Mexwell equations** (which has nothing to do with the meaning of covariant, is just to use a tensor sounding name).

Differential Forms

A **Differential form**
A p-form is a (0,p) tensor which is **completely antisymmetric** (antisymmetric in every pair of indexes).

Thus, scalars are 0 forms and dual vectors are 1-forms. We have also met the 2-form $F_{\mu\nu}$ and the 4-form $\epsilon_{\mu\nu\rho\sigma}$.

The space of all p-forms is denoted Λ^p, and the space of all p-form fields over a manifold M is denoted $\Lambda^p(M)$.
The number of linearly independent p-forms on a n-dimensional vector space is $n!/(p!(n-p)!)$
So in 4-dimensional space there is one LI 0-form, four 1-forms, six 2-forms, four 3-forms, and one 4-form. There are no p forms forms with $p > n$, since all components would be 0 by antisymmetry.

Wedge Product: Given a p-form A and a q-form B, we can form a $(p+q)$ form known as the **wedge product** $A \wedge B$ by taking the antisymmetrized tensor product:

$$(A \wedge B)_{\mu_1 \cdots \mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}]}$$

Thus for example, the wedge product of two 1-forms is:

$$(A \wedge B)_{\mu\nu} = 2A_{[\mu}B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu$$

Note that:

$$A \wedge B = (-1)^{pq} B \wedge A$$

Exterior derivative 'd': It takes a p-form field and gives a $p+1$ form field. It is defined as a normalized antisymmetric partial derivative:

$$(dA)_{\mu_1 \cdots \mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}]}$$

The simplest example is the gradient, which is the exterior derivative of a 1-form:

$$(d\phi)_\mu = \partial_\mu \phi$$

The exterior derivative turns out to be a tensor no matter if we are in flat space or not.

An interesting fact is that for any form A, we have:

$$d(dA) = 0$$

Which is often written as $d^2 = 0$. This identity is a consequence of the definition of d and the commutative property of partial derivatives $\partial_a \partial_b = \partial_b \partial_a$

Closed: A p-form is closed if $dA = 0$

Exact: A p-form A is exact if $A = dB$ for some $(p-1)$ form B.

All exact forms are closed, but the converse is not always true.

On a manifold M, closed p-forms are a vector space $Z^p(M)$ and exact forms are a vector space $B^p(M)$. We define a new vector space as their quotient:

$$H^p(M) = Z^p(M)/B^p(M)$$

called the pth **de Rham Cohomology** vector space. Which depends only on the topology of the Manifold M. In Minkowki space M, we have $H^0(M) = \mathbb{R}$ and all others for $p > 0$ are 0.

Therefore, in Minkosky space, all closed forms are exact except for 0 forms (because there are no -1 forms).

The dimension b_p of the space $H^p(M)$ is called the pth Betti number of M, and the Euler characteristic of the manifold is found as:

$$\chi(M) = \sum_{p=0}^{n} (-1)^p b_p$$

Hodge Duality: We define the Hodge star operator on a n-dimensional manifold as a map from p-forms to $(n-p)$ forms,

$$(*A)_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \epsilon^{\nu_1 \cdots \nu_p}{}_{\mu_1 \cdots \mu_{n-p}} A_{\nu_1 \cdots \nu_p}$$

Mapping A to A dual.
There is the property:

$$* * A = (-1)^{s+p(n-p)} A$$

Electrodynamics provides an especially compelling example of the use of differential forms. We have that $\partial_{[\mu} F_{\nu\lambda]} = 0$, that is:

$$dF = 0$$

Therefore (in Minkowsky space all closed forms are exact), there is a one form A_μ such that:

$$F = dA$$

this is the **vector potential**

Again Relativity

We define the **line element** or infinitesimal interval as:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

If $x^\mu(\lambda)$ is some path of a particle (defined as the map from $\mathbb{R}$ to M).

Then, for spacelike paths ($ds^2 > 0$) we have that the **path length** is defined as:

$$\Delta s = \int \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

Where the integral is taken over the path.

For timelike paths ($ds^2 < 0$), the **proper time is**:

$$\Delta \tau = \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

τ actually measures the time elapsed on a physical clock carried along the path

Four velocity: If a particle moves in some path, we can parametrize its motion by the proper time τ and define the **four velocity** as:

$$U^\mu = \frac{dx^\mu}{d\tau}$$

since $d\tau^2 = -\eta_{\mu\nu}dx^\mu dx^\nu$, the four velocity is automatically normalized:

$$\eta_{\mu\nu}U^\mu U^\nu = -1$$

In the rest frame of a particle, its four velocity has components $U^\mu = (1, 0, 0, 0)$.

The **Energy momentum four vector** is defined as:

$$p^\mu = mU^\mu$$

Where m is the mass of the particle (the rest mass)
The **energy** is simply p^0 and therefore is not invariant under Lorentz.
In the particle's rest frame we have $p^0 = m$ (recalling that $c = 1$)
And in a moving frame we have:

$$p^\mu = (\gamma m, v_x\gamma m, v_y\gamma m, v_z\gamma m)$$

For small v, this terms reduce to the classic ones.

Force: we Should have something analogous to the second law, so we define:

$$f^\mu = m\frac{d^2}{d\tau^2}x^\mu(\tau) = \frac{d}{d\tau}p^\mu(\tau)$$

For example, we consider the Lorentz force $\vec{f} = q(\vec{E} + \vec{v} \times \vec{B})$ and the tensorial generalization is:

$$f^\mu = qU^\lambda F^\mu_\lambda$$

Fluids

A continuum matter described by macroscopic quantities. We will consider **perfect fluids** which have no viscosity and no heat conduction between parts.

Dust:

Dust is the simplest example, it is a fluid in which all particles are at rest in some frame.
In some other frame, the particles move with velocity U.
We can define the **number flux 4-vector** to be:

$$N^\mu = nU^\mu$$

Where n is the **number density** of the particles in the rest frame.
Then N^0 is the number density as measured in another frame, N^i is the flux of particles in the ith direction.

Let's imagine all the particles have the same mass m, then the **energy density in the rest frame** is:

$$\rho = nm$$

We can see that ρ is the $\mu = 0, \nu = 0$ component of a tensor $p \otimes N$ as measured in its rest frame.
Therefore, in general, in any frame, we have:

$$T^{\mu\nu}_{dust} = \rho U^\mu U^\nu$$

Perfect fluid
Back to the perfect fluid, it must be isotropic in its rest frame, so T must be diagonal, there is no net flux of any component of momentum in an orthogonal direction (no viscosity). Furthermore all the nonzero spacelike components must be equal, we call this number p, the pressure. Therefore, in the rest frame:

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

We can see that this is equal to:

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p\nu^{\mu\nu}$$

Which is the general definition in any system.
$T^{\mu\nu}$ is conserved, in the sense that:

$$\partial_\mu T^{\mu\nu} = 0$$

Which account for energy and momentum conservation.

Manifolds

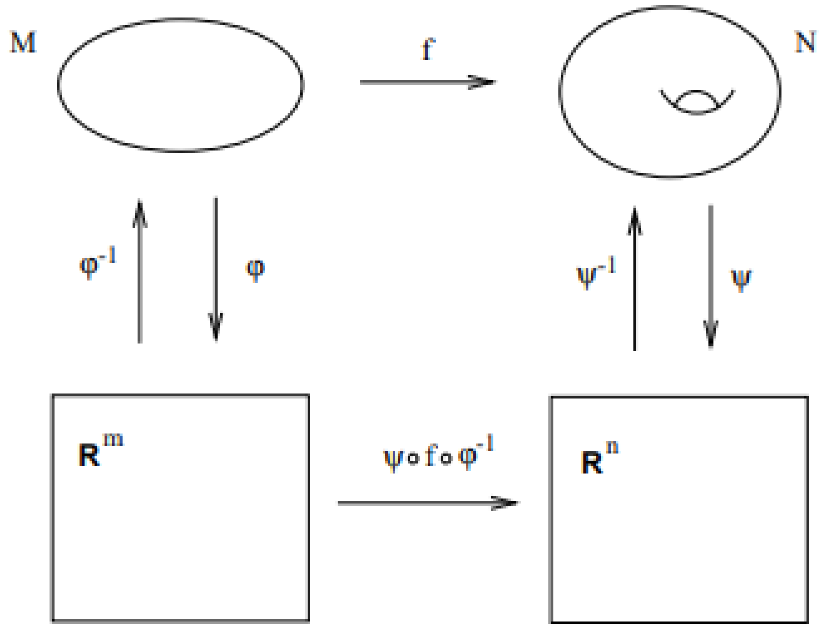
- **Map:** A function between two sets $\phi : M \to N$
- **One to one (injective):** If each element of N has at most one element of M mapped into it.
- **Onto (surjective):** If each element of N has at least one element of M mapped into it.
  A map that is both is invertible.

  M is the **Domain** and $\phi(M)$ the image of ϕ. For $U \subset N$, the preimage of U is $\phi^{-1}(U)$
- A map $\phi : \mathbb{R}^m \to \mathbb{R}^n$ is **smooth** if each of its component functions are C^∞.
- Two sets M, N are **diffeomorphic** if there exists a smooth function with smooth inverse between them.
  Chain rule: $\dfrac{\partial}{\partial x^a} = \dfrac{\partial y^b}{\partial x^a}\dfrac{\partial}{\partial y^b}$
- **Open Ball (radius r around y):** The set of all points $x \in \mathbb{R}^n$ such that $|x - y| < r$
- **Open Set:** Is a union of open balls in $\mathbb{R}^n$
- **Chart:** A chart consists of a subset U of a set M, along with a one-to-one map $\phi : U \to \mathbb{R}^n$, such that the image $\phi(U)$ is open in $\mathbb{R}$ (so it is invertible).
  We can say that U is an open set of M, so we have given M an induced topology.
- A C^∞ **atlas** is an indexed collection of charts $\{U_\alpha, \phi_\alpha\}$, which satisfies:
  - The union of the U_α is equal to M (U_α covers M)
  - The charts are smoothly sewn together. More precisely, if $U_\alpha \cap U_\beta \neq \emptyset$, then the map $(\phi_a \circ \phi_\beta^{-1})$ takes points in $\phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ onto $\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$
    And all these maps should be C^∞

  So a chart defines a coordinate system on all open sets of M.
- **Manifold:** A C^∞ n-dimensional manifold is a set M along with a **maximal atlas** (one that contains every possible compatible chart).
  The requirement of maximal atlas is so that equivalent spaces equipped with different atlases don't count as a different manifold
  This definition captures the definition of a set that locally looks like $\mathbb{R}^n$.

  Example: The sphere S^2 is a 2-manifold. We can define the stereographic projection that takes all the sphere except the north pole to $\mathbb{R}^2$. We can now make a similar projection but from the south pole, to get a complete chart. It can be seen that this two maps are C^∞ compatible when sewing them together.

The fact that manifolds locally look like $\mathbb{R}^n$ allows us to define functions and things between manifolds by analogy to functions between flat spaces.



For example, if we have a function between the sets M, N such as $f : M \to N$,
we can see the corresponding function between flat spaces is $\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ (where the maps are defined).
With this, we can now differentiate f, for example:

$$\frac{\partial f}{\partial x^\mu} := \frac{\partial}{\partial x^\mu}(\psi \circ f \circ \phi^{-1})(x^\mu)$$

Where x^μ represents $\mathbb{R}^m$

Tangent Spaces

We can now define tangent spaces (the set of vectors at a single point in spacetime, it is an abstract vector space associated with each point)

Suppose we want to construct a tangent space to a point p in manifold M only using things intrinsic to M.
We might consider the set of all parameterized curve through p, that is, all nondegenerate maps $\gamma : \mathbb{R} \to M$ such that p is in the image of γ

We define F to be the space of all smooth funcitons on M (that is, C^∞ maps $f : M \to \mathbb{R}$). Then we notice that each curve through p defines an operator on this space, the directional derivative, which maps $f \to df/d\lambda$ (at p).

We will make the following **claim:** The tangent space T_p can be identified with the space of directional derivative operators along curves through p.

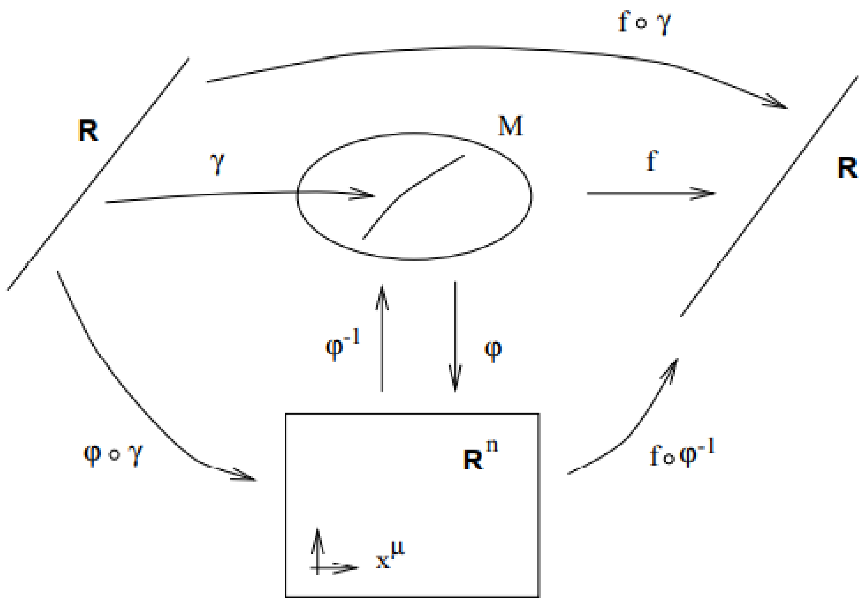
We must demonstrate that the space of directional derivatives is a vector space, and that it is the one we want (same dimensionality as M, yields a natural idea of a vector pointing along a certain direction).

- **Is a vector space**: Imagine two operators $\frac{d}{d\lambda}$ and $\frac{d}{d\eta}$ representing derivatives along two curves through p (evaluated at p). We can scale them and add them to get a new operator $a\frac{d}{d\lambda} + b\frac{d}{d\eta}$.
It is not obvious that the resulting operator is itself a derivative operator. A good derivative operator is one that acts linearly on functions and obeys the Leibniz product rule, linearity is obvious, and Leibniz rule:
$$\begin{aligned}\left(a\frac{d}{d\lambda} + b\frac{d}{d\eta}\right)(fg) &= af\frac{dg}{d\lambda} + ag\frac{df}{d\lambda} + bf\frac{dg}{d\eta} + bg\frac{df}{d\eta} \\ &= \left(a\frac{df}{d\lambda} + b\frac{df}{d\eta}\right)g + \left(a\frac{dg}{d\lambda} + b\frac{dg}{d\eta}\right)f\end{aligned}$$
So it satisfies the product rule.
Therefore, the space of this linear operators is a linear space.

- **Correct Space:** Consider a coordinate chart with coordinates x^μ. Then, there is an obvious set of n directional derivatives at p, namely, the partial derivatives ∂_μ evaluated at p

Therefore, we claim that the partial derivative operators $\{\partial_\mu\}$ at p form a basis for the tangent space T_p (therefore it is n dimensional).

To see this, consider an n-manifold M, a coordinate chart $\phi : M \to \mathbb{R}^n$, a curve $\gamma : \mathbb{R} \to M$ and a function $f : M \to \mathbb{R}$, which leads to:



If λ is the parameter along γ, we want to expand the vector / operator $\dfrac{d}{d\lambda}$ in terms of the partials ∂_μ. Using the chain rule, we have:

$$\begin{aligned}\frac{d}{d\lambda}f &= \frac{d}{d\lambda}(f \circ \gamma) \\ &= \frac{d}{d\lambda}[(f \circ \phi^{-1}) \circ (\phi \circ \gamma)] \\ &= \frac{d(\phi \circ \gamma)^\mu}{d\lambda}\frac{\partial(f \circ \phi^{-1}}{\partial x^\mu} \\ &= \frac{dx^\mu}{d\lambda}\partial_\mu f\end{aligned}$$

The first line simply takes the informal expression on the left hand side and rewrites it as an honest derivative of the function $(f \circ \gamma) : \mathbb{R} \to \mathbb{R}$.
Therefore we have:

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu$$

Thus, the partials $\{\partial_\mu\}$ represent a good basis for the vector space of directional derivatives.

So our basis is $\widehat{e}_\mu = \partial_\mu$ and is known as the **coordinate basis** of T_p

Then, the transformation law for another basis with coordinate system $x^{\mu'}$ is immediate:

$$\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu$$

And if we have a vector $V = V^{\mu}\partial_{\mu}$, we can see the transformation into coordinates $V^{\mu'}$ in base $\partial_{\mu'}$:

$$\begin{aligned} V^{\mu}\partial_{\mu} &= V^{\mu'}\partial_{\mu'} \\ &= V^{\mu'}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\partial_{\mu} \end{aligned}$$

Hence (since the matrix $\partial x^{\mu'}/\partial x^{\mu}$ is the inverse of the matrix $\partial x^{\mu}/\partial x^{\mu'}$), we get:

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}V^{\mu}$$

This rule encompasses the behaviour of vector components under arbitrary changes of coordinates (not only linear transformations).

We now consider dual vectors. The cotangent space T_p^* is the set of linear maps $\omega : T_p \to \mathbb{R}$. The canonical example of a one-form is the gradient of a function f, denoted df. Its action on a vector $\frac{d}{d\lambda}$ is:

$$df\left(\frac{d}{d\lambda}\right) = \frac{df}{d\lambda}$$

Just as the partial derivatives along coordinate axes provide a natural basis for the tangent space, the gradiantes of the coordinate functions x^{μ} provide a natural basis for the cotangent space. Recall that in flat space we constructed a basis for T_p^* by demanding that $\widehat{\theta}^{\mu}(\widehat{e}_{\nu}) = \delta^{\mu}_{\nu}$. We find this leads to:

$$dx^{\mu}(\partial_{\nu}) = \frac{\partial x^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu}$$

Therefore, the gradients $\{dx^{\mu}\}$ are an appropriate set of basis one-forms.
An arbitrary one form is $\omega = \omega_{\mu}dx^{\mu}$

The transformation of dual vectors and components can be found as usual:

$$\begin{aligned} dx^{\mu'} &= \frac{\partial x^{\mu'}}{\partial x^{\mu}}dx^{\mu} \\ \omega_{\mu'} &= \frac{\partial x^{\mu}}{\partial x^{\mu'}}\omega_{\mu} \end{aligned}$$

Now, a (k, l) **tensor** T can be written as:

$$T = T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_k}\otimes dx^{\nu_1}\otimes\cdots\otimes dx^{\nu_l}$$

And the transformation law is:

$$T^{\mu_1'\cdots\mu_k'}{}_{\nu_1'\cdots\nu_l'} = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}}\cdots\frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}}\frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}}\cdots\frac{\partial x^{\nu_l}}{\partial x^{\nu_l'}}T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}$$

1.0.1 Resumen espacios Tangentes

- Sea M una variedad. Digamos que tiene un sistema de coordenadas locales $\{x^1, \cdots, x^n\}$ y otro sistema de coordenadas $\{y^1, \cdots, y^n\}$.
- Definimos $\Omega^0(M)$ como el conjunto de todas las funciones $f : M \to \mathbb{R}$ que son continuas (Es decir que son continuas al componer con las funciones coordenadas y ver como una función real)
- **Espacio Tangente**: T_pM Es el conjunto de todos los operadores que toman una función f y dan un real. Y cumplen con linealidad y regla de Leibniz.
  La **base** son elementos de la forma $\left.\frac{\partial}{\partial x^i}\right|_p$, que toman una función de $\Omega^0(M)$ y devuelven su derivada evaluada en p.
  En general sus elementos son **vectores** de la forma $X_p = X^i \left.\frac{\partial}{\partial x^i}\right|_p$.
  O bien, en la otra base es de la forma $X_p = Y^j \left.\frac{\partial}{\partial y^j}\right|_p$
- **Espacio Cotangente:** T_p^*M es el espacio dual a T_pM.
  La **base** son elementos de la forma dx^i. Estos elementos pertenecen a $\Omega^0(M)$. Toman los puntos de M y los proyectan sobre el eje x^i.
  Es el dual porque $\langle \frac{\partial}{\partial x^i}, dx^j \rangle = \delta_i^j$, donde el producto punto consiste en derivar la función dx^j usando $\frac{\partial}{\partial x^i}$ y por eso la delta de Kroenecker.
  En general, son **covectores** de la forma $\alpha_p = a_i dx^i$
  O bien, en la otra base es de la forma $\alpha_p = b_i dy^i$
- **Transformaciones:**
  - **la Base** Tenemos la base $\frac{\partial}{\partial x^i}$ y la base $\frac{\partial}{\partial y^j}$ del espacio tangente, entonces por la regla de la cadena, se transforman **covariantemente:**

    $$\frac{\partial}{\partial y^j} = \left.\frac{\partial}{\partial x^i}\right|_p \left.\frac{\partial x^i}{\partial y^j}\right|_p$$

    Donde el operador $\frac{\partial}{\partial x^i}$ se deja indicado.
  - **La Cobase**: Si tenemos la cobase $\{dx^i\}$ y la cobase $\{dy^i\}$ que vienen de las bases coordenadas $\{x^i\}, \{y^j\}$. Enotnces, la cobase se transforma **contravariantemente**

    $$dy^j = \left.\frac{\partial y^j}{\partial x^i}\right|_p dx^i$$

- **Coordenadas de vectores:** Digamos que $X_p = X^i \frac{\partial}{\partial x^i}\Big|_p$ y $X_p = Y^j \frac{\partial}{\partial y^j}\Big|_p$. Entonces las coordenadas de los vectores se transforman **contravariantemente** como:

$$Y^j = X^i \frac{\partial y^j}{\partial x^i}\Big|_p$$

- **Coordenada de covectores:** Digamos que $\alpha_p = a_i dx^i$ y también $\alpha_p = b_j dy^j$. Entonces, las coordenadas se transforman **covariantemente:**

$$a_i = b_j \frac{\partial x^i}{\partial y^j}\Big|_p$$

Es decir, las cosas se transforman como $\mathbf{e}' = \mathbf{e}\mathbf{A}$, $\theta' = \mathbf{A}^{-1}\theta$, $\mathbf{v}' = \mathbf{A}^{-1}\mathbf{v}$, $\mathbf{f}' = \mathbf{f}\mathbf{A}$. Donde $A = \frac{\partial x^i}{\partial y^j}$

Campos: Podemos definir campos como asignaciones que a cada punto p le dan un vector tangente y un covector y así.

1.0.2 Campos Tensoriales En Variedades

Consideramos el espacio tangente $T_{x_0}(\mathbb{R}^n)$ en $x_0 = (x_0^1, \cdots, x_0^n)$ en $\mathbb{R}^n$ (correspondiente a $T_p(M)$) Y el espacio cotangente consiste de las funciones lineales de $T_{x_0}(\mathbb{R}^n)$ a $\mathbb{R}$. El espacio $T_p(M)$ tiene como base:

$$\vec{e}_{p\ i} = \frac{\partial}{\partial x^i}\Big|_p$$

Y el espacio cotangente contiene las funciones lineales $f : M \to \mathbb{R}$ y se define la **forma natural** entre ambos como $\langle \frac{\partial}{\partial x^i}, f \rangle = \frac{\partial f}{\partial x^i}(p)$
Una base de $T_p^*(M)$ son las proyecciones dx^i que dan la iésima coordenadas de p.

Tensor: Un tensor en un punto $p \in M$ se puede ver ya sea como un elemento de un espacio de producto tensorial o como un mapeo multilineal. En cualquier caso, un tensor en $p \in M$ es:

$$\Psi = \Psi^{i_1 \cdots i_r}_{j_1 \cdots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}$$

O bien es un mapeo multilineal que toma r elementos de $T_p^* M$ y s elementos de $T_p M$ como $(\alpha_1, \cdots, \alpha_r, X_1, \cdots, X_s)$ y aplica $\Psi^{i_1 \cdots i_r}_{j_1 \cdots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}$ a esto (aplicando cada uno por parejas usando el producto interno en el que el elemento de $T_p M$ se aplica a $T_p^* M$.

Campo Tensorial. Es una expresión del tipo:

$$\Psi = \Psi^{i_1\cdots i_r}_{j_1\cdots j_s}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_s}$$

Donde $\Psi^{i_1\cdots i_r}_{j_1\cdots j_s}$ es una expresión que depende del punto $p\in M$. Es decir, a cada punto p (o bien a cada coordenadas x) le da un tensor de $(T_pM)^{\otimes r}\otimes(T_p^*M)^{\otimes s}$.
Y ahora las derivadas se deben de evaluar en p
Este es un **Tensor de tipo** (r,s).
Este tensor se tiene que aplicar a un elemento de $\Pi_r T_p^*(M)\times\Pi_s T_p(M)\to\mathbb{R}$. Es decir, toma r funciones lineales de M en $\mathbb{R}$ y s operadores diferenciales.
Luego aplica la forma natural a cada par como es de esperar de un tensor y realize las sumas sobre los índices repetidos.

Transformación de coordenadas. Si tenemos unas coordenadas $\{y^1,\cdots,y^n\}$, entonces vimos que las bases se transforman como $\dfrac{\partial}{\partial x^i}=\dfrac{\partial y^k}{\partial x^i}\dfrac{\partial}{\partial y^k}$ y $dx^i=\dfrac{\partial x^i}{\partial y^k}dy^k$. Entonces, en estas nuevas coordenadas, los componentes del tensor se transforman como:

$$\Psi^{i_1'\cdots i_r'}_{j_1'\cdots j_s'}(y(x))=\Psi^{i_1\cdots i_r}_{j_1\cdots j_s}(x)\frac{\partial y^{i_1'}}{\partial x^{i_1}}\cdots\frac{\partial y^{i_r'}}{\partial x^{i_r}}\frac{\partial x^{j_1}}{\partial y^{j_1'}}\cdots\frac{\partial x^{j_s}}{\partial y^{j_s'}}$$

Y ahora para cada punto p con coordenadas y, se le asigna un tensor.

Se define una suma entre tensores como es de esperar y un producto tensorial también.

Example

Consider a symmetric $(0,2)$ tensor S on a 2-dimensional manifold, whose components in a coordinate system $(x^1=x\ ,\ x^2=y)$ are given by:

$$S_{\mu\nu}=\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix}$$

This can be written equivalently as:

$$\begin{aligned}S &= S_{\mu\nu}(dx^\mu\otimes dx^\nu)\\ &= x(dx)^2+(dy)^2 \qquad (2.21)\end{aligned}$$

Where in the last line the tensor product symbols are suppressed for brevity. $(dx)^2:=dx\otimes dx$
Now consider the new coordinates:

$$\begin{aligned}x' &= x^{1/3}\\ y' &= e^{x+y}\end{aligned}$$

This leads directly to:

$$\begin{aligned} x &= (x')^3 \\ y &= \log(y') - (x')^3 \\ dx &= 3(x')^2 dx' \\ dy &= \frac{1}{y'} dy' - 3(x')^2 dx' \end{aligned}$$

We plug this in 2.21 to get the tensor in terms of the new coordinates:

$$S = 9(x')^4[1 + (x')^3](dx')^2 - 3\frac{(x')^2}{y}(dx'dy' + dy'dx') + \frac{1}{(y')^2}(dy')^2$$

So:

$$S_{\mu'\nu'} = \begin{pmatrix} 9(x')^4[1 + (x')]^3 & -3\dfrac{(x')^2}{y'} \\ -3\dfrac{(x')^2}{y'} & \dfrac{1}{(y')^2} \end{pmatrix}$$

We did not need to use the transformation laws directly, but the result would have been the same.

Most things we defined in flat space (contraction, symmetrization, etc) are unchanged in this general setting.
But there are some exceptions (partial derivatives, the metric, Levi-Civita tensor).

- **Partial Derivative:** The partial derivative of a tensor is not in general a tensor. The gradient, which is the partial derivative of a scalar, is an honest $(0, 1)$ tensor, as we have seen.
  But the partial derivative of a higher-rank tensors is not tensorial.

  For example, consider W_ν and its partial derivative: $\partial_\mu W_{|\nu}$. This should be a (0,2) tensor. But it isn't, we see it isn't by transforming the basis:

$$\begin{aligned} \frac{\partial}{\partial x^{\mu'}} W_{\nu'} &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \left(\frac{\partial x^\nu}{\partial x^{\nu'}} W_\nu \right) \\ &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \left(\frac{\partial}{\partial x^\mu} W_\nu \right) + W_\nu \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^{\nu'}} \\ &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \left(\frac{\partial}{\partial x^\mu} W_\nu \right) + W_\nu \frac{\partial^2 x^\nu}{\partial x^{\mu'} \partial x^{\nu'}} \end{aligned}$$

  The last term should not be there if we wanted $\partial_\mu W_\nu$ to transform as a $(0, 2)$ tensor. It arises because the derivative of the transformation matrix does not vanish as it does in the flat space.
  We need a new substitute for the partial derivative for curved space.

- **Metric:** The metric in curved space is given a new symbol $g_{\mu\nu}$.
  There a few restrictions on the components of $g_{\mu\nu}$ other that it is a symmetric $(0, 2)$ tensor.
  It is usually taken to be nondegenerate, so that it has an inverse $g^{\mu\nu}$ defined via:

  $$g^{\mu\nu} g_{\nu\sigma} = \delta^{\mu}_{\sigma}$$

  Just as before, it is used to lower and raise indexes.
  The metric is used for a lot of things we will see later:
  1) supplies a notion of past and future, 2) metric allows the computation of path length and proper time, 3) the metric determines the shortest distance between two points, 4) the metric replaces the Newtonian gravitational field, 5) the metric provides a notion of locally inertial frames, 6) The metric determines causality, by defining the speed of light, 7) the metric replaces the dot product.

  As with flat space, we can now write:

  $$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$$

  Example: In euclidean 3-D space, the line element is:

  $$ds^2 = (dx)^2 + (dy)^2 + (dz)^2$$

  We now change to other coordinates, for example, in spherical coordinates:

  $$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$

  Which leads to:

  $$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

  A good example of a space with curvature is the two sphere, which has $r = 1$ and $dr = 0$, so it has a line element of:

  $$ds^2 = d\theta^2 + \sin^2\theta d\phi^2$$

  Canonical form: We write the components of the metric $g_{\mu\nu}$ as:

  $$g_{\mu\nu} = diag(-1, -1, \cdots, -1, 1, 1 \cdots, 1, 0, 0, \cdots, 0)$$

  If it has s +1s and t -1s, we call $s - t$ the **signature** of the metric.
  And $s + t$ the **rank** of the metric.
  Euclidean metric: If all the signs are positive (t=0)

Lorenzian or pesudo Riemannian metric: If there are some +1s and some −1s.

It results that it is always possible to put a metric into canonical form at some point $p \in M$ (only in the point, not even in a neighborhood) and also the first derivatives $\partial_\sigma g_{\mu\nu}$ vanishes at p.
Such coordinate are known as **Riemann normal coordinates**

Curvature

In flat space in Cartesian coordinates, the partial derivative operator ∂_μ is a map from (k,l) tensor fields to $(k,l+1)$ tensor fields, which acts linearly on its arguments and obeys the Leibniz rule on tensor products.
We would like a **covariant derivative** operator ∇ to perform the functions of the partial derivative, buy in a way independent of coordinates.
We therefore require that ∇ be a map from (k,l) tensor fields to $(k,l+1)$ tensor fields which has the two properties:

- **Linearity:** $\nabla(T+S) = \nabla T + \nabla S$
- **Leibniz Product Rule:** $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$

If ∇ is going to obey the Leibniz rule, it can always be written as the partial derivative plus some linear transformation.
That is, to take the covariant derivative, we take the partial derivative and then apply a correction to make the result covariant.

Let's consider what whis means for the covariant derivative of a vector V^ν. It means that, for each direction μ, the covariant derivative ∇_μ will be given by the partial derivative ∂_μ plus a correction specified by a matrix $(\Gamma_\mu)^\rho_\sigma$ (an $n \times n$ matrix, where n is the dimensionality of the manifold, for each μ).
This matrices are known as the **connection coefficients**. We therefore have:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$$

Notice that the free index on V was moved to Γ and a new index is summed over.
We want this **Connection coefficients** to fulfill a couple of things:

1) It is linear $\nabla(T+S) = \nabla T + \nabla S$

2) Product law: $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$
   This two conditions are what led us to define it as $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$

3) **It is a covariant derivative:** That is, the result above is a $(1,1)$ tensor.

   For this, we want the transformation law to be:

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu$$

   We can expand the left side using the definition of ∇V and that the connection coefficients are constant at a given point, so:

$$\begin{aligned}\nabla_{\mu'} V^{\nu'} &= \partial_{\mu'} V^{\nu'} + \Gamma^{\nu'}_{\mu'\lambda'} V^{\lambda'} \\ &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} V^\nu \frac{\partial}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu} + \Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda\end{aligned}$$

Meanwhile, the right side can be expanded as $\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\nabla_{\mu}V^{\nu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\partial_{\mu}V^{\nu} + \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}_{\mu\lambda}V^{\lambda}$
We now equate this two expressions and do some algebra, to get the **transformation law of the coefficients:**

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}_{\mu\lambda} - \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\lambda}}$$

So, the connection coefficients are not components of a tensor, but that's OK, they are only built to make the covariant derivative a tensor.

We can also hope for the derivative of a covector to be:

$$\nabla_{\mu}\omega_{\nu} = \partial_{\mu}\omega_{\nu} + \tilde{\Gamma}^{\lambda}_{\mu\nu}\omega_{\lambda}$$

Where the $\tilde{\Gamma}$ elements are not yet known in relation to the normal Γ coefficients.

4) **Commutes with contractions:** $\nabla_{\mu}(T^{\lambda}_{\lambda\rho}) = (\nabla T)_{\mu}{}^{\lambda}{}_{\lambda\rho}$
5) **Reduces to the partial derivative on scalars** $\nabla_{\mu}\phi = \partial_{\mu}\phi$
We demand this properties to be true.
Given some one form field ω_{μ} and vector field V^{μ}, we can take the covariant derivative of $\omega_{\lambda}V^{\lambda}$ (which is a scalar). We then use the two properties just stated and derive that the rule for covectors is:

$$\nabla_{\mu}\omega_{\nu} = \partial_{\mu}\omega_{\nu} - \Gamma^{\lambda}_{\mu\nu}\omega_{\lambda}$$

And in general, the covariant derivative for any tensor is:

$$\begin{aligned}\nabla_{\sigma}T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} = {} & \partial_{\sigma}T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} \\ & +\Gamma^{\mu_1}_{\sigma\lambda}T^{\lambda\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} + \Gamma^{\mu_2}_{\sigma\lambda}T^{\mu_1\lambda\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} + \cdots \\ & -\Gamma^{\lambda}_{\sigma\nu_1}T^{\mu_1\mu_2\cdots\mu_k}{}_{\lambda\nu_2\cdots\nu_l} - \Gamma^{\lambda}_{\sigma\nu_2}T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\lambda\cdots\nu_l} - \cdots \, .\end{aligned}$$

Sometimes semicolons are used for notation of covariant derivative.

We can see some properties, for example:

The difference of two connection coefficients is a tensor.

Therefore, we define the **Torsion tensor** as:

$$T^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} = 2\Gamma^{\lambda}_{[\mu\nu]}$$

We now define an specific connection on a manifold with metric $g_{\mu\nu}$ by introducing two additional properties (which aren't necessary for a general connection):

6) **Torsion Free** $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$

7) **Metric Compatibility:** $\nabla_{\rho} g_{\mu\nu} = 0$

This implies a couple of nice properties. First, the inverse is also compatible:

$$\nabla_{\rho} g^{\mu\nu} = 0$$

Second, a metric-compatible covariant derivative commutes with raising and lowering of indices. Thus, for some vector field V^{λ},

$$g_{\mu\lambda} \nabla_{\rho} V^{\lambda} = \nabla_{\rho}(g_{\mu\lambda} V^{\lambda}) = \nabla_{\rho} V_{\mu}$$

Theorem (Fundamental theorem of Riemannian differential Geometry): There is exactly one torsion free connection on a given manifold which is compatible with some given metric on that manifold.

To demonstrate this, we can find the connection components in terms of the metric.
to accomplish this, we write $\nabla_{\rho} g_{\mu\nu} = \partial_{\rho} g_{\mu\nu} - \Gamma^{\lambda}_{\rho\mu} g_{\lambda\nu} - \Gamma^{\lambda}_{\rho\nu} g_{\mu\lambda} = 0$ and we permute the indexes to get 3 equations. With some algebra, we get:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\rho\mu} - \partial_{\rho} g_{\mu\nu})$$

This specific connection is defined as the **Christoffel connection or Levi Civita connection**.
And the associated coefficients are called the **Christoffel Symbols**.

Divergence:
The divergence of a vector is:

$$\nabla_{\mu} V^{\mu} = \partial_{\mu} V^{\mu} + \Gamma^{\mu}_{\mu\lambda} V^{\lambda}$$

It's easy to show that:

$$\nabla_{\mu} V^{\mu} = \frac{1}{\sqrt{|g|}} \partial_{\mu}(\sqrt{|g|} V^{\mu})$$

Example: In plane polar coordinates, with metric:

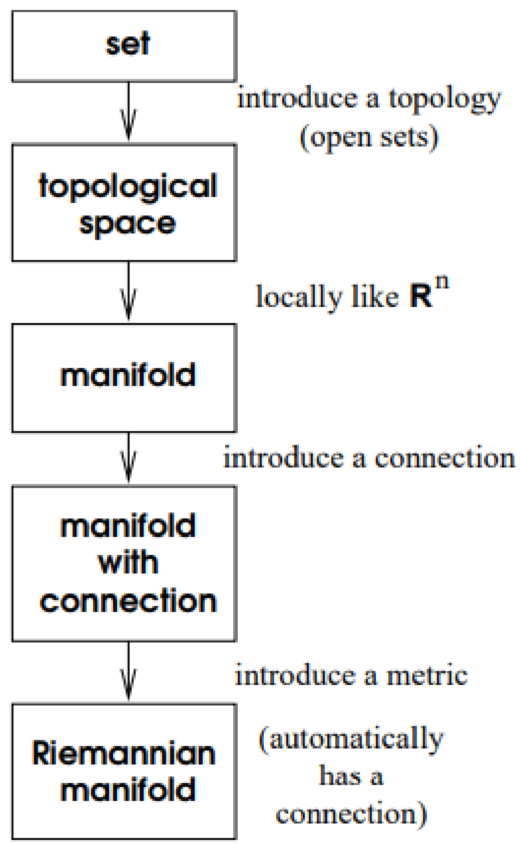
$$ds^2 = dr^2 + r^2 d\theta^2$$

The components of the metric are therefore $g^{rr} = 1$, $g^{\theta\theta} = r^{-2}$. We can compute a typical connection coefficient, for example:

$$\Gamma^{r}_{rr} = \frac{1}{2} g^{r\rho}(\partial_{r} g_{r\rho} + \partial_{r} g_{\rho r} - \partial_{\rho} g_{rr}) = \cdots = 0$$

Calculating all of them, we have:

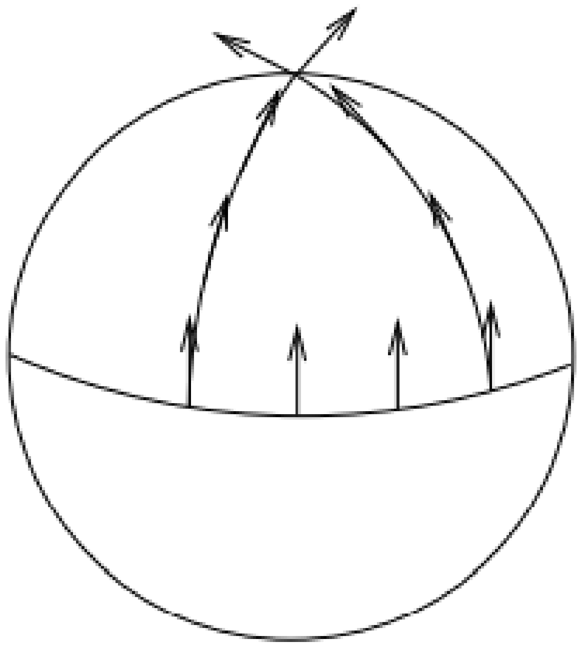
$$\Gamma^r_{\theta r} = \Gamma^r_{r\theta} = 0$$
$$\Gamma^\theta_{rr} = 0$$
$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}$$
$$\Gamma^\theta_{\theta\theta} = 0$$



Parallel transport

In flat space, it is easy to compare vectors at different points, because we just move a vector from one point to the other and compare them, the vectors don't change when transporting.

The concept of moving a vector along a path, keeping constant all the while, is known as parallel transport.
In curved space *the result of parallel transporting a vector from one point to another depends on the path taken.*
We can see this in a sphere, starting with a vector on the equator and transport it in two different ways to the north pole.



So we cannot really compare vectors in different points, since we would not know how to transport one vector to the other tangent space.
So two particles at different points on do not have any well defined notion of relative velocity.

Parallel transport is supposed to ' keep the vector constant ' as we move it.
Given a curve $x^\mu(\lambda)$, the requirement of constancy of a tensor T along this curve in *flat space* is simply $\dfrac{dT}{d\lambda} = \dfrac{dx^\mu}{d\lambda}\dfrac{\partial T}{\partial x^\nu} = 0$.

We therefore define the **coovariant** derivative along the path to be given by an operator:

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda}\nabla_\mu$$

We then define the **parallel transport** of the tensor T along the path $x^\mu(\lambda)$ to be the requirement that, along the path:

$$\left(\frac{D}{d\lambda}T\right)^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l} := \frac{dx^\sigma}{d\lambda}\nabla_\sigma T^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l} = 0$$

This is the **equation of parallel transport**.
For a vector it takes the form:

$$\frac{d}{d\lambda}V^\mu + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}V^\rho = 0$$

This is a first order differential equation defining an initial value problem. Given a tensor at some point along the path, there will be a unique continuation of the tensor to other points along the path such that the continuation solves the parallel transport equation.

If the connection is metric-compatible, the metric is always parallel transported with respect to itself:

$$\frac{D}{d\lambda}g_{\mu\nu} = \frac{dx^\sigma}{d\lambda}\nabla_\sigma g_{\mu\nu} = 0$$

It follows that the inner product of two parallel transported vectors is preserved. That is, if V^μ and W^ν are parallel-transported along a curve $x^\sigma(\lambda)$, we have:

$$\frac{D}{d\lambda}(g_{\mu\nu}V^\mu W^\nu) = \left(\frac{D}{d\lambda}g_{\mu\nu}\right)V^\mu W^\nu + g_{\mu\nu}\left(\frac{D}{d\lambda}V^\mu\right)W^\nu + g_{\mu\nu}V^\mu\left(\frac{D}{d\lambda}W^\nu\right) = 0$$

So parallel transport in a metric-compatible connection preserves norms, orthogonality, etc.

You can write an explicit and general solution to the parallel transport equation, but it is a little bit difficult and formal.

Geodesic

A geodesic is a generalization of straight line. A straight line is a path which parallel transports its own tangent vector, so we define a geodesic as such (for a manifold with a Christoffel connection).
The tangent vector to a path $x^\mu(\lambda)$ is $dx^\mu/d\lambda$.
The condition of it to be parallel transported is:

$$\frac{D}{d\lambda}\frac{dx^\mu}{\lambda} = 0$$

Or alternatively:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

This is the **Geodesic equation**.
For Euclidian space, the connection symbols are all 0 and the equation reduces to $d^2x^\mu/d\lambda^2 = 0$ as expected.

Schutz Preface to Curvature

On the relation of gravitation and Curvature

One important ingredient of SR is the existence of inertial frames that fill all spacetime, we can describe all of spacetime by a single frame, all of whose coordinate points are at rest respect to the origin.
WE are led to the idea of the interval Δs^2, which gives an invariant meaning to physical statementes.

A timelike interval between two events is the time elapsed on a clock which passes through the two events.
A spacelike interval is the length of a rod that joins two events in a frame in which they are simultaneous.
The mathematical function that calculates the interval is the metric, and is defined physically by lengths of rods and readings of clocks.

But, is it really possible to construct a frame in which the clocks all run at the same rate? Not in gravitational fields.
Gravitational fields are incompatible with global SR. We can construct local SR in places with small nonuniformities of the gravitational field.

The Gravitational Redshift experiment

Let a tower of height h be constructed on the surface of Earth. Begin with a particle of rest mass m at the top.
The particle is dropped and falls freely, it reaches a velocity on the ground of $v = \sqrt{2gh}$.
So, on the ground the total energy is $m + \frac{1}{2}mv^2 + O(v^4) = m + mgh + O(v^4)$
The experimenter on the ground has some magical device to convert all this energy into a single photon which he directs upwards.
Upon arrival at the top of the tower with energy E', the photon is again magically changed into a particle of rest mass $m' = E'$. It must be that $m' = m$, otherwise we have perpetual motion with energy gained.

But we have:

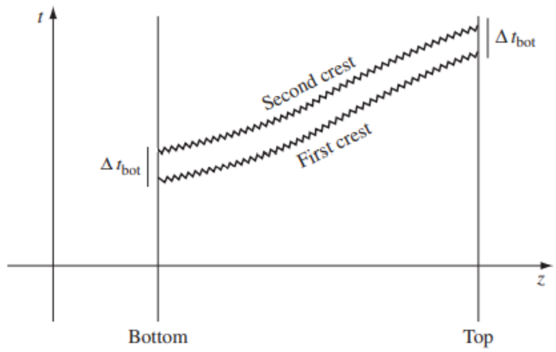
$$\frac{E'}{E} = \frac{h\nu'}{h\nu} = \frac{m}{m + mgh + O(v^4)} = 1 - gh + O(v^4)$$

So a photon climbing the Earth's gravitational field will lose energy and be redshifted.

It is possible to measure this redshift, and it is even central in the functioning of GPS.

Nonexistence of Lorentz frame at rest on Earth

If SR is valid in a gravitational field, it is natural to assume that the ' laboratory ' frame at rest on Earth is a Lorentz frame. But this is false.



In a time-independent gravitational field, two successive 'crests' of an electromagnetic wave must travel identical paths. Because of the redshift (Eq. (5.1)) the time between them at the top is larger than at the bottom. An observer at the top therefore 'sees' a clock at the bottom running slowly.

We draw a spacetime diagram in the hypothetical frame at rest on Earth, in which z is the height from earth.
Consider light as a wave, and look at two succesive crests as the move upward in the Pound-Rebka-Snider experiment. Light is drawn in a wiggly way to allow the possibility that gravity may act in an unknown way.
But no matter how it is deflected, the effect must be the same for both crests.
So we conclude from the Minkowsky geometry that $\Delta t_{top} = \Delta t_{bottom}$, but that contradicts the Pound-Recka experiment.
Therefore, the reference frame at rest in Earth is not a Lorentz frame, the gravitational field has some effects.

Principle of Equivalence

In an inertial frame a particle stays at rest if no forces act on it.
But gravity is distinguished from all other forces: All bodies have the same trajectory in a gravitational field, and it is with gravity that is impossible to define an inertial frame.

But there is a frame in which particles do keep a uniform velocity. This is a frame that falls freely in the field.
Since this frame accelerates at the same rate as free particles, all such particles will maintain a uniform velocity relative to the frame. This frame is a candidate for an inertial frame.

Consider in empty space free from gravity, a uniformly accelerating rocket. From the point of view of an observer inside, it appears that there is a gravitational field in the rocket towards the rear, since all objects fall there with the same acceleration.

Moreover, an object stationary relative to the rocket has a weight equal to the force required to keep it accelerating with the ship and is proportional to the mass of the object.
A true inertial frame is one that falls freely toward the rear of the ship, at the same acceleration as particles.
So uniform gravitational fields are equivalent to frames that accelerate uniformly relative to inertial frames.

This is the **weak principle of equivalence** between gravity and acceleration.

Notes:

- The argument is only valid locally, for uniform gravitational fields.
- There are infinite freely falling frames at any point, with different velocities and orientations.

The importance of Gravity is that all particles accelerate at the same rate. So the frame that accelerates with them is really inertial, because it sees all particles not accelerating when no forces act on them (except gravity).

The Redshift Experiment again

Let us now revise the Pound Rebka experiment. We view it in a freely falling frame, which we have seen has at least some of the characteristics of an inertial frame.
We take the particular frame that is at rest when the photon begins its upward journey and falls freely after that.
Since the photon moves a distance h, it takes time $\Delta t = h$ to arrive at the top.
In this time the frame has acquired velocity gh downward relative to the experimental apparatus. So the photon's frequency relative to the freely falling frame can be obtained by the redshift formula:

$$\frac{v(freely\ falling)}{v'(apparatus\ at\ top)} = \frac{1+gh}{\sqrt{1-g^2h^2}} = 1 + gh + O(v^4)$$

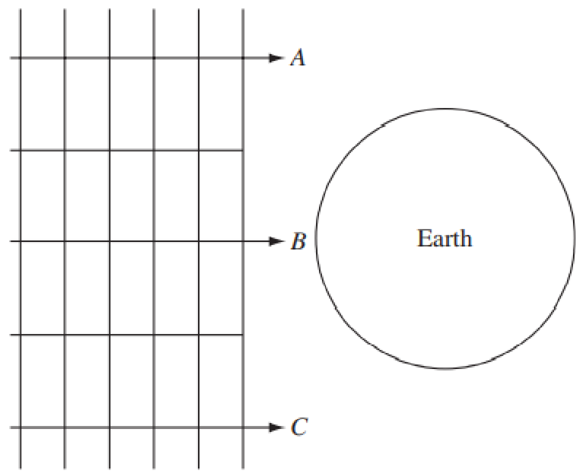
From the original equation for the experiment, we see that there is no redshift in a freely falling frame.

Local Inertial Frames

So we see a form of saving SR. Perhaps, we simply have to use freely falling frames instead of one at rest in Earth.
But this doesn't save SR, for the simple reason that the freely falling frames on different sides of Earth fall in different directions: There is no single global frame which is everywhere freely falling in Earth's gravitational field and which is still rigid (the distances between

coordinate points are constant).
It is still impossible to construct a global inertial frame, so the most we can salvage is a local inertial frame.

Consider a freely falling frame in Earth's gravitational field. An inertial frame in SR fills all of spacetime, but this freely falling frame would not be inertial if it were extended too far horizontally.



A rigid frame cannot fall freely in the Earth's field and still remain rigid.

In the figure we have a frame that is freely falling at B but at A and C the motion is not along the trajectory of a test particle.
And since the gravity changes with height, the frame cannot remain inertial if extended too far a vertical distance (if it where falling with particles at some height, it isn't at another).

All of these limitations are due to nonuniformities in the gravitational field, if the nonuniformities can be neglected, the freely falling frame can be regarded as inertial.
Any gravitational field can be regarded as uniform over a small enough region of space and time.

Inertial frames should be ones in which free particles (no forces, seeing gravity as not a force) move at constant velocities.
Therefore, for a particle moving in a uniform gravitational field, it should have constant velocity, so the frame should also accelerate at g.
This inertial frames cannot be infinite, because of nonuniformities in the field as we saw earlier, so we cannot extend SR to all space.
Therefore, in gravitational fields, it is impossible to construct a global inertial frame.

The role of curvature

In SR, two world lines which begin parallel remain parallel and this is what makes Minkowsky space flat.
In a nonuniform gravitational field, the world lines of two nearby particles which begin parallel do not generally remain parallel, this space is not flat.
This is true for al Riemannian spaces: They are locally flat but the locally straight lines (geodesics) do not remain parallel usually.

Einstein saw the similarity of trajectories of freely falling particles and geodesics in curved geometry.
So we seek a theory of gravity that uses curved spacetime.

Tensor algebra in polar coordinates

We will study the euclidian plane still, but now for different coordinates.

We consider the Euclidean plane but with coordinates given by:

$$r = (x^2 + y^2)^{1/2} \quad , \quad \theta = \arctan(y/x)$$
$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

If we produce small increments in r and θ, we have as a consequence small increments in x, y:

$$\Delta r = \frac{x}{r}\Delta x + \frac{y}{r}\Delta t = \cos\theta\Delta x + \sin\theta\Delta y$$
$$\Delta\theta = -\frac{y}{r^2}\Delta x + \frac{x}{r^2}\Delta y = -\frac{1}{r}\sin\theta\Delta x + \frac{1}{r}\cos\theta\Delta y$$

In general, if we have a coordinate system χ, η (which are functions of the variables x, y), then a change in them has the relation:

$$\chi = \chi(x, y) \quad , \quad \eta = \eta(x, y)$$
$$\Delta\chi = \frac{\partial\chi}{\partial x}\Delta x + \frac{\partial\chi}{\partial y}\Delta y$$
$$\Delta\eta = \frac{\partial\eta}{\partial x}\Delta x + \frac{\partial\eta}{\partial y}\Delta y$$

For χ, η to be good coordinates, we need the relation to be biyective. This is true if the Jacobian is nonzero:

$$\det\begin{pmatrix} \partial\chi/\partial x & \partial\chi/\partial y \\ \partial\eta/\partial x & \partial\eta/\partial y \end{pmatrix} \neq 0$$

If the Jacobian vanishes at some point, the transformation is singular there.

For small $\Delta x, \Delta y$, there will be a corresponding small $\Delta\chi, \Delta\eta$ related by:

$$\begin{pmatrix} \Delta\chi \\ \Delta\eta \end{pmatrix} = \begin{pmatrix} \partial\chi/\partial x & \partial\chi/\partial y \\ \partial\eta/\partial x & \partial\eta/\partial y \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

We can define the matrix of transformation from the x, y space to the χ, η one:

$$\Lambda^{\alpha'}_{\beta} = \begin{pmatrix} \partial\chi/\partial x & \partial\chi/\partial y \\ \partial\eta/\partial x & \partial\eta/\partial y \end{pmatrix}$$

Then, for a given vector with components V^{β} in the x, y space, the *same vector* will have components $V^{\alpha'}$ in the χ, η space given by:

$$V^{\alpha'} = \Lambda^{\alpha'}_{\beta} V^{\beta}$$

Unprimed indices refer to (x, y) space and prime ones are the (χ, η) space.

Consider a scalar field ϕ. Given coordinates (χ, η), it is possible to form the derivatives $\partial\phi/\partial\chi$, $\partial\phi/\partial\eta$.
We define the **one form** $\tilde{d}\phi$ to be the geometrical object with components:

$$\tilde{d}\phi \to (\partial\phi/\partial\chi\ ,\ \partial\phi/\partial\eta)$$

In the (χ, η) coordinate system.
To see this is a covector, we see how it acts on vectors:

Given a vector (χ_0, η_0), the action of $\tilde{d}\phi$ on the vector is:

$$\tilde{d}\phi((\chi_0, \eta_0)) = \chi_0 \frac{\partial\phi}{\partial\chi} + \eta_0 \frac{\partial\phi}{\partial\eta}$$

Evaluated at some specific point obviously.
We see that the gradient covector acting in a vector gives the derivative of ϕ in the direction of the vector.

We can also see how this covectors transform, we see that:

$$\frac{\partial\phi}{\partial\chi} = \frac{\partial x}{\partial\chi}\frac{\partial\phi}{\partial x} + \frac{\partial y}{\partial\chi}\frac{\partial\phi}{\partial y}$$

And similartly for $\partial\phi/\partial\eta$. We can write this using row vectors:

$$\begin{pmatrix}\partial\phi/\partial\chi & \partial\phi/\partial\eta\end{pmatrix} = \begin{pmatrix}\partial\phi/\partial x & \partial\phi/\partial y\end{pmatrix}\begin{pmatrix}\partial x/\partial\chi & \partial x/\partial\eta \\ \partial y/\partial\chi & \partial y/\partial\eta\end{pmatrix}$$

So we define the inverse matrix:

$$\Lambda^{\alpha}_{\beta'} = \begin{pmatrix}\partial x/\partial\chi & \partial x/\partial\eta \\ \partial y/\partial\chi & \partial y/\partial\eta\end{pmatrix}$$

Then:

$$(\tilde{d}\phi)_{\beta'} = \Lambda^{\alpha}_{\beta'}(\tilde{d}\phi)_{\alpha}$$

In SR we didn't have to worry about row vectors, because the Lorentz transformations were symmetric.

We can see that this matrices are inverses:

$$\begin{pmatrix} \partial\xi/\partial x & \partial\xi/\partial y \\ \partial\eta/\partial x & \partial\eta/\partial y \end{pmatrix} \begin{pmatrix} \partial x/\partial\xi & \partial x/\partial\eta \\ \partial y/\partial\xi & \partial y/\partial\eta \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\partial\xi}{\partial x}\frac{\partial x}{\partial\xi} + \frac{\partial\xi}{\partial y}\frac{\partial y}{\partial\xi} & \frac{\partial\xi}{\partial x}\frac{\partial x}{\partial\eta} + \frac{\partial\xi}{\partial y}\frac{\partial y}{\partial\eta} \\ \frac{\partial\eta}{\partial x}\frac{\partial x}{\partial\xi} + \frac{\partial\eta}{\partial y}\frac{\partial y}{\partial\xi} & \frac{\partial\eta}{\partial x}\frac{\partial x}{\partial\eta} + \frac{\partial\eta}{\partial y}\frac{\partial y}{\partial\eta} \end{pmatrix}.$$

atrix is

$$\begin{pmatrix} \partial\xi/\partial\xi & \partial\xi/\partial\eta \\ \partial\eta/\partial\xi & \partial\eta/\partial\eta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

Curves and Vectors

A curve is a mapping from an interval of the real line into the plane such as:

$$\{\chi = f(s) \;\;,\;\; \eta = g(s) \;\;, a \leq s \leq b\}$$

The derivative of a scalar field ϕ along a curve is $d\phi/ds$.
If we have a vector $\vec{V}$ with components $(d\chi/ds, d\eta/ds)$ (tangent to the curve).
Then the derivative of the scalar along the curve is:

$$\begin{aligned} d\phi/ds &= \langle \tilde{d}\phi, \vec{V} \rangle \\ &= \begin{pmatrix} \partial\phi/\partial\chi & \partial\phi/\partial\eta \end{pmatrix} \begin{pmatrix} \frac{d\chi}{ds} \\ \frac{d\eta}{ds} \end{pmatrix} \\ &= \frac{\partial\phi}{\partial\chi}\frac{d\chi}{ds} + \frac{\partial\phi}{\partial\eta}\frac{d\eta}{ds} \end{aligned}$$

Which is just the chain rule.
Here we can see again how the gradient is a covector and acts on the vector, to give as a result the derivative in the direction of the vector.
And similarly, a vector acts on a function ϕ to produce $d\phi/ds$ (it actually acts on the gradient covector).

A curve has a unique tangent vector, is we change the parameter, the vector will have a different length, but we actually think of it as a different curve.

To sum up, the modern view is that a vector is a tangent to some curve, and is the function that gives $d\phi/ds$ when it takes $\tilde{d}\phi$ as an argument.

Polar Coordinate basis one forms and vectors

The basis coordinates can be found by the transformation rules.
If $\vec{e}_\beta$ are the basis vectors in x, y space, the vectors in χ, η are:

$$\vec{e}_{\alpha'} = \Lambda^{\beta}_{\alpha'} \vec{e}_\beta$$

Or, in polar coordinates:

$$\begin{aligned}\vec{e}_r &= \Lambda^x_r \vec{e}_x + \Lambda^y_r \vec{e}_y \\ &= \frac{\partial x}{\partial r}\vec{e}_x + \frac{\partial y}{\partial r}\vec{e}_y \\ &= \cos\theta \vec{e}_x + \sin\theta \vec{e}_y\end{aligned}$$

And similarly:

$$\begin{aligned}\vec{e}_\theta &= \frac{\partial x}{\partial \theta}\vec{e}_x + \frac{\partial y}{\partial \theta}\vec{e}_y \\ &= -r\sin\theta \vec{e}_x + r\cos\theta \vec{e}_y\end{aligned}$$

We have used $\Lambda^x_r = \dfrac{\partial x}{\partial r}$ and $\Lambda^r_x = \dfrac{\partial r}{\partial x}$ among other things.

For the **one form**, we use also the transformation rules. Let $\tilde{\omega}^\beta$ be the original covectors $\tilde{d}x$ and $\tilde{d}y$. And $\tilde{\omega}^{\alpha'}$ are the covectors in the (χ, η) space.
They are related by the transformation rules

$$\tilde{\omega}^{\alpha'} = \Lambda^{\alpha'}_{\beta} \tilde{\omega}^\beta$$

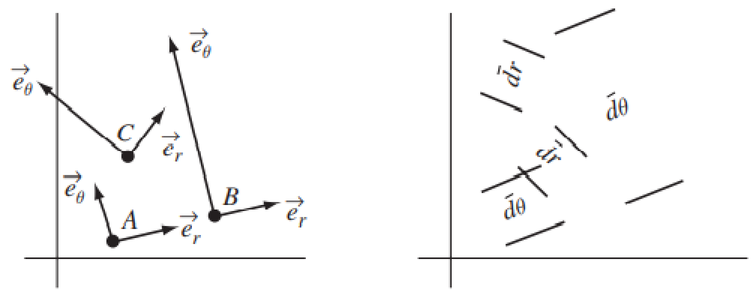
so, for example:

$$\begin{aligned}\tilde{d}\theta &= \Lambda^\theta_x \tilde{d}x + \Lambda^\theta_y \tilde{d}y \\ &= \frac{\partial \theta}{dx}\tilde{d}x + \frac{\partial \theta}{\partial y}\tilde{d}y \\ &= -\frac{1}{r}\sin\theta \tilde{d}x + \frac{1}{r}\cos\theta \tilde{d}y\end{aligned}$$

Notice this is just as $\Delta\theta$ in terms of $\Delta x, \Delta t$.
Similarly, we find:

$$\tilde{d}r = \cos\theta \tilde{d}x + \sin\theta \tilde{d}y$$

We can note that the bases change of direction and size in different points, (they are not unitary).



Basis vectors and one-forms for polar coordinates.

Metric Tensor

We can find the components using:

$$g_{\alpha'\beta'} = \mathbf{g}(\vec{e}_{\alpha'}, \vec{e}_{\beta'}) = \vec{e}_{\alpha'} \cdot \vec{e}_{\beta'}$$

And we get:

$$g_{rr} = 1 \ \ , \ \ g_{\theta\theta} = r^2 \ \ , \ \ g_{r\theta} = g_{\theta r} = 0$$

So:

$$(g_{\alpha\beta})_{polar} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

We can display this components using the line element:

$$\begin{aligned} d\vec{l} \cdot d\vec{l} = ds^2 &= |dr\vec{e}_r + d\theta\vec{e}_\theta|^2 \\ &= dr^2 + r^2 d\theta^2 \end{aligned}$$

Here $dr, d\theta$ are not the basis one forms, they mean infinitesimal.

There is another way of deriving this, which just consists on writing the terms of g:

$$g = g_{\alpha\beta}\tilde{d}x^\alpha \otimes \tilde{d}x^\beta = \tilde{d}r \otimes \tilde{d}r + r^2 \tilde{d}\theta \otimes \tilde{d}\theta$$

The metric has an inverse given by:

$$\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}$$

so we have $g^{rr} = 1$, $g^{r\theta} = 0$, $g^{\theta\theta} = 1/r^2$.

Tensor Calculus in polar coordinates

The fact that basis vectors are not uniform gives some problems when trying to differentiate. Since $\vec{e}_x, \vec{e}_y$ are constant vector fields, we can find:

$$\frac{\partial}{\partial r}\vec{e}_r = \frac{\partial}{\partial r}(\cos\theta\vec{e}_x + \sin\theta\vec{e}_y) = 0$$
$$\frac{\partial}{\partial \theta}\vec{e}_r = \frac{\partial}{\partial \theta}(\cos\theta\vec{e}_x + \sin\theta\vec{e}_y) = -\sin\theta\vec{e}_x + \cos\theta\vec{e}_y = \frac{1}{r}\vec{e}_\theta$$

Similarly:

$$\frac{\partial}{\partial r}\vec{r}_\theta = \frac{1}{r}\vec{e}_\theta$$
$$\frac{\partial}{\partial \theta}\vec{e}_\theta = -r\vec{r}_r$$

Derivatives of general Vectors

And for example, we have:

$$\vec{e}_x = \cos\theta\vec{e}_r - \frac{1}{r}\sin\theta\vec{e}_\theta$$

We have that:

$$\begin{aligned}\frac{\partial}{\partial \theta}\vec{e}_x &= \frac{\partial}{\partial \theta}(\cos\theta)\vec{e}_r + \cos\theta\frac{\partial}{\partial \theta}(\vec{e}_r) - \frac{\partial}{\partial \theta}\left(\frac{1}{r}\sin\theta\right)\vec{e}_\theta - \frac{1}{r}\sin\theta\frac{\partial}{\partial \theta}(\vec{e}_\theta)\\ &= -\sin\theta\vec{e}_r + \cos\theta\left(\frac{1}{r}\vec{e}_\theta\right) - \frac{1}{r}\cos\theta\vec{e}_\theta - \frac{1}{r}\sin\theta(-r\vec{e}_r)\\ &= \cdots = 0\end{aligned}$$

Now consider a general vector (V^r, V^θ) on the polar basis. Its derivative is:

$$\begin{aligned}\frac{\partial\vec{V}}{\partial r} &= \frac{\partial}{\partial r}(V^r\vec{e}_r + V^\theta\vec{e}_\theta)\\ &= \frac{\partial V^r}{\partial r}\vec{e}_r + V^r\frac{\partial\vec{e}_r}{\partial r} + \frac{\partial V^\theta}{\partial r}\vec{e}_\theta + V^\theta\frac{\partial\vec{e}_\theta}{\partial r}\end{aligned}$$

Similarly for the derivative with respecto to θ.
Written in index notation, it is:

$$\frac{\partial\vec{V}}{\partial r} = \frac{\partial}{\partial r}(V^\alpha\vec{e}_\alpha) = \frac{\partial V^\alpha}{\partial r}\vec{e}_\alpha + V^\alpha\frac{\partial\vec{e}_\alpha}{\partial r}$$

Or we can generalize it to:

$$\frac{\partial\vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\vec{e}_\alpha + V^\alpha\frac{\partial\vec{e}_\alpha}{\partial x^\beta}$$

The Christoffel symbols

We introduce the Christoffel symbol as:

$$\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \Gamma^\mu{}_{\alpha\beta} \vec{e}_\mu$$

$\Gamma^\mu_{\alpha\beta}$ is the μ th component of the derivative of the vector $\vec{e}_\alpha$ with respect to x^β.
The first lower α index is the vector being differentiated, the second β is the coordinate with respect to which it is being differentiated, and the third μ is the component of the resulting vector.
We have already calculated a few.

$$\left.\begin{array}{ll}
(1) & \partial \vec{e}_r / \partial r = 0 \Rightarrow \Gamma^\mu{}_{rr} = 0 \quad \text{for all } \mu, \\
(2) & \partial \vec{e}_r / \partial \theta = \frac{1}{r}\, \vec{e}_\theta \Rightarrow \Gamma^r{}_{r\theta} = 0, \quad \Gamma^\theta{}_{r\theta} = \frac{1}{r}, \\
(3) & \partial \vec{e}_\theta / \partial r = \frac{1}{r}\, \vec{e}_\theta \Rightarrow \Gamma^r{}_{\theta r} = 0, \quad \Gamma^\theta{}_{\theta r} = \frac{1}{r}, \\
(4) & \partial \vec{e}_\theta / \partial \theta = -r\, \vec{e}_r \Rightarrow \Gamma^r{}_{\theta\theta} = -r, \quad \Gamma^\theta{}_{\theta\theta} = 0.
\end{array}\right\}$$

The covariant derivative

Using the definition of the Christoffel symbols, the derivative we had calculated is:

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta} \vec{e}_\alpha + V^\alpha \Gamma^\mu_{\alpha\beta} \vec{e}_\mu$$

We can relabel the coordinates and write:

$$\frac{\partial \vec{V}}{\partial x^\beta} = \left(\frac{\partial V^\alpha}{\partial x^\beta} + V^\mu \Gamma^\alpha_{\mu\beta} \right) \vec{e}_\alpha$$

So the vector field $\partial \vec{V} / \partial x^\beta$ has components $\dfrac{\partial V^\alpha}{\partial x^\beta} + V^\mu \Gamma^\alpha_{\mu\beta}$
We define a new symbol for the covariant derivative:

$$\boxed{V^\alpha_{;\beta} := V^\alpha_{,\beta} + V^\mu \Gamma^\alpha_{\mu\beta}}$$

Or, with the other notation:

$$\boxed{(\nabla_\beta V)^\alpha = \partial_\beta V^\alpha + V^\mu \Gamma^\alpha_{\mu\beta}}$$

And we have:

$$\boxed{\frac{\partial \vec{V}}{\partial x^\beta} = V^\alpha_{;\beta} \vec{e}_\alpha}$$

We can regard $\partial \vec{V}/\partial x^{\beta}$ as a $(1,1)$ tensor field that maps every vector $\vec{e}_{\beta}$ into the vector $\partial \vec{V}/\partial x^{\beta}$

The covariant derivative of a scalar is equal to the normal derivative:

$$\nabla_{\alpha} f = \partial_{\alpha} f$$

this is because f doesn't depend on the vectors.

Divergence and Laplacian

The divergence of a vector shouldn't depend on coordinates. So if α represent cartesian components, and β are other coordinates, then:

$$V^{\alpha}_{,\alpha} = V^{\beta'}_{;\beta'}$$

For polar coordinates, we have:

$$\begin{aligned} V^{\alpha}_{;\alpha} &= \frac{\partial V^{\alpha}}{\partial x^{\alpha}} + \Gamma^{\alpha}_{\mu\alpha} V_{\mu} \\ &= \cdots = \frac{1}{r}\frac{\partial}{\partial r}(rV^{r}) + \frac{\partial}{\partial \theta} V^{\theta} \end{aligned}$$

Derivative of one-forms and tensors of higher types

We can use that the covariant derivative of a scalar is equal to the partial derivative, and calculate the derivative of $\phi = p_{\alpha} V^{\alpha}$
Where p_{α} are the components of a covector $\tilde{p}$ and V^{α} of a vector, and $\phi = \langle \tilde{p}, \vec{V} \rangle$.
Then, it can be proven that:

$$(\nabla_{\beta} \tilde{p})_{\alpha} = p_{\alpha\,,\beta} - p_{\mu} \Gamma^{\mu}_{\alpha\beta}$$

And in general, for bigger tensors:

$$\begin{aligned} \nabla_{\beta} T_{\mu\nu} &= T_{\mu\nu,\beta} - T_{\alpha\nu} \Gamma^{\alpha}{}_{\mu\beta} - T_{\mu\alpha} \Gamma^{\alpha}{}_{\nu\beta}; \\ \nabla_{\beta} A^{\mu\nu} &= A^{\mu\nu}{}_{,\beta} + A^{\alpha\nu} \Gamma^{\mu}{}_{\alpha\beta} + A^{\mu\alpha} \Gamma^{\nu}{}_{\alpha\beta}; \\ \nabla_{\beta} B^{\mu}{}_{\nu} &= B^{\mu}{}_{\nu,\beta} + B^{\alpha}{}_{\nu} \Gamma^{\mu}{}_{\alpha\beta} - B^{\mu}{}_{\alpha} \Gamma^{\alpha}{}_{\nu\beta}. \end{aligned}$$

Chridtoffel and the metric

If $\vec{V}$ is an arbitrary vector and $\tilde{V} = g(\vec{V}, \cdot)$ is its related one form, then in cartesian coordinates :

$$\nabla_{\beta} \vec{V} = g(\nabla_{\beta} \vec{V}, \cdot)$$

But this is a tensor equation, so it must be valid in all coordinates. We conclude:

$$V_{\alpha\,;\beta} = g_{\alpha\mu} V^{\mu}{}_{;\beta}$$

This means the metric is compatible to the covariant.

After some algebra, we get that:

$$g_{\alpha'\mu'\,;\beta'} = 0$$

Calculate the Christoffel symbols from the Metric

We first prove **symmetry**:

In cartesian coordinates, for scalar ϕ we have:
$\phi_{,\beta;\alpha} = \phi_{,\alpha;\beta}$ in any basis (because they are simple derivatives)
Therefore $\phi_{,\beta,\alpha} - \phi_{,\mu}\Gamma^{\mu}_{\beta\alpha} = \phi_{,\alpha,\beta} - \phi_{,\mu}\Gamma^{\mu}_{\alpha\beta}$
Therefore, in any coordinate system:

$$\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$$

We can use some algebra as before and get the relation:

$$\boxed{\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha})}$$

So we can calculate the Christophel from the metric, we can do this in polar coordinates for example.

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Non Coordinate bases

Sometimes we have non coordinate bases (pretty rarely) and some things change.

Polar coordinate basis

The basis vectors for our polar coordinate system were defined by:

$$\vec{e}_{\alpha'} = \Lambda^{\beta}_{\alpha'} \vec{e}_{\beta}$$

Where primed are polar coordinates and unprimed are cartesian.
Moreover, we had:

$$\Lambda^{\beta}_{\alpha'} = \partial x^{\beta} / \partial x^{\alpha'}$$

We found that:

$$\vec{e}_{\alpha'} \cdot \vec{e}_{\beta'} := g_{\alpha'\beta'} \neq \delta_{\alpha'\beta'}$$

So they are not unit vectors.

Plar unit basis

The unit vectors are:

$$\vec{e}_{\widehat{r}} = \vec{e}_r \ \ , \ \ \vec{e}_{\widehat{\theta}} = \frac{1}{r}\vec{e}_\theta$$

And the corresponding unit one form basis:

$$\tilde{\omega}^{\widehat{r}} = \tilde{d}r \quad , \quad \tilde{\omega}^{\widehat{\theta}} = r\tilde{d}\theta$$

We can now verify:

$$\vec{e}_{\widehat{\alpha}} \cdot \vec{e}_{\widehat{\beta}} := g_{\widehat{\alpha}\widehat{\beta}} = \delta_{\widehat{\alpha}\widehat{\beta}}$$
$$\tilde{\omega}^{\widehat{\alpha}} \cdot \tilde{\omega}^{\widehat{\beta}} := g^{\widehat{\alpha}\widehat{\beta}} = \delta^{\widehat{\alpha}\widehat{\beta}}$$

So these are orthonormal bases for the vectors and one-forms.

Now the question arises if there exists a pair of coordinates (ξ, η) such that they naturally give this vectors:

$$\vec{e}_{\widehat{r}} = \vec{e}_\xi = \frac{\partial x}{\partial \xi}\vec{e}_x + \frac{\partial y}{\partial \xi}\vec{e}_y$$
$$\vec{e}_{\widehat{\theta}} = \vec{e}_\eta = \frac{\partial x}{\partial \eta}\vec{e}_x + \frac{\partial y}{\partial \eta}\vec{e}_y$$

Then, $\{\vec{e}_{\widehat{r}}, \vec{e}_{\widehat{\theta}}\}$ are the basis for the coordinates (ξ, η) and so can be called a coordinate basis. If so (ξ, η) don't exist, then these vectors are a **noncoordinate basis**.

The question is more easily answered if we look at the basis one forms. Thus, we seek (ξ, η) such that:

$$\tilde{\omega}^{\widehat{r}} = \tilde{d}\xi = \frac{\partial \xi}{\partial x}\tilde{d}x + \frac{\partial \xi}{\partial y}\tilde{d}y$$
$$\tilde{\omega}^{\widehat{\theta}} = \tilde{d}\eta = \frac{\partial \eta}{\partial x}\tilde{d}x + \frac{\partial \eta}{\partial y}\tilde{d}y$$

Since we know $\tilde{\omega}^{\widehat{r}}$ and $\tilde{\omega}^{\widehat{\theta}}$ in terms of $\tilde{d}r$ and $\tilde{d}\theta$, we have:

$$\tilde{\omega}^{\widehat{r}} = \tilde{d}r = \cos\theta\tilde{d}x + \sin\theta\tilde{d}y$$
$$\tilde{\omega}^{\widehat{\theta}} = r\tilde{d}\theta = -\sin\theta\tilde{d}x + \cos\theta\tilde{d}y$$

The orthonormality of these is obvious. Thus if (ξ, η) exist, we have:

$$\frac{\partial \eta}{\partial x} = -\sin\theta \quad , \quad \frac{\partial \eta}{\partial y} = \cos\theta$$

If this were true, then mixed derivatives are equal:
$\dfrac{\partial}{\partial y}\dfrac{\partial \eta}{\partial x} = \dfrac{\partial}{\partial x}\dfrac{\partial \eta}{\partial y}$
This would imply that $\dfrac{\partial}{\partial y}(-\sin\theta) = \dfrac{\partial}{\partial x}(\cos\theta)$
That is, $\dfrac{\partial}{\partial y}\left(\dfrac{y}{\sqrt{x^2+y^2}}\right) + \dfrac{\partial}{\partial x}\left(\dfrac{x}{\sqrt{x^2+y^2}}\right) = 0$
This is not true, so ξ, η do not exist. We have a **noncoordinate basis**.

In textbooks, we usually use the unit vectors instead of coordinate basis.

General Remarks

The principal difference between coordinate and non coordinate basis arise from the following.
Consider an scalar field ϕ and the number $\tilde{d}\phi(\vec{e}_\mu)$, where $\vec{e}_\mu$ is a basis vector of some basis. We have used the notation:

$$\tilde{d}\phi(\vec{e}_\mu) = \phi_{,\mu} \quad 5.88$$

Now, if $\vec{e}_\mu$ is a member of a coordinate basis, then $\tilde{d}\phi(\vec{e}_\mu) = \partial\phi/\partial x^\mu$, and we have:

$$\phi_{,\mu} = \frac{\partial \phi}{\partial x^\mu} \quad \text{coordinate basis}$$

But if no coordinates exist for $\{\vec{e}_\mu\}$, then the equation fails. For example, if we let 5.88 define $\phi_{,\widehat{\mu}}$, we have:

$$\phi_{,\widehat{\theta}} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

And in general we get:

$$\nabla_{\widehat{\alpha}}\phi := \phi_{,\widehat{\alpha}} = \Lambda^{\beta}_{\widehat{\alpha}}\nabla_\beta\phi = \Lambda^{\beta}_{\widehat{\alpha}}\frac{\partial \phi}{\partial x^\beta}$$

for any coordinate system $\{x^\beta\}$ and noncoordinate basis $\{\vec{e}_{\widehat{\alpha}}\}$.
It is thus convenient to continue with the notation of eq 5.88 and to make the rule that $\phi_{,\mu} = \partial\phi/\partial x^\mu$ only in a coordinate basis.

The Christoffel symbols are defined just as before:

$$\nabla_{\widehat{\beta}}\vec{e}_{\widehat{\alpha}} = \Gamma^{\widehat{\mu}}_{\widehat{\alpha}\widehat{\beta}}\vec{e}_{\widehat{\mu}}$$

But now:

$$\nabla_{\widehat{\beta}} = \Lambda^{\alpha}_{\widehat{\beta}}\frac{\partial}{\partial x^{\alpha}}$$

Where $\{x^{\alpha}\}$ is any coordinate system and $\{\vec{e}_{\widehat{\beta}}\}$ any basis (coordinate or not).
Now however we cannot prove that $\Gamma^{\widehat{\mu}}_{\widehat{\alpha}\widehat{\beta}} = \Gamma^{\widehat{\mu}}_{\widehat{\beta}\widehat{\alpha}}$, since that proof used $\phi_{,\widehat{\alpha},\widehat{\beta}} = \phi_{,\widehat{\beta},\widehat{\alpha}}$, which was true in a coordinate basis but is not otherwise.
Hence, writing Γ in terms of g applies only in a coordinate basis.

What is the general reason for the nonexistence of coordinates for a basis?

If $\{\tilde{\omega}^{\bar{\alpha}}\}$ is a coordinate one-form basis, then its relation to any other one $\{\tilde{d}x^{\alpha}\}$ is:

$$\tilde{\omega}^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_{\beta}\tilde{d}x^{\beta} = \frac{\partial x^{\bar{\alpha}}}{\partial x^{\beta}}dx^{\beta}$$

The key point is that $\Lambda^{\bar{\alpha}}_{\beta}$, which is generally a function of position, must actually be the partial derivative $\partial x^{\bar{\alpha}}/\partial x^{\beta}$ everywhere. Thus we have:

$$\frac{\partial}{\partial x^{\gamma}}\Lambda^{\bar{\alpha}}_{\beta} = \frac{\partial^2 x^{\bar{\alpha}}}{\partial x^{\gamma}\partial x^{\beta}} = \frac{\partial^2 x^{\bar{\alpha}}}{\partial x^{\beta}\partial x^{\gamma}} = \frac{\partial}{\partial x^{\beta}}\Lambda^{\bar{\alpha}}_{\gamma}$$

These conditions must be satisfied by all elements $\Lambda^{\bar{\alpha}}_{\beta}$ in order for $\tilde{\omega}^{\bar{\alpha}}$ to be a coordinate basis.

Exercises

- **7) Calculate all elements of the transformation matrices $\Lambda^{\alpha'}_{\beta}$ and $\Lambda^{\mu}_{\nu'}$ for the transformation from Cartesian (x, y) - the unprimed indices - to polar (r, θ) - the primed indices -**

We could first calculate $\Lambda^{\mu}_{\nu'}$. We write the (x, y) in terms of the (r, θ) variables.

$$(\Lambda^{\mu}_{\nu'}) = \begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

The matrix $\Lambda^{\alpha'}_{\beta}$ can be found by inverting, so:

$$(\Lambda^{\alpha'}_{\beta}) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\dfrac{\sin\theta}{r} & \dfrac{\cos\theta}{r} \end{pmatrix} = \begin{pmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ & -\dfrac{y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{pmatrix}$$

11) **For the vector field $\vec{V}$ whose cartesian components are $(x^2 + 3y, y^2 + 3x)$, compute $V^{\alpha}_{,\beta}$**

$$V^{\alpha}_{,\beta} = \rightarrow_{Car} \begin{pmatrix} \dfrac{\partial V^x}{\partial x} & \dfrac{\partial V^x}{\partial y} \\ \dfrac{\partial V^y}{\partial x} & \dfrac{\partial V^y}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 3 \\ 3 & 2y \end{pmatrix} = \begin{pmatrix} 2r\cos\theta & 3 \\ 3 & 2r\sin\theta \end{pmatrix}$$

The last components are still Cartesian, but written using polar.

b) The transformation $\Lambda^{\mu'}_{\alpha}\Lambda^{\beta}_{\nu'}V^{\alpha}_{,\beta}$ to polars

We can calculate them directly:

-
$$\begin{aligned} V^{r}_{;r} &= \Lambda^{1'}_{\alpha}\Lambda^{\beta}_{1'}V^{\alpha}_{,\beta} \\ &= \Lambda^{1'}_{1}\Lambda^{1}_{1'}V^{1}_{,1} + \Lambda^{1'}_{2}\Lambda^{1}_{1'}V^{2}_{,1} + \Lambda^{1'}_{1}\Lambda^{2}_{1'}V^{1}_{,2} + \Lambda^{1'}_{2}\Lambda^{2}_{1'}V^{2}_{,2} \\ &= 2r(\cos^3\theta + \sin^3\theta) + 6\sin\theta\cos\theta \end{aligned}$$

- $V^{r}_{;\theta} = 2r^2\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3r(\cos^2\theta - \sin^2\theta)$
- $V^{\theta}_{;r} = 2\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3(\cos^2\theta - \sin^2\theta)/r$
- $V^{\theta}_{;\theta} = 2r\sin\theta\cos\theta(\sin\theta + \cos\theta) - 6\sin\theta\cos\theta$

The components $V^{\mu'}_{;\nu'}$ in polar coordinates using directly Christoffel:
First we write the vector in terms of polar: $(x^2+3y)\vec{e}_x + (y^2+3x)\vec{e}_y$
$= (r^2\cos^2\theta + 3\sin\theta)(\cos\theta\vec{e}_r - \dfrac{\sin\theta}{r}\vec{e}_\theta) + (r^2\sin^2\theta + 3r\cos\theta)(\sin\theta\vec{e}_r + \dfrac{\cos\theta}{r}\vec{e}_\theta)$
$= (r^2\cos^3\theta + r^2\sin^3\theta + 6r\sin\theta\cos\theta)\vec{e}_r + (r\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3(\cos^2\theta - \sin^2\theta))\vec{e}_\theta$
Now we use the formulas for derivatives:

- $V^r_{;r} = \partial_r V^r + V^\mu \Gamma^r_{\mu r} = \partial_r V^r + V^r \Gamma^r_{rr} + V^\theta \Gamma^r_{\theta r}$
  $= 2r(\cos^3\theta + \sin^3\theta) + 6\sin\theta\cos\theta + 0 + 0$
- $V^r_{;\theta} = \partial_\theta V^r + V^\mu \Gamma^r_{\mu\theta} = \partial_\theta V^r + V^r \Gamma^r_{r\theta} + V^\theta \Gamma^r_{\theta\theta}$
  $= -3r^2\cos^2\theta\sin\theta + r^2\sin^2\cos\theta + 6r\cos^2(\theta) - 6r\sin^2(\theta) + 0 - r(r\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3(\cos^2\theta - \sin^2\theta))$
  $= \cdots = 2r^2\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3r(\cos^2\theta - \sin^2\theta)$

And similarly for the other derivatives.
We get the same results as earlier.

d) The divergence $V^\alpha_{,\alpha}$ using results in a)

$$V^\alpha_{,\alpha} = 2x + 2y = 2r(\cos\theta + \sin\theta)$$

e) calculate the divergence $V^{\mu'}_{;\mu'}$ using the polar results

$$\begin{aligned} V^{\mu'}_{;\mu'} &= V^r_{;r} + V^\theta_{;\theta} \\ &= 2r(\cos^3\theta + \sin^3\theta) + 6\cos\theta\sin\theta + 2r\sin\theta\cos\theta(\sin\theta + \cos\theta) - 6\sin\theta\cos\theta \\ &= 2r(\cos\theta + \sin\theta) \end{aligned}$$

We get the result as expected.

12) **For the one-form field $\tilde{p}$ whose cartesian components are (x^2+3y, y^2+3x), compute $p_{\alpha,\beta}$ in Cartesian**

We do it directly:

$$p_{\alpha,\beta} =\!\!\rightarrow_{car} \begin{pmatrix} \dfrac{\partial p^x}{\partial x} & \dfrac{\partial p^x}{\partial y} \\ \dfrac{\partial p^y}{\partial x} & \dfrac{\partial p^y}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 3 \\ 3 & 2y \end{pmatrix} = \begin{pmatrix} 2r\cos\theta & 3 \\ 3 & 2r\sin\theta \end{pmatrix}$$

b) The transformation $p_{\mu';\nu'} = \Lambda^\alpha_{\mu'}\Lambda^\beta_{\nu'} p_{\alpha,\beta}$ to polars:
We calculate them directly:

- $p_{r;r} = \Lambda^{\alpha}_{r}\Lambda^{\beta}_{r}p_{\alpha,\beta} = \Lambda^{x}_{r}\Lambda^{x}_{r}p_{x,x} + \Lambda^{y}_{r}\Lambda^{x}_{r}p_{y,x} + \Lambda^{x}_{r}\Lambda^{y}_{r}p_{x,y} + \Lambda^{y}_{r}\Lambda^{y}_{r}p_{y,y}$
  $= \cdots = 2r(\cos^3\theta + \sin^3\theta) + 6\cos\theta\sin\theta$
- $p_{r;\theta} = 2r^2\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3r(\cos^2\theta - \sin^2\theta)$
- $p_{\theta;r} = 2r^2\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3r(\cos^2\theta - \sin^2\theta)$
- $p_{\theta;\theta} = 2r^3\sin\theta\cos\theta(\sin\theta + \cos\theta) - 6r^2\sin\theta\cos\theta$

Calculate $p_{\mu';\nu'}$ directly in polars using the Christoffel symbols

First we write it in polar. We use the transformation law for the covector bases $\tilde{\omega}^{\alpha} = \Lambda^{\alpha}_{\beta'}\tilde{\omega}^{\beta'}$
Therefore: $\tilde{d}x = \Lambda^{x}_{r}\tilde{d}r + \Lambda^{x}_{\theta}\tilde{d}\theta = \dfrac{\partial x}{\partial r}\tilde{d}r + \dfrac{\partial x}{\partial \theta}\tilde{d}\theta = \cos\theta\tilde{d}r - r\sin\theta\tilde{d}\theta$
$\tilde{d}y = \Lambda^{y}_{r}\tilde{d}r + \Lambda^{y}_{\theta}\tilde{d}\theta = \dfrac{\partial y}{\partial r}\tilde{d}r + \dfrac{\partial y}{\partial \theta}\tilde{d}\theta = \sin\theta\tilde{d}r + r\cos\theta\tilde{d}\theta$

$p = (x^2 + 3y)\tilde{d}x + (y^2 + 3x)\tilde{d}y$
$= (r^2\cos^2\theta + 3r\sin\theta)(\cos\theta\tilde{d}r - r\sin\theta\tilde{d}\theta) + (r^2\sin^2\theta + 3r\cos\theta)(\sin\theta\tilde{d}r + r\cos\theta\tilde{d}\theta)$
$= [r^2(\cos^3\theta + \sin^3\theta) + 6r\cos\theta\sin\theta]\tilde{d}r + [3r^2(\cos^2\theta - \sin^2\theta) + r^3\cos\theta\sin^2\theta - r^3\cos^2\theta\sin\theta]\tilde{d}\theta$

Now we can calculate the derivatives using the Christoffel symbols.
For example:

- $p_{r;r} = \partial_r p_r - p_\mu\Gamma^{\mu}_{rr} = [2r(\cos^3\theta + \sin^3\theta) + 6\cos\theta\sin\theta] - p_r\Gamma^{r}_{rr} - p_\theta\Gamma^{\theta}_{rr}$
  $= [2r(\cos^3\theta + \sin^3\theta) + 6\cos\theta\sin\theta] - 0 - 0$
- $p_{r;\theta} = \partial_\theta p_r - p_\mu\Gamma^{\mu}_{r\theta} = \cdots = 2r^2\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3r(\cos^2\theta - \sin^2\theta)$
- $p_{\theta;r} = \partial_r p_\theta - p_\mu\Gamma^{\mu}_{r\theta} = \cdots = 2r^2\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3r(\cos^2\theta - \sin^2\theta)$
- $p_{\theta;\theta} = \partial_\theta p_\theta - p_\mu\Gamma^{\mu}_{\theta\theta} = \cdots = 2r^3\sin\theta\cos\theta(\sin\theta + \cos\theta) - 6r^2\sin\theta\cos\theta$

13) **Show that $g_{\mu'\alpha'}V^{\alpha'}_{;\nu'} = p_{\mu';\nu'}$**
We can show it from the things we calculated.

Curved Manifolds

Differentiable Manifolds and Tensors

A manifold is basically a set that is continously parametrized, the number of independent parameters is the dimension of the manifold, and the parameters themselves the coordinates. Mathematically, the association of points in the set with the values of their parameters can be thought as a mapping of points from the manifold to $\mathbb{R}^n$.
It has a local correspondence with an Euclidean space, but not necessarily global.

Differential Structure

It is roughly a manifold that in the neighborhood of each point it is possible to define a smooth map to Euclidean space that preserves derivatives of scalar functions at that point.

The assumption of differentiability means that we can define one-forms (gradients) and vectors.
In a certain coordinate system on the manifold, the members of the set $\{\phi_{,\alpha}\}$ are the components of the one-form $\tilde{d}\phi$. And any set of the form $\{a\phi_{,\alpha} + b\psi_{,\alpha}\}$ is also a one form field (where a, b are scalar functions).

Similarly, every curve has a tangent vector $\vec{V}$ defined as the linear function that takes a one form $\tilde{d}\phi$ and gives out the derivative of ϕ along the curve, $d\phi/d\lambda$:

$$d\phi/d\lambda = \langle \tilde{d}\phi, \vec{V} \rangle = \vec{V}(\tilde{d}\phi) = \nabla_{\vec{V}}\phi$$

Review

1) A tensor field defines a tensor at every point.

2) Vectors and one-forms are linear operators on each other, producing real numbers:

$$\langle \tilde{p}, a\vec{V} + b\vec{W} \rangle = a\langle \tilde{p}, \vec{V} \rangle + b\langle \tilde{p}, \vec{W} \rangle$$
$$\langle a\tilde{p} + b\tilde{q}, \vec{V} \rangle = a\langle \tilde{p}, \vec{V} \rangle + b\langle \tilde{q}, \vec{V} \rangle$$

3) Tensors (k, l) are similarly linear operator on k one-forms and l vectors.

4) If two tensors of the same type have equal components in a given basis, they have equal components in all base and are identical. If a tensor's components are all zero in one basis, they are zero in all.

5) The next manipulations of components of tensor fields are 'permissible' because they produce components of new tensors:

   - Multiplication by a scalar field produces components of a new tensor.

- Addition of components of two tensors of the same type gives components of a new tensor of the same type.
- Multiplication of components of two tensors of arbitrary type gives components of a new tensor of the sum of the types, the outer product of the two tensors.
- Covariant differentiation of the components of a tensor (N,M) gives a tensor of the type (N,M+1)
- Contraction of a pair of indices of a tensor (N, M) gives a tensor $(N-1, M-1)$
- If an equation is formed using components of tensors combined in permissible ways and if the equation is true in some basis, it is true in all of them.

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Riemannain Manifolds

Metric: A metric is a symmetric $(0, 2)$ tensor field **g**

Riemannian Manifold: A differentiable manifold on which a symmetric $(0, 2)$ tensor field g has been singled out to act as the metric at each point.
Strictly speaking, in a Riemannian manifold the metric has to be *positive definite* - That is $g(\vec{V}, \vec{V}) > 0$ for all $\vec{V} \neq 0$.
If the metric is not positive definite, we have a **pseudo Riemannian manifold**, which is what we want in SR, GR.

In picking out a metric, we add structure and completely define the curvature of the manifold. Different curvatures on a manifold give it different curvatures.
From now on we will study Riemannian manifolds, on which a metric g is assumed to be defined at every point.

It is actually to define curvature on a manifold without a metric (affine manifold), but we don't care.

Metric and Local Flatness

The metric provides a mapping between vectors and one forms at every point.

For example, given a vector field $\vec{V}(P)$ (it gives a vector $\vec{V}$ at every point P of the manifold). Then we define a unique one-form field:

$$\tilde{V}(P) = g(\vec{V}(P), \cdot)$$

The mapping is invertible, so that associated with $\tilde{V}(P)$, there is a unique $\vec{V}(P)$.

The components of g are $g_{\alpha\beta}$ and its inverse are $g^{\alpha\beta}$.
The metric permits raising and lowering of indices in the same way as in SR:

$$V_\alpha = g_{\alpha\beta} V^\beta$$

In general $g_{\alpha\beta}$ will be complicated functions of position, so the relation is not simple.

In SR we only studied Lorentz (inertial) frames, but gravity prevents such frames from being global, so we shall have to allow all coordinates, and hence all coordinate transformations that are nonsingular (with an inverse).

Now, the matrix $(g_{\alpha\beta})$ is symmetric matrix by definition.
It is well known from linear algebra that for symmetric matrices we can find a transformation matrix that turns it into a diagonal matrix with each entry on the main diagonal either $+1, -1, 0$.
The number of $+1$ entries equals the number of positive eigenvalues of g and the amount of -1 the number of negative eigenvalues.
So if g has 3 positive ev and one negative eval, then we can always find a $\Lambda^{\alpha'}_{\beta}$ to make the metric components become:

$$(g_{\alpha'\beta'}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} := \eta_{\alpha\beta} \qquad (6.2)$$

Which is the metric of SR.

The sum of the $+1$ and the -1 is called the **signature of the metric**. For SR and GR it is $+2$.

The fact that we can construct local inertial frames finds its representation in 6.2, that we can transform it into $\eta_{\alpha\beta}$ at a point. So the metric has to have signature $+2$ to describe spacetime with gravity.

The second remark is that the matrix $\Lambda^{\alpha'}_{\beta}$ that produces Eq 6.2 may not be a coordinate transformation.
That is, the set $\tilde{\omega}^{\alpha'} = \Lambda^{\alpha'}_{\beta} \tilde{d}x^\beta$ may not be a coordinate basis.
By our earlier discussion of noncoordinate bases, it would be a coordinate transfomation only if:

$$\frac{\partial \Lambda^{\alpha'}_{\beta}}{\partial x^\gamma} = \frac{\partial \Lambda^{\alpha'}_{\gamma}}{\partial x^\beta}$$

holds.

In a general gravitational field this will be impossible, because it would imply the existence of coordinates for which equation 6.2 is true everywhere, that is a global Lorentz frame.

However, having found a basis at a particular point P for which equation 6.2 is true, it is possible to find coordinates such that in the neighborhood of P, equation 6.2 is ' nearly ' true. This is the **theorem of local flatness** which we prove later.

Choose any point P of the manifold, a coordinate system $\{x^\alpha\}$ can be found whose origin is at P and in which:

$$g_{\alpha\beta}(x^\mu) = \eta_{\alpha\beta} + O[(x^\mu)^2]$$

That is, the metric near P is approximately that of SR, with difference only of second order. This is a *local Lorenz frame or Local inertial frame.*
We can write it in a more precise way as:

$$g_{\alpha\beta}(P) = \eta_{\alpha\beta} \quad \forall\, \alpha, \beta$$
$$\frac{\partial}{\partial x^\gamma} g_{\alpha\beta}(P) = 0 \quad \forall\, \alpha, \beta, \gamma$$

But generally, second derivatives are not 0:

$$\frac{\partial^2}{\partial x^\gamma \partial x^\mu} g_{\alpha\beta}(P) \neq 0$$

The existence of local Lorentz frames is merely the statement that any curved space has a flat space 'tangent' to it at any point. The proof is at the end of this chapter.

Lengths and Volumes

The metric gives a way to define lengths of curves. Let $d\vec{x}$ be a small vector displacement on some curve.
Then $d\vec{x}$ has squared length of $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$ (this is the *line element* of the metric).
If we take the absolute value and square root, we get a measure of length $dl := |g_{\alpha\beta} dx^\alpha dx^\beta|^{1/2}$
Then integrating it gives:

$$l = \int_{alongcurve} |g_{\alpha\beta} dx^\alpha dx^\beta|^{1/2}$$
$$= \int_{\lambda_0}^{\lambda_1} \left| g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right|^{1/2} d\lambda$$

Where λ is the parameter of the curve from λ_0 to λ_1.
But since the tangent vector $\vec{V}$ has components $V^\alpha = dx^\alpha / d\lambda$, we finally have:

$$\boxed{l = \int_{\lambda_0}^{\lambda_1} |\vec{V} \cdot \vec{V}|^{1/2} d\lambda}$$

Volume:
We mean by 'volume' the four dimensional volume element we used for integrations in Gauss' law.
Let us go to a local Lorentz frame, where we know that a small four dimensional region has four volume $dx^0dx^1dx^2dx^3$, where $\{x^\alpha\}$ are the coordinates which at this point give the nearly Lorentz metric.
Now, in any coordinate system $\{x^{\alpha'}\}$ it is a well known result that:

$$dx^0dx^1dx^2dx^3 = \frac{\partial(x^0,x^1,x^2,x^3)}{\partial(x^{0'},x^{1'},x^{2'},x^{3'})}dx^{0'}dx^{1'}dx^{2'}dx^{3'}$$

There the factor $\partial(\cdot)/\partial(\cdot)$ is the Jacobian of the transformation from $\{x^{\alpha'}\}$ to $\{x^\alpha\}$,

$$\frac{\partial(x^0,x^1,x^2,x^3)}{\partial(x^{0'},x^{1'},x^{2'},x^{3'})} = \det\begin{pmatrix} \partial x^0/\partial x^{0'} & \partial x^0/\partial x^{1'} & \cdots \\ \partial x^1/\partial x^{0'} & \cdots & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \det(\Lambda^\alpha_{\beta'})$$

There is an easier way to calculate the Jacobian.
In matrix terminology, the transformation of the metric components is:

$$(g) = (\Lambda)(\eta)(\Lambda)^T$$

It follows that:

$$\det(g) = \det(\Lambda)\det(\eta)\det(\Lambda^T)$$

But $\det(\Lambda) = \det(\Lambda^T)$ for any matrix and $\det(\eta) = -1$, therefore, we get:

$$\det(g) = -[\det(\Lambda)]^2$$

And we define also:

$$g := \det(g_{\alpha'\beta'})$$

So we conclude that:

$$\det(\Lambda^\alpha_{\beta'}) = (-g)^{1/2}$$

Therefore:

$$\boxed{dx^0dx^1dx^2dx^3 = [-\det(g_{\alpha'\beta'})]^{1/2}dx^{0'}dx^{1'}dx^{2'}dx^{3'} = (-g)^{1/2}dx^{0'}dx^{1'}dx^{2'}dx^{3'}}$$

This is the expression for the true volume in a curved space at any point in any coordinates, this is the **proper volume element**

Example (in 3D): Here proper volume is $(g)^{1/2}$ since in 3D the metric is positive definite and then $\det(\eta) = 1$. In spherical coordinates, the line element is $dl^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$, so the metric is:

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

so $(g)^{1/2} = r^2\sin\theta drd\theta d\phi$

Proof of the local Flatness theorem

Let $\{x^\alpha\}$ be an arbitrary coordinate system and $\{x^{\alpha'}\}$ the one which is desired, it reduces to the inertial system at a certain fixed point P. Then there is some relation:

$$x^\alpha = x^\alpha(x^{\mu'})$$
$$\Lambda^\alpha_{\mu'} = \partial x^\alpha / \partial x^{\mu'}$$

Expanding $\Lambda^\alpha_{\mu'}$ in a Taylor series about P (whose coordinates are $x_0^{\mu'}$) gives the transformation at an arbitrary point $\vec{x}$ near P:

$$\Lambda^\alpha_{\mu'}(\vec{x}) = \Lambda^\alpha_{\mu'}(P) + (x^{\gamma'} - \gamma_0^{\gamma'})\frac{\partial \Lambda^\alpha_{\mu'}}{\partial x^{\gamma'}}(P) + \frac{1}{2}(x^{\gamma'} - x_0^{\gamma'})(x^{\lambda'} - x_0^{\lambda'})\frac{\partial^2 \Lambda^\alpha_{\mu'}}{\partial x^{\lambda'}\partial x^{\gamma'}}(P)$$

Expanding the metric in the same way gives:

$$g_{\alpha\beta}(\vec{x}) = g_{\alpha\beta}|_P + (x^{\gamma'} - x_0^{\gamma'})\left.\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma'}}\right|_P + \frac{1}{2}(x^{\gamma'} - x_0^{\gamma'})(x^{\lambda'} - x_0^{\lambda'})\left.\frac{\partial^2 g_{\alpha\beta}}{\partial x^{\lambda'}\partial x^{\gamma'}}\right|_P + \cdots$$

We put this into the transformation:

$$g_{u'v'} = \Lambda^\alpha_{\mu'}\Lambda^\beta_{\nu'} g_{\alpha\beta}$$

Etc.

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Covariant Differentiation

By definition, the derivative of a vector field involves the difference between vectors at two different points.
But in curved space, vectors at different points must be handled with care because we can only compare vectors that start in the same point. However the local flatness of the Riemannian manifold helps out, since we only need to compare vectors in the limit as they get infinitesimally close together.

So in a small region the manifold looks flat, and it is then natural to say that the derivative of a vector whose components are constant in this coordinate system is zero at that point. In particular, the derivatives of the basis vectors of a locally inertial coordinate system are zero at P

This is the *definition* of the covariant derivative. The local inertial frame is a frame in which everything is locally like SR, and in SR the derivatives of these basis vectors are zero.
Therefore, in these coordinates at this point, the covariant derivative of a vector has components given by the partial derivatives of the components (that is, Christoffel symbols vanish):

$$V^\alpha{}_{;\beta} = V^\alpha{}_{,\beta} \qquad \text{at P in this frame}$$

This is true for any other tensor, including the metric:

$$g_{\alpha\beta;\gamma} = g_{\alpha\beta,\gamma} = 0 \quad \text{at P}$$

Now, the equation $g_{\alpha\beta;\gamma} = 0$ is true in one frame (the locally inertial one), and is a valid tensor equation, so it is true in any basis:

$$g_{\alpha\beta;\gamma} = 0 \quad \text{any basis}$$

Then, using that we know $\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2}g^{\alpha\gamma}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha})$, we have, for *any metric* if $\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$, then:

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta})$$

We assumed at the start that at P in a locally inertial frame, $\Gamma^{\alpha}_{\mu\nu} = 0$. But importantly, the derivatives of $\Gamma^{\alpha}_{\mu\nu}$ at P in this frame are not all zero generally, since they involve $g_{\alpha\beta,\gamma\mu}$. This means that though coordinates can be found where $\Gamma = 0$, these symbols don't generally vanish elsewhere.

Given $g_{\alpha\beta}$, we can calculate $\Gamma^{\alpha}_{\mu\nu}$ everywhere. We can therefore calculate all covariant derivatives given g, with the formulas:

$$V^{\alpha}_{;\beta} = V^{\alpha}_{,\beta} + \Gamma^{\alpha}_{\mu\beta}V^{\mu}$$
$$P_{\alpha;\beta} = P_{\alpha,\beta} - \Gamma^{\mu}_{\alpha\beta}P_{\mu}$$

Divergence Formula

The divergence of an arbitrary vector field V^{α} is:

$$V^{\alpha}_{;\alpha} = V^{\alpha}_{,\alpha} + \Gamma^{\alpha}_{\mu\alpha}V^{\mu}$$

The formula involves a sum in the Christoffel symbol, which is:

$$\begin{aligned}\Gamma^{\alpha}_{\mu\alpha} &= \frac{1}{2}g^{\alpha\beta}(g_{\beta\mu,\alpha} + g_{\beta\alpha,\mu} - g_{\mu\alpha,\beta}) \\ &= \frac{1}{2}g^{\alpha\beta}(g_{\beta\mu,\alpha} - g_{\mu\alpha,\beta}) + \frac{1}{2}g^{\alpha\beta}g_{\alpha\beta,\mu}\end{aligned}$$

Notice that the term in parentheses is antisymmetric in α and β, while it is contracted in α, β with $g^{\alpha\beta}$, which is symmetric. Therefore, the first term vanishes and we get:

$$\Gamma^{\alpha}_{\mu\alpha} = \frac{1}{2}g^{\alpha\beta}g_{\alpha\beta,\mu}$$

Since $g^{\alpha\beta}$ is the inverse, it can be shown that the derivative of the determinant g is:

$$g_{,\mu} = gg^{\alpha\beta}g_{\alpha,\mu}$$

Therefore, we find:

$$\boxed{\Gamma^{\alpha}_{\mu\alpha} = (\sqrt{-g})_{,\mu}/\sqrt{-g}}$$

Then, we can write the divergence as:

$$V^{\alpha}_{;\alpha} = V^{\alpha}_{,\alpha} + \frac{1}{\sqrt{-g}} V^{\alpha}(\sqrt{-g})_{,\alpha}$$

or:

$$\boxed{V^{\alpha}_{;\alpha} = \frac{1}{\sqrt{-g}}(\sqrt{-g}V^{\alpha})_{,\alpha}}$$

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Parallel Transport, geodesics and curvature

There are two different types of curvatures: Intrinsic and extrinsic.
For example, since a cylinder is round in one direction, we consider it as curved, this is **extrinsic curveture**. The curvature it has in relation to the flat 3D space it is part of.
On the other hand, a cylinder can be made by rolling a flat piece of paper without crumpling it, so the **intrinsic** geometry is that of the original paper, it is flat.
This means that the distance in the surface of the cylinder between any two points is the same as it was in the original paper; parallel lines remain parallel when continued and all Euclid axioms are valid.

The intrinsic geometry of an n-dimensional manifold considers only the relationships between its points on paths that remain in the manifold, the extrinsic curvature comes from considering it as a surface in a space of higher dimension, and relies on the existance of this higher dimensional space.
In this book we talk about intrinsic curvature only, because world lines are confined to remain in space-time.

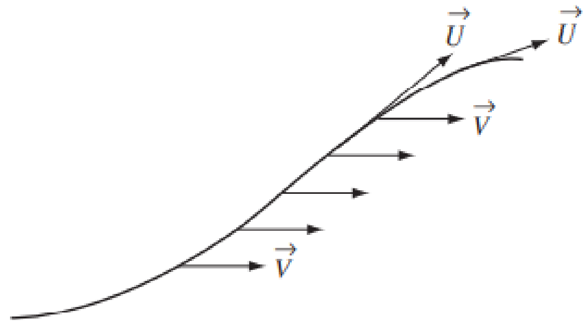
For example, a sphere has an intrinsically curved surface.
To see this, we consider two points A, B in the equator and draw two initially parallel lines that go to the north pole. When continued, this lines intersect even though they are parallel at the start, so parallel lines, when continue, do not remain parallel.

There is a more striking illystration of the curvature. We transport a vector from A back to A in a way that it is always locally parallel, and magically the vector has changed direction when coming back.

Therefore, on a curved manifold, it is impossible to define globally parallel vector fields, we can only define parallelisms when vectors have almost the same point of beginning.

Parallel Transport

The construction we have just made is called parallel-transport.
Suppose a vector field $\vec{V}$ is defined on the sphere and we examine how it changes along a curve



Parallel transport of $\vec{V}$ along $\vec{U}$.

If the vectors $\vec{V}$ at infinitesimally close points of the curve are parallel and of equal length, then $\vec{V}$ is said to be parallel transported along the curve.
If $\vec{U} = d\vec{x}/d\lambda$ is the tangent to the curve, then in a locally inertial coordinate system at point P, the components of V must be constant along the curve at P:

$$\frac{dV^\alpha}{d\lambda} = 0 \quad \text{at P}$$

This can be written as:

$$\frac{dV^\alpha}{d\lambda} = U^\beta V^\alpha_{,\beta} = U^\beta V^\alpha_{;\beta} = 0 \quad \text{at P}$$

The first equaality is the definition of the derivative of a function (in this case V^α) along a curve; the second equality comes from the fact that $\Gamma = 0$ at P in these coordinates. But the third equality is a frame invariant expression and holds for any basis, so it is the frame invariant **definition of the parallel transport of $\vec{V}$ along $\vec{U}$**

$$U^\beta V^\alpha_{;\beta} = 0 \Leftrightarrow \frac{d}{d\lambda}\vec{V} = 0 = \nabla_{\vec{U}}\vec{V} = 0$$

Geodesic

The most important curves in flat space are straight lines. The fifth Euclid axiom says that initially parallel lines remain parallel when extended.
but what does he mean by 'extended'? It means that the line doesn't change direction, which more precisely means that the tangent to the curve at one point is parallel to the tangent at the previous point.
A line is a curve that parallel transports its own tangent vector.

So, in curve space we define a **geodesic** as:

$$(\vec{U} \text{ is tangent to a geodesic}) \Leftrightarrow \nabla_{\vec{U}}\vec{U} = 0$$

In component notation:

$$U^\beta U^\alpha_{;\beta} = U^\beta U^\alpha_{,\beta} + \Gamma^\alpha_{\mu\beta} U^\mu U^\beta = 0$$

Now let λ be the parameter of the curve, then $U^\alpha = dx^\alpha/d\lambda$ and $U^\beta \partial/\partial x^\beta = d/d\lambda$, so:

$$\boxed{\frac{d}{d\lambda}\left(\frac{dx^\alpha}{d\lambda}\right) + \Gamma^\alpha_{\mu\beta}\frac{dx^\mu}{d\lambda}\frac{dx^\beta}{d\lambda} = 0}$$

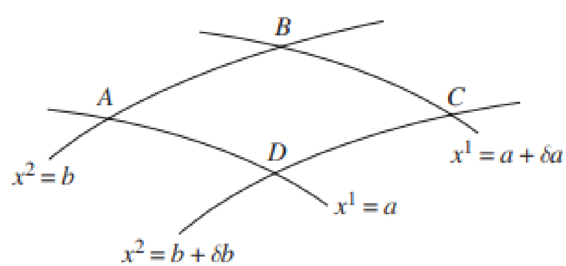
Since the Christoffel symbols are known functions of the coordinates (x^α), this is a nonlinear (quasilinear) second order differential equation for $x^\alpha(\lambda)$.
It has a unique solution when initial conditiona at $\lambda = \lambda_0$ are given, $x^\alpha_0 = x^\alpha(\lambda_0)$ and $U^\alpha_0 = (dx^\alpha/d\lambda)_{\lambda_0}$ (a initial position and direction give a unique geodesic).

A geodesic is also a curve of extremal length between two points: Its length is changed to first order in small changes to the curve.
.

The curvature tensor

Now we can define a tensor that embodies the curvature of a manifold using parallel transport of a vector around a closed loop.

Let us imagine a very small closed loop in our loop whose four sides are the coordinate lines $x^1 = a, x^1 = a + \delta a, x^2 = b, x^2 = b + \delta b$



Small section of a coordinate grid

A vector $\vec{V}$ is defined at A and parallel transported to B/ The parallel transport law $\nabla_{\vec{e}_1}\vec{V} = 0$ has the components form:

$$\frac{\partial V^\alpha}{\partial x^1} + \Gamma^\alpha_{\mu 1} V^\mu = 0$$
$$\Rightarrow \quad \frac{\partial V^\alpha}{\partial x^1} = -\Gamma^\alpha_{\mu 1} V^\mu$$

Integrating this from A to B gives:

$$V^\alpha(B) = V^\alpha(A) - \int_{x^2=b} \Gamma^\alpha_{\mu 1} V^\mu dx^1$$

Similarly, the transport from B to C and C to D gives:

$$V^\alpha(C) = V^\alpha(B) - \int_{x^1=a+\delta a} \Gamma^\alpha_{\mu 2} V^\mu dx^2$$
$$V^\alpha(D) = V^\alpha(C) + \int_{x^2=b+\delta b} \Gamma^\alpha_{\mu 1} V^\mu dx^1$$

The last integral has a different sign because of the direction. And finally, from D to A:

$$V^\alpha(A_{final}) = V^\alpha(D) + \int_{x^1=a} \Gamma^\alpha_{\mu 2} V^\mu dx^2$$

So, the net change of V^α is:

$$\begin{aligned}\delta V^\alpha &= V^\alpha(A_{final}) - V^\alpha(A_{initial}) \\ &= \int_{x^1=a} \Gamma^\alpha_{\mu 2} V^\mu dx^2 - \int_{x^1=a+\delta a} \Gamma^\alpha_{\mu 2} V^\mu dx^2 + \int_{x^2=b+\delta b} \Gamma^\alpha_{\mu 1} V^\mu dx^1 - \int_{x^2=b} \Gamma^\alpha_{\mu 1} V^\mu dx^1\end{aligned}$$

Notice that these would cancel in pairs if Γ and V^μ where constant in the loop, as they are in flat space. But in curved space they are not.
So if we combine the integrals over similar integration variables and work in first order in the separation in the paths, we get to lowest order:

$$\begin{aligned}\delta V^\alpha &\simeq -\int_b^{b+\delta b} \delta a \frac{\partial}{\partial x^1}(\Gamma^\alpha_{\mu 2} V^\mu) dx^2 + \int_a^{a+\delta a} \delta b \frac{\partial}{\partial x^2}(\Gamma^\alpha_{\mu 1} V^\mu) dx^1 \\ &\simeq \delta a \delta b \left[-\frac{\partial}{\partial x^1}(\Gamma^\alpha_{\mu 2} V^\mu) + \frac{\partial}{\partial x^2}(\Gamma^\alpha_{\mu 1} V^\mu) \right]\end{aligned}$$

The derivatives of V^α can be eliminated by using what we had first $\dfrac{\partial V^\alpha}{\partial x^1} = -\Gamma^\alpha_{\mu 1} V^\mu$. (and the one with 2 instead of 1)
So we get:

$$\boxed{\delta V^\alpha = \delta a \delta b [\Gamma^\alpha_{\mu 1,2} - \Gamma^\alpha_{\mu 2,1} + \Gamma^\alpha_{\nu 2}\Gamma^\nu_{\mu 1} - \Gamma^\alpha_{\nu 1}\Gamma^\nu_{\mu 2}] V^\mu}$$

Notice that this is just a number times V^μ and summed over μ.
The indices $1, 2$ appear because the path chosen was along those coordinates.
In is antisymmetric in 1 and 2 beacuse the change of δV^α would have to have the opposite sign if we went around the loop in the opposite direction (interchanging the roles of 1 and 2).
If we use the general coordinates x^σ, x^λ, we would find:

$$\begin{aligned}\delta V^\alpha &= \text{change in } V^\alpha \text{ due to transport, first} \delta a \vec{e}_\sigma \text{ then } \delta b \vec{e}_\lambda \text{ , then } - \delta a \vec{a}_\sigma \text{ then } - \delta b \vec{e}_\lambda \\ &= \delta a \delta b [\Gamma^\alpha_{\mu\sigma,\lambda} - \Gamma^\alpha_{\mu\lambda,\sigma} + \Gamma^\alpha_{\nu\lambda}\Gamma^\nu_{\mu\sigma} - \Gamma^\alpha_{\nu\sigma}\Gamma^\nu_{\mu\lambda}] V^\mu\end{aligned}$$

Now, δV^α depends on $\delta a \delta b$ (the 'area' of the loop), so if the length of one direction is double, δV^α is too.
So ΔV^α is linear on $\delta a \vec{e}_\sigma$ and $\delta b \vec{e}_\lambda$. And it also depends linearly on V^α and on $\tilde{\omega}^\alpha$, which is the basis one-form that gives δV^α from the vector $\delta \vec{V}$.

Hence, we define:

$$\boxed{R^\alpha_{\beta\mu\nu} := \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu}}$$

Then $R^\alpha_{\beta\mu\nu}$ are the components of the $(1,3)$ tensor.
Which takes as arguments $\tilde{\omega}^\alpha$, $\vec{V}, \delta a \vec{e}_\mu, \delta b \vec{e}_\nu$ and gives δV^α, the component of the change in $\vec{V}$ after parallel transport around a loop given by $\delta a \vec{e}_\mu$ and $\delta b \vec{e}_\nu$.
This is called the **Riemann Curvature Tensor R**

We can look at the components of $\mathbf{R}$ in a locally inertial frame at a point P, we have $\Gamma = 0$ at P, but we can find its derivative from some equation:

$$\Gamma^\alpha_{\mu\nu,\sigma} = \frac{1}{2} g^{\alpha\beta}(g_{\beta\mu,\nu\sigma} + g_{\beta\nu,\mu\sigma} - g_{\mu\nu,\beta\sigma})$$

Since the second derivatives of $g_{\alpha\beta}$ don't vanish, we get at P:

$$R^\alpha_{\beta\mu\nu} = \frac{1}{2} g^{\alpha\sigma}(g_{\sigma\beta,\nu\mu} + g_{\sigma\nu,\beta\mu} - g_{\beta\nu,\sigma\mu} - g_{\sigma\beta,\mu\nu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu})$$

Using the symmetry of $g_{\alpha\beta}$ and that second derivatives commute, we find that Locally at P:

$$\boxed{R^\alpha_{\beta\mu\nu} = \frac{1}{2} g^{\alpha\sigma}(g_{\sigma\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\sigma\mu})}$$

If we lower the index α, we get:

$$\boxed{R_{\alpha\beta\mu\nu} := g_{\alpha\lambda} R^\lambda_{\beta\mu\nu} = \frac{1}{2}(g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu})}$$

We can verify then the following identities:

- $R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}$
- $R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0$

Thus, $R_{\alpha\beta\mu\nu}$ is antisymmetric on the first and second pair of insices and symmetric on exchange of the two pairs.
Since these equations are valid tensor equations, they are true in all bases. This reduces the amount of independent components of R to just 20 in 4D.

A **flat** manifold is one which has a **global** definition of parallelism, so:

$$R^{\alpha}_{\beta\mu\nu} = 0 \;\Leftrightarrow\; \text{flat manifold}$$

.

An important use of the curvature tensor comes when we exaine the consequences of taking two covariant derivatives of a vector field $\vec{V}$. We found that the first derivatives were like flat space ones, since we could find coordinates in which the metric was flat to first order, but the second derivatives are different:

$$\begin{aligned}\nabla_\alpha\nabla_\beta V^\mu &= \nabla_\alpha(V^\mu_{;\beta}) \\ &= (V^\mu_{;\beta})_{,\alpha} + \Gamma^\mu_{\sigma\alpha}V^\sigma_{;\beta} - \Gamma^\sigma_{\beta\alpha}V^\mu_{;\sigma}\end{aligned}$$

In locally inertial coordinates whose origin is at P, all the Γ are zero, but their derivatives not, so at P:

$$\nabla_\alpha\nabla_\beta V^\mu = V^\mu_{,\beta\alpha} + \Gamma^\mu_{\nu\beta,\alpha}V^\nu$$

This expression is valid only in this specially chose coordinates system, and that is also true for the next equations. This males things easier. When we change α, β, we get:

$$\nabla_\beta\nabla_\alpha V^\mu = V^\mu_{,\alpha\beta} + \Gamma^\mu_{\nu\alpha,\beta}V^\nu$$

If we subtract these, we get the **commutator** of the covariant derivative operators:

$$\begin{aligned}[\nabla_\alpha, \nabla_\beta]V^\mu &:= \nabla_\alpha\nabla_\beta V^\mu - \nabla_\beta\nabla_\alpha V^\mu \\ &= (\Gamma^\mu_{\nu\beta,\alpha} - \Gamma^\mu_{\nu\alpha,\beta})V^\nu\end{aligned}$$

The terms involving second derivatives drop out here since $V^\mu_{,\alpha\beta} = V^\mu_{,\beta\alpha}$
So the double covariant derivative generally is not symmetric. Now, in this frame where $\Gamma = 0$ at P, we can compare with the defintion of R to get:

$$\boxed{[\nabla_\alpha, \nabla_\beta]V^\mu = R^\mu_{\nu\alpha\beta}V^\nu}$$

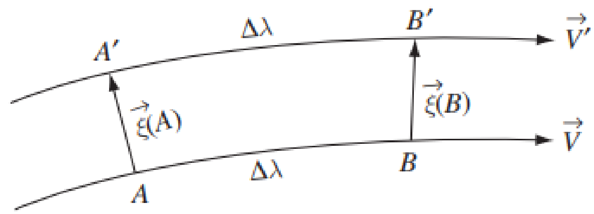
Now this is a valid tensor equation, so it is true in any coordinate system, so covariante derivatives do not commute and R is the tensor that symbolizes this.
For a (1,1) tensor we have to sum:

$$[\nabla_\alpha, \nabla_\beta]F^\mu_\nu = R^\mu_{\sigma\alpha\beta}F^\sigma_\nu + R_\nu{}^\sigma{}_{\alpha\beta}F^\mu_\sigma$$

.

Geodesic Deviation

We have often mentioned that in a curved space, parallel lines don't remain parallel. This can now be formulated mathematically in terms of the Riemann tensor.
Consider two geodesics (with tangent $\vec{V}, \vec{V'}$) that begin parallel and near each other at points A, A'. Let the affine parameter on the geodesics be called λ



We define a connecting vector $\vec{\xi}$ which reaches frome one geodesic to another connecting points at equal intervals in λ.
Let us adopt a locally inertial coordinate system at A in which coordinates x^0 points along the geodesics and advances at the same rate as λ there.
Then because $V^\alpha = dx^\alpha/d\lambda$, we have at A that $V^\alpha = \delta^\alpha_0$. The equation of the geodesic at A is:

$$\left.\frac{d^2x^\alpha}{d\lambda^2}\right|_A = 0$$

Since all Christofell symbols vanish at A.
The christoffel symbols don't vanish at A', so the equation for the geodesic $\vec{V'}$ at A' is:

$$\left.\frac{d^2x^\alpha}{d\lambda^2}\right|_{A'} + \Gamma^\alpha_{00}(A') = 0$$

Where again at A' we have arranged the coordinates so that $V^\alpha = \delta^\alpha_0$.
But since A and A' are separated by $\vec{\xi}$, we have:

$$\Gamma^\alpha_{00}(A') \simeq \Gamma^\alpha_{00,\beta}\chi^\beta$$

The right side evaluated at A. With the last equation, this becomes:

$$\left.\frac{d^2x^\alpha}{d\lambda^2}\right|_{A'} = -\Gamma^\alpha_{00,\beta}\xi^\beta$$

Now, the difference x^α (λ , geodesic $\vec{V'}$) $-x^\alpha$(λ, geodesic $\vec{V}$) is just the component of ξ^α of the vector $\vec{\xi}$. Therefor, at A, we have:

$$\frac{d^2\xi^\alpha}{d\lambda^2} = \left.\frac{d^2x^\alpha}{d\lambda^2}\right|_{A'} - \left.\frac{d^2x^\alpha}{d\lambda^2}\right|_{A} = -\Gamma^\alpha_{00,\beta}\xi^\beta$$

Then this gives how the components of $\vec{\xi}$ change. But since the coordinates are to some extent arbitrary, we want to have the full second covariant derivative of ξ.

Finally, we get:

$$\boxed{\nabla_V \nabla_V \xi^\alpha = R^\alpha{}_{\mu\nu\beta} V^\mu V^\nu \xi^\beta}$$

This shows how the diverging of geodesics is caused by curvature.

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Bianchi identities: Ricci and Einstein tensors

Let us return to the definition of $R^\alpha_{\beta\mu\nu} := \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu}$. If we differentiate it with respect to x^λ (just partial derivative) and evaluate in locally inertial coordinates, we find:

$$R_{\alpha\beta\mu\nu,\lambda} = \frac{1}{2}(g_{\alpha\nu,\beta\mu\lambda} - g_{\alpha\mu,\beta\nu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda})$$

From this equation, the symmetry $g_{\alpha\cdot\cdot} = g_{\beta\alpha}$ and the fact that partial derivatives commute, we can show that:

$$R_{\alpha\cdot\cdot\mu\nu,\lambda} + R_{\alpha\beta\lambda\mu,\nu} + R_{\alpha\beta\nu\lambda,\mu} = 0$$

Since in our coordinates $\Gamma = 0$ at this point, this equaiton is equivalent to:

$$\boxed{R_{\alpha\cdot\cdot\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0}$$

But this is a tensor equation, valid in any system. It is called the **Bianchi identities**.

Ricci Tensor

We define a new tensor:

$$\boxed{R_{\alpha\beta} := R^\mu_{\sigma\mu\beta} = R_{\beta\alpha}}$$

Because of the antisymmetry of R^μ_{afasf}, this is the only contractions that doesn't vanish or just changes signs.

Similarly, the **Ricci scalar** is defined as:

$$\boxed{R := g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} g^{\alpha\beta} R_{\alpha\mu\beta\nu}}$$

The Einstein Tensor

We apply the Ricci contraction to the Bianchi identities:

$$g^{\alpha\mu}[R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu}] = 0$$

Or:

$$R_{\beta\nu;\lambda} + (-R_{\beta\lambda;\nu}) + R^{\mu}_{\beta\nu\lambda;\mu} = 0$$

To derive this result we need two facts. First, we have $g_{\alpha\beta;\mu} = 0$
Since $g^{\alpha\mu}$ is a function only of $g_{\alpha\beta}$ it follow that $g^{\alpha\beta}_{;\mu} = 0$
Therefore, $g^{\alpha\mu}$ and $g_{\beta\nu}$ can be takes in and out of covariant derivatives at will: index raising and lowering commutes with covariant differentiation.

The second fact is that $g^{\alpha\mu}R_{\alpha\beta\lambda\mu;\nu} = -g^{\alpha\mu}R_{\alpha\beta\mu\lambda;\nu} = -R_{\beta\lambda;\nu}$
We can contract again on the indices β and ν to get
$g^{\beta\nu}[R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + R^{\mu}_{\beta\nu\lambda;\mu}] = 0$
Or $R_{;\lambda} - R^{\mu}_{\lambda;\mu} + (-R^{\mu}_{\lambda;\mu}) = 0$
The antisymmetry of $\mathbf{R}$ has been used. Since R is a scalar, $R_{;\lambda} = R_{,\lambda}$ in all coordinates, so we have the bianchu identity:
$(2R^{\mu}_{\lambda} - \delta^{\mu}_{\lambda}R)_{;\mu} = 0$
These are the Twice contracted Bianchi identities.

.

We define the new tensor:

$$\boxed{G^{\alpha\beta} := R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R = G^{\beta\alpha}}$$

And the **Bianchi identity** says that:

$$\boxed{G^{\alpha\beta}_{;\beta} = 0}$$

The tensor G is constructed only from the Riemann tensor and the metric, and is automatically divergenec free as an identity. It is called the **Einstein tensor**

Curvature In Perspective

We review some things:

- We work on Riemannian manifolds, which are smooth spaces with a metric defined on them.

- The metric has signature +2, and there always exists a coordinate system in which, at a single point we have:

$$g_{\alpha\beta} = \eta_{\alpha\beta}$$
$$g_{\alpha\beta,\gamma} = 0 \;\Rightarrow\; \Gamma^{\alpha}_{\beta\gamma} = 0$$

- The element of proper volume is:

$$|g|^{1/2} d^4 x$$

  Where g is the determinant of $g_{\alpha\beta}$

- The covariant derivative is simply the ordinary derivative in locally inertial coordinates. Because of curvature ($\Gamma^{\alpha}_{\beta\gamma,\sigma} \neq 0$) these derivatives do not commute.

- The definition of parallel transport is that the covariant derivative along the curve is 0. A geodesic parallel transports its own tangent vector.

- The Riemann tensor is the characterization of the curvature. If it vanishes identically , the manifold is intrinsecally flat. It has 20 independent components in 4 dimensions, and satisfies the Bianchi identities, which are differential equations.
  The Riemann tensor in a general coordinate system depends on $g_{\alpha\beta}$ and its first and second partial derivatives.
  The Ricci tensor, Ricci scalar and Einstein tensor are contractions of the Riemann tensor.
  In particular, the Einstein tensor is symmetric and of second rank, so it has ten independent components. It satisfies the Four differential equations $G^{\alpha\beta}_{\ \ ;\beta} = 0$

Curvature Carrol

Parallel Transport: Given a curve $x^\mu(\lambda)$, the requirement of constancy of a tensor T along this curve in flat space is simply $\dfrac{dT}{d\lambda} = \dfrac{dx^\mu}{d\lambda}\dfrac{\partial T}{\partial x^\mu} = 0$.
We therefore define the covariant derivative along the path to be given by an operator:

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda}\nabla_\mu$$

We then define the **parallel transport** of the tensor T along the path $x^\mu(\lambda)$ to be the requirement that along the path:

$$\left(\frac{D}{d\lambda}T\right)^{\mu_1\mu_2\cdots\mu_k}_{\qquad\quad \nu_1\nu_2\cdots\nu_l} := \frac{dx^\sigma}{d\lambda}\nabla_\sigma T^{\mu_1\cdots\mu_k}_{\qquad \nu_1\cdots\nu_l} = 0$$

This is a tensor equation. For a vector, the **equation** is:

$$\frac{d}{d\lambda}V^\mu + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}V^\rho = 0$$

Geodesic: It is a curve such that the tangent vector is parallel transported. Then, the equation is:

$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

This is the **geodesic equation**.

Shortest distance definition: We can also define a geodesic as a curve with stationary distance.
The infinitesimal distance between points is $dl = \sqrt{ds^2} = |g_{\alpha\beta}dx^\alpha dx^\beta|^{1/2}$. Then, integrating we get the length between to points:

$$l = \int (-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda})^{1/2}d\lambda$$

We use now calculus of variations.
We consider the change in l due to a infinitesimal variation of the path:

$$x^\mu \to x^\mu + \delta x^\mu$$
$$g_{\mu\nu} \to g_{\mu\nu} + \delta x^\sigma \partial_\sigma g_{\mu\nu}$$

We plug this into l to get $l + dl$:

$$l + dl = \int [-(g_{\mu\nu} + \delta x^\sigma \partial_\sigma g_{\mu\nu})\frac{dx^\mu + \delta x^\mu}{d\lambda}\frac{dx^\nu + \delta x^\nu}{d\lambda}]^{1/2}d\lambda$$

We can calculate dl using this and get some simplifications.
We equal this to 0 because we want no variation, after some things we get the geodesic equation (but with Γ written out explicitly in terms of g)

Geodesics are the paths followed by unaccelerated particle. It is like a generalization of $\vec{f} = m\vec{a}$ for the case $\vec{f} = 0$.

Curvature: Curvature is quantified by the Riemann tensor which is derived from the connection.
A consequence of curvature is that a parallel transport of a vector around a infinitessimal closed loop.
Say we begin at the point $(0,0)$ and then go through vector A^μ to the point $(\delta a, 0)$ and then through B^ν to $(\delta a, \delta b)$, then to $(0, \delta b)$ and then back to 0.
The paralel transport is independent of coordinates, so there should be some tensor which tells us how the vector changes when we come back to the origin; it will be a linear transformation of the vector transforming it into another vector (therefore, 1 upper index and 1 lower), but it also depends on A, an B (3 lower 1 upper).

Furthermore, it should be antisimmetric in the indices corresponding to A, B, since that changes direction of the loop.
Therefore, the expression for the change of the vector should be:

$$\delta V^\rho = (\delta a)(\delta b) A^\nu B^\mu R^\rho{}_{\sigma\mu\nu} V^\sigma$$

This is the **Riemann tensor**
And it has:
$R^\rho_{\sigma\mu\nu} = -R^\rho_{\sigma\nu\mu}$

We can calculate the actual change in V to obtain the tensor R from this definition.
And it is defined as:

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

Also, it is the result of commuting covariant derivatives:

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho_{\sigma\mu\nu} V^\sigma$$

Assuming it is tensor free as always.

We now examine the Riemann tensor with lower indices:

$$R_{\rho\sigma\mu\nu} = g^{\rho\lambda} R^\lambda{}_{\sigma\mu\nu}$$

We can consider the components of this tensor in coordinates established at p where Chirstofell vanishes (but not their derivatives), so:

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda}(\partial_\mu \Gamma^\lambda_{\nu\sigma} - \partial_\nu \Gamma^\lambda_{\mu\sigma}$$

After substituting Γ, we get the following properties:

$$\begin{aligned} R_{\rho\sigma\mu\nu} &= -R_{\sigma\rho\mu\nu} \\ R_{\rho\sigma\mu\nu} &= R_{\mu\nu\rho\sigma} \\ R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\rho} + R_{\rho\nu\sigma\mu} &= 0 \\ R_{[\cdot\sigma\mu\nu]} &= 0 \end{aligned}$$

Bianchi identity: We use **Riemann normal coordinates** (coordinates where $\Gamma = 0$ but derivatives aren't zero), to write $\nabla_\lambda R_{\rho\sigma\mu\nu} = \partial_\lambda R_{\rho\sigma\mu\nu}$
We then use the definition of R and some things to get:

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} = 0$$

We use the antisymmetry to write this as:

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$$

Ricci tensor:
We define:

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$$

This is the only contraction that gives an independent tensor. If fulfills:
$R_{\mu\nu} = R_{\nu\mu}$

Ricci Scalar:

$$R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu}$$

An especially useful form of the **Bianchi identity** is:

$$\nabla^{\mu} R_{\rho\mu} = \frac{1}{2} \nabla_{\rho} R$$

Einstein tensor
We define it as:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

And now the **Bianchi identity** is:

$$\nabla^{\mu} G_{\mu\nu} = 0$$

Properties Tensor

: **Riemann Curvature**

- **Skewness:** $R_{abcd} = -R_{abdc}$
- **More Skew** $R_{abcd} = -R_{bacd}$
- **First Bianchi**: $R_{abcd} + R_{acdb} + R_{adbc} = 0$
- **Interchange Symmetry:** $R_{abcd} = R_{cdab}$
- **Differential Bianchi:** $R_{acbd;e} + R_{abde;c} + R_{abec;d} = 0$

Ricci:

- $R_{\mu\nu} := R^{\lambda}_{\mu\lambda\nu}$
- This is the only contraction that gives an independent tensor.
- **Ricci scalar:** $R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu}$
- **Bianchi Identity:**

$$\nabla^{\mu} R_{\rho\mu} = \frac{1}{2} \nabla_{\rho} R$$

- Symmetry: $R_{\mu\nu} = R_{\nu\mu}$

Einstein Tensor:

- $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$
- **Bianchi Identity:** $\nabla^{\mu} G_{\mu\nu} = 0$
- Symmetry: $G_{\mu\nu} = G_{\nu\mu}$

Physics in Curved Spacetime

The transition from differential geometry to gravity

We have some things:

1) Spacetime (the set of all events) is a four dimensional manifold with a metric

2) The metric is measurable by rods and clocks. The distance along a rod between two nearby points is $|d\vec{x} \cdot d\vec{x}|^{1/2}$ and the time measured by a clock that experiences two events closely separated in time is $|-d\vec{x} \cdot d\vec{x}|^{1/2}$

   We have also shown that there do not exist global inertial frames in the presence of nonuniform gravitational fields. So there do not generally exist coordinates in which $d\vec{x} \cdot d\vec{x} = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^3$ everywhere.
   But such frames do exist locally, therefore, we make the further requirement:

3) The metric of spacetime can be put in the Lorentz form $\eta_{\alpha\beta}$ at any particular event by an appropriate choice of coordinates.

   Now we only need to know how particles move in curved spacetime and how objects determine the curvature.
   In Newtonian gravity, particles experience forces $\vec{F} = -m\nabla\phi$ where $\nabla^2\phi = 4\pi G\rho$ and they move according to $\vec{F} = m\vec{a}$.
   Since we know that the acceleration of a particle in a gravitational field is independent of mass, we can go to a freely falling frame in which all nearby particles will have no acceleration. This is a locally inertial frames. Freely falling particles have no acceleration in that frame, so they follow straight lines locally. But straight lines locally are the definition of geodesics in the full curved manifold. Therefore

4) **Weak equivalence Principle:** Freely falling particles move on timelike geodesics of the spacetime

   Freely falling means unaffected by other forces.

4') **Einstein Equivalence Principle:** Any local physical experiment not involving gravity will have the same result if performed in a freely falling inertial frame as if it were performed in the flat spacetime of special gravity

Local means that the experiment does not involve fields that may extend over large regions and therefore extend outside the validity of the local inertial frame.
All the local physics is the same in a freely falling inertial frame as it is in special relativity. Gravity introduces nothing new locally, all effects of gravity are felt over extended regions

of spacetime.

When we fall freely on a geodesic, we feel no gravity and we can dispose of the Newtonian concept of a gravitational force. What we feel when on Earth is the Earth preventing us from following the geodesic path.

The true measure of gravity on the Earth are its tides, which are nonlocal effects caused by nonuniformities of the gravitational field.
Mathematically, what the Einstein Equivalence Principle means is that if we have a local law of physics that is expressed in tensor notation in SR, the its mathematical form should be the same in a locally inertial frame of a curved spacetime.

This principle is called 'Comma-goes-to-semicolon-rule'
Because if a law contains derivatives in its special-relativistic form ('commas'), then it has these same derivatives in the local inertial frame. To convert the law into an expression valid in any coordinate frame, we simply make the derivatives covariant ('semicolons').

As an **example** of how (4') translates into math, we discuss fluid dynamics. The law of conservation of particles in SR is:

$$(nU^\alpha)_{,\alpha} = 0 \qquad 7.1$$

Where n is the density of particles in the MCRF and U^α is the 4-velocity of a fluid element.
In a curved spacetime, at any event, we can find a locally inertial frame comoving momentarily with the fluid element at that event and define n in the same way. Similarly, we can define $\vec{U}$ to be the time basis vector of that frame, just as in SR.
Then, according to the Einstein equivalence principle, the law of conservation of particles in the locally inertial frame is exactly the same.
But because the Christoffel symbols are 0 at the given event because it is the origin of the locally inertial frame, this is equivalent to:

$$(nU^\alpha)_{;\alpha} = 0 \qquad 7.2$$

This form of the law is **valid in all frames**. We have therefore generalized the law of particle conservation to a curved spacetime.

But we could generalize the SR law to a different form in curved spacetime. For example, the law in curved spacetime could instead be $(nU^\alpha)_{;\alpha} = kR^2$ with k a constant and R the Ricci scalar.
This would imply that curvature creates particles. And it still reduces to 7.1 when taking away curvature.
The Einstein equivalence principle asserts that we should generalize 7.1 in the simplest possible manner (that is 7.2), but it is really a matter of experiment to see this is correct (and it is).

Similarly, the law of conservation of entropy in SR is:

$$U^{\alpha}S_{,\alpha}=0$$

Since S is a scalar and there are no Christoffel symbols in the covariant derivative of a scalar like S, this law is **unchanged** in curved spacetime.

Finally, conservation of four momentum in SR is:

$$T^{\mu\nu}_{,\nu}=0$$

And the generalization is:

$$T^{\mu\nu}_{;\nu}=0$$

With the definition:

$$T^{\mu\nu}=(\rho+p)U^{\mu}U^{\nu}+pg^{\mu\nu}$$

Exactly as before (with g whose components in the local inertial frame equal the flat space metric tensor η).

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1.1 Physics in slightly curved spacetimes

Since we have not yet studied how the metric is generated, we shall assume it takes the following expression in weak field (which we will prove later). The ordinary Newtonian potential ϕ completely determines the metric, which takes the form:

$$ds^2=-(1+2\phi)dt^2+(1-2\phi)(dx^2+dy^2+dz^2) \qquad 7.8$$

The sign of ϕ is chosen negative, so that far from a source mass M, the potential is $\phi=-GM/r$.

We can compute the motion of a freely falling particle. We denote its four momentum by $\vec{p}$. This is $m\vec{U}$ where $\vec{U}=d\vec{x}/d\tau$ (except for massless particles).
Now, by 4) the particle moves in a geodesic, therefore $\vec{U}$ must satisfy the geodesic equation:

$$\nabla_{\vec{U}}\vec{U}=0$$

We note that dividing by m is an affine parameter, so we can write it in terms of $\vec{p}$ instead of $\vec{U}$, that is:

$$\nabla_{\vec{p}}\vec{p}=0$$

If the particle has nonrelativistic velocity in the coordinates of 7.8, we can find an approximate form form $\nabla_{\vec{p}}\vec{p} = 0$.
First, we consider the zero component of the equation, noting that the ordinary derivative along $\vec{p}$ is m times the ordinary derivarive along $\vec{U}$, or in other words $md/d\tau$:
The equation is:

$$\begin{aligned} &\nabla_{\vec{p}}\vec{p} = 0 \\ \Rightarrow \quad & p^\beta p^\alpha_{;\beta} = 0 \\ \Rightarrow \quad & p^\beta \partial_\beta p^\alpha + \Gamma^\alpha_{\mu\beta} p^\mu p^\beta = 0 \end{aligned}$$

And the first component of this is (we solve for now only the **time component**):

$$m\frac{d}{d\tau}p^0 + \Gamma^0_{\alpha\beta}p^\alpha p^\beta = 0$$

Because the particle has a nonrelativistic velocity, we have $p^0 >> p^1$, so it is approximately equal to :

$$m\frac{d}{d\tau}p^0 + \Gamma^0_{00}(p^0)^2 = 0 \qquad 7.12$$

We need the Christoffel symbol for this metric:

$$\Gamma^0_{00} = \frac{1}{2}g^{0\alpha}(g_{\alpha 0,0} + g_{\alpha 0,0} - g_{00,\alpha})$$

But $g_{\alpha\beta}$ is diagonal, so $g^{\alpha\beta}$ is also diagonal with the terms being the reciprocals of the original. Therefore, $g^{0\alpha}$ is nonzero only when $\alpha = 0$. So the equation becomes:

$$\begin{aligned} \Gamma^0_{00} &= \frac{1}{2}g^{00}g_{00,0} = \frac{1}{2}\frac{1}{-(1+2\phi)}(-2\phi)_{,0} \\ &= \phi_{,0} + O(\phi^2) \end{aligned}$$

So, we substitute in equation 7.12 and use that in the lowest order in the velocity of the particle and in ϕ, we can replace $(p^0)^2$ by m^2, so we get:

$$\boxed{\frac{d}{d\tau}p^0 \frac{\partial\phi}{\partial\tau}}$$

Since p^0 is the energy of the particle in this frame, this means the energy is conserved unless the gravitational field depends on time, this is not true in Newtonian. Here, however, we must note that p^0 is the energy of the particle with respect to this frame only.

So that is the temporal part of the geodesic equation.
For the **Spatial components** of the Geodesic equation, the equation is:

$$p^\alpha \partial_\alpha p^i + \Gamma^i_{\alpha\beta}p^\alpha p^\beta = 0$$

Or, to lowest order in velocity ($p^0 >> p^i$):

$$m\frac{dp^i}{d\tau} + \Gamma^i_{00}(p^0)^2 = 0$$

Consistent with this approximation, $(p^0)^2 = m^2$, so the equation is:

$$\frac{dp^i}{d\tau} = -m\Gamma^i_{00} \qquad 7.18$$

We calculate the Christoffel symbol:

$$\Gamma^i_{00} = \frac{1}{2}g^{i\alpha}(g_{\alpha 0,0} + g_{\alpha 0,0} - g_{00,\alpha})$$

Now, since $[g^{\alpha\beta}]$ is diagonal, we can write:

$$g^{i\alpha} = (1-2\phi)^{-1}\delta^{i\alpha}$$

We therefore get:

$$\Gamma^i_{00} = \frac{1}{2}(1-2\phi)^{-1}\delta^{ij}(2g_{j0,0} - g_{00,j})$$

We changed α to j because $\delta^{i0} = 0$. Now we notice that $g_{j,0} \equiv 0$, so we get:

$$\begin{aligned}\Gamma^i_{00} &= -\frac{1}{2}g_{00,j}\delta^{ij} + O(\phi^2)\\ &= -\frac{1}{2}(-2\phi)_{,j}\delta^{ij}\end{aligned}$$

Therefore, the equation 7.18 becomes:

$$\boxed{\frac{dp^i}{d\tau} = -m\partial_j\phi\ \delta^{ij}}$$

This is the usual equation in Newtonian gravity, since the force of the gravitational field is $-m\nabla\phi$.
So General relativity predicts the Keplerian motion of planets, so long as the higher order effects neglected here are too small, this is true for most planets, but not Mercury.

The limits we used were $|\phi| << 1$ and $p^0 >> p^i$

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Curved Intuition

Although the predictions are the same, the concepts of curved spacetime are very different to those of Newton's theory.

The first difference is the absence of a preferred frame. In Newtonian physics and in SR, inertial frames are preferred.
Since 'velocity' cannot be measured locally but acceleration can be, both theories single out special clases of coordinate systems for spacetimes in which particles which have no physical acceleration ($d\vec{U}/d\tau = 0$) also have no coordinate acceleration ($d^2x^i/dt^2 = 0$)
IN our new picture, there is no coordinate system which is inertial everywhere (in which $d^2x^i/dt^2 = 0$ for every particle for which $d\vec{U}d\tau = 0$
Therefore, we have to allow all coordinates on an equal footing, therefore we don't develop coordinate dependent ways of thinking.

Second difference: In Newtonian physics, SR and our geometrical gravity theory, each particle has a definit energy and momentum whose values depend on the frame.
In SR the total momentum is $\sum_i \vec{p}_i$.
But in curved spacetime, we cannot add vectors that are defined at different points, and even if we parallel transport them, that depends on the curve taken. So the definition of total 4-momentum must be very different.

Conserved quantities

In Newtonian physics, gravity is a force, so it changes a particle's momentum and energy. Therefore, In our new viewpoint, we cannot expect to find a coordinate system in which the components of $\vec{p}$ are constants along the trajectory , except in one notable example:

The geodesic equatoin is $\nabla_{\vec{p}}\vec{p} = 0 \quad \Rightarrow \quad p^\alpha p^\beta_{;\alpha} = 0$. Or, lowering the index:

$$\begin{aligned} & p^\alpha p_{\beta;\alpha} = 0 \\ \Rightarrow \quad & p^\alpha p_{\beta,\alpha} - \Gamma^\gamma_{\beta\alpha} p^\alpha p_\gamma = 0 \\ \Rightarrow \quad & m\frac{dp_\beta}{d\tau} = \Gamma^\gamma_{\beta\alpha} p^\alpha p_\gamma \end{aligned}$$

The right hand side is:

$$\begin{aligned} \Gamma^\gamma_{\alpha\beta} p^\alpha p_\gamma &= \frac{1}{2} g^{\gamma\nu}(g_{\nu\beta,\alpha} + g_{\nu\alpha,\beta} - g_{\alpha\beta,\nu}) p^\alpha p_\gamma \\ &= \frac{1}{2}(g_{\nu\beta,\alpha} + g_{\nu\alpha,\beta} - g_{\alpha\beta,\nu}) g^{\gamma\nu} p^\alpha p_\gamma \\ &= \frac{1}{2}(g_{\nu\beta,\alpha} + g_{\nu\alpha,\beta} - g_{\alpha\beta,\nu}) p^\nu p^\alpha \end{aligned}$$

The product $p^\nu p^\alpha$ is symmetric on α, ν, while the first and third terms inside parenthesis are, together, antisymmetric on α, ν. Therefore they cancel, leaving only the middle term:

$$\Gamma^\gamma_{\beta\alpha} p^\alpha p_\gamma = \frac{1}{2} g_{\nu\alpha,\beta} p^\nu p^\alpha$$

The **geodesic equation**, can thus, in complete generality, be written:

$$\boxed{m\frac{dp_\beta}{d\tau} = \frac{1}{2} g_{\nu\alpha,\beta} p^\nu p^\alpha}$$

So, *if all components $g_{\alpha\nu}$ are independent of some x^β for some fixed index β, then p_β is a constant along any particle's trajectory.*

For instance, if we have a time independent gravitational field, then a coordinate system can be found in which the metric components are time independent and therefore p_0 is conserved. Therefore p_0 is usually called the ' energy ' of the particle, without the qualification ' in this frame '.
The frame in which the metric components are time independent is special and is the usual 'lab frame' on Earth.
Consider the equation:

$$\begin{aligned}\vec{p}\cdot\vec{p} &= -m^2 = g_{\alpha\beta} p^\alpha p^\beta \\ &= -(1+2\phi)(p^0)^2 + (1-2\phi)[(p^x)^2 + (p^y)^2 + (p^z)^2]\end{aligned}$$

Where we have used the metric. This can be solved to give:

$$(p^0)^2 = [m^2 + (1-2\phi)(p^2)](1+2\phi)^{-1}$$

Where, we denote by $p^2 = (p^x)^2 + (p^y)^2 + (p^z)^2$. Keeping with the approximation $|\phi| << 1, |p| << m$, we can simplify it to:

$$\begin{aligned}(p^0)^2 &\simeq m^2(1 - 2\phi + p^2/m^2) \\ \Rightarrow \quad p^0 &\simeq m(1 - \phi + p^2/2m^2)\end{aligned}$$

Now we lower the index to get:

$$p_0 = g_{0\alpha} p^\alpha = g_{00} p^0 = -(1+2\phi) p^0$$

That is:

$$\boxed{-p_0 \simeq m(1 + \phi + p^2/2m^2) = m + m\phi + p^2/2m}$$

More Exercises

$$x = r \operatorname{sen}\theta \cos\phi \qquad y = r \operatorname{sen}\theta \operatorname{sen}\phi \qquad z = r\cos\theta$$

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\nu}} dx^{\nu} \qquad ds^2 = dx^2 + dy^2 + dz^2$$

- $dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \phi} d\phi$
  $= \operatorname{sen}\theta \cos\phi \, dr + r\cos\theta \cos\phi \, d\theta - r \operatorname{sen}\theta \operatorname{sen}\phi \, d\phi$
- $dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \phi} d\phi$
  $= \operatorname{sen}\theta \operatorname{sen}\phi \, dr + r\cos\theta \operatorname{sen}\phi \, d\theta + r\operatorname{sen}\theta\cos\phi \, d\phi$
- $dz = \cos\theta \, dr - r\operatorname{sen}\theta \, d\theta$

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 = \\ &(\operatorname{sen}\theta\cos\phi \, dr + r\cos\theta\cos\phi \, d\theta - r\operatorname{sen}\theta\operatorname{sen}\phi \, d\phi)^2 \\ &+ (\operatorname{sen}\theta\operatorname{sen}\phi \, dr + r\cos\theta\operatorname{sen}\phi \, d\theta + r\operatorname{sen}\theta\cos\phi \, d\phi)^2 \\ &+ (\cos\theta \, dr - r\operatorname{sen}\theta \, d\theta)^2 \\ &= [\operatorname{sen}^2\theta\cos^2\phi + \operatorname{sen}^2\theta\operatorname{sen}^2\phi + \cos^2\theta]dr^2 + [r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\operatorname{sen}^2\phi + r^2\operatorname{sen}^2\theta]d\theta^2 + [r^2\operatorname{sen}^2\theta\operatorname{sen}^2\phi + r^2\operatorname{sen}^2\theta\cos^2\phi]d\phi^2 \\ &+ 2[r\cos\theta\operatorname{sen}\theta\cos^2\phi + r\operatorname{sen}\theta\cos\theta\operatorname{sen}^2\phi - r\cos\theta\operatorname{sen}\theta]dr\,d\theta + 2[-r\operatorname{sen}^2\theta\cos\phi\operatorname{sen}\phi + r\operatorname{sen}^2\theta\cos\phi\operatorname{sen}\phi]dr\,d\phi \\ &+ 2[-r^2\cos\theta\operatorname{sen}\theta\cos\phi\operatorname{sen}\phi + r^2\cos\theta\operatorname{sen}\phi\operatorname{sen}\theta\cos\phi]d\theta\,d\phi \\ &= dr^2 + r^2 d\theta^2 + r^2\operatorname{sen}^2\theta \, d\phi^2 \end{aligned}$$

(a) Transporte paralelo

$$\nabla_{\vec{U}}\vec{V} = 0 \rightarrow U^{\alpha}\nabla_{\alpha}V^{\beta} = 0 \rightarrow U^{\alpha}(\partial_{\alpha}V^{\beta} + \Gamma^{\beta}_{\alpha\lambda}V^{\lambda}) = 0$$

$$U^{\alpha}\partial_{\alpha}V^{\beta} + \Gamma^{\beta}_{\alpha\lambda}V^{\lambda}U^{\alpha} = 0$$

$$\frac{d}{d\lambda}V^{\beta} + \Gamma^{\beta}_{\alpha\lambda}V^{\lambda}\frac{dx^{\alpha}}{d\lambda} = 0$$

Geodésica

$$\nabla_{\vec{U}}\vec{U} = 0 \rightarrow U^{\alpha}\nabla_{\alpha}U^{\beta} = 0$$

$$\rightarrow U^{\alpha}\partial_{\alpha}U^{\beta} + \Gamma^{\beta}_{\alpha\lambda}U^{\lambda}U^{\alpha} = 0$$

$$\rightarrow \frac{d}{d\lambda}U^{\beta} + \Gamma^{\beta}_{\alpha\lambda}\frac{dx^{\lambda}}{d\lambda}\frac{dx^{\alpha}}{d\lambda} = 0$$

Geodésicas en Esfera

Esfera: $x = \sin\theta\cos\phi \quad y = \sin\theta\sin\phi \quad z = \cos\theta \quad (r=1)$

as we saw, $ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2 = d\theta^2 + \sin^2\theta\, d\phi^2$

$$\Rightarrow g_{ij} = \begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \rightarrow g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \csc^2\theta \end{pmatrix} = \begin{pmatrix} g^{\theta\theta} & \\ & g^{\phi\phi} \end{pmatrix}$$

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)$$

- $\Gamma^{\theta}_{\theta\theta} = \frac{1}{2} g^{\theta\rho}\left(\partial_\theta g_{\theta\rho} + \partial_\theta g_{\rho\theta} - \partial_\rho g_{\theta\theta}\right) = \frac{1}{2}\left(\partial_\theta(1)\right) = 0$
- $\Gamma^{\theta}_{\theta\phi} = \Gamma^{\theta}_{\phi\theta} = \frac{1}{2} g^{\theta\rho}\left(\partial_\theta g_{\phi\rho} + \partial_\phi g_{\rho\theta} - \partial_\rho g_{\theta\phi}\right) = \frac{1}{2} g^{\theta\rho}\partial_\theta g_{\phi\rho} = \frac{1}{2} g^{\theta\theta}\partial_\theta g_{\phi\theta} + \frac{1}{2} g^{\theta\phi}\partial_\theta g_{\phi\phi} = 0$
- $\Gamma^{\theta}_{\phi\phi} = \frac{1}{2} g^{\theta\rho}\left(\partial_\phi g_{\phi\rho} + \partial_\phi g_{\rho\phi} - \partial_\rho g_{\phi\phi}\right)$
  $= \frac{1}{2} g^{\theta\theta}\left(\partial_\phi g_{\phi\theta} + \partial_\phi g_{\theta\phi} - \partial_\theta g_{\phi\phi}\right) + \frac{1}{2} g^{\theta\phi}\left(\partial_\phi g_{\phi\phi} + \partial_\phi g_{\phi\phi} - \partial_\phi g_{\phi\phi}\right)$
  $= \frac{1}{2}(1)\left(-2\sin\theta\cos\theta\right) + \frac{1}{2}(0) = \underline{-\sin\theta\cos\theta}$

- $\Gamma^{\phi}_{\theta\theta} = \frac{1}{2} g^{\phi\rho}\left(\partial_\theta g_{\theta\rho} + \partial_\theta g_{\rho\theta} - \partial_\rho g_{\theta\theta}\right) = \underline{0}$
- $\Gamma^{\phi}_{\phi\phi} = \frac{1}{2} g^{\phi\rho}\left(\partial_\phi g_{\phi\rho} + \partial_\phi g_{\rho\phi} - \partial_\rho g_{\phi\phi}\right) = \left(-\frac{1}{2} g^{\phi\theta}\partial_\theta g_{\phi\phi} - \frac{1}{2} g^{\phi\phi}\partial_\phi g_{\phi\phi}\right) = \underline{0}$
- $\Gamma^{\phi}_{\theta\phi} = \frac{1}{2} g^{\phi\rho}\left(\partial_\theta g_{\phi\rho} + \partial_\phi g_{\rho\theta} - \partial_\rho g_{\theta\phi}\right)$
  $= \frac{1}{2} g^{\phi\rho}\partial_\theta g_{\phi\rho} = \frac{1}{2} g^{\phi\theta}\partial_\theta g_{\phi\theta} + \frac{1}{2} g^{\phi\phi}\partial_\theta g_{\phi\phi} = \frac{1}{2}\csc^2\theta\left(2\sin\theta\cos\theta\right) = \underline{\cot\theta}$

$\Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi}$

Geodesic: $(\theta(\lambda), \phi(\lambda))$

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^{\mu}_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

- $\frac{d^2\theta}{d\lambda^2} + \Gamma^{\theta}_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \rightarrow \frac{d^2\theta}{d\lambda^2} + \Gamma^{\theta}_{\phi\phi}\frac{d\phi}{d\lambda}\frac{d\phi}{d\lambda} = 0$

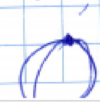
$$\rightarrow \underline{\theta'' - \cos\theta\sin\theta\,(\phi')^2 = 0}$$

- $\frac{d^2\phi}{d\lambda^2} + \Gamma^{\phi}_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \rightarrow \phi'' + \Gamma^{\phi}_{\theta\phi}\theta'\phi' + \Gamma^{\phi}_{\phi\theta}\phi'\theta' = 0$

$$\rightarrow \underline{\phi'' + 2\cot\theta\,\theta'\phi' = 0}$$

When solved with $\theta(0) = \phi(0) = 0$, $\theta'(0) = \phi'(0) = 0$

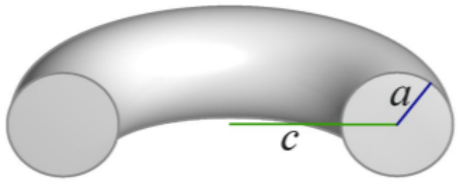
nos queda



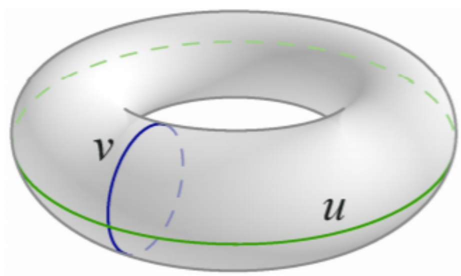
Curvature and Geodesic of the Torus

1. The Line Element and Metric

Our model of a torus has major radius *c* and minor radius *a*. We only consider the ring torus, for which $c>a$.



We use a *u*,*v* coördinate system for which planes of constant *u* pass through the torus's axis.



We parameterize the surface $\boldsymbol{x}$ by $\boldsymbol{x}(u,v)=\left\{\begin{array}{l} x=(c+a\cos v)\cos u \\ y=(c+a\cos v)\sin u \\ z=a\sin v \end{array}\right\}$.

We begin by calculating the coefficients E, F, and G of the first fundamental form.

$$\boldsymbol{x}_u = (-(c+a\cos v)\sin u, (c+a\cos v)\cos u, 0)$$
$$\boldsymbol{x}_v = (-a\cos u\sin v, -a\sin u\sin v, a\cos v)$$

$$\begin{aligned} E &= \boldsymbol{x}_u \cdot \boldsymbol{x}_u \\ &= (-(c+a\cos v)\sin u)^2 + ((c+a\cos v)\cos u)^2 + 0 \\ &= (c+a\cos v)^2 \end{aligned}$$

$$\begin{aligned} F &= \boldsymbol{x}_u \cdot \boldsymbol{x}_v \\ &= (-(c+a\cos v)\sin u)(-a\sin v\cos u) + ((c+a\cos v)\cos u)(-a\sin v\sin u) + (0)(a\cos v) \\ &= a\sin v\cos u\sin u(c+a\cos v) - a\sin v\cos u\sin u(c+a\cos v) \\ &= 0 \end{aligned}$$

$$\begin{aligned} G &= \boldsymbol{x}_v \cdot \boldsymbol{x}_v \\ &= (-a\sin v\cos u)^2 + (-a\sin v\sin u)^2 + (a\cos v)^2 \\ &= a^2\sin^2 v\cos^2 u + a^2\sin^2 v\sin^2 u + a^2\cos^2 v \\ &= a^2\sin^2 v + a^2\cos^2 v \\ &= a^2 \end{aligned}$$

This gives us the line element $ds^2 = (c+a\cos v)^2 du^2 + a^2 dv^2$ and metric:

$$g_{ij} = \begin{bmatrix} (c+a\cos v)^2 & 0 \\ 0 & a^2 \end{bmatrix}, \ g^{ij} = \begin{bmatrix} \frac{1}{(c+a\cos v)^2} & 0 \\ 0 & \frac{1}{a^2} \end{bmatrix}.$$

For later computations we'll need the partial derivatives of the metric:

$$g_{ij,u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } g_{ij,v} = \begin{bmatrix} -2a\sin v(c+a\cos v) & 0 \\ 0 & 0 \end{bmatrix}$$

3. The Curvature Tensor

The Christoffel symbols of the second kind

$$\Gamma^u_{uu} = \frac{1}{2}[g^{uu}(g_{uu,u} + g_{uu,u} - g_{uu,u}) + g^{uv}(g_{vu,u} + g_{vu,u} - g_{uu,v})]$$
$$= \frac{1}{2}[g^{uu}(0 + 0 - 0) + 0(g_{vu,u} + g_{vu,u} - g_{uu,v})]$$
$$= 0$$

$$\Gamma^u_{uv} = \frac{1}{2}[g^{uu}(g_{uv,u} + g_{uu,v} - g_{uv,u}) + g^{uv}(g_{vv,u} + g_{vu,v} - g_{uv,v})]$$
$$= \frac{1}{2}[g^{uu}(0 + g_{uu,v} - 0) + 0(g_{vv,u} + g_{vu,v} - g_{uv,v})]$$
$$= \frac{1}{2}g^{uu}g_{uu,v}$$
$$= \frac{1}{2}\left(\frac{-2a\sin v(c + a\cos v)}{(c + a\cos v)^2}\right)$$
$$= -\frac{a\sin v}{(c + a\cos v)}$$

$$\Gamma^u_{vu} = \frac{1}{2}[g^{uu}(g_{uu,v} + g_{uv,u} - g_{vu,u}) + g^{uv}(g_{vu,v} + g_{vv,u} - g_{vu,v})]$$
$$= \frac{1}{2}[g^{uu}(g_{uu,v} + 0 - 0) + 0(g_{vu,v} + g_{vv,u} - g_{vu,v})] = \frac{1}{2}g^{uu}g_{uu,v} = \Gamma^u_{uv}$$
$$= -\frac{a\sin v}{(c + a\cos v)}$$

$$\Gamma^u_{vv} = \frac{1}{2}[g^{uu}(g_{uv,v} + g_{uv,v} - g_{vv,u}) + g^{uv}(g_{vv,v} + g_{vv,v} - g_{vv,v})]$$
$$= \frac{1}{2}[g^{uu}(0 + 0 - 0) + 0(g_{vv,v} + g_{vv,v} - g_{vv,v})]$$
$$= 0$$

$$\begin{aligned}\Gamma^{v}_{uu} &= \frac{1}{2}[0(g_{uu,u}+g_{uu,u}-g_{uu,u})+g^{vv}(0+0-g_{uu,v})] \\ &= -\frac{1}{2}g^{vv}g_{uu,v} \\ &= -\frac{1}{2}\frac{1}{a^2}(-2a\sin v(c+a\cos v)) \\ &= \frac{1}{a}\sin v(c+a\cos v)\end{aligned}$$

$$\begin{aligned}\Gamma^{v}_{uv} &= \frac{1}{2}[g^{vu}(g_{uv,u}+g_{uu,v}-g_{uv,u})+g^{vv}(g_{vv,u}+g_{vu,v}-g_{uv,v})] \\ &= \frac{1}{2}[0(g_{uv,u}+g_{uu,v}-g_{uv,u})+g^{vv}(0+0-0)] \\ &= 0\end{aligned}$$

$$\begin{aligned}\Gamma^{v}_{vu} &= \frac{1}{2}[g^{vu}(g_{uu,v}+g_{uv,u}-g_{vu,u})+g^{vv}(g_{vu,v}+g_{vv,u}-g_{vu,v})] \\ &= \frac{1}{2}[0(g_{uu,v}+g_{uv,u}-g_{vu,u})+g^{vv}(0+0-0)] \\ &= 0\end{aligned}$$

$$\begin{aligned}\Gamma^{v}_{vv} &= \frac{1}{2}[g^{vu}(g_{uv,v}+g_{uv,v}-g_{vv,u})+g^{vv}(g_{vv,v}+g_{vv,v}-g_{vv,v})] \\ &= \frac{1}{2}[0(g_{uv,v}+g_{uv,v}-g_{vv,u})+g^{vv}(0+0-0)] \\ &= 0\end{aligned}$$

Partial derivatives of the nonzero Christoffel symbols:

$$\begin{aligned}\Gamma^{u}_{uv,v} = \Gamma^{u}_{vu,v} &= -\left[(a\sin v)(-1)(c+a\cos v)^{-2}(-a\sin v)+(c+a\cos v)^{-1}(a\cos v)\right] \\ &= -(a\sin v)^2(c+a\cos v)^{-2}-(c+a\cos v)^{-1}(a\cos v) \\ &= -\frac{(a\sin v)^2}{(c+a\cos v)^2}-\frac{a\cos v}{(c+a\cos v)}\end{aligned}$$

$$\Gamma^{v}_{uu,v} = \frac{1}{a}[\sin v(-a\sin v)+(c+a\cos v)\cos v] = \frac{1}{a}[c\cos v+a\cos^2 v-a\sin^2 v]$$

The Riemann tensor

Throughout this section we use the identity $R^i{}_{jkl} = -R^i{}_{jlk}$.

$$
\begin{aligned}
R^u{}_{uuu} &= \Gamma^u_{uu,u} - \Gamma^u_{uu,u} - \Gamma^u_{uu}\Gamma^u_{uu} - \Gamma^u_{vu}\Gamma^v_{uu} + \Gamma^u_{uu}\Gamma^u_{uu} + \Gamma^u_{vu}\Gamma^v_{uu} \\
&= 0 - 0 - 0 - \Gamma^u_{vu} \times 0 + 0 + 0 = 0
\end{aligned}
$$

$$
\begin{aligned}
R^u{}_{uuv} = -R^u{}_{uvu} &= \Gamma^u_{uv,u} - \Gamma^u_{uu,v} - \Gamma^u_{uv}\Gamma^u_{uu} - \Gamma^u_{vv}\Gamma^v_{uu} + \Gamma^u_{uu}\Gamma^u_{uv} + \Gamma^u_{vu}\Gamma^v_{uv} \\
&= 0 - 0 - 0 - 0 + 0 + 0 = 0
\end{aligned}
$$

$$
\begin{aligned}
R^u{}_{uvv} &= \Gamma^u_{uv,v} - \Gamma^u_{uv,v} - \Gamma^u_{uv}\Gamma^u_{uv} - \Gamma^u_{vv}\Gamma^v_{uv} + \Gamma^u_{uv}\Gamma^u_{uv} + \Gamma^u_{vv}\Gamma^v_{uv} \\
&= \Gamma^u_{uv,v} - \Gamma^u_{uv,v} - (\Gamma^u_{uv})^2 - 0 + (\Gamma^u_{uv})^2 + 0 = 0
\end{aligned}
$$

$$
\begin{aligned}
R^u{}_{vuu} &= \Gamma^u_{vu,u} - \Gamma^u_{vu,u} - \Gamma^u_{uu}\Gamma^u_{vu} - \Gamma^u_{vu}\Gamma^v_{vu} + \Gamma^u_{uu}\Gamma^u_{vu} + \Gamma^u_{vu}\Gamma^v_{vu} \\
&= 0 - 0 - 0 - 0 + 0 + 0 = 0
\end{aligned}
$$

$$
\begin{aligned}
R^u{}_{vuv} = -R^u{}_{vvu} &= \Gamma^u_{vv,u} - \Gamma^u_{vu,v} - \Gamma^u_{uv}\Gamma^u_{vu} - \Gamma^u_{vv}\Gamma^v_{vu} + \Gamma^u_{uu}\Gamma^u_{vv} + \Gamma^u_{vu}\Gamma^v_{vv} \\
&= 0 - \Gamma^u_{vu,v} - (\Gamma^u_{vu})^2 - 0 + 0 - 0 \\
&= -\left[-\left(\frac{a\sin v}{c + a\cos v}\right)^2 - \frac{a\cos v}{c + a\cos v}\right] - \left(\frac{a\sin v}{c + a\cos v}\right)^2 \\
&= \left(\frac{a\sin v}{c + a\cos v}\right)^2 + \frac{a\cos v}{c + a\cos v} - \left(\frac{a\sin v}{c + a\cos v}\right)^2 \\
&= \frac{a\cos v}{c + a\cos v}
\end{aligned}
$$

$$
\begin{aligned}
R^u{}_{vvv} &= \Gamma^u_{vv,v} - \Gamma^u_{vv,v} - \Gamma^u_{uv}\Gamma^u_{vv} - \Gamma^u_{vv}\Gamma^v_{vv} + \Gamma^u_{uv}\Gamma^u_{vv} + \Gamma^u_{vv}\Gamma^v_{vv} \\
&= 0 - 0 - 0 - 0 + 0 + 0 = 0
\end{aligned}
$$

$$
\begin{aligned}
R^v{}_{uuu} &= \Gamma^v_{uu,u} - \Gamma^v_{uu,u} - \Gamma^v_{uu}\Gamma^u_{uu} - \Gamma^v_{vu}\Gamma^v_{uu} + \Gamma^v_{uu}\Gamma^u_{uu} + \Gamma^v_{vu}\Gamma^v_{uu} \\
&= 0 - 0 - 0 - 0 + 0 + 0 = 0
\end{aligned}
$$

$$
\begin{aligned}
R^v{}_{uuv} = -R^v{}_{uvu} &= \Gamma^v_{uv,u} - \Gamma^v_{uu,v} - \Gamma^v_{uv}\Gamma^u_{uu} - \Gamma^v_{vv}\Gamma^v_{uu} + \Gamma^v_{uu}\Gamma^u_{uv} + \Gamma^v_{vu}\Gamma^v_{uv} \\
&= 0 - \Gamma^v_{uu,v} - 0 - 0 + \Gamma^v_{uu}\Gamma^u_{uv} + 0 \\
&= -\frac{1}{a}(\sin v(-a\sin v) + \cos v(c + a\cos v)) + \frac{1}{a}\sin v(c + a\cos v)\left(-\frac{a\sin v}{c + a\cos v}\right) \\
&= -\frac{1}{a}[-a\sin^2 v + \cos v(c + a\cos v) + a\sin^2 v] \\
&= -\frac{1}{a}\cos v(c + a\cos v)
\end{aligned}
$$

$$
\begin{aligned}
R^v{}_{uvv} &= \Gamma^v_{uv,v} - \Gamma^v_{uv,v} - \Gamma^v_{uv}\Gamma^u_{uv} - \Gamma^v_{vv}\Gamma^v_{uv} + \Gamma^v_{uv}\Gamma^u_{uv} + \Gamma^v_{vv}\Gamma^v_{uv} \\
&= 0 - 0 - 0 - 0 + 0 + 0 = 0
\end{aligned}
$$

$$
\begin{aligned}
R^v{}_{vuu} &= \Gamma^v_{vu,u} - \Gamma^v_{vu,u} - \Gamma^v_{uu}\Gamma^u_{vu} - \Gamma^v_{vu}\Gamma^v_{vu} + \Gamma^v_{uu}\Gamma^u_{vu} + \Gamma^v_{vu}\Gamma^v_{vu} \\
&= 0 - 0 - \Gamma^v_{uu}\Gamma^u_{vu} - 0 + \Gamma^v_{uu}\Gamma^u_{vu} + 0 = 0
\end{aligned}
$$

$$
\begin{aligned}
R^v{}_{vuv} = -R^v{}_{vvu} &= \Gamma^v_{vv,u} - \Gamma^v_{vu,v} - \Gamma^v_{uv}\Gamma^u_{vu} - \Gamma^v_{vv}\Gamma^v_{vu} + \Gamma^v_{uu}\Gamma^u_{vv} + \Gamma^v_{vu}\Gamma^v_{vv} \\
&= 0 - 0 - 0 - 0 + 0 + 0 = 0
\end{aligned}
$$

$$
\begin{aligned}
R^v{}_{vvv} &= \Gamma^v_{vv,v} - \Gamma^v_{vv,v} - \Gamma^v_{uv}\Gamma^u_{vv} - \Gamma^v_{vv}\Gamma^v_{vv} + \Gamma^v_{uv}\Gamma^u_{vv} + \Gamma^v_{vv}\Gamma^v_{vv} \\
&= 0 - 0 - 0 - 0 + 0 + 0 = 0
\end{aligned}
$$

The Ricci tensor

$$R_{ij} = R^m{}_{imj}$$

$$R_{uu} = R^m{}_{umu} = \frac{1}{a}\cos v(c + a\cos v)$$

$$R_{vv} = R^m{}_{vmv} = \frac{a\cos v}{c + a\cos v}$$

$$R_{ij} = \begin{bmatrix} \frac{1}{a}\cos v(c + a\cos v) & 0 \\ 0 & \frac{a\cos v}{(c + a\cos v)} \end{bmatrix}$$

The Ricci scalar, a.k.a. the curvature scalar

$$
\begin{aligned}
R &= g^{ij}R_{ij} = g^{uu}R_{uu} + g^{vv}R_{vv} \\
&= \left(\frac{1}{(c + a\cos v)^2}\right)\left(\frac{1}{a}\cos v(c + a\cos v)\right) + \left(\frac{1}{a^2}\right)\left(\frac{a\cos v}{c + a\cos v}\right) \\
&= \frac{\cos v}{a(c + a\cos v)} + \frac{\cos v}{a(c + a\cos v)} \\
R &= \frac{2\cos v}{a(c + a\cos v)}
\end{aligned}
$$

R is twice the Gaussian curvature, as expected.

4. The Geodesic Equation

Let's look at the geodesic equation $\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$. Plugging in Christoffel symbols and components of the Riemann tensor yields two equations.

(i) $\ddot{u} + 2\Gamma^u_{uv}\dot{u}\dot{v} = \ddot{u} - \frac{2a\sin v}{c+a\cos v}\dot{u}\dot{v} = 0$
(ii) $\ddot{v} + \Gamma^v_{uu}\dot{u}^2 = \ddot{v} + \frac{1}{a}\sin v(c + a\cos v)\dot{u}^2 = 0$

.

Exercises 5

- **7) Calculate all elements of the transformation matrices $\Lambda^{\alpha'}_{\beta}$ and $\Lambda^{\mu}_{\nu'}$ for the transformation from Cartesian (x, y) - the unprimed indices - to polar (r, θ) - the primed indices -**

We could first calculate $\Lambda^{\mu}_{\nu'}$. We write the (x, y) in terms of the (r, θ) variables.

$$(\Lambda^{\mu}_{\nu'}) = \begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

The matrix $\Lambda^{\alpha'}_{\beta}$ can be found by inverting, so:

$$(\Lambda^{\alpha'}_{\beta}) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\dfrac{\sin\theta}{r} & \dfrac{\cos\theta}{r} \end{pmatrix} = \begin{pmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ -\dfrac{y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{pmatrix}$$

8) **Calculate all elements of the transformation matrices $\Lambda^{\alpha'}_{\beta}$ and $\Lambda^{\mu}_{\nu'}$ for the transformation from Cartesian (x, y) -unprimed - to polar (r, θ) -primed**

$$\Lambda^{\alpha'}_{\beta} = \begin{pmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial\theta/\partial x & \partial\theta/\partial y \end{pmatrix} = \begin{pmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\dfrac{\sin\theta}{r} & \dfrac{\cos\theta}{r} \end{pmatrix}$$

$$\Lambda^{\mu}_{\nu'} = \begin{pmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

8a) **Let $f = x^2 + y^2 + 2xy$. Compute f as a function of r and θ. Find $\tilde{d}f$ in Cartesian and polar coordinates.**

$$\tilde{d}f = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = (2x+2y)dx + (2x+2y)dy$$

So $\tilde{d}f_x = 2x + 2y$ and $\tilde{d}f_y = 2x + 2y$.
We can find directly the polars:

$$(\tilde{d}f)_r = \frac{\partial f}{\partial r} = 2r + 2\cos\theta\sin\theta \quad , \quad (\tilde{d}f)_\theta = \frac{\partial f}{\partial \theta} = 2r^2\cos 2\theta$$

Or with transformations:

$$(\tilde{d}f)_r = \Lambda^\mu_r(\tilde{d}f)_\mu = \Lambda^x_r\tilde{d}f_x + \Lambda^y_r\tilde{d}f_y = \cos\theta(2r\cos\theta + 2r\sin\theta) + \sin\theta(2r\cos\theta + 2r\sin\theta)$$
$$(\tilde{d}f)_\theta = \Lambda^\mu_\theta(\tilde{d}f)_\mu = \Lambda^x_\theta\tilde{d}f_x + \Lambda^y_\theta\tilde{d}f_y = \cdots = 2r^2\cos 2\theta$$

8a) Let $\vec{V} \to (x^2 + 3y, y^2 + 3x)$, compute f as a function of r, θ and find the components of $\vec{V}$ and $\vec{W}$ on the polar basis
We have that:

$$V^r = \Lambda^r_x V^x + \Lambda^r_y V^y = \frac{\partial r}{\partial x}V^x + \frac{\partial r}{\partial y}V^y = \cdots = r^2(\cos^3\theta + \sin^3\theta) + 6r\sin\theta\cos\theta$$
$$V^\theta = \Lambda^\theta_x V^x + \Lambda^\theta_y V^y = \frac{\partial \theta}{\partial x}V^x + \frac{\partial \theta}{\partial y}V^y = \cdots = r\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3(\cos^2\theta - \sin^2\theta)$$

11) **For the vector field $\vec{V}$ whose cartesian components are (x^2+3y, y^2+3x), compute $V^\alpha_{,\beta}$**

$$V^\alpha_{,\beta} = \rightarrow_{Car} \begin{pmatrix} \dfrac{\partial V^x}{\partial x} & \dfrac{\partial V^x}{\partial y} \\ \dfrac{\partial V^y}{\partial x} & \dfrac{\partial V^y}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 3 \\ 3 & 2y \end{pmatrix} = \begin{pmatrix} 2r\cos\theta & 3 \\ 3 & 2r\sin\theta \end{pmatrix}$$

The last components are still Cartesian, but written using polar.

b) The transformation $\Lambda^{\mu'}_\alpha \Lambda^\beta_{\nu'} V^\alpha_{,\beta}$ to polars

We can calculate them directly:

–

$$\begin{aligned} V^r_{;r} &= \Lambda^{1'}_\alpha \Lambda^\beta_{1'} V^\alpha_{,\beta} \\ &= \Lambda^{1'}_1 \Lambda^1_{1'} V^1_{,1} + \Lambda^{1'}_2 \Lambda^1_{1'} V^2_{,1} + \Lambda^{1'}_1 \Lambda^2_{1'} V^1_{,2} + \Lambda^{1'}_2 \Lambda^2_{1'} V^2_{,2} \\ &= 2r(\cos^3\theta + \sin^3\theta) + 6\sin\theta\cos\theta \end{aligned}$$

- $V^r_{;\theta} = 2r^2 \sin\theta\cos\theta(\sin\theta - \cos\theta) + 3r(\cos^2\theta - \sin^2\theta)$
- $V^\theta_{;r} = 2\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3(\cos^2\theta - \sin^2\theta)/r$
- $V^\theta_{;\theta} = 2r\sin\theta\cos\theta(\sin\theta + \cos\theta) - 6\sin\theta\cos\theta$

The components $V^{\mu'}_{;\nu'}$ in polar coordinates using directly Christoffel:
First we write the vector in terms of polar: $(x^2+3y)\vec{e}_x + (y^2+3x)\vec{e}_y$
$= (r^2\cos^2\theta + 3\sin\theta)(\cos\theta\vec{e}_r - \frac{\sin\theta}{r}\vec{e}_\theta) + (r^2\sin^2\theta + 3r\cos\theta)(\sin\theta\vec{e}_r + \frac{\cos\theta}{r}\vec{e}_\theta)$
$= (r^2\cos^3\theta + r^2\sin^3\theta + 6r\sin\theta\cos\theta)\vec{e}_r + (r\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3(\cos^2\theta - \sin^2\theta))\vec{e}_\theta$
Now we use the formulas for derivatives:

- $V^r_{;r} = \partial_r V^r + V^\mu\Gamma^r_{\mu r} = \partial_r V^r + V^r\Gamma^r_{rr} + V^\theta\Gamma^r_{\theta r}$
  $= 2r(\cos^3\theta + \sin^3\theta) + 6\sin\theta\cos\theta + 0 + 0$
- $V^r_{;\theta} = \partial_\theta V^r + V^\mu\Gamma^r_{\mu\theta} = \partial_\theta V^r + V^r\Gamma^r_{r\theta} + V^\theta\Gamma^r_{\theta\theta}$
  $= -3r^2\cos^2\theta\sin\theta + r^2\sin^2\cos\theta + 6r\cos^2(\theta) - 6r\sin^2(\theta) + 0 - r(r\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3(\cos^2\theta - \sin^2\theta))$
  $= \cdots = 2r^2\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3r(\cos^2\theta - \sin^2\theta)$

And similarly for the other derivatives.
We get the same results as earlier.

d) The divergence $V^\alpha_{,\alpha}$ using results in a)

$$V^\alpha_{,\alpha} = 2x + 2y = 2r(\cos\theta + \sin\theta)$$

e) calculate the divergence $V^{\mu'}_{;\mu'}$ using the polar results

$$\begin{aligned} V^{\mu'}_{;\mu'} &= V^r_{;r} + V^\theta_{;\theta} \\ &= 2r(\cos^3\theta + \sin^3\theta) + 6\cos\theta\sin\theta + 2r\sin\theta\cos\theta(\sin\theta + \cos\theta) - 6\sin\theta\cos\theta \\ &= 2r(\cos\theta + \sin\theta) \end{aligned}$$

We get the result as expected.

12) **For the one-form field $\tilde{p}$ whose cartesian components are (x^2+3y, y^2+3x), compute $p_{\alpha,\beta}$ in Cartesian**

We do it directly:

$$p_{\alpha,\beta} =\rightarrow_{car} \begin{pmatrix} \frac{\partial p^x}{\partial x} & \frac{\partial p^x}{\partial y} \\ \frac{\partial p^y}{\partial x} & \frac{\partial p^y}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 3 \\ 3 & 2y \end{pmatrix} = \begin{pmatrix} 2r\cos\theta & 3 \\ 3 & 2r\sin\theta \end{pmatrix}$$

b) The transformation $p_{\mu';\nu'} = \Lambda^{\alpha}_{\mu'}\Lambda^{\beta}_{\nu'}p_{\alpha,\beta}$ to polars:
We calculate them directly:

- $p_{r;r} = \Lambda^{\alpha}_{r}\Lambda^{\beta}_{r}p_{\alpha,\beta} = \Lambda^{x}_{r}\Lambda^{x}_{r}p_{x,x} + \Lambda^{y}_{r}\Lambda^{x}_{r}p_{y,x} + \Lambda^{x}_{r}\Lambda^{y}_{r}p_{x,y} + \Lambda^{y}_{r}\Lambda^{y}_{r}p_{y,y}$
  $= \cdots = 2r(\cos^3\theta + \sin^3\theta) + 6\cos\theta\sin\theta$
- $p_{r;\theta} = 2r^2\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3r(\cos^2\theta - \sin^2\theta)$
- $p_{\theta;r} = 2r^2\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3r(\cos^2\theta - \sin^2\theta)$
- $p_{\theta;\theta} = 2r^3\sin\theta\cos\theta(\sin\theta + \cos\theta) - 6r^2\sin\theta\cos\theta$

Calculate $p_{\mu';\nu'}$ directly in polars using the Christoffel symbols

First we write it in polar. We use the transformation law for the covector bases $\tilde{\omega}^{\alpha} = \Lambda^{\alpha}_{\beta'}\tilde{\omega}^{\beta'}$
Therefore: $\tilde{d}x = \Lambda^{x}_{r}\tilde{d}r + \Lambda^{x}_{\theta}\tilde{d}\theta = \frac{\partial x}{\partial r}\tilde{d}r + \frac{\partial x}{\partial\theta}\tilde{d}\theta = \cos\theta\tilde{d}r - r\sin\theta\tilde{d}\theta$
$\tilde{d}y = \Lambda^{y}_{r}\tilde{d}r + \Lambda^{y}_{\theta}\tilde{d}\theta = \frac{\partial y}{\partial r}\tilde{d}r + \frac{\partial y}{\partial\theta}\tilde{d}\theta = \sin\theta\tilde{d}r + r\cos\theta\tilde{d}\theta$

$$
\begin{aligned}
p &= (x^2 + 3y)\tilde{d}x + (y^2 + 3x)\tilde{d}y \\
&= (r^2\cos^2\theta + 3r\sin\theta)(\cos\theta\tilde{d}r - r\sin\theta\tilde{d}\theta) + (r^2\sin^2\theta + 3r\cos\theta)(\sin\theta\tilde{d}r + r\cos\theta\tilde{d}\theta) \\
&= [r^2(\cos^3\theta + \sin^3\theta) + 6r\cos\theta\sin\theta]\tilde{d}r + [3r^2(\cos^2\theta - \sin^2\theta) + r^3\cos\theta\sin^2\theta - r^3\cos^2\theta\sin\theta]\tilde{d}\theta
\end{aligned}
$$

Now we can calculate the derivatives using the Christoffel symbols.
For example:

- $p_{r;r} = \partial_r p_r - p_\mu\Gamma^{\mu}_{rr} = [2r(\cos^3\theta + \sin^3\theta) + 6\cos\theta\sin\theta] - p_r\Gamma^{r}_{rr} - p_\theta\Gamma^{\theta}_{rr}$
  $= [2r(\cos^3\theta + \sin^3\theta) + 6\cos\theta\sin\theta] - 0 - 0$
- $p_{r;\theta} = \partial_\theta p_r - p_\mu\Gamma^{\mu}_{r\theta} = \cdots = 2r^2\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3r(\cos^2\theta - \sin^2\theta)$
- $p_{\theta;r} = \partial_r p_\theta - p_\mu\Gamma^{\mu}_{r\theta} = \cdots = 2r^2\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3r(\cos^2\theta - \sin^2\theta)$
- $p_{\theta;\theta} = \partial_\theta p_\theta - p_\mu\Gamma^{\mu}_{\theta\theta} = \cdots = 2r^3\sin\theta\cos\theta(\sin\theta + \cos\theta) - 6r^2\sin\theta\cos\theta$

13) **Show that $g_{\mu'\alpha'}V^{\alpha'}_{;\nu'} = p_{\mu';\nu'}$**
We can show it from the things we calculated.

.

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Exercises 6

3) **For any symmetric matrix A, there exists a matrix H such that $H^T AH$ is diagonal, whose entries are eigenvalues of A**
a) There exists a matrix Λ such that if A is symmetric, then $\Lambda^T A\Lambda$ is is a diagonal matrix with $-1, 0, 1$
.

5) **Prove that $\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}$ in any coordinate system in a curved Riemannian space**

6.5 (a) Prove that

$$\Gamma^{\mu}{}_{\alpha\beta} = \Gamma^{\mu}{}_{\beta\alpha} \tag{6.16}$$

in any coordinate system in a curved Riemannian space.

Hint: In principle a solution would be to use the expression for the Christoffel symbol in terms of the metric:

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}). \qquad \text{Schutz Eq. (6.32)} \tag{6.17}$$

From the symmetry of the metric, $g_{\mu\nu} = g_{\nu\mu}$, and the invariance of $(g_{\beta\mu,\nu} + g_{\beta\nu,\mu})$ under an exchange $\mu \leftrightarrow \nu$, it immediately follows that $\Gamma^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\nu\mu}$. But that misses the spirit of the exercise! Recall in §6.3 Schutz stated "It is left to Exer. 6.5 ... to demonstrate, by repeating the flat-space argument now in the locally inertial frame, that $\Gamma^{\mu}{}_{\beta\alpha}$ is indeed symmetric in any coordinate system, so that Eq. (6.32) is correct in any coordinates." This exercise is so important one really must do it. By the local flatness theorem on a general Riemann manifold, see Schutz §6.2, there is a local inertial (Lorentz) reference frame wherein the local physics is indistinguishable from that of SR. In a Lorentz frame spacetime is locally flat and one can construct a coordinate system with basis vectors that do not change with position, so the Christoffel symbols are zero. This is all one needs to reproduce the argument of §5.4 leading to Schutz Eq. (5.74), $\Gamma^{\mu}{}_{\alpha\beta} = \Gamma^{\mu}{}_{\beta\alpha}$.

19) **Prove that $R^\alpha_{\beta\mu\nu} = 0$ for polar coordinates in the Euclidean plane. Use the Christoffel symbols.**

Firs we see how many symbols we really need. We know that $R_{r\theta cd} = -R_{\theta rcd}$ and $R_{\alpha\alpha bc} = 0$, so there is just one degree of freedom in the first two indices. Similarly for the last two.
Furthermore, these two are related, $R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$, so there is just one component we need to consider, and many others will be the same.

We can now calculate the Christoffel symbols using the polar metric and then calculate R:

Starting with the definition of the Riemann tensor in terms of the Christoffel symbols, and using eqn. (5.25) for the Christoffel symbols of polar coordinates, we find (underlined terms are zero):

$$\begin{aligned}
R_{r\theta r\theta} &= \Gamma^r{}_{\theta\theta,r} - \underline{\Gamma^r{}_{\theta r,\theta}} + \Gamma^\sigma{}_{\theta\theta}\Gamma^r{}_{\sigma r} - \Gamma^\sigma{}_{\theta r}\Gamma^r{}_{\sigma\theta} && \text{used eqn. (6.52)}\\
&= \Gamma^r{}_{\theta\theta,r} + \Gamma^r{}_{\theta\theta}\underline{\Gamma^r{}_{rr}} - \Gamma^\theta{}_{\theta r}\Gamma^r{}_{\theta\theta} && \text{used } \Gamma^r{}_{\theta r} = \Gamma^\theta{}_{\theta\theta} = 0\\
&= \Gamma^r{}_{\theta\theta,r} - \Gamma^\theta{}_{\theta r}\Gamma^r{}_{\theta\theta} && \text{used } \Gamma^\mu{}_{rr} = 0\\
&= \frac{\partial(-r)}{\partial r} - \frac{1}{r}(-r) = -1 + 1 = 0. && \text{used eqn. (5.25)}
\end{aligned} \tag{6.56}$$

And this is of course what we expect since (despite the polar coordinates) we are in Euclidean space, which is flat. A necessary and sufficient condition for space to be flat is that the Riemann tensor vanishes, cf. Schutz Eq. (6.71).

18) **We can show that $R_{\alpha\beta\mu\nu}$ in 4 dimensions has 21 independent components instead of the expected 4x4x4x4 = 256 because of symmetry**

25) **Prove that the Ricci tensor is the only independent contraction of R^a_{bcd}; all others are multiples of it**

Solution: We simply step through the possibilities and determine their values based upon the symmetry relations

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}. \qquad \text{Schutz Eq. (6.69)} \qquad (6.66)$$

An important principle here is that we can only use the Riemann tensor symmetry relations when the indices are all in the same position (either all lower or all upper); if you're not sure why, see Exercise 3.24(b)!

Exercises

The contraction of the first and second indices gives

$$\begin{aligned} R^{\alpha}{}_{\alpha\mu\nu} &= g^{\alpha\beta} R_{\alpha\beta\mu\nu} = -g^{\alpha\beta} R_{\beta\alpha\mu\nu} && \text{used eqn. (6.66)} \\ &= -R^{\alpha}{}_{\alpha\mu\nu} = 0, \quad \forall \mu, \nu, && (6.67) \end{aligned}$$

since zero is the only number equal to its own negative. Furthermore, this also implies $R_{\alpha}{}^{\alpha}{}_{\mu\nu} = 0$, see SP3.2; or you can see this quickly via $R^{\alpha}{}_{\alpha\mu\nu} = g^{\alpha\beta} R_{\alpha\beta\mu\nu} = R_{\alpha}{}^{\alpha}{}_{\mu\nu} = 0$. By similar reasoning contracting the last two indices gives

$$R_{\alpha\beta}{}^{\mu}{}_{\mu} = 0, \quad \forall \alpha, \beta.$$

It remains to consider $R^{\alpha}{}_{\mu\nu\alpha}, R_{\alpha}{}^{\mu}{}_{\mu\beta}, R_{\alpha}{}^{\mu}{}_{\beta\mu}$. These candidates were identified by stepping through the possibilities systematically: first and second, first and third, first and fourth, (that is all for those involving the first index), second and third (first and second already considered), second and fourth, third and fourth. That is all.

Two of the remaining candidates give -1 times the Ricci tensor. Contracting the first and last indices we have

$$\begin{aligned} -R^{\mu}{}_{\alpha\beta\mu} &= -g^{\sigma\mu} R_{\sigma\alpha\beta\mu} = g^{\sigma\mu} R_{\sigma\alpha\mu\beta} && \text{by symmetry in eqn. (6.66)} \\ &= R^{\mu}{}_{\alpha\mu\beta} = R_{\alpha\beta}. && \text{definition of Ricci tensor} \quad (6.68) \end{aligned}$$

And contracting the second and third indices gives

$$\begin{aligned} -R_{\alpha}{}^{\mu}{}_{\mu\beta} &= -g^{\sigma\mu} R_{\alpha\sigma\mu\beta} = g^{\sigma\mu} R_{\sigma\alpha\mu\beta}, && \text{by symmetry in eqn. (6.66)} \\ &= R^{\mu}{}_{\alpha\mu\beta} = R_{\alpha\beta}. && \text{definition of Ricci tensor} \quad (6.69) \end{aligned}$$

Finally contracting the second and fourth indices gives the same result as the standard first and third:

$$R_{\alpha}{}^{\mu}{}_{\beta\mu} = -R_{\alpha}{}^{\mu}{}_{\mu\beta} \qquad \text{by symmetry in eqn. (6.66)}$$

29) **In polar coordinates, calculate the Riemann curvature tensor of the sphere of unit radius, whose metric we know (note that in 2D, the only independent components is $R_{\theta\phi\theta\phi}$**

Solution: This is a great exercise. Working through this covers several key ideas we need for GR in 4D spacetime but in the much less computationally demanding and easily visualized setting of 2D.

To calculate the Riemann tensor we need the Christoffel symbols. Now we could calculate these using their definition involving the partial derivatives of the basis vectors eqn. (5.48), or from the metric using eqn. (6.17) above. In this case it's much easier to use eqn. (6.17) since, as pointed out in the question, we already have the metric for the surface of a sphere in spherical coordinates. Our 2D manifold is the surface of the unit sphere but we keep r as a variable since it's no extra effort and it gains us a more general result. From Exercise 6.28, $(g_{\alpha\beta}) = \mathrm{diag}(r^2, r^2\sin^2\theta)$ in the coordinates $x^\alpha = (\theta, \phi)$. As a word of warning, you might get the false impression that in general we can infer the metric on a lower-dimensional submanifold by simply ignoring the unused dimensions; that works here, ignoring the r dimension, but in general one must be cautious; see SP7.7. We also need the inverse metric; fortunately this is easy for a diagonal metric: $(g^{\alpha\beta}) = \mathrm{diag}(r^{-2}, r^{-2}\sin^{-2}\theta)$, (see Exercise 6.3 if that's not obvious.)

It is easiest to use eqn. (6.17) above to calculate the Christoffel symbols for this metric. Only three are non-zero. The first is

$$\begin{aligned}\Gamma^\theta{}_{\phi\phi} &= \frac{1}{2}g^{\theta\sigma}\left(2g_{\sigma\phi,\phi} - g_{\phi\phi,\sigma}\right) = \frac{1}{2}g^{\theta\theta}\left(\cancelto{0}{2g_{\theta\phi,\phi}} - g_{\phi\phi,\theta}\right) \quad \text{used diagonal metric}\\ &= -\frac{1}{2}r^{-2}\frac{\partial r^2\sin^2\theta}{\partial\theta} = -\sin\theta\,\cos\theta. \end{aligned} \tag{6.84}$$

Consider next

$$\begin{aligned}\Gamma^\phi{}_{\phi\theta} &= \frac{1}{2}g^{\phi\sigma}\left(g_{\sigma\phi,\theta} + g_{\sigma\theta,\phi} - g_{\phi\theta,\sigma}\right) = \frac{1}{2}g^{\phi\phi}\left(g_{\phi\phi,\theta}\right) \quad \text{used diagonal metric}\\ &= \frac{1}{2}r^{-2}\sin^{-2}\theta\frac{\partial r^2\sin^2\theta}{\partial\theta} = \cot\theta = \Gamma^\phi{}_{\theta\phi}. \quad \text{used eqn. (6.16)}\end{aligned} \tag{6.85}$$

Substitute these into the general expression for the Riemann curvature tensor, eqn. (6.52), to find the only independent, non-zero component

$$\begin{aligned}R^\theta{}_{\phi\theta\phi} &= \Gamma^\theta{}_{\phi\phi,\theta} - \Gamma^\theta{}_{\phi\theta,\phi} + \Gamma^\theta{}_{\sigma\theta}\Gamma^\sigma{}_{\phi\phi} - \Gamma^\theta{}_{\sigma\phi}\Gamma^\sigma{}_{\phi\theta}\\ &= \frac{\partial(-\sin\theta\cos\theta)}{\partial\theta} + 0 + 0 - (-\sin\theta\cos\theta)\frac{\cos\theta}{\sin\theta} = \sin^2\theta\\ R_{\theta\phi\theta\phi} &= g_{\alpha\theta}R^\alpha{}_{\phi\theta\phi} = g_{\theta\theta}R^\theta{}_{\phi\theta\phi} = r^2\sin^2\theta.\end{aligned} \tag{6.86}$$

From this and the symmetry relations, eqn. (6.66), we can find the other components:

$$\begin{aligned}&R_{\phi\theta\theta\phi} = -r^2\sin^2\theta \qquad R_{\theta\phi\phi\theta} = -r^2\sin^2\theta \qquad R_{\phi\theta\phi\theta} = r^2\sin^2\theta\\ &R_{\alpha\alpha\mu\nu} = 0 \qquad R_{\alpha\beta\mu\mu} = 0,\end{aligned} \tag{6.87}$$

which agrees with that found with Maple™, see accompanying worksheet.

6.31 Show that covariant differentiation obeys the usual product rule, e.g.

$$(V^{\alpha\beta} W_{\beta\gamma})_{;\mu} = V^{\alpha\beta}{}_{;\mu} W_{\beta\gamma} + V^{\alpha\beta} W_{\beta\gamma;\mu}.$$

Hint: Use a locally inertial frame.

Solution: In a locally inertial frame, the Christoffel symbols vanish and covariant derivatives equal partial derivatives, so

$$\begin{aligned}
(V^{\alpha\beta} W_{\beta\gamma})_{;\mu} &\underset{\text{LIF}}{=} (V^{\alpha\beta} W_{\beta\gamma})_{,\mu} && \text{in a locally inertial frame}\\
&= \frac{\partial}{\partial x^\mu}\left(\sum_\beta V^{\alpha\beta} W_{\beta\gamma}\right) && \text{suspend summation convention}\\
&= \sum_\beta \frac{\partial}{\partial x^\mu}(V^{\alpha\beta} W_{\beta\gamma}) && \text{partial derivative commutes with sum}\\
&= \sum_\beta \left(W_{\beta\gamma}\frac{\partial}{\partial x^\mu}V^{\alpha\beta} + V^{\alpha\beta}\frac{\partial}{\partial x^\mu}W_{\beta\gamma}\right) && \text{regular product rule}\\
&= \sum_\beta (W_{\beta\gamma} V^{\alpha\beta}{}_{,\mu} + V^{\alpha\beta} W_{\beta\gamma,\mu}) && \text{notation change only}\\
&= W_{\beta\gamma} V^{\alpha\beta}{}_{,\mu} + V^{\alpha\beta} W_{\beta\gamma,\mu} && \text{reinvoke summation convention}\\
&= W_{\beta\gamma} V^{\alpha\beta}{}_{;\mu} + V^{\alpha\beta} W_{\beta\gamma;\mu}. && \text{in a locally inertial frame}
\end{aligned}$$

The last equality is a valid tensor equation, valid in all reference frames.

6.31)

6.33 (b) Show that the metric of the three-sphere of radius r has components in these coordinates $g_{\chi\chi} = r^2$, $g_{\theta\theta} = r^2 \sin^2\chi$, $g_{\phi\phi} = r^2 \sin^2\chi \sin^2\theta$, all other components vanishing. (Use the same method as in Exercise 6.28.)

Solution: There are only six independent terms (because of symmetry, $g_{\bar{\alpha}\bar{\beta}} = g_{\bar{\beta}\bar{\alpha}}$). We will use an overbar to indicate indices on the basis in (θ, ϕ, χ), with $x^{\bar{1}} = \theta$, $x^{\bar{2}} = \phi$, $x^{\bar{3}} = \chi$. And indices without overbar indicate the original coordinates in (x, y, z, w). Then in general

$$g_{\bar{\alpha}\bar{\beta}} = g_{\alpha\beta} \Lambda^{\alpha}{}_{\bar{\alpha}} \Lambda^{\beta}{}_{\bar{\beta}}, \tag{6.89}$$

where $\Lambda^{\alpha}{}_{\bar{\alpha}} = \partial x^{\alpha}/\partial x^{\bar{\alpha}}$. The metric tensor in the 4D Euclidean space in the Cartesian coordinates (x, y, z, w) is

$$g_{\alpha\beta} = \begin{cases} +1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta. \end{cases} \tag{6.90}$$

The calculus is tedious but straightforward. For instance,

$$\begin{aligned} g_{\bar{1}\bar{1}} \equiv g_{\theta\theta} &= g_{xx}\left(\frac{\partial x}{\partial\theta}\right)^2 + g_{yy}\left(\frac{\partial y}{\partial\theta}\right)^2 + g_{zz}\left(\frac{\partial z}{\partial\theta}\right)^2 + g_{ww}\left(\frac{\partial w}{\partial\theta}\right)^2 \\ &= r^2 \sin^2\chi \cos^2\theta \sin^2\phi + r^2 \sin^2\chi \cos^2\theta \cos^2\phi + r^2 \sin^2\chi \sin^2\theta \\ &= r^2 \sin^2\chi. \end{aligned} \tag{6.91}$$

In a similar manner one can easily show the off-diagonal terms are zero.

6.33)

Exercises 7

- **Calculate all Christoffel sybols for the metric in eq 7.8 assume** $\phi = \phi(t, x, y, z)$

Where $ds^2 = -(1 + 2\phi)dt^2 + (1 - 2\phi)(dx^2 + dy^2 + dz^2)$
We can see that to first order, the inverse is $g^{00} = \dfrac{1}{-(1 + 2\phi)} = -(1 - 2\phi)$ and $g^{ij} = \delta^{ij}(1 + 2\phi)$

Solution: This exercise requires a great deal of algebra but given the importance of this metric, a complete set of Christoffel symbols will prove to be very useful later. All non-zero Christoffel symbols can be found for this metric in the accompanying Maple™ worksheet but here we reduce them to first order in ϕ. First count the number of independent Christoffel symbols $\Gamma^{\alpha}{}_{\mu\nu}$ to calculate. For each α there are only ten independent terms because $\Gamma^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\nu\mu}$ in any coordinate basis. Hereinafter we ignore redundant ones. Given the metric and inverse metric, see Exercise 7.2, we can calculate the Christoffel symbols using eqn. (6.17),

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}). \qquad \text{eqn. (6.17)}$$

The calculation simplifies tremendously because $(g^{\alpha\beta})$ is diagonal. Thus we need only consider the $\beta = \alpha$ contribution in eqn. (6.17).

First consider $\Gamma^{0}{}_{\mu\nu}$ (the first equalities below introduces an alternative notation):

$$\Gamma^{0}{}_{00} = \Gamma^{t}{}_{tt} = \frac{1}{2} g^{tt} g_{tt,t} = \frac{-\phi_{,t}}{-(1+2\phi)} \simeq (1-2\phi)\,\phi_{,t} = \phi_{,t} + O(\phi^2). \tag{7.7}$$

Similarly,

$$\Gamma^{0}{}_{01} = \Gamma^{t}{}_{tx} = \frac{1}{2} g^{tt} g_{tt,x} = \frac{-\phi_{,x}}{-(1+2\phi)} \simeq (1-2\phi)\,\phi_{,x} = \phi_{,x} + O(\phi^2). \tag{7.8}$$

Note x, y, and z play identical roles in the metric eqn. (7.5), so eqn. (7.8) implies:

$$\Gamma^{0}{}_{02} = \Gamma^{t}{}_{ty} = \phi_{,y} + O(\phi^2), \qquad \Gamma^{0}{}_{03} = \Gamma^{t}{}_{tz} = \phi_{,z} + O(\phi^2). \tag{7.9}$$

Now consider:

$$\Gamma^{0}{}_{11} = \Gamma^{t}{}_{xx} = \frac{1}{2} g^{tt}(-g_{xx,t}) = \frac{-(-\phi_{,t})}{-(1+2\phi)} \simeq -(1-2\phi)\,\phi_{,t} = -\phi_{,t} + O(\phi^2). \tag{7.10}$$

And eqn. (7.10) implies:

$$\Gamma^{0}{}_{22} = \Gamma^{t}{}_{yy} = -\phi_{,t} + O(\phi^2), \qquad \Gamma^{0}{}_{33} = \Gamma^{t}{}_{zz} = -\phi_{,t} + O(\phi^2). \tag{7.11}$$

For a general diagonal metric the Christoffel symbols vanish when all the indices are different:

$$\Gamma^0{}_{12} = \Gamma^t{}_{xy} = 0, \qquad \Gamma^0{}_{13} = \Gamma^t{}_{xz} = 0, \qquad \Gamma^0{}_{23} = \Gamma^t{}_{yz} = 0. \tag{7.12}$$

Now consider $\Gamma^1{}_{\mu\nu}$:

$$\Gamma^1{}_{00} = \Gamma^x{}_{tt} = \frac{1}{2} g^{xx}(-g_{tt,x}) = \frac{-(-\phi_{,x})}{(1-2\phi)} \simeq (1+2\phi)\,\phi_{,x} = \phi_{,x} + O(\phi^2), \tag{7.13}$$

and

$$\Gamma^1{}_{01} = \Gamma^x{}_{tx} = \frac{1}{2} g^{xx}(g_{xx,t}) = \frac{-\phi_{,t}}{(1-2\phi)} \simeq -(1+2\phi)\,\phi_{,t} = -\phi_{,t} + O(\phi^2). \tag{7.14}$$

And when the indices are all different (see above):

$$\Gamma^1{}_{02} = \Gamma^x{}_{ty} = 0, \qquad \Gamma^1{}_{03} = \Gamma^x{}_{tz} = 0. \tag{7.15}$$

Furthermore,

$$\begin{aligned}
\Gamma^1{}_{11} = \Gamma^x{}_{xx} &= \frac{1}{2} g^{xx}(g_{xx,x}) = \frac{-\phi_{,x}}{(1-2\phi)} \simeq -(1+2\phi)\,\phi_{,x} = -\phi_{,x} + O(\phi^2),\\
\Gamma^1{}_{12} = \Gamma^x{}_{xy} &= \frac{1}{2} g^{xx}(g_{xx,y}) = \frac{-\phi_{,y}}{(1-2\phi)} \simeq -(1+2\phi)\,\phi_{,y} = -\phi_{,y} + O(\phi^2),\\
\Gamma^1{}_{13} = \Gamma^x{}_{xz} &= \frac{1}{2} g^{xx}(g_{xx,z}) = \frac{-\phi_{,z}}{(1-2\phi)} \simeq -(1+2\phi)\,\phi_{,z} = -\phi_{,z} + O(\phi^2),\\
\Gamma^1{}_{22} = \Gamma^x{}_{yy} &= \frac{1}{2} g^{xx}(-g_{yy,x}) = \frac{-(-\phi_{,x})}{(1-2\phi)} \simeq (1+2\phi)\,\phi_{,x} = \phi_{,x} + O(\phi^2).
\end{aligned} \tag{7.16}$$

$$\Gamma^1{}_{23} = \Gamma^x{}_{yz} = 0. \qquad \text{indices all different} \tag{7.17}$$

$$\Gamma^1{}_{33} = \Gamma^x{}_{zz} = \frac{1}{2} g^{xx}(-g_{zz,x}) = \frac{-(-\phi_{,x})}{(1-2\phi)} \simeq (1+2\phi)\,\phi_{,x} = \phi_{,x} + O(\phi^2). \tag{7.18}$$

The rest, $\Gamma^2{}_{\mu\nu}$ and $\Gamma^3{}_{\mu\nu}$, can be inferred from the above by noting that y and z play the same role as x. They can be represented succinctly through:

$$\begin{aligned}
&\Gamma^i{}_{00} = \phi_{,i} + O(\phi^2), \qquad \Gamma^i{}_{0j} = -\phi_{,t}\delta^i{}_j + O(\phi^2),\\
&\Gamma^i{}_{jk} = \delta_{jk}\delta^{il}\phi_{,l} - \delta^i_j\phi_{,k} - \delta^i_k\phi_{,j} + O(\phi^2).
\end{aligned} \tag{7.19}$$

The Einstein Field Equations

Purpose and Justification of the field equations

In Newton's theory of gravity, there is a source ρ and a potential ϕ given by:

$$\nabla^2\phi = 4\pi G\rho$$

.

There must be a similar equation for **General Relativity**. For general relativity, the only thing we have as a possible source is T (the whole T, since if we consider only $\rho = T^{00}$, it is frame dependent).
And the effect of this source must be felt on g, so that:

$$O(g) = kT$$

Where k is a constant.

$O(g)$ is a function applied on g that must have as a result a 2-contravariant tensor, and it must involve rotation:

$$O^{\alpha\beta} = R^{\alpha\beta} + \mu g^{\alpha\beta}R + \Lambda g^{\alpha\beta}$$

.

To determine μ, Λ, we use the conservation of energy $T^{\alpha\beta}_{,\beta} = 0$, which by **Einstein Equivalence principle** becomes $T^{\alpha\beta}_{;\beta} = 0$.
This then implied $O^{\alpha\beta}{}_{;\beta} = 0$

Since $g^{\alpha\beta}_{;\mu} = 0$, we now find that:

$$(R^{\alpha\beta} + \mu g^{\alpha\beta}R)_{;\beta} = 0$$

We see that if we put $\mu = -1/2$
Then the thing in parenthesis is $G^{\alpha\beta}$, and the equation is assured by Bianchi identity $G^{\alpha\beta}_{;\beta} = 0$.

Therefore, we are led to the equation:

$$\boxed{G^{\alpha\beta} + \Lambda g^{\alpha\beta} = kT^{\alpha\beta}}$$
$$\boxed{G + \Lambda g = kT}$$

What we have done to get here is:

- Resemble the Newtonian equation
- Introduce no preferred coordinate system

- Guarantee local conservation of energy momentum for any metric tensor.

This equations is unchallenged.
.

Geometrized units

Units that have $c = G = 1$
.

Einstein's equations

We will put $\Lambda = 0$ for now (suppose it is valid by experiment)
And we will take $k = 8\pi$ and later see this is the correct value so that the equation reduces to Newton's equation. So that:

$$G^{\alpha\beta} = 8\pi T^{\alpha\beta}$$

The constant Λ is the **Consmological constant**

This equation are 10 coupled differential equations (the thing to solve for are the components of g, which form part of Γ and therefore of R)

.

Einstein's Equations for Weak gravitational Fields

Nearly Lorentz coordinate system

Since in the absence of gravity, the space is flat and $g_{\alpha\beta} = \eta_{\alpha\beta}$.
Then, in the presence of a weak field, there must be coordinates such that g is:

$$\boxed{g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}}$$

Where:

$$|h_{\alpha\beta}| << 1$$

.
The solutions when considering this coordinates such that g has this form must give the same physical results than using any other method.
If a coordinate system exists such that this is true, then many other such systems must exist.

We can **transform between nearly Lorentzian systems in two different ways**

- **Background Lorentz transformation**

  Consider a usual Lorentz transformation $\Lambda^{\bar{\alpha}}_{\beta} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

  We define a background Lorentz transformation as one that has the form:

  $$x^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_{\beta} x^{\beta}$$

  Where Λ is identical to a normal Lorentz transformation as in SR (of course we are not in SR, so this is only one type of transformation).

  It has a nice feature, that is:

  $$g_{\bar{\alpha}\bar{\beta}} = \Lambda^{\mu}_{\bar{\alpha}} \Lambda^{\nu}_{\bar{\beta}} g_{\mu\nu} = \Lambda^{\mu}_{\bar{\alpha}} \Lambda^{\nu}_{\bar{\beta}} \eta_{\mu\nu} + \Lambda^{\mu}_{\bar{\alpha}} \Lambda^{\nu}_{\bar{\beta}} h_{\mu\nu}$$

  Now, the Lorentz transformation fulfills (it was actually defined to fulfill):

  $$\Lambda^{\mu}_{\bar{\alpha}} \Lambda^{\nu}_{\bar{\beta}} \eta_{\mu\nu} = \eta_{\bar{\alpha}\bar{\beta}}$$

  So we get:

  $$\boxed{g_{\bar{\alpha}\bar{\beta}} = \eta_{\bar{\alpha}\bar{\beta}} + h_{\bar{\alpha}\bar{\beta}}}$$
  $$h_{\bar{\alpha}\bar{\beta}} := \Lambda^{\mu}_{\bar{\alpha}} \Lambda^{\nu}_{\bar{\beta}} h_{\mu\nu}$$

  So that h (defined as what η need to become g) transforms as if it were a tensor in SR. Then, we can think of slightly curved spacetime as a flat spacetime with a 'tensor' h defined on it.

  .

- **Gauge:**
  A gauge transformation is one in which we change slightly the coordinates:

  $$x^{\alpha'} = x^{\alpha} + \chi^{\alpha}(x^{\beta})$$

  Were χ^{α} is a vector such that $|\chi^{\alpha}_{,\beta}| << 1$.
  Then, we have that:

  $$\Lambda^{\alpha'}_{\beta} = \frac{\partial x^{\alpha'}}{\partial x^{\beta}} = \delta^{\alpha}_{\beta} + \chi^{\alpha}_{,\beta}$$
  $$\Lambda^{\alpha}_{\beta'} = \delta^{\alpha}_{\beta} - \chi^{\alpha}_{,\beta} + O(|\chi^{\alpha}_{,\beta}|^2)$$

  In can be verified that, to first order, this means that:

  $$g_{\alpha'\beta'} = \eta_{\alpha\beta} + h_{\alpha\beta} - \chi_{\alpha,\beta} - \chi_{\beta,\alpha}$$

Where we define: $\chi_\alpha := \eta_{\alpha\beta}\chi^\beta$

This means, that the change in coordinates caused a change in h given by:

$$\boxed{h_{\alpha\beta} \rightarrow h_{\alpha\beta} - \chi_{\alpha,\beta} - \chi_{\beta,\alpha}}$$

If $|\chi^\alpha_{,\beta}|$ are small, then the new $h_{\alpha\beta}$ is still small, and we are still in an acceptable coordinate system.
This change is called a **Gauge transformation**. This is because it leaves us some liberty as on how to choose χ, and we can decide to choose it in a way that simplifies equations.

Riemann Tensor

Using equation $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, we can find the Riemann tensor in terms of only h:

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(h_{\alpha\nu,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu})$$

Theorem: This components of R are independent of the Gauge. That is, for $h_{\alpha\beta}$ and $h_{\alpha\beta} - \chi_{\alpha,\beta} - \chi_{\beta,\alpha}$ for any vector χ, the Riemann tensor is the same (in first order).

Therefore, making a gauge transformation doesn't change the physics described.

Weak Field Einstein Equation

We shall adopt the viewpoint that $h_{\alpha\beta}$ is a tensor on a 'background' Minkowsky spacetime (i.e. a tensor in SR). Therefore, all our equations will be expected to be valid tensor equations when interpreted in SR.
Gauge equations will be allowed, and with the benefit that they don't change the physics of the problem.

- We define the index raised quantities:

$$h^\mu_\beta := \eta^{\mu\alpha}h_{\alpha\beta}$$
$$h^{\mu\nu} := \eta^{\nu\beta}h^\mu_\beta$$

- We define the trace:

$$h := h^a_a$$

- We define a 'tensor' called the trace reverse of $h_{\alpha\beta}$:

$$\bar{h}^{\alpha\beta} := h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}h$$

  It has this name because:

$$\bar{h} := \bar{h}^{a}_{a} = -h$$

  Moreover, we can obtain back the original:

$$h^{\alpha\beta} = \bar{h}^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}\bar{h}$$

- If we have some ' tensor' $h_{\alpha\beta}$ and make a small coordinate transform $x^{\alpha'} = x^{\alpha} + \chi^{\alpha}(x^{\beta})$, then, the inverse transforms as:

$$\bar{h}^{new}_{\mu\nu} = \bar{h}^{old}_{\mu\nu} - \chi_{\mu,\nu} - \chi_{\nu,\mu} + \eta_{\mu\nu}\chi^{\alpha}_{,\alpha}$$

  And the divergence:

$$\bar{h}^{(new)}{}^{\mu\nu}{}_{,\nu} = \bar{h}^{(old)}{}^{\mu\nu}{}_{,\nu} - \chi^{\mu,\nu}{}_{,\nu}$$

Theorem: With this definitions, we can show that:

$$G_{\alpha\beta} = -\frac{1}{2}\left[\bar{h}_{\alpha\beta,\mu}{}^{,\mu} + \eta_{\alpha\beta}\bar{h}_{\mu\nu}{}^{,\mu\nu} - \bar{h}_{\alpha\mu,\beta}{}^{,\mu} - \bar{h}_{\beta\mu,\alpha}{}^{,\mu} + O(h^2_{\alpha\beta})\right]$$

Recall that $f'^{\mu} := \eta^{\mu\nu}f_{,v}$

.

We can see that it would be nice to require that:

$$\bar{h}^{ab}{}_{,b} = 0$$

So we ask this as our **Lorentz Gauge condition**.

Suppose we have some arbitrary $\bar{h}^{old}_{\mu\nu}$ that doesn't satisfy the Gauge condition (but that is the right answer to the problem and gives the correct curvature)
Can we gauge it so that the physics don't change but h^{new} now satisfies the Gauge conditions and makes the equations easier??
We have proven that when applying a small transformation:

$$\bar{h}^{(new)}{}^{\mu\nu}{}_{,\nu} = \bar{h}^{(old)}{}^{\mu\nu}{}_{,\nu} - \chi^{\mu,\nu}{}_{,\nu}$$

We want this to be 0 in our new coordinates. Therefore, χ^{μ} is determined by the equation:

$$\Box^2\chi^{\mu} := \left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\chi^{\mu} = \chi^{\mu,\nu}{}_{,\nu} = \bar{h}^{(old)}{}^{\mu\nu}{}_{,\nu}$$

This equation is the **Inohomogeneous** wave equation.
According to this equation, we can always find a Gauge to a solution h such that it greatly simplifies Eisteins equation.
To find it, we apply a change of coordinates with the components of χ given by the solution to the inhomogeneous wave equation.

What's more, if κ^μ is a vector such that $\Box^2 \kappa^\mu = 0$
Then, we have the added gauge liberty to add this κ to χ and still get a solution that satisfies the Wave equation:

$$\nabla(\chi^\mu + \eta^\mu) = \bar{h}^{(old)\mu\nu}_{\quad\;\; ,\nu}$$

.

Knowing that this Gauge is possible to find, the Einstein tensor expression reduces to:

$$G^{\alpha\beta} = \bar{h}^{\mu\nu}$$

And therefore, the Einstein equation is:

$$\boxed{\Box\bar{h}^{\mu\nu} = 16\pi T^{\mu\nu}}$$

.

Summary:

- We look for a solution of Einstein equations to get the metric $g_{\alpha\beta}$.
- In weak fields, we propose a solution of the type $g_{ab} = \eta_{ab} + h_{ab}$
- A gauge transformation is one of the type $x^{a'} = x^a + \chi^a(x^b)$ (where $\chi^a_{,b}| << 1$)
- This transformation changes the metric to $g_{a'b'} = \eta_{ab} + h_{ab} - \chi a, b - \chi_{b,a}$, so $h_{a'b'} = h_{ab} - \chi a, b - \chi_{b,a}$ And we are still in the weak field approximation.
- We define $h := h^a_a$ and $\bar{h}^{ab} := h^{ab} - \dfrac{1}{2}\eta^{ab} h$
- With this in mind, we can find a new expression for R_{abcd} involving h.
  We can see that this expression stays the same under a Gauge transformation no matter the vector χ_a
- This leads us to a complicated Einstein equation. The best way to simplify it is by asking that $\bar{h}^{mn}_{\quad ,n} = 0$
- So we would like to find a χ_a such that h simplifies this way. It can be shown that this χ always exists. If we have a h^{old}_{ab}, the χ to convert it to a good h is given by $\Box\chi^m = \bar{h}^{(old)\ \ mn}_{\quad\ ,n}$

- So the first Gauge freedom has led us to a way to assure that we only need to find a solution to the simplified Einstein equation $G^{ab} = -\frac{1}{2}\Box^2\bar{h}^{ab}$
- We actually have an extra freedom to choose χ, we can add to it any vector η^μ fulfilling $\nabla\eta^\mu = 0$,
  The result will still be a good χ that will give rise to a good h. So this extra liberty allows us to simplify other problems.

Newtonian Gravitational Fields

In Newtonian approximation, the velocities are small so that $|T^00| >> |T^{0i}| >> |T^{ij}|$
According to Einsteins equations, this should transfer to $\bar{h}$, so that $|\bar{h}^{00}| >> |\bar{h}^{0i}| >> |\bar{h}^{ij}|4$

Then, the Newtonian equations reduce to just one:

$$\Box^2\bar{h}^{00} = -16\pi\rho$$

Also, if the fields change only because sources move at v, then $\partial/\partial t$ is of the same order as $v\partial/\partial x$ so that:

$$\Box^2 = \nabla^2 + O(v^2\nabla^2)$$

Thus, our equation is:

$$\nabla^2\bar{h}^{00} = -16\pi\rho$$

We should solve this equation for , and then use this to solve for $\bar{h}^{00}$. Then, get back $h^{\alpha\beta}$ from this. Finally, change the metric according to $g = \eta + h$.

In this case, we can compare the equation to Newton's $\nabla^2\phi = 4\pi\rho \quad (G = 1)$
Then, we identify that $\bar{h}^{00} = -4\phi$

All other components of $\bar{h}$ are negligible in comparison, so we have that:

$$h = h^\alpha_\alpha = -\bar{h}^a_a = \bar{h}^{00}$$

This implies that:

$$h^{00} = -2\phi$$
$$h^{xx} = h^{yy} = h^{zz} = -2\phi$$

(or we could have just used the inverse equations from $\bar{h}$ to h).

Therefore:

$$\begin{aligned} ds &= \eta^{\alpha\beta} + h^{\alpha\beta} \\ &= -(1 + 2\phi)dt^2 + (1 - 2\phi)(dx^2 + dy^2 + dz^2) \end{aligned}$$

We already studied this.

Far Field of stationary relativistic sources

We try to solve Einstein's equations in a place far far away from the sources.
Here gravity is small and all, but we cannot directly use the last result ds^2 for this fields, since to get there we assumed that the field was small even when inside the source, which is not true now.

Still, we can get **the same answer**
We assume T^{ab} is stationary and that $h_{\mu\nu}$ is independent of time.
Then, the Einstein equation $\Box^2 \bar{h}^{\mu\nu} = -16\pi T^{\mu\nu}$ becomes:

$$\nabla^2 \bar{h}^{\mu\nu} = 0$$

far from the source.

This has solutions:

$$\bar{h}^{\mu\nu} = A^{\mu\nu}/r + O(r^{-2})$$

Where $A^{\mu\nu}$ are constants.

In addition, we must demand the gauge condition (which makes the simple Einstein equation written above plausible in the first place).
That is:

$$0 = \bar{h}^{\mu\nu}{}_{,\nu} = \bar{h}^{\mu j}{}_{j} = -A^{\mu j} n_j / r^2 + O(r^{-3})$$

Where the sum on v reduces to the spatial index j because $\bar{h}^{ab}$ is independent of time.
Where n_j is the unit radial normal $n_j = x_j/r$

The consequence of this is that:

$$A^{\mu j} = 0$$

This means that only $\bar{h}^{00}$ survives far from the source.

These conditions assure that the gravitational field behaves like Newton. And we can identify them no make the definition:

$$(\phi)_{relativistic\ far\ field} = -\frac{1}{4}(\bar{h}^{00})_{far\ field}$$

Definition of the mass of relativistic body

Now, far from a Newtonian source, the potential is:

$$(\phi)_{relativistic\ far\ field} = -M/r + O(r^2)$$

Where M is the mass of the source.
According to what we have before, we rename A^{00} to be $4M$, so that the last eqautions says:

$$(\phi)_{relativistic\ far\ field} = -M/r$$

Then, the field far from any stationary source is that:

$$ds^2 = -[1 - 2M/r + O(r^{-2})]dt^2 + [1 + 2M/r + O(r^{-2})](dx^2 + dy^2 + dz^2)$$

Physics in Slightly curved

In slightly curved spacetime:

$$\boxed{ds^2 = -(1 + 2\phi)dt^2 + (1 - 2\phi)(dx^2 + dy^2 + dz^2)}$$

Motion of a freely falling particle

We denote the four momentum by $\vec{p}$. For all except massless particles, this is $m\vec{U}$, where $\vec{U} = d\vec{x}/d\tau$.
The particle's path is a geodesic, so:

$$\nabla_{\vec{U}}\vec{U} = 0$$

Or in terms of momentum:

$$\nabla_{\vec{p}}\vec{p} = 0$$

.

Now we want to solve the geodesic equation. The zero component of the equation would be:

$$m\frac{d}{d\tau}p^0 + \Gamma^0_{\alpha\beta}p^\alpha p^\beta = 0$$

But for non relativistic speed, $p^0 >> p^1$. So:

$$m\frac{d}{d\tau}p^0 + \Gamma^0_{00}(p^0)^2 = 0$$

We need to compute Γ^0_{00}

$$\Gamma^0_{00} = \frac{1}{2}g^{0\alpha}(g_{\alpha 0,0} + g_{\alpha 0,0} - g_{00,\alpha})$$

Now, because g_{ab} is diagonal, the elements of g^{ab} are the reciprocals.
Therefore, we get that:

$$\begin{aligned}\Gamma^0_{00} &= \frac{1}{2}g^{00}g_{00,0} = \frac{1}{2}\frac{1}{-(1+2\phi)}(-2\phi)_{,0}\\ &= \phi_{,0} + O(\phi^2)\end{aligned}$$

To lower order, we can replace $(p^0)^2$ to m^2 in the geodesic equation, obtaining:

$$\frac{d}{d\tau}p^0 = -m\frac{\partial\phi}{\partial\tau}$$

Since p^0 is the energy in this frame, this means it is conserved unless the gravitational field depends on time.

The spatial component of the geodesic equation are:

$$p^a p^i_{,a} + \Gamma^i_{ab}p^a p^b = 0$$

Or, to lowest order in velocity:

$$m\frac{dp^i}{d\tau} + \Gamma^i_{00}(p^0)^2 = 0$$

Again, we have neglected p^i compared to p^0.
And now, we also use $(p^0)^2 = m^2$. To first approximation we get then:

$$\frac{dp^i}{d\tau} = -m\Gamma^i_{00}$$

We calculate the Christoffel symbol as:

$$\Gamma^i_{00} = \frac{1}{2}g^{i\alpha}(g_{\alpha 0,0} + g_{\alpha 0,0} - g_{00,\alpha})$$

Since g is diagonal:

$$g^{i\alpha} = (1-2\phi)^{-1}\delta^{i\alpha}$$

And we get:

$$\Gamma^i_{00} = \frac{1}{2}(1-2\phi)^{-1}\delta^{ij}(2g_{j0,0} - g_{00,j})$$

Where we have changed α to j because $\delta^{i0} = 0$. Now we notice that $g_{j0} == 0$ and so we get:

$$\begin{aligned}\Gamma^i_{00} &= -\frac{1}{2}g_{00,j}\delta^{ij} + O(\phi^2) \\ &= -\frac{1}{2}(-2\phi)_{,j}\delta^{ij}\end{aligned}$$

Therefore, the equation becomes:

$$\frac{dp^i}{d\tau} = -m\phi_{,j}\delta^{ij}$$

This is the usual equation in Newtonian theory.

Since we get the correct equation, we can conclude that the value $k = 8\pi$ when obtaining the equation was correct.

Gravitational Radiation

Propagation of Gravitational waves

The **Einsteins equations** in vacuum $T^{ab} = 0$ far outside from the field are:

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\bar{h}^{\alpha\beta} = 0$$

This is a 3D wave equation. We can suppose a solution of the form:

$$\bar{h}^{ab} = A^{ab} e^{ik_a x^a}$$

The Einsteins equation can be written as:

$$\eta^{mn}\bar{h}^{ab}{}_{,mn} = 0$$

When we put the possible solution in, we get $\bar{h}^{ab}{}_{,mn} = -\eta^{mn}k_m k_n \bar{h}^{ab} = 0$
This is true only if:

$$\eta^{mn}k_m k_n = k^n k_n = 0$$

This means that k must be a **Null vector**.

The **hypersurface** of constant value of $\bar{h}^{\alpha\beta}$ is given by:

$$k_a x^a = k_0 t + \mathbf{k}\cdot\mathbf{x} = cte$$

Where $\mathbf{k} = \{k^i\}$. It is conventional to refer to k^0 as ω, so that:

$$\vec{k} \to (\omega, \mathbf{k})$$

.

Theorem: Gravitational wave move at the speed of light:

- Suppose we have a photon moving in the direction of the null vector $\vec{k}$. It travels the curve:

  $$x^m(\lambda) = k^m\lambda + l^m$$

  Where λ is a parameter and l^m is a constant (the position at $\lambda = 0$). Now, since k is null, we multiply by k_m and get:

  $$k_m x^m(\lambda) = k_m l^m = cte$$

  Comparing this with the last equation we got for the wave, we see that the wave travels as a photon.

The nullity of $\vec{k}$ implies:

$$\omega^2 = |\mathbf{k}|^2$$

Which immediately means that the wave has a phase velocity and group velocity of 1.
It is important to note that the solution thus far is just the simplest solution to the wave equation, the real solution for h would be a combination of these waves.

We need also to require the Gauge equation, which is what makes possible the simple Einstein equation to begin with. Therefore:

$$\bar{h}^{ab}{}_{,b} = 0$$

When using the possible expression for $\bar{h}$ we find this implies that:

$$A^{ab}k_b = 0$$

So A must be orthogonal to $\vec{k}$.

The solution $A^{ab}e^{ik_m x^m}$ is called a **plane wave**.
But any solution to the wave equation is a combination of these solutions.

Transverse Traceless gauge

So far we have only one constraint on A.
Recall now that we have some liberty in choosing the gauge while remaining within the Lorentz class of Gauges. We can add a vector to the Gauge that fulfills:

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\chi_\alpha = 0$$

Let us choose a solution of the type:

$$\chi_\alpha = B_\alpha e^{ik_m x^m}$$

This produces a change in h^{ab} of:

$$h_{ab}^{(new)} = h_{ab}^{(old)} - \chi_{a,b} - \chi_{b,a}$$

And a consequent change in $\bar{h}_{ab}$ given by:

$$\bar{h}_{ab}^{(new)} = \bar{h}_{ab}^{(old)} - \chi_{a,b} - \chi_{b,a} + \eta_{ab}\chi^m{}_{,m}$$

We use the definition of χ and divide by the exponential to get:

$$A_{ab}^{(new)} = A_{ab}^{(Old)} - iB_a k_b - iB_b k_a + i\eta_{ab}B^m k_m$$

With this, it can be shown that B can be shown in such a way as to impose two further restrictions on $A_{ab}^{(new)}$:

$$\boxed{A^a_a = 0}$$

$$\boxed{A_{ab}U^b = 0}$$

Where $\vec{U}$ is some fixed four velocity. i.e any constant timelike unit vector.

These two conditions are called the **transverse traceless (TT) gauge conditions**.

Now we have used all of our gauge freedom. Notice that the trace condition implies:

$$\bar{h}^{TT}_{ab} = h^{TT}_{ab}$$

Let us go to a Lorentz frame for the background Minkowsky spacetime, in which the vector $\vec{U}$ upon which we have based the TT gauge is the time basis vector $U^b = \delta^b_0$.
Then, the second condition on A implies $A_{a0} = 0$ for all a.
In this frame, let us orient our spatial coordinate axes so that the wave is traveling in the z direction, $\vec{k} \to (\omega, 0, 0, \omega)$.
Then, this implies $A_{az} = 0$ for all a.
These two restrictions mean only $A_{xx}, A_{yy}, A_{xy} = A_{yx}$ are non zero.
So, in this frame:

$$(A^{TT}_{ab}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are only two independent constants.

.

The effect of waves on free particles

Consider a situation in which a particle is initally in a wave free region of spacetime and encounters a gravitational wave.
Choose a background Lorentz frame in which the particle is initially at rest and a TT gauge referred to this frame (i.e the four velocity U^α referred to earlier is the initial four velocity of the particle).
A free particle obeys the geodesic equation:

$$\frac{d}{d\tau}U^a + \Gamma^a_{mn}U^m U^n = 0$$

Since the particle is initially at rest, the initial value of its acceleration is:

$$(dU^a\ d\tau)_0 = -\Gamma^a_{00} = -\frac{1}{2}\eta^{ab}(h_{b0,0} + h_{0b,0} - h_{00,b})$$

But h^{TT}_{b0} vanishes, so initially the acceleration vanishes.
This means the particle will still be at rest a moment later, and then, by the same argument, the acceleration will be zero a moment later and so on.
So the particle remains at rest forever, regardless of the wave.

However, this rest simply means that it remains at a constant coordinate position. All we have discovered is that by choosing the TT gauge -which means making a particular adjustment in the 'wiggles' of our coordinates- we have found a coordinate system that stays attached to individual particles. This in itself has no invariant geometrical meaning.

To get a better measure of the effect of the wave. let us consider two nearby particle. One at the origin and another at $x = \epsilon, y = z = 0$ both beginning at rest. Both then remain at these coordinate positions, and the proper distance between them is:

$$\begin{aligned}\Delta l &:= \int |ds^2|^{1/2} = \int |g_{ab}dx^a dx^b|^{1/2} \\ &= \int_0^\epsilon |g_{xx}|^{1/2}dx \simeq |g_{xx}(x=0)|^{1/2}\epsilon \\ &\simeq [1 + \frac{1}{2}h^{TT}_{xx}(x=0)]\epsilon\end{aligned}$$

Since h^{TT}_{xx} is not generally zero, the proper distance does change with time (as opposed to the coordinate distance).
This is an illustration of the difference of computing a coordinate-dependent number (position of a particle) and a coordinate independent one (proper distance).

We see that the change in the proper distance between two particles is proportional to their initial separation ϵ, and the effect is small, proportional to h^{TT}_{ij}

Spherical Solutions for Stars

Coordinates for Spherically Symmetric Spacetimes

Flat Space in spherical coordinates

We define the usual coordinates (r, θ, ϕ), the line element of Minkowski space can be written as:

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Each surface of constant r and t is a sphere (a two sphere). Distances along curves confined to such a sphere are given by putting $dt = dr = 0$:

$$dl^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) := r^2 d\Omega^2$$

Which defines the **element of solid angle** $d\Omega^2$.

Two spheres in a curved spacetime

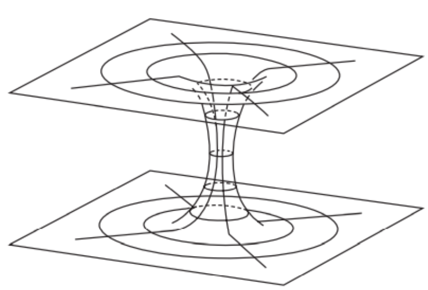
Spacetime is spherically symmetric if every point of spacetime is on a two surface which is a two sphere, i.e, whose line element is:

$$dl^2 = f(r', t)(d\theta^2 + \sin^2\theta d\phi^2)$$

Where $f(r', t)$ is an unknown function of the two other coordinates of our mainfold r' and t.
The area of each sphere is $4\pi f(r', t)$.
We define the radial coordinate r of our spherical geometry such that $f(r', t) = r^2$.
This is a coordinate transformation from (r', t) to (r, t).

Then any surface $r = cte, t = cte$ is a two sphere of area $4\pi r^2$ and circumference $2\pi r$.
This coordinate r is the **curvature coordinate** or area coordinate, because it defined the radius of curvature and area of the spheres.
There is no a priori relation between r and the distance from the center to the surface. r is defined only by properties of the spheres themselves.

For example, in the below spacetime



Two plane sheets connected by a circular throat: there is circular (axial) symmetry, but the center of any circle is not in the two-space.

There are circle but the centers themselves are not in the spacetime.
Still, there is an angle ϕ about the axis and the line element of the circles is $r^2 d\phi^2$ for some r that we now use to label the circles.

Meshing the two spheres into a three space for $t = cte$

Consider the spheres at r and $r + dr$. Each has a coordinate system (θ, ϕ), but up to know we have not asked for a relation between them.
We want the lines $\theta = cte, \phi = cte$ to be orthogonal to the two spheres, such a line would have by definition a tangent $\vec{e}_r$.
Since the vectors $\vec{e}_\theta$ and $\vec{e}_\phi$ lie in the spheres, we require that $\vec{e}_r \cdot \vec{e}_\theta = \vec{e}_r \cdot \vec{e}_\phi = 0$. This means $g_{r\theta} = g_{r\phi} = 0$.
We thus have restricted the metric in GR to the form:

$$ds^2 = g_{00}dt^2 + 2g_{0r}drdt + 2g_{0\theta}d\theta dt + 2g_{0\phi}d\phi dt + g_{rr}dr^2 + r^2 d\Omega^2$$

Spherically Symmetric spacetime

Since not only the spaces $t = cte$ are spherically symmetric, but also the whole spacetime, we must have that a line $r = cte$, $\theta = cte$, $\phi = cte$ is also orthogonal to the two spheres. Otherwise there would be a preferred direction in space. This means that $\vec{e}_t$ is orthogonal to $\vec{e}_\theta$ and $\vec{e}_\phi$, or $g_{t\theta} = g_{t\phi} = 0$. So now we have:

$$\boxed{ds^2 = g_{00}dt^2 + 2g_{0r}drdt + g_{rr}dr^2 + r^2 d\Omega^2}$$

This is the general metric of a spherically symmetric spacetime, where g_{00}, g_{0r} and g_{rr} are functions of r and t.

1.2 Static Spherically Symmetric Spacetimes

The metric

The simplest physical situation is a quiescent star or black hole -a static system
We define a static spacetime as one in which we can find a coordinate t such that:

- All metric components are independent of t
- The geometry is unchanged by time reversal $t \to -t$

If the second condition is not fulfilled, the system is stationary. This second condition is what prohibits us from calling a rotating sphere 'static'

From condition 2, the coordinate transforation $(t, r, \theta, \phi) \to (-t, r, \theta, \phi)$ has $\Lambda^{0'}_0 = -1$, $\Lambda^i_j = \delta^i_j$ and we find:

$$\begin{aligned} g_{0'0'} &= (\Lambda^0_{0'})^2 g_{00} = g_{00} \\ g_{0'r'} &= \Lambda^0_{0'}\Lambda^r_{r'} g_{0r} = -g_{0r} \\ g_{r'r'} &= (\Lambda^r_{r'})^2 g_{rr} = g_{rr} \end{aligned}$$

Since the geometry must be unchanged for a static system, then $g_{a'b'} = g_{ab}$. Therefore, $g_{0r} = 0$.

Thus, the **Static** spherically symmetric spacetime is:

$$\boxed{ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2 d\Omega^2}$$

Where we have introduces $\Phi(r)$ and $\Lambda(r)$ in place of the two unknowns $g_{00}(r)$ and $g_{rr}(r)$. This replacement is valid as long as $g_{00} < 0$ and $g_{rr} > 0$ everywhere. We shall see that this conditions do hold inside stars but not black holes.

Since stars are bounded systems, we impose that spacetime is essentially flat far from the star, that is:

$$\lim_{r\to\infty} \Phi(r) = \lim_{r\to\infty} \Lambda(r) = 0$$

So that spacetime is **Asymptotically flat**.

Physical Interpretation of metric terms

The proper radial distance from any radius r_1 to another radius r_2 is $(dt = 0, d\theta = d\phi = 0)$:

$$\begin{aligned} l_{12} &= \int |ds^2|^{1/2} = \int |g_{ab}dx^a dx^b|^{1/2} \\ &= \int_{r_1}^{r_2} e^{\Lambda} dr \end{aligned}$$

since the curve is one on which $dt = d\theta = d\phi = 0$.

More important is the significance of g_{00}. Since the metric is independent of t, the particle following a geodesic has a constant momentum component p_0, which we define to be the constant $-E$:

$$p_0 := -E$$

But a local **inertial** observer at rest (momentarily) at any radius r of the spacetime measures a different energy. Her four velocity must have $U^i = dx^i/d\tau = 0$ (since she is momentarily at rest), and the condition $\vec{U} \cdot \vec{U} = 1$ implies (remember $dr = d\phi = d\theta = 0$):

$$\begin{aligned} 1 = \vec{U} \cdot \vec{U} &= g_{ab} U_a U^b = g_{00} U_0 U^0 \\ &= -e^{2\Phi}(-U^0 U^0) \\ \Rightarrow \quad U^0 = e^{-\Phi} & \end{aligned}$$

The energy she measures is then:

$$E^* = -\vec{U} \cdot \vec{p} = e^{-\Phi} E$$

We have found that a particle whose geodesic is characterized by the constant E has energy $e^{-\Phi}E$ relative to a locally inertial observer at rest in the spacetime. Since $e^{-\Phi} = 1$ far away, we see that E is the energy a distant observer would measure if the particle gets far away. We call it the energy at infinity. Since $e^{-\Phi} > 1$ everywhere else (we will see this later), we see that the particle has larger energy relative to inertial observes that it passes elsewhere. This extra energy is just the kinetic energy it gains by falling in a gravitational field.

For Photons:
Consider a photon emitted at radius r_1 and received very far away. If its frequency in the local inertial frame is v_{em} (which is determined by the process emitting it, e.g a spectral line), then its local energy is hv_{em}. And its conserve constant E is $hv_{em}e^{\Phi(r_1)}$. When it reaches the distant observer it is measured to have energy E and hence frequency $E/h = v_{rec} = v_{em}e^{\Phi(r_1)}$. The **redshift** of the photon is defined by:

$$z = \frac{\lambda_{rec} - \lambda_{em}}{\lambda_{em}} = \frac{v_{em}}{v_{rec}} - 1$$

is therefore:

$$z = e^{-\Phi(r_1)} - 1$$

.

The Einstein tensor

Given the metric we found:

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2 d\Omega^2$$

We can use it to calculate the Christoffel symbols, and then calculate the curvature tensor and then the Einstein tensor:

$$\begin{aligned} G_{00} &= \frac{1}{r^2} e^{2\Phi} \frac{d}{dr}[r(1 - e^{-2\Lambda})] \\ G_{rr} &= -\frac{1}{r^2} e^{2\Lambda}(1 - e^{-2\Lambda}) + \frac{2}{r}\Phi' \\ G_{\theta\theta} &= r^2 e^{-2\Lambda}[\Phi'' + (\Phi')^2 + \Phi'/r - \Phi'\Lambda' - \Lambda'/r] \\ G_{\phi\phi} &= \sin^2\theta G_{\theta\theta} \end{aligned}$$

Where $\Phi' := d\Phi/dr$ and $\Lambda' = d\Lambda/dr$. All other components vanish.

Static Perfect Fluid Einstein Equations

Stress energy tensor

We are interested in static stars, in which the fluid has no motion. The only nonzero component of $\vec{U}$is therefore U^0. What is more, the normalization condition $\vec{U} \cdot \vec{U} = -1$ implies, as we saw before that:

$$U^0 = e^{-\Phi} \quad , \quad U_0 = -e^{\Phi}$$

Then, T has components given by $T = (\rho + p)\vec{U} \otimes \vec{U} + pg^{-1}$:

$$\begin{aligned} T_{00} &= \rho e^{2\Phi} \\ T_{rr} &= pe^{2\Lambda} \\ T_{\theta\theta} &= r^2 p \\ T_{\phi\phi} &= \sin^2\theta T_{\theta\theta} \end{aligned}$$

.

Equation of state

The Stress energy tensor has both p and ρ. but these may be related by an equation of state $p = p(\rho, S)$
Since we are going to consider S to be almost constant, this means:

$$p = p(\rho)$$

.

Equations of Motion

The conservation laws are:

$$T^{ab}_{;a} = 0$$

These are four equations (for every a), but because of symmetries, only one does not vanish ($a = r$). It implies:

$$\boxed{(\rho + p)\frac{d\Phi}{dr} = -\frac{dp}{dr}}$$

Einstein equations

The $(0,0)$ component of Einstein's equations can be found by using $G_{00} = \frac{1}{r^2}e^{2\Phi}\frac{d}{dr}[r(1 - e^{-2\Lambda})]$ and that $T_{00} = \rho e^{2\Phi}$.
First, we will define a new unknown function $m(r)$ defined as:

$$\boxed{m(r) := \frac{1}{2}r(1 - e^{-2\Lambda})}$$

Therefore:

$$g_{rr} = e^{2\Lambda} = \left(1 - \frac{2m(r)}{r}\right)^{-1}$$

Then, Einsteins (0,0) equation is:

$$\begin{aligned}
&G^{00} = 8\pi T^{00} \\
\Rightarrow\quad &\frac{1}{r^2}e^{2\Phi}\frac{d}{dr}[r(1 - e^{-2\Lambda})] = 8\pi\rho e^{2\Phi} \\
\Rightarrow\quad &\frac{1}{r^2}e^{2\Phi}2\frac{dm(r)}{dr} = 8\pi\rho e^{2\Phi} \\
\Rightarrow\quad &\boxed{\frac{dm(r)}{dr} = 4\pi r^2\rho}
\end{aligned}$$

This has the same form as the Newtonian equation, which calls $m(r)$ the mass inside the sphere of radius r.
Therefore, in relativity we call $m(r)$ the mass function, but it cannot be interpreted as the mass energy inside r, since total energy is not localizable in GR.

Now, the (r, r) equation is:

$$\begin{aligned}
&G^{rr} = 8\pi T^{rr} \\
\Rightarrow\quad &-\frac{1}{r^2}e^{2\Lambda}(1 - e^{-2\Lambda}) + \frac{2}{r}\Phi' = 8\pi p e^{2\Lambda} \\
\Rightarrow\quad &\cdots \Rightarrow \boxed{\frac{d\Phi}{dr} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]}}
\end{aligned}$$

.

Therefore, if we have an equation of state of the form $p = p(\rho)$, then we have the following equations:

- Equation of state: $p = p(\rho)$
- Conservation of energy $(\rho + p)\frac{d\Phi}{dr} = -\frac{dp}{dr}$

- Einstein (0,0) equation: $\dfrac{dm(r)}{dr} = 4\pi r^2 \rho$
- Einstein (r,r) equation: $\dfrac{d\Phi}{dr} = \dfrac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]}$

For the four unknowns Φ, m, ρ, p.

.

Exterior Geometry

Schwarzchild metric

In the region outside the star ($\rho = p = 0$), we get only two equations:

- $\dfrac{dm}{dr} = 0$
- $\dfrac{d\Phi}{dr} = \dfrac{m}{r(r - 2m)}$

These have the solutions given by:

$$m(r) = M = ctee^{2\Phi} = 1 - \frac{2M}{r}$$

Where the requirement that $\Phi \to 0$ as $r \to \infty$ has been applied.

Therefore, outside the sphere we get the **exterior metric**, the **Schwarzchild metric:**

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2 d\Omega^2$$
$$\Rightarrow \quad ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$
$$\boxed{ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2}$$

For large r, this becomes:

$$ds^2 \simeq -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 + \frac{2M}{r}\right)dr^2 + r^2 d\Omega^2$$

We can find coordinates (x, y, z) such that this becomes:

$$ds^2 \simeq -\left(1 - \frac{2M}{R}\right)dt^2 + \left(1 + \frac{2M}{R}\right)(dx^2 + dy^2 + dz^2)$$

Where $R := (x^2 + y^2 + z^2)^{1/2}$.
We see that this is the far field metric of a star of total mass M (from the equations of far fields we had earlier). This justifies the definition of M.

Birkhoff's theorem: The Schwarzschild solution is the only spherically symmetric, asymptotically flat solution to Einstein's vacuum field equations.

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The interior Structure of the Star

Inside the star, we have that $\rho \neq 0, p \neq 0$.
As before, the equations are:

- $p = p(\rho)$
- $(\rho + p)\dfrac{d\Phi}{dr} = -\dfrac{dp}{dr}$
- $\dfrac{dm(r)}{dr} = 4\pi r^2 \rho$
- $\dfrac{d\Phi}{dr} = \dfrac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]}$

Since $\rho \neq 0, p \neq 0$, we can divide the energy equation by $(\rho + p)$ and use it to eliminate Φ from Einstein (r, r) equatoin.
The result is the **T-O-V equation**

$$\boxed{\frac{dp}{dr} = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r(r - 2m)}}$$

Combined with the other equations, this gives three equations for the variables m, ρ, p. And Φ has a subsidiary position: It can be found once the other things have been solved.

The equations are therefore:

$$\begin{aligned} p &= p(\rho) \\ \frac{dm(r)}{dr} &= 4\pi r^2 \rho \\ \frac{dp}{dr} &= -\frac{(\rho + p)(m + 4\pi r^3 p)}{r(r - 2m)} \end{aligned}$$

General Rules for integrating the equations

The two lase eqautions require two constants of integration, one being $m(r=0)$ and the other $p(r=0)$.

We now argue that $m(r=0)=0$: A tiny sphere of radius $r=\epsilon$ has circunference $2\pi\epsilon$ and proper radius $|g_{rr}|^{1/2}\epsilon$ (from the line element). Thus, a small circle about $r=0$ has ratio of circunference to radius of $2\pi|g_{rr}|^{-1/2}$. But if spacetime is locally flat at $r=0$, as it must be at any point of the manifold, then a small circle about $r=0$ must have ratio of circunference to radius of 2π.
Therefore $g_{rr}(r=0)=1$ and so as r goes to 0, $m(r)$ must go to 0 too.

The other constant of integration $p(r=0):=p_c$, or equivalently, ρ_c, from the equation of state, simply defines the stellar model.
For a given equation of state $p=p(\rho)$, the set of all spherically symmetric static stellar models forms a one parameter sequence, the parameter being the value of density at the center $\rho_c=\rho(r=0)$.

Once $m(r), p(r), \rho(r)$ are known, the surface of the star is defined as the place where $p=0$ (notice that from TOV, the pressure decreases monotonically outwards). Since $p=0$ in the vacuum outside the star, the surface must also have $p=0$ because of continuity of p.

Therefore, we stop integrating the inside solution there and we use the Schawzchild metric outside.
Let the radius of the surface be R. Then in order to have a smooth geometry, the metric functions must be continuous at $r=R$.

Inside the star, we have:

$$g_{rr}=\left(1-\frac{2m(r)}{r}\right)^{-1}$$

And outside:

$$g_{rr}=\left(1-\frac{2M}{r}\right)^{-1}$$

Where $M:=m(R)$

Thus, the total mass of the star as determined by distant orbits is found to be the integral (from the dm/dr equation):

$$\boxed{M=\int_0^R 4\pi r^2\rho dr}$$

just as in Newtonian theory.

However, since the integral is over the volume element $4\pi r^2 dr$, which is not the element of proper volume. Proper volume in the hypersurface $t = cte$ is given by:

$$|g|^{1/2} d^3x = e^{\Lambda} r^2 \sin\theta dr d\theta d\phi$$

Which after doing the (θ, ϕ) integration, is just $4\pi r^2 e^{\Phi+\Lambda} dr$.
Thus M is not in any sense just the sum of all the proper energies of the fluid elements.

Having obtained M, this determined g_{00} outside the star, and hence g_{00} at its surface:

$$g_{00}(r = R) = -\left(1 - \frac{2M}{R}\right)$$

This serves as the integration constant for the final differential equation, the one which determines Φ inside the star: the (r, r) Einstein equation.

Notice that when solving for the interior of the start, this was the first time using that $r = 0$ is part of the spacetime. However, we didn't use this to solve for the outside of the sphere.

The structure of Newtonian stars

We shall briefly look at the Newtonian limit of these equations. In Newtonian situations we have $p << \rho$, so we also have $4\pi r^3 p << m$.
Moreover, the metric must be nearly flat, so in $g_{rr} = e^{2\Lambda} = \left(1 - \frac{2m(r)}{r}\right)^{-1}$ we require that $m << r$.
These inequalities simplify the TOV equation to:

$$\frac{dp}{dr} = -\frac{\rho m}{r^2}$$

This is the same as the equation of hydrostatic equilibrium in Newtonian stars.
Something that should not surprise us in view of our eaelier interpretation of m and of the trivial fact that the Newtonian limit of ρ is just the mass density.

The actual equation (not simplified) is the TOV equation. In that one, we can see that the relativistic corrections tend to steepen the pressure gradient relative to the Newtonian one. In other words, for a fluid to remain static it must have stronger internal forces in GR than in Newtonian gravity.

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1.3 Exact Interior Solution

The exact solutions are hard even in Newtonian theory, let alone relativistic.

The Schwarzschild constant-density interior solution

We have the interior equations:

- $\frac{dm(r)}{dr} = 4\pi r^2 \rho$
- $p = p(\rho)$
- $\frac{dp}{dr} = -\frac{(\rho + p)(m + 4\pi r^3 \rho)}{r(r - 2m)}$

To simplify the task of solving these, we use the assumption:

$$\rho = cte$$

Which replaces the question of state. This is not really physical, but it is almost true for dense Neutron stars for example.
We can integrate then $\frac{dm(r)}{dr} = 4\pi r^2 \rho$ immediately:

$$m(r) = 4\pi \rho r^3/3 \quad , \quad r \leq R$$

Where R is the star's yet undetermined radius. Outside R, the density vanishes, so $m(r)$ is constant.
By demanding continuity of g_{rr}, we find that $m(r)$ must be continuous at R, this implies:

$$m(r) = 4\pi \rho R^3/3 := M \quad , \quad r \geq M$$

WE denote this constant by M, the **Schwarzschild mass**

We can now solve the TOV equation:

$$\frac{dp}{dr} = -\frac{4}{3}\pi r \frac{(\rho + p)(\rho + 3p)}{1 - 8\pi r^2 \rho/3}$$

This can be easily integrated from an arbitrary central pressure p_c to give:

$$\frac{\rho + 3p}{\rho + p} = \frac{\rho + 3p_c}{\rho + p_c}\left(1 - 2\frac{m}{r}\right)^{1/2} \qquad 10.49$$

From this, it follows that:

$$R^2 = \frac{3}{8\pi\rho}[1 - (\rho + p_c)^2/(\rho + 3p_c)^2]$$

Or:

$$p_c = \rho[1 - (1 - 2M/R)^{1/2}]/[3(1 - 2M/R)^{1/2} - 1] \qquad 10.51$$

Replacing p_c in 10.49 gives:

$$p_c = \rho \frac{(1 - 2Mr^2/R^3)^{1/2} - (1 - 2M/R)^{1/2}}{3(1 - 2M/R)^{1/2} - (1 - 2Mr^2/R^3)^{1/2}}$$

Notice that 10.51 implies that $p_c \to \infty$ as $M/R \to 4/9$.

We complete the uniform density case by solving Φ using $(p + \rho)\frac{d\Phi}{dr} = -\frac{dp}{dr}$.
Here we know the value of Φ at R since it is implied by continuity of g_{00}:

$$g_{00}(R) = -1(1 - 2M/R)$$

Therefore, solving for Φ we find that:

$$e^{\Phi} = \frac{3}{2}(1 - 2M/R)^{1/2} - \frac{1}{2}(1 - 2Mr^2/R^3)^{1/2} \quad , \quad r \leq R$$

note that Φ and m are monotonically increasing functoins of r, while p decreases monotonically.

.

Realistic Stars and Collapse

Buchdahl's theorem

We have seen in the previous section that there are no uniform density stars with radii smaller than $9/4M$, because to support them in a static configuration requires pressures larger than infinite.
This is actually true for any stellar model, and is known as Buchdahl's theorem.

Suppose we have a star with $R = 9M/4$ and we radially push it inward. It has no choice but to collapse inwards; it cannot reach a static state again.]] But during the collapse, the metric outside it is just the Schwarzchild metric. What it leaves then, is the vacuum Schwarzschild geometry outside. This is the metric of a black hole, and we will study it in detail in the next chapter.

First we look at some causes of gravitational collapse:

1.4 Formation of stellar mass black holes

An ordinary star like the sun derives its luminosity from nuclear reactions taking place in its core.
Because a star is always radiating energy, it needs the nuclear reactions to replace that energy in order to remain static.

Astronomers have a name for such steady stars: they call them 'main sequence stars' because they all fall in a fairly narrow band when we plot their surface temperatures against luminosity.

When the star runs out of H (converted to He), the core stars to shrink as it radiates energy away. This shrinking actually heats the core!
So, as they loose energy, they become hotter, so it has a negative specific heat, it is a thermodinamically unstable system.

Eventually, the core gets so heated that it converts He in C and O, releasing more energy. In order to cope with this new energy flux, the outer layers have to expand.
Its surface area is typically so large that it cooa lot. This star is called a red giant, because its lower surface temperature makes it radiate more energy in the red part of the spectrum.

Eventually the star exhausts its He as well. Then, it may begin to cool off and contract into a white dwarf, supported forever by quantum mechanical pressure.
Or, if its core has a higher temeprature, it may turn carbon into Si and then into Fe. Eventually, it cannot convert Fe into anything, because it is a very stable nuclei.
The next part depends on the star's mass, rotation, magnetic field and chemical composition:

- Not so massive (up to 20 sun masses): the strong nuclear repulsion forces may stop the collapse and the collapse bounces back into a supernova explosion of type 2. The compact core left behind is a neutron star
- Massive: The collapse cannot be reversed and it forms a black hole

This picture can be substantially altered by rotation and magnetic fields

Quantum mechanical pressure

We shall give an elementary discussion of the forces that support white dwarfs and neutron stars.

Consider an electron in a box of Volume V. Because of Heisenberg uncertainty principle, its momentum is uncertain by an amount of the order:

$$\Delta p = hV^{-1/3}$$

If its momentum has magnitude between p and $P + dp$, it is a region of momentum space of volume $4\pi p^2 dp$. The number of 'cells' in this region of volume Δp is:

$$dN = 4\pi p^2 dp/(\Delta p)^3 = \frac{4\pi p^2 dp}{h^3} V$$

Since it is impossible to define the momentum of the electron more precisely than Δp, this is the number of possible momentum states with momentum between p and $P + dp$ in a box of volume V.
Electrons are Fermi particles, so they cannot occupy exactly the same state. Electrons have spin 1/2, which means that for each momentum state, there are two spin states, so there are a total of:

$$V \frac{8\pi p^2 dp}{h^3}$$

states. Which is the maximum number of electrons that can have momentabetween p and $p + dp$ in a box of volume V.

Now we cool off a gas of electrons as far as possible. Which means reducing each electron's momentum as far as possible. If there is a total of N electrons, then they are as cold as possible when they fill all the momentum states from p up to a upper limit p_f determined by the equation:

$$\frac{N}{V} = \int_0^{p_f} \frac{8\pi p^2 dp}{h^3} = \frac{8\pi p_f^2}{3h^3}$$

Since $N/V := n$ is the number density, we get that a cold electron gas obeys the relation:

$$n = \frac{8\pi p_f^3}{3h^3}$$

$$p_f = \left(\frac{3h^3}{8\pi}\right)^{1/3} n^{1/3}$$

The number p_f is the **Fermi momentum** and depends only on the number of molecules per unit volume, not on their masses.

Each electron has mass m and energy $E = (p^2 + m^2)^{1/2}$. Therefore, the total energy density in such a gas is:

$$\rho = \frac{E_{tot}}{V} = \int_0^{p_f} \frac{8\pi p^2}{h^3} (m^2 + p^2)^{1/2} dp$$

The pressure can be found from $\Delta Q = \Delta E + p\Delta V$ with $\Delta Q = 0$, since it is a closed system:

$$p = -\frac{d}{dV}(E_{tot}) = -V \frac{8\pi \rho_f^2}{h^3} (m^2 + p_f^2)^{1/2} \frac{dp_f}{dV} - \rho$$

For a constant number of particles N, we have:

$$V\frac{dp_f}{dV} = -n\frac{dp_f}{dn} = \frac{1}{3}\left(\frac{3h^3}{8\pi}\right)^{1/3} n^{1/3} = \frac{1}{3}p_f$$

And we get:

$$p = \left(\frac{8\pi}{3h^3}\right) p_f^2(m^2 + p_f^2)^{1/2} - \rho$$

For a very relativistic gas, where $p_f >> m$ we have:

$$\rho \simeq \frac{2\pi p_f^4}{h^3}$$
$$p \simeq \frac{1}{3}\rho$$

This is the equation of state for a 'cold' electron gas.

White dwarfs

When an ordinary star is compressed, it reaches a stage where the electrons are free from the nuclei and we have two gases, one of electrons and one of nuclei.
Since they have the same temperatures (hence same particle speeds), the less massive electrons have far less momentum per particles.
Upon compression, the Fermi momentum rises until it becomes comparable with the momentum of the electrons. Then they are effectively a cold electron gas, and supply the pressure for the star.

The nuclei have momenta well above p_f, so they are a classical gas, but they supply little pressure. On the other hand, nuclei support most of the gravity, since they are more massive. So the mass density for Newtonian gravity is:

$$\rho = \mu m_p n_e$$

$\mu =$ ratio of number of nucleons to number of electrons.
m_p proton mass
n_e number density of electrons.

The relation between pressure and density for the whole gas when the electrons are relativsitic is therefore:

$$p = k\rho^{4/3}$$

$$k = \frac{2\pi}{3h^3}\left(\frac{3h^3}{9\pi\mu m_p}\right)^{4/3}$$

The Newtonian structure equations for the star are:

$$\frac{dm}{dr} = 4\pi r^2 \rho$$
$$\frac{dp}{dr} = -\rho \frac{m}{r^2}$$

Doing some calculation, this takes us to the conclusion that a star supported by electron pressure cannot have a mass greater than 1.3 times the sun mass.

Neutron Star

If the material is compressed further than that characteristic of a white dwarf. The kinetic energy of the electrons is so large that they may combine with a proton to form a neutron and release energy in the form of a neutrino.
So compression results in the loss of electrons from the gas which is providing the pressure, the star is therefore unstable.

Then it becomes a gas of pure neutrons. There are no protons as before to provide the total energy density, it is provided by the neutrons themselves. The equation of state at high compression is then

$$p = \frac{1}{3}\rho$$

Trajectories in the Schwarzschild Spacetime

The Schwarzschild geometry is the geometry of the vacuum spacetime outside a spherical star. It is determined by the parameter M, and has the line element:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

In the coordinate system developed in the previous chapter.
This is not only the gravitational field of a star, but also the geometry of a spherical black hole.

Black Holes in Newtonian Geometry

A star is visible because light escapes from its surface. Also, we need to remember that all objects in a gravitational field move the same way (Equivalence principle).
By conservation of energy, a particle launched from the surface of a star with mass M and radius R will just barely escape if its gravitational potential energy balances its kinetic energy:

$$\frac{1}{2}v^2 = \frac{GM}{R}$$

.

In the 1700s, Laplace speculated that there could be some radius at which not even particles at the speed of light could escape.
Setting $v = c$, we find that for a given mass M, the radius of a star that makes even particles at the speed of light impossible to escape is:

$$R = \frac{2GM}{c^2}$$

Amazingly, even though this is Newtonian, it is actually correct.
For the sun, using this equation, the radius would have to be a few kilometers.
A sun with this characteristic would not emit light (since it doesn't overcome the escape velocity) so it is a **black hole**.

.

We can calculate the mean density of an object (in Newtonian terms) as:

$$\bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3c^6}{32\pi G^3 M^2}$$

We see that the necessary density decreases with mass, so for a very massive star, not so much density is needed for it to be a black hole.

Main difference between Newtonian and Relativistic Black Hole: In a Newtonian, the star shines light but gravity pulls it back and it never escapes. While on the relativistic view, light never leaves the 'surface' and the surface itself is not the edge of a body but just empty space.

Conserved Quantities

We have a lot of symmetries (time independence and spherical symmetry), so we have a lot of conserved quantities that will actually determine the trajectory of bodies completely.

- Time independence means that the energy $-p_0$ is constant on the trajectory. For particles with rest mass $m \neq 0$, we define the energy per unit mass (specific energy) $\tilde{E}$, while for photons, we consider only the energy:

$$\begin{aligned} \text{particle:} \quad \tilde{E} &:= -p_0/m \\ \text{Photon:} \quad E &= -p_0 \end{aligned}$$

- Independence of the angle ϕ about the axis implied angular momentum p_ϕ is conserved. We again define the specific angular momentum $\tilde{L}$:

$$\begin{aligned} \text{PArticle}: \quad \tilde{L} &:= p_\phi/m \\ \text{photon}: \quad L &= p_\phi \end{aligned}$$

- Independence of the angle θ
  Because of spherical symmetry, motion is confined to a single plane, we choose the equatorial plane. Then θ is constant $\pi/2$ and $d\theta/d\lambda = 0$ (λ is the parameter of the orbit).
  But p^θ is proportional to this, so it also vanishes

$$p^\theta = 0$$

In general, the components of momentum are:

- **Particle:**

$$\begin{aligned} p^0 = g^{00}p_0 = m\left(1 - \frac{2M}{r}\right)^{-1} \tilde{E} \\ p^r = mdr/d\tau \\ p^\phi = g^{\phi\phi}p_\phi = m\frac{1}{r^2}\tilde{L} \end{aligned}$$

- **Photon:**

$$\begin{aligned} p^0 = \left(1 - \frac{2M}{r}\right)^{-1} E \\ p^r = dr/d\lambda \\ p^\phi = d\phi/d\lambda = L/r^2 \end{aligned}$$

  The equation for a photon's p^r should be regarded as defining the affine parameter λ.

The equation $\vec{p}\cdot\vec{p} = -m^2$ implies:

- **Particle:**

$$
\begin{aligned}
\vec{p}\cdot\vec{p} &= g_{ab}p^a p^b \\
&= g_{aa}p^a p^a = -\left(1-\frac{2M}{r}\right)(p^0)^2 + \left(1-\frac{2M}{r}\right)^{-1}(p^r)^2 + r^2(p^\theta)^2 + r^2\sin^2\theta(p^\phi)^2 \\
&= -\left(1-\frac{2M}{r}\right)m^2\tilde{E}^2\left(1-\frac{2M}{r}\right)^{-2} + \left(1-\frac{2M}{r}\right)^{-1}m^2\left(\frac{dr}{d\tau}\right)^2 + r^2(0)^2 + r^2\sin^2\theta m^2\tilde{L}^2\frac{1}{r^4} \\
&= -m^2\tilde{E}^2\left(1-\frac{2M}{r}\right)^{-1} + m^2\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \frac{m^2\tilde{L}^2}{r^2} = -m^2
\end{aligned}
$$

- **Photon:**

$$
-E^2\left(1-\frac{2M}{r}\right)^{-1} + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2} = 0
$$

These equations can be solved to give the basic equations for orbits:

- **Particle:**

$$
\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right)
$$

- **Photon**

$$
\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \left(1-\frac{2M}{r}\right)\frac{L^2}{r^2}
$$

.

Types of Orbits

These equations can be solved to give the basic equations for orbits:

- **Particle:**

$$
\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right)
$$

- **Photon**

$$
\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \left(1-\frac{2M}{r}\right)\frac{L^2}{r^2}
$$

We define the effective potentiales as:

- **Particle**

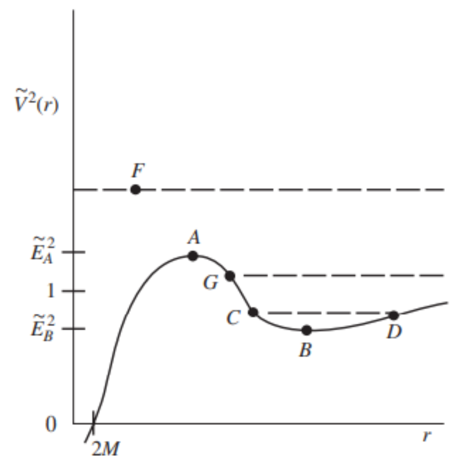
$$\tilde{V}^2(r) = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right)$$

- **Photon**

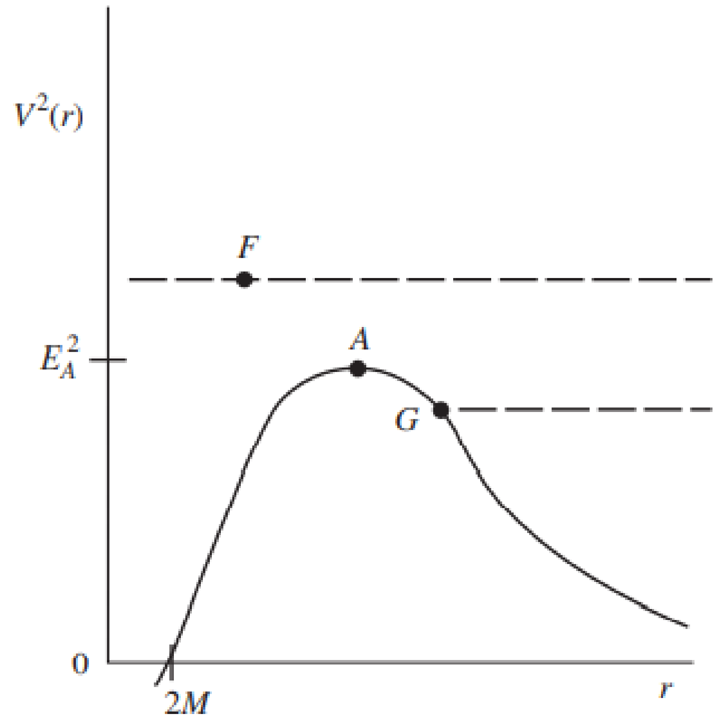
$$V^2(r) = \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2}$$

Because the equations of motion are equaled to $-m^2$, the left side must be negative.
So the energy of a trajectory should not be less than the potential.

So for a given E, the radial range is restricted to those radii for which V is smaller than E



Typical effective potential for a massive particle of fixed specific angular momentum in the Schwarzschild metric.



The same as Fig. 11.1 for a massless particle.

For instance, for a trayectory that has energy given by the point G (in either diagram). If it comes in from $r = \infty$, then it cannot reach a smaller r than where the dotted line hits the V^2 curve, at point G, since then the potential at A would be grater than the totla energy. G is called a **turning point**.
At G, since $E^2 = V^2$, we must have $(dr/d\lambda)^2 = 0$.

We now differentiate the equation $\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \tilde{V}^2(r)$ with respect to τ:

$$2\left(\frac{dr}{d\tau}\right)\left(\frac{d^2r}{d\tau^2}\right) = -\frac{d\tilde{V}^2(r)}{dr}\frac{dr}{d\tau}$$

That is:

- **Particles**

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2}\frac{d}{dr}\tilde{V}^2(r)$$

- **Photons:**

$$\frac{d^2r}{d\lambda^2} = -\frac{1}{2}\frac{d}{dr}V^2(r)$$

these are the relativistic analogues to $m\vec{a} = -\nabla\phi$

It is clear that a circular orbit ($r = cte$) is possible only at max or min of V^2. These occur at A and B for particles and A for photons.
So for particles, there is one stable circular orbit (B) and one unstable (A) for this given value of $\tilde{L}$. While for photons there is only one unstable orbit for this L.
We can be quantitative by searching for the maximum or minimum (the circular orbits):

- **Particle**

$$0 = \frac{d}{dr}(V(r)) = \frac{d}{dr}$$
$$\Rightarrow \quad r = \frac{\tilde{L}^2}{2M}\left[1 \pm \left(1 - \frac{12M^2}{\tilde{L}^2}\right)^{1/2}\right]$$

- **Photon:**

$$0 = \frac{d}{dr}\left[\left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2}\right]$$
$$\Rightarrow \quad r = 3M$$

For particles, there are two radii, but only if $\tilde{L} > 12M^2$. The two radii are identical for $\tilde{L}^2 = 12M^2$ and nonexistent if $\tilde{L}^2 < 12M^2$. Which indicates a qualitative change in the curve $V(r)$ depending on L.
Then, the minimum angular momentum needed to have a circular orbit is $\tilde{L}^2 = 12M^2$. The corresponding radius is:

$$\text{particle :} \quad r_{MIN} = 6M$$

.
For photons, the unstable circular orbit is always at $r = 3M$ regardless of L.

Of course, for some stars, the orbits $r = 3M$ or $6M$ are inside the star itself, but orbits are only valid outside the star of course.

Perihelion Shift

A particle in a stable ciruclar orbit will make one complete orbit in time P (period).
We can determine it as follows.
From the equation for radius of circular orbits found in previous subsubsection, for a stable orbit (taking the plus sign) we get:

$$\tilde{L}^2 = \frac{Mr}{1 - 3M/r}$$

And since $\tilde{E}^2 = \tilde{V}^2$ for a circular orbit, it also has energy:

$$\tilde{E}^2 = \left(1 - \frac{2M}{r}\right)^2 / \left(1 - \frac{3M}{r}\right)$$

Now, we have $\frac{d\phi}{d\tau} := U^\phi = \frac{p^\phi}{m} = g^{\phi\phi}\frac{p_\phi}{m} = g^{\phi\phi}\tilde{L} = \frac{1}{r^2}\tilde{L}$
And $\frac{dt}{d\tau} = U^0 = \frac{p^0}{m} = g^{00}\frac{p_0}{m} = g^{00}(-\tilde{E}) = \frac{\tilde{E}}{1 - 2M/r}$
We obtain the angular velocity by the reciprocal of:

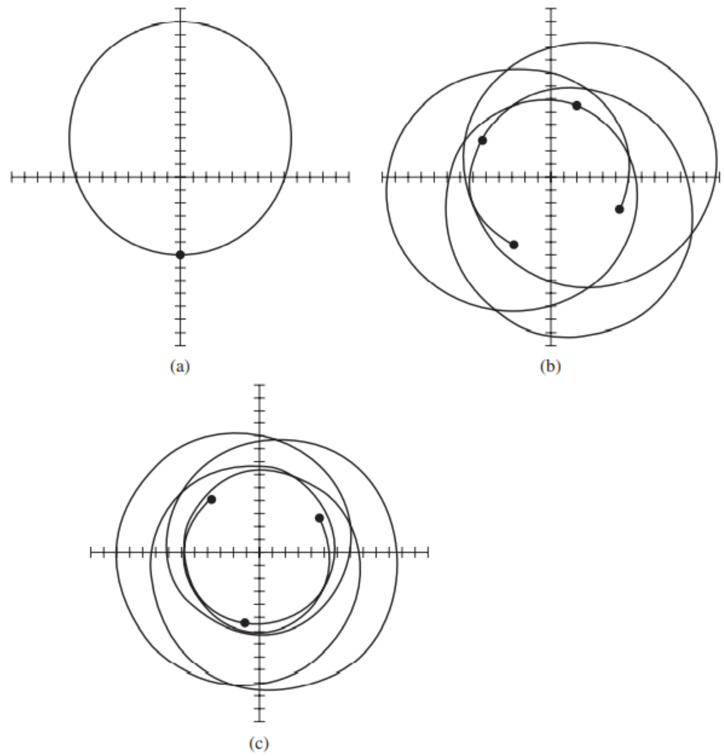
$$\frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = \left(\frac{r^3}{M}\right)^{1/2}$$

The period is then:

$$P = 2\pi\left(\frac{r^3}{M}\right)^{1/2}$$

This is the coordinate, not the proper time, of course. This is identical to the Third law of Kepler actually.

For a noncircular orbit,the distance r will oscillate about a center radius r.
In Newtonian gravity, the orbit is a perfect ellipse, and therefore it is closed.
This does not happen in GR, the orbits outside circular ones are not generally closed and they look as we see here:



e 11.4 (a) A Newtonian orbit is a closed ellipse. Grid marked in units of *M*. (b) An orbit in the Schwarzschild metric with pericentric and apcentric distances similar to those in (a). Pericenters (heavy dots) advance by about 97° per orbit. (c) A moderately more compact orbit than in (b) has a considerably larger pericenter shift, about 130°.

For particles far away the orbit is almost closed and the correction form ellipses is small. To put it into numbers, we look at the angular movement of the **perihelion** in time.

For Mercury, the angular movement of the perihelion was measured to be 5600" / century. Out of this, 5557" where accounted by the presence of other planets but the remaining 43" was unknown.
Then, Einstein proved that his theory accounted for this 43" shift every century, in great favor for his theory. Other planets don't feel it so much because far away from the sun, relativistic effects fade.

Calculating the Perihelion precession

We know $dr/d\tau$ and $d\phi/d\tau$ for the orbit, we divide it to get a implicit equation for the orbit:

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{\tilde{E}^2 - (1-2M/r)(1+\tilde{L}^2/r^2)}{\tilde{L}^2/r^4}$$

It is convenient to define $u := 1/r$ and obtain:

$$\left(\frac{du}{d\phi}\right)^2 = \frac{\tilde{E}^2}{\tilde{L}^2} - (1-2Mu)\left(\frac{1}{\tilde{L}^2} + u^2\right)$$

.

While the Newtonian orbit is found by neglecting u^3 (doing all the process to find the orbit predicted by Newton, we find):

$$\text{Newtonian:} \quad \left(\frac{du}{d\phi}\right)^2 = \frac{\tilde{E}^2}{\tilde{L}^2} - \frac{1}{\tilde{L}^2}(1-2Mu) - u^2$$

A circular orbit in Newtonian theory has $u = M/\tilde{L}^2$ according to his theory, so we can define $y = u - \dfrac{M}{\tilde{L}^2}$
So that y represents the deviation from circularity. We get then:

$$\text{Newton} \quad \left(\frac{dy}{d\phi}\right)^2 = \frac{\tilde{E}^2-1}{\tilde{L}^2} + \frac{M^2}{\tilde{L}^4} - y^2$$

It is easy to see this is satisfied by:

$$\text{Newton:} \quad y = \left[\frac{\tilde{E}^2 + M^2/\tilde{L}^2 - 1}{\tilde{L}^2}\right]^{1/2} \cos(\phi + B)$$

Where B is arbitrary. This is clearly periodic.

We now consider again the **Relativistic case**
We define $y = u - \dfrac{M}{\tilde{L}^2}$ the same way but don't disregard the u^3 terms.
However, we assume the orbit is nearly circular and disregard the y^3 terms, we get:

$$\left(\frac{dy}{d\phi}\right)^2 = \frac{\tilde{E}^2 + M^2/\tilde{L}^2 - 1}{\tilde{L}^2} + \frac{2M^4}{\tilde{L}^6} + \frac{6M^3}{\tilde{L}^2}y - \left(1 - \frac{6M^2}{\tilde{L}^2}\right)y^2$$

The result is the solution:

$$y = y_0 + A\cos(k\phi + B)$$

Where:

$$k = \left(1 - \frac{6M^2}{\tilde{L}^2}\right)^{1/2}$$
$$y_0 = 3M^3/k^2\tilde{L}^2$$
$$A = \frac{1}{k}\left[\frac{\tilde{E}^2 + M^2/\tilde{L}^2 - 1}{\tilde{L}^2} + \frac{2M^4}{\tilde{L}^6} - y_0^2\right]^{1/2}$$

We see that the orbit oscillates not about $y = 0$ but $y = y_0$, the amplitude A is also different, and the factor k is not 1. Then, the orbit returns to the same r only when $k\phi$ goes through

2π.
So, the change in ϕ from one perihelion to the next is:

$$\Delta\phi = \frac{2\pi}{k} = 2\pi\left(1 - \frac{6M^2}{\tilde{L}^2}\right)^{-1/2}$$

Which for Newtonian orbits is $\Delta\phi \simeq 2\pi\left(1 + \frac{3M^2}{\tilde{L}^2}\right)$

The **perihelion advance** from one orbit to the next is:

$$\Delta\phi = 6\pi M^2/\tilde{L}^2$$

radians per orbit.

We can approximate $\tilde{L}^2 = \frac{Mr}{1 - 3M/r} \simeq Mr$ and get $\Delta\phi \simeq 6\pi\frac{M}{r}$
For mercury's orbit, $r = 5.55 \times 10^7 km$ and $M = 1.47km$, so that:

$$(\Delta\phi)_{Mercury} = 4.99 \times 10^{-7} rad/orbit = 43''/century$$

Considering each orbit takes $0.24yr$

Gravitaion post Newton

The pericenter shift is an example of corrections that GR makes to Newton in a first order, but we could go further and approximate less (which we won't)

Gravitational Deflection of Light

Photons do not have bound orbits in Newtonian gravity, they move in straight lines.
However, in GR, they deflect from straight lines.

We begin by calculating the trajectory of a photon in the Schwarzschild metric under the assumption that M/r is everywhere small along the trajectory. The equation of the orbit can be found by dividing the equation of $dr/d\tau$ and the one of $d\phi/d\tau$ as we did for particles:

$$\frac{d\phi}{dr} = \pm\frac{1}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\right]^{-1/2}$$

Where we have defined the **impact parameter** $b := L/E$

b would be the minimum value of r in Newtonian theory, where there is no deflection. It therefore represents the 'offset' of the photon's initial trajectory from a parallel one moving purely radially. An incoming photon with $L > 0$ obeys the equation:

$$\frac{d\phi}{du} = \left(\frac{1}{b^2} - u^2 + 2Mu^3\right)^{-1/2}$$

With the same definition as before, $u = 1/r$

If we neglect the u^3 term, all effects of M disappear and the solution is:

$$r \sin(\phi - \phi_0) = b$$

a straight line, the Newtonian result.

Now suppose $Mu << 1$ but not completely negligible. Then we define:

$$y := u(1 - Mu) \quad , \quad u = y(1 + My) + O(M^2u^2)$$

So the equation becomes:

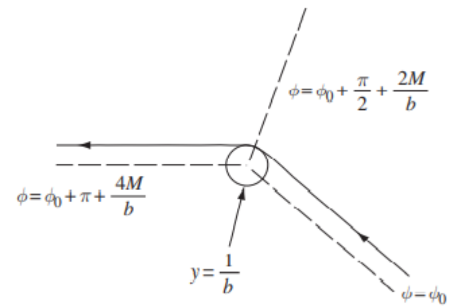
$$\frac{d\phi}{dy} = \frac{(1 + 2My)}{(b^{-2} - y^2)^{1/2}} + O(M^2u^2)$$

This can be integrated to give:

$$\phi = \phi_0 + \frac{2M}{b} + \arcsin(by) - 2M\left(\frac{1}{b^2} - y^2\right)^{1/2}$$

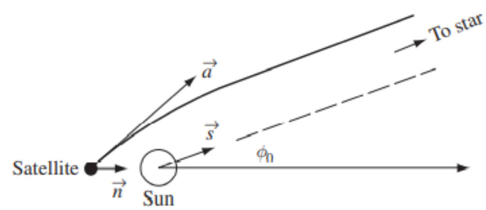
The initial trajectory has $y \to 0$, so $\phi \to \phi_0$: ϕ_0 is the incoming direction. The photon reaches its smallest r when $y = 1/b$, as we can see from setting $dr/d\lambda = 0$ and this occurs at the angle $\phi = \phi_0 + 2M/b + \pi/2$.
It has thus passed through an angle $\pi/2 + 2M/b$ as it travels to its point of closest approach. By symmetry, it passes through a further angle of the same size as it moves outwards from its point of closest approach.
It thus passes through a total angle of $\pi + 4M/b$. If it were going on a straight line, this angle would be π, so the net deflection is:

$$\Delta\phi = 4M/b$$



Deflection of a photon.

11.1 Trajectories in the Schwarzschild spacetime



An observation from Earth of a star not at the limb of the Sun does not need to correct for the full deflection of Fig. 11.5.

In that way, massive objects act as lenses.

Nature of the Surface $r = 2M$

Coordinate Singularities

From the equation of line element:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

we see that at $r = 2M$ there is some problem with the coordinates.
These are called coordinate singularities, places where the coordinates don't describe the geometry. It is a very subtle question.

Infalling particles

Let a particle fall to the surface $r = 2M$ from a finite radius R. How much proper time does that take?
The simplest case is a particle that falls radially, $d\phi = 0, \tilde{L} = 0$ and therefore, from equations we had earlier:

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - 1 + \frac{2M}{r}$$
$$\Rightarrow \quad d\tau = -\frac{dr}{(\tilde{E}^2 - 1 + 2M/r)^{1/2}}$$

The minus sign because the particle falls inward.

If $\tilde{E}^2 > 1$ (unbound particle) the integral from R to $2M$ is finite.
If $\tilde{E} = 1$ (particle falling from rest at ∞) the integral is $\Delta\tau = \frac{4M}{3}\left[\left(\frac{r}{2M}\right)^{3/2}\right]_{2M}^{R}$ which is finite.
And if $\tilde{E} < 1$ there is again no problem, since the particle cannot be t larger r than where $1 - \tilde{E}^2 = 2M/r$. So the answer is that any particle can reach the horizon in a finite amount of proper time.
In fact, we can put the lower limit at a value less than $2M$ and there won't be a problem either.

We now ask how much coordinate time elapses as the particle falls in. For this, we use:

$$U^0 = \frac{dt}{d\tau} = g^{00}U_0 = g^{00}\frac{p_0}{m} = -g^{00}\tilde{E} = \left(1 - \frac{2M}{r}\right)^{-1}\tilde{E}$$

Therefore, we have:

$$dt = \frac{\tilde{E}d\tau}{1 - 2M/r} = -\frac{\tilde{E}dr}{(1 - 2M/r)(\tilde{E}^2 - 1 + 2M/r)^{1/2}}$$

For simplicity, we again consider the case $\tilde{E} = 1$ and examine this near $r = 2M$ by defining a new variable $\epsilon := r - 2M$
We get:

$$dt = \frac{-(\epsilon + 2M)^{3/2}d\epsilon}{(2M)^{1/2}\epsilon}$$

It is clear that as $\epsilon \to 0$, the integral goes like $\log\epsilon$ and diverges. We would also find this if $\tilde{E} \neq 1$.

Therefore, the proper time is finite but the coordinate time is infinite, so the coordinates must be behaving badly.

Inside $r = 2M$

Let's see what happens after it reaches $r = 2M$. It must clearly pass to smaller r, it will just keep on going.
If we look at the geometry inside but near $r = 2M$, by introducing $\epsilon := 2M - r$, then the line element is:

$$ds^2 = \frac{\epsilon}{2M - \epsilon}dt^2 - \frac{2M - \epsilon}{\epsilon}d\epsilon^2 + (2M - \epsilon)^2 d\Omega^2$$

Since $\epsilon > 0$ inside $r = 2M$, we see that a line on which t, θ, ϕ are constants has $ds^2 < 0$, it is timelike.
Therefore ϵ (and hence r) is a timelike coordinate, while t has become spacelike!!!
Since the infallingg particle must follow a timelike world line, it must constantly change r (decrease r). So a particle inside will inevitably reach $r = 0$ and there is a true curvature singularity there.

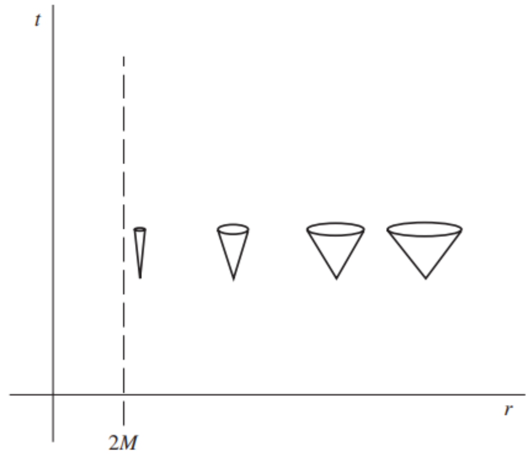
But what happens if a particle inside tries to send out a photon. The photon, must also go forward in 'time' (as seen locally by the particle) no matter the direction it was sent in, which means it will decrease r. So the photon won't get out either.
Everything inside $r = 2M$ is trapped and doomed to encounter $r = 0$, since $r = 0$ is in the future of every timelike and null world line inside $r = 2M$.
Once a particle crosses the surface $r = 2M$, it cannot be seen by an external observer, since it can not send out photons.

This surface is therefore called a **horizon**.

So $r = 2M$ is the **Schwarzschild horizon**

Coordinate Systems

To develop a picture, we will first draw a coordinate diagram in Schwarzschild coordinates, and on it we will draw the light cones, or at least the paths of the radially ingoing and outgoing null lines emanating from certain events.
These light cones can be found by solving $ds^2 = 0$ with $\theta, \phi = cte$



Light cones drawn in Schwarzschild coordinates close up near the surface $r = 2M$.

$$\frac{dt}{dr} = \pm\left(1 - \frac{2M}{r}\right)^{-1}. \tag{11.64}$$

$$\frac{dt}{dr} = \pm\left(1 - \frac{2M}{r}\right)^{-1}$$

They have a slope ± 1 far from the star (standard SR light cone) but their slope approaches $\pm\infty$ as $r \to 2M$.
Since particle world lines are confined within the local light cone, this closing forces the particle's world lines to become more vertical. If they reach $r = @M$, they reach it at $t = \infty$. This is the picture behind the result that a particle takes an infinite coordinate time to reach the horizon. No particle world line reaches $r = 2M$ in any finite t.
This suggests that $r = 2M, -\infty \leq t \leq \infty$ is not really a line, but a single point.
This suggests that the problem with the coordinates is that it somehow expanded this point into a line (just as polar coordiantes expand the origin into a line of $r = 0$)

The Light cones don't really close up, particles reach the r=2M in a finite time as we saw earlier (finite proper time). The diagram represents the coordinates, but not the true geometry.
The remedy is to find coordinates where the cones doesn't close up.

Kruskal Szekeres Coordinates

The good coordinates are known as Kruskal Szekeres coordinates, they are called u, v and are defined by:

For $r > 2M$**:**

$$u = (r/2M - 1)^{1/2} e^{r/4M} \cosh(t/4M)$$
$$v = (r/2M - 1)^{1/2} e^{r/4M} \sinh(t/4M)$$

.

For $r < 2M$**:**

$$u = (1 - r/2M)^{1/2} e^{r/4M} \sinh(t/4M)$$
$$v = (1 - r/2M)^{1/2} e^{r/4M} \cosh(t/4M)$$

.

The metric in these coordinates can be found to be:

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M}(dv^2 - du^2) + r^2 d\Omega^2$$

Where now r is not a coordinate, it is a function of u, v defined implicitly by the transformations, which we can simplify to:

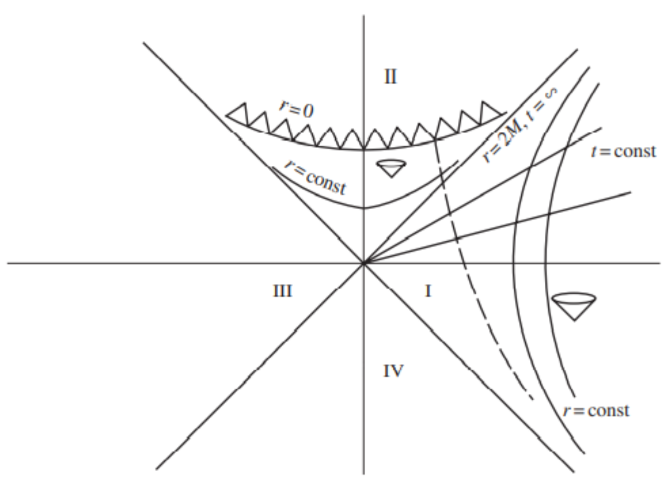
$$\left(\frac{r}{2M} - 1\right) e^{r/2M} = u^2 - v^2 \qquad 11.68$$

.

Notice that there is nothing singular about the metric at $r = 2M$.
A radial null line ($d\theta = d\phi = ds = 0$) is a line with:

$$dv = \pm du$$

This last result means that all light cones are as open as in SR.
We can draw a (u, v) diagram:



Kruskal–Szekeres coordinates keep the light cones at 45° everywhere. The singularity at $r = 0$ (toothed line) bounds the future of all events inside (above) the line $r = 2M$, $t = +\infty$. Events outside this horizon have part of their future free of singularities.

About the image:

- We see that any 45 degree line is a radial null line.
- Only u, v are plotted (ϕ, θ) are suppressed; therefore each point is really a two sphere of events.
- Lines at constant r are hyperbolae as is clear from 11.68
- For $r > 2M$, theses hyperbolae run roughly vertically, being asymptotic to the 45 degree line from the origin $u = v = 0$.
- For $r < 2M$, the hyperbolae run roughly horizontally, with the same asymptotes. This means that for $r < 2M$, a timelike line (confined within the light cone) cannot remain at constant r
- Note that although $r = 0$ is 'point' in ordinary space, it is a whole hyperbolae here
- Lines of constant t, being orthogonal to lines of constant r are straight lines radiating from the origin. In the limit as $t \to \infty$, these lines approach a 45 degree line.
  A world line crossing into this $t = \infty$ line enters a region where r is a time coordinate, and so cannot get out again.
- For a distant observer, t really does measure proper time, and an object that falls to the horizon crosses all the lines $t = cte$ up to $t = \infty$, a distant observer would conclude it takes an infinite time for the infalling object to reach the horizon. But the object reaches it in a finite time in his own clock.
- The horizon is a null line.
- The 45 degree lines divide the space time into 4 regions. Region I is the 'exterior' $r > 2M$, region II is the interior. The other regions are beyond our scope

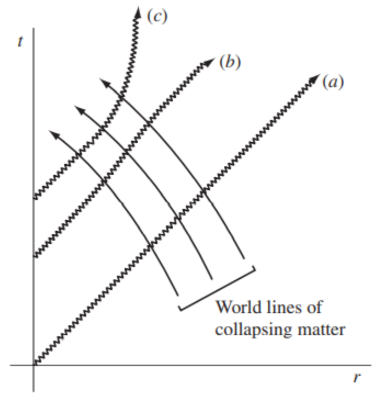
General Black Holes

Formation of black holes in general

The phenomenon of the formation of a horizon has to do with the collapse of matter to such small dimensions that the gravitational field traps everything within a certain region, called the interior of the horizon.
Black holes generally form from complicated dynamical circumstances.

An Event horizon: is the boundary in **spacetime** between events that can communicate with distant observers and events that cannot.

The test to whether events can communicate to distant observers is whether they can send light rays, that is, whether there are null rays that can go abitrarily far away.
As the boundary between null rays that can escape and null rays that cannot, the horizon itself is composed of null world lines, this null rays neither go ways to infinity nor fall inwards and stay on the horizon forever.



igure 11.12 **Schematic spacetime diagram of spherical collapse. Light ray (*a*) hardly feels anything, (*b*) is delayed, and (*c*) is marginally trapped. The horizon is defined as the ray (*c*), so it grows continuously from zero radius as the collapse proceeds.**

If gas falls in the hole:

—

gas is perfectly spherical. Let the mass of the hole before the gas falls in be called M_0. The surface $r = 2M_0$ is static and appears to be the event horizon: photons inside it fall towards the singularity at the center, and photons infinitesimally outside it gradually move further and further away from it. But then the new gas falls across this surface and increases the mass of the hole to M_1. Clearly, the new final state will be a Schwarzschild solution of mass M_1, where the larger surface $r = 2M_1$ looks like an event horizon. It consists of null rays neither falling in nor diverging outwards. Now, what is the history of these rays? What happens if we trace backwards in time along a ray that just stays at $r = 2M_1$? We would find that, before the new gas arrived, it was one of those null world lines just *outside* the surface $r = 2M_0$, one of the rays that were very gradually diverging from it. The extra mass has added more gravitational attraction (more curvature) that stopped the ray from moving away and now holds it exactly at $r = 2M_1$. Moreover, the null rays that formed the static surface $r = 2M_0$ before the gas fell in are now inside $r = 2M_1$ and are therefore falling toward the singularity, again pulled in by the gravitational attraction of the extra mass. The boundary in *spacetime* between what is trapped and what is not consists therefore of the null rays that in the end sit at $r = 2M_1$, including their continuation backwards in time. The null rays on $r = 2M_0$ were not actually part of the true horizon of spacetime even at the earlier time: they are just trapped null rays that took a long time to find out that they were trapped! Before the gas fell in, what looked like a static event horizon ($r = 2M_0$) was not an event horizon at all, even though it was temporarily a static collection of null rays. Instead, the true boundary between trapped and untrapped was even at that early time gradually expanding outwards, traced by the null world lines that eventually became the surface $r = 2M_1$.

This illustrates that the horizon is not a location in space but a boundary in spacetime. To konw it we should know the entire evolution of the black hole.

To get around this, we define the **locally trapped surface** to be a 2D surface at any particular time whose outwardly directed null rays are neither expanding nor contracting at that moment.

General Properties of black holes

Some important theorems and conjectures:

1) It is believed that every event horizon will eventually become stationary, provided that it is not constantly disturbed by outside effects.
   A stationary black hole is characterized by two numbers, the total mass M and the angular momentum J. The unique stationary vacuum black hole is the Kerr solution. If the angular momentum is 0, the Kerr solution becomes the Schwarzschild one.

2) If the BH is not in vacuum it may carry an electric charge Q, and in principle, a magnetic monopole moment F

3) If gravitational collapse is nearly spherical, then all nonespherical parts of the mass distribution are radiated away in gravitational waves and a stationary BH of the Kerr type is left behind.

4) **Area theorem of Hawking:** In any dynamical process involving BH, the total area of all the horizons cannot decrease in time.
   For example, the area actually increases when infalling gas reaches the hole.
   So, this implies that a black hole cannot get divided in two for example.
   However, Hawking radiation violates this, and this is because in QM energies are not always required to be positive.

5) Inside the Schwarzchild and Kerr horizons, there are curvature singularities where the curvature becomes infinite.

6) It is believed that there exists no singularities outside black holes (naked singularities)

The first point is very impressive, we only need to numbers to fully determine a black hole exactly.

Kerr BH

The Kerr BH is axially symmetric but not spherically summetric (rotationally symmetric about one axis only) and is caracterized by M and J only.
We define:

$$a := J/M$$

Which has the same dimensions as M.
The line element is:

$$ds^2 = -\frac{\Delta - a^2 \sin^2\theta}{\rho^2} dt^2 - 2a\frac{2Mr\sin^2\theta}{\rho^2} dt d\phi + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

Where:

$$\Delta := r^2 - 2Mr + a^2$$
$$\rho^2 := r^2 + a^2\cos^2\theta$$

The coordinates are called **Boyer Lindquist** coordinates
ϕ is the angle around the axis of symmetry
t is the time coordinate in which everything is stationary.
r, θ are similar to the spherically symmetric r, θ but are not readily associated to any geometrical definition. The following points are important:

- Surfaces $t = cte, r = cte$. Do not have the metric of a 2-sphere
- The metric for $a = 0$ is identically the Schwarschild metric
- There is an off diagonal term in the metric, in contrast to Schwarzschild:

  $$g_{t\phi} = -a\frac{2Mr\sin^2\theta}{\rho^2}$$

  Which is 1/2 the coefficient of $dtd\phi$ in the line element, because the line element contains two terms, $g_{t\phi}dtd\phi + g_{\phi t}d\phi dt = 2g_{t\phi}d\phi dt$ by the symmetry of the metric.

Dragging of inertial frames

The presence of $g_{t\phi} \neq 0$ introduces new effects on particle trajectories. Because g is independent of ϕ, a particle's trajectory still conserves p_ϕ, but now we have:

$$p^\phi = g^{\phi\alpha}p_\alpha = g^{\phi\phi}p_\phi + g^{\phi t}p_t$$

And similarly for time components:

$$p^t = g^{t\alpha}p_\alpha = g^{tt}p_t + g^{t\phi}p_\phi$$

.

Consider a zero angular momentum particle, $p_\phi = 0$. then, using the definitions (for nonzero rest mass):

$$p^t = m dt/d\tau \ \ , \ \ p^\phi = m d\phi/d\tau$$

We find the particle's trajectory has

$$\frac{d\phi}{dt} = \frac{p^\phi}{p^t} = \frac{g^{\phi t}}{g^{tt}} := \omega(r, \theta)$$

This equation defines what we mean by ω, the angular velocity of zero angular momentum particle.

So we have the result that a particle just dropped straight in ($p_\phi = 0$) from infinity is dragged just by the influence of gravity so that it acquires an angular velocity in the same sense as that of the source of the metric This effect weakens with distance.
This is similar to magnetism, we have the gravitational analog, a spinning mass causes a drag of particles, it is called **gravitomagnetism**.

Real BH in astronomy

The unique stationary solution for a BH is the Kerr metric.
hey only have two degrees of freedom that define them completely

This simplicity has made it possible to identify systems containing black holes based only on indirect evidence.

BH of Stellar mass

An isolated black hole, formed by the collapse of a massive star, would be very difficult to identify. It might accrete a small amount of gas as it moves through the interstellar medium, but this gas would not emit much X-radiation before being swallowed. No such candidates have been identified. All known stellar-mass black holes are in binary systems whose companion star is so large that it begins dumping gas on to the hole. Being in a binary system, the gas has angular momentum, and so it forms a disk around the black hole. But within this disk there is friction, possibly caused by turbulence or by magnetic fields. Friction leads material to spiral inwards through the disk, giving up angular momentum and energy. Some of this energy goes into the internal energy of the gas, heating it up to temperatures in excess of 10^6K, so that the peak of its emission spectrum is at X-ray wavelengths.
Many such X-ray binary systems are known

Hawking Radiation

In QM we have that $\Delta E \Delta t \geq \hbar/2$, where ΔR is the minimum uncertainity in a particle's energy which resides in a QM state for a time Δt.
According to QFT, space is filled with vacuum fluctuations in EM fields, which consist of pairs of photons being produced at one event and recombining in another. Such pairs violate conservation of energy, unless they last less than $\Delta t = \hbar/2\Delta E$ where ΔR is the amount of

violation.

Consider a fluctuation which produces two photons of energy $E, -E$. In flat space, both photons would eventually recombine. But if produced just outside the horizon, it has a chance of crossing the horizon before the time $\hbar/2E$ elapses; once inside the horizon, it can propagate freely

.

Super resumen

Basic Riemannian Geometry

Definition: An n dimensional **smooth manifold** is a set M and a collection of subsets $\{O_a\}$ of M such that:

- $\bigcup_a O_a = M$
- For each a, there is a bijection (1-1 and onto) map $\phi_a : O_a \to U_a$ where U_a is an open set of $\mathbb{R}^n$. The maps (O_a, ϕ_a) are called **charts**.
- If $O_a \cap O_b \neq \emptyset$, then the map $\phi_b \circ \phi_a^{-1}$ from $\phi_a(O_a \cap O_b) \subset U_a \to \phi_b(O_A \cap O_b) \subset U_b$ has to be a smooth map from $\mathbb{R}^n$ to $\mathbb{R}^n$

Definition (Smooth Function on Manifold): A function $f : M \to \mathbb{R}$ is smooth iff for any charg, the composition $f \circ \phi^{-1} : U \to \mathbb{R}$ is a smooth function.

The set of all smooth functions on M is often denoted $C^\infty(M)$

Vectors

Motivation: At any point $p \in S$ (a surface embedded in $\mathbb{R}^n$) we define the **tangent plane** T_pS to be the collection of vectors in $\mathbb{R}^n$ that are tangent to S at p. Note that T_pS and T_qS are different vector spaces.

If $\gamma(t)$ is a curve that lies on a surface, then its tangent vector $\gamma'(t)$ at p belongs to T_pS. By considering all the possible tangent vectors of possible curves, we construct T_pS.

Definition (Smooth curve): A smooth curve on a differentiable manifold is a smooth function $\gamma : I \to M$ where M is an open interval in $\mathbb{R}$. This means the composition $\phi_a \circ \gamma$ is a smooth function $I \to \mathbb{R}^n$ for all charts ϕ_a

Now consider a smooth function $f : M \to \mathbb{R}$. Then, the composition $f \circ \gamma : I \to \mathbb{R}$ defines a smooth 1-variable function $f(\gamma(t))$.
We can differentiate this at $t = 0$:

$$\frac{d}{dt}[(f \circ \gamma)(t)]\bigg|_{t=0}$$

In calculus, this would be $\gamma'(0) \cdot \nabla f$. Or, if $\vec{v} = \gamma'(0)$ is the tangent to the curve γ at $p = \gamma(0)$, we can define the directional derivative: $D_{\vec{v}} = \vec{v} \cdot \nabla f$, this is a scalar.

Definition (Tangent vector): Suppose $\gamma(t) : I \to M$ is a smooth curve with $\gamma(0) = p \in M$. The **tangent vector** to γ at p is the linear map $X_p : f \to \mathbb{R}$ given by:

$$X_p(f) = \frac{d}{dt}[(f \circ \gamma)(t)]\bigg|_{t=0}$$

Note that X_p is linear.

Let us suppose we are working in a chart ϕ with coordinates $x^a = (x^1, x^2, \cdots, x^n)$. The composition $\phi \circ \gamma = x^a(t)$ represents a curve in our coordinate system. And the composition $f(x^1, x^2, \cdots, x^n) = f \circ \phi^{-1}$ represents a function in this chart. Thus:

$$X_p(f) = \frac{d}{dt}[(f \circ \gamma)(t)]\bigg|_{t=0} = \frac{dx^a(t)}{dt}\bigg|_{t=0} \left(\frac{\partial f(x^b)}{\partial x^a}\right)\bigg|_{\phi(p)}$$

.

Proposition 1: The set of all tangent vectors at p forms an n-dimensional vector space, referred as the tangent space to M at p, denoted $T_p(M)$.

Basis: A natural basis is $\{\partial_a\}$ and then, any vector is:

$$V = V^a \partial_a$$

The derivatives are evaluated at p.

The coordinate basis $(\partial_{x^1}, \partial_{x^2}, \cdots, \partial_{x^n})$ is the tangent vectors to coordinate curves that pass through p.

Proposition: Let (O_1, ϕ_1) and (O_2, ϕ_2) be two overlapping charts with coordinates x^a, y^a respectively such that in the overlap region, the change of coordinates is $y^a = y^a(x^b)$. Suppose that V is a vector based at $p \in O_1 \cup O_2$ and $V = V^a \partial_{x^a}$ in the first coordinates. Then, the components V'^a for the vector in the other coordinates $V'^a \partial_{y^a}$ is:

$$V'^a = \frac{\partial y^a}{\partial x^b} V^b$$

Basis vectors transform as:

$$\frac{\partial}{\partial x^b} = \frac{\partial y^a}{\partial x^b} \frac{\partial}{\partial y^a}$$

.

Covectors

Definition(Dual): The dual space V^* of a vector space V is the vector space of linear maps of V to $\mathbb{R}$

Proposition: If $\dim V = n$, then $\dim V^* = n$. If e_a, is a basis for V, the **dual basis** is θ^a defined by $\theta^a(e_b) = \delta^a_b$

Definition (Cotangent space): Let T_pM be the tangent space at p. The cotangent space T^*_pM is the space of all dual vectors at p, elements are called covectors. A one form is written in components as $\omega = \omega_a \theta^a$
To determine the components, we have that $\omega_a = \omega(e_a)$

Definition (gradient) Let $f : M \to \mathbb{R}$ be a smooth function. The one form $df \in T^*_pM$ defined by $(df)(X) = X(f)$, for any $X \in T_pM$, is called the gradient or differential of f at p. The action of df on a tangent vector X is the directional derivative of f in the direction X.

Propositino: Let (O, ϕ) be a chart on M with coordinates x^a and $p \in O$. The n one forms $\{dx^a\}$ is the **dual basis** of T^*_pM associated to the coordinate basis vectors $\{\partial_{x^a}\}$ of T_pM.

We have:

$$dx^a(\partial_{x^b})\Big|_p = \delta^a_b$$
$$df(\partial_{x^b}) = \frac{\partial f}{\partial x^b}$$

Transformation rules:
If y^a and x^a are coordinates for two intersecting charts such that $y^a = y^a(x^b)$.
Then, the basis one forms transforms as:

$$dy^a = \frac{\partial y^a}{\partial x^b} dx^b$$

And, for any covector $\omega = \omega_a dx^a = \omega'_a dy^a$ then, on overlap regions:

$$\omega'_a = \frac{\partial x^b}{\partial y^a} \omega_b$$

Tensors

Definition: A (r, s) tensor at $p \in M$ is a multilinear map $T^*_pM \times \cdots \times T^*_pM \times T_pM \times \cdots \times T_pM \to \mathbb{R}$ where there are r factors T^*_pM and s factors T_pM.

Vector field: A vector field X is a map that associates any point $p \in M$ to a vector $X_p \in T_pM$. It is smooth if, given any smooth function f, the scalar function $X(f) : M \to \mathbb{R}$ defined by $X(f) : p \to X_p(f)$ is itself smooth.

Metric Tensor: It is a $(0, 2)$ tensor with the two properties:

- $g(X, Y) = g(Y, X)$ for all $X, Y \in T_pM$. Thus $g_{ab} = g(e_a, e_b) = g(e_b, e_a) = g_{ba}$
- $g(X, Y) = 0$ for all $Y \in T_pM$ iff $X = 0$

Note that:

$$g = g_{ab} dx^a \otimes dx^b$$

It is conventional to write the line element as:

$$ds^2 = g_{ab} dx^a dx^b$$

Riemannian Manifold: A smooth manifold with a metric tensor field.

Norm: The norm of a vector is $|X| = (g(X,X))^{1/2}$

Like: A vector X is

- Timelike if $g(X,X) < 0$
- Null if $g(X,X) = 0$
- Spacelike if $g(X,X) > 0$

Transformation: We have coordinate x^a and y^a of intersecting charts. The components of g in x^a are g_{ab} and in y^a are g'_{ab}. These are related by:

$$g_{ab} = \frac{\partial y^c}{\partial x^a} \frac{\partial y^d}{\partial x^b} g'_{cd}$$

Geodesics

Let us consider curves $\gamma : (a,b) \to M$ on (M,g). Such a curve has a tangent vector T at each point.

Definition (Timelike curve): If the tangent vector T is everywhere timelike.

Length of a curve: We can define the length of a curve C parametrized by $\gamma : (a,b) \to M$ with tangent vector X to be:

$$Length = \int_C ds = \int_a^b \sqrt{g(X,X)} dt$$

Variational Calculus

Functional: A functional S takes a function $x(t)$ and returns a number. In general, for a smooth function $x(t)$ with boundary $x(t_2) = x_2, x(t_1) = x_1$, a functional may have the form:

$$S[x] = \int_{t_1}^{t_2} L(x, \dot{x}, t) dt$$

The function $x(t)$ that extremizes S must satisfy **Euler Lagrange:**

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

Propositions

- Suppose the Lagrangian is independent of x, then, $\dfrac{\partial L}{\partial \dot{x}} = cte$
- If L has no explicit dependence on t ($\partial L/\partial t = 0$) Then, for solutions to EL equations, we define the Hamiltonian, and it is constant:
$$\frac{\partial L}{\partial \dot{x}}\dot{x} - L = cte$$
- **Multiple functions:** The functional may depend on a set of functions $\{x^a(t)\}$ and their derivatives:
$$S[\{x^a\}] = \int_{t_1}^{t_2} L(x^a, \dot{x}^a, t)dt$$
With the ends fixed. Then there are n EL equations, each of the form:
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = \frac{\partial L}{\partial x^a}$$

Geodesics

We now apply this to our extremization problem for the length functional:

$$S[x^a] = \int_{t_1}^{t_2} L(x^a, \dot{x}^a)dt \quad , \quad L = \sqrt{g_{ab}\dot{x}^a\dot{x}^b}$$

Note that it doesn't depend on t, so the Hamiltonian is conserved.

Proposition: The E-L equations following from the functional for length are:

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^a\dot{x}^b = 0$$

Known as the **Geodesic equation**.
Where the **Christoffel** symbols are:

$$\Gamma^a_{bc} = \frac{g^{ad}}{2}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc})$$

.

Timelike and Null Geodesics

In GR, freely falling particles fall in geodesics (this is not a postulate, but a consequence of the Einstein field equations).
In a geodesic, the Hamiltonian is preserved, but it is actually a multiple of $L = g_{ab}\dot{x}^a\dot{x}^b$, so L is preserved.
Thus, the trajectories of a particle are given by $x^a(\lambda)$, where:

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^a\dot{x}^b = 0$$
$$g_{ab}\dot{x}^a\dot{x}^b = \epsilon = cte$$

Where $\epsilon = 0, -C, +C$ for null, timleike and spacelike geodesics.

We recall that **Proper time** is defined as the parameter of the curve with the property that:

$$g_{ab}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau} = -1$$

.
.

.

2 Cosmology and Schwarzchild

The most general Einstein equation would be:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

Where Λ is the **cosmological constant**.

The study of the evolution of the universe as a space time with Einstein's equatoin is called **cosmology**.

According to legend,in 1917 when Einstein tried to solve the equation with $\Lambda = 0$, he concluded that the universe is expanding. Einstein didn't like that and astronomers didn't think the universe was expanding, so Einstein found a value of Λ that made the universe stable.
In 1931, Hubble made the first measurement that confirmed the expansion of the universe, so $\Lambda = 0$ was taken back. Nevertheless, a value of Λ different to 0 has been proposed once again for different reasons.

The result that the universe expands (and actually accelerates its expansion) led to the postulation of **dark energy**, which needs a positive Λ.
Also, **dark matter** was proposed to explain the rotation curves of some galaxies that cannot be explained only with normal matter.

The second reason for a new protagonism of Λ has been in holography, string theory

Schwarzschild Solution

We take the Einstein equation without cosmological constant:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}$$

Minkowski

When we are in empty space, $T_{\mu\nu}$, our equation should inlcude the normal Minkowsky solution. That is, the metric $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$.

The Minkowsky metric has constant coefficients, so the Christoffel symbols are constant, and therefore $R_{\mu\nu} = 0$, $R = 0$, and the Minkosky metric satisfies:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$

To proved this, we used an specific coordinate system, but Einstein's equations are tensorial, so they are true in any other coordinates

However, this is not the only solution for a space time in empty space.

Schwarszchild solution

History

In 1915 Schwarzchild found the first non empty space solution while on WW1, literally in the trenches.

Schwarzchild looked for a solution generated with an object with spherical symmetry.

Coordinates

The first step is to fix the coordinates of space time. But the coordinates may only cover some part of the manifold (just as coordinates can only cover some part of a sphere but not all). We won't know if the coordinates are **global** or not until the end.

Because of symmetry, the best choice for the coordinates are:

$$x^\mu = \{t, r, \theta, \phi\}$$

With a temporal coordinates $-\infty < t < \infty$ and spherical spatial coordinates $0 < r < \infty$, $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$

Stress Energy tensor

If R is the radius of the sphere, to calculate the gravitational field for $r > R$ we only need to know the total mass M, because the density nullifies at $r > R$.
On the other hand, $T_{\mu\nu}$ may be complicated when $r < R$, but it is zero for $r > R$.

Therefore, the equation we will try to solve is:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0$$

This will be the **exterior Schwarzschild** solution.
The solution will be curved.

Ansatz

The symmetries are:

- Space time is spherically symmetric
- Space time is static

Condition 2 comes from Newton's gravitation and the fact that the gravitational field of Earth doesn't change with time.
We can write the metric generally as:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

Where there are 10 independent coordinates. We know that in $\mathbb{R}^3$, the spherical coordinates metric is:

$$ds^2_{\mathbb{R}^3} = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

So $\mathbb{R}^3$ is formed of many spheres, each with a metric of:

$$ds^2_{S^2} = r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

.
So, for our full space time we will propose as metric:

$$ds^2 = g_{tt}dt^2 + 2g_{ti}dtdx^i + g_{rr}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

.

Here we note the difference between static and stationary. A static system is one that doesn't change with time. In a stationary one, the system looks the same at any time, buy it changes if we reverse time $t \to -t$.

The only way this can happen is if the components g_{ti} nullify.
So that we only have two free components of the metric. This ones cannot depend on ϕ, θ, t due to the symmetries, therefore:

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

.

Solution

We now take the Ansatz and substitute it in Einstein's equations.

We should substitute this into Einstein's equation to gen a differential equation for f, h.
In this case, the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$
Will be diagonal, therefore the equation:

$$G_{\mu\nu} = 0$$

Will give us 4 differential equations for two unknowns $f(r), h(r)$ (but they aren't all independent).
Explicitly, we get 3 diff equations, that are:

18) $-rh(r)f'(r)^2 + f(r)[2h(r)(rf''(r) + f'(r)) - rf'(r)h'(r)] - 2f(r)^2h'(r) = 0$

19) $rh'(r) + h(r)^2 - h(r) = 0$

20) $rf'(r) + f(r)(-h(r)) + f(r) = 0$

From 19 and 20 we conclude that:

$$h(r)f(r) = k$$

For k an arbitrary constant.
So there is actually only one independent function, and the metric is now:

$$ds^2 = -\frac{k}{h(r)}dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Note that the constant only appears in the time component, so we can change $t \to \sqrt{k}t$.
So we get the metric:

$$ds^2 = -\frac{1}{h(r)}dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

And the equations to determine $h(r)$ are:

$$rh'(r) + h(r)^2 - h(r) = 0$$
$$h(r)(rh''(r) + 2h'(r)) - 2rh'(r)^2 = 0$$

These are independent and the general solution turns out to be:

$$h(r) = \left(1 + \frac{C}{r}\right)^{-1}$$

The only free parameter now is C, so we think it might be related to the **mass** M. Therefore, our solution is:

$$\boxed{ds^2 = -\left(1 + \frac{C(M)}{r}\right)dt^2 + \frac{dr^2}{1 + \frac{C(M)}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)}$$

The **Schwarzschild metric**.
We can see that far from the source, when $C(M) << r$ or $r >> 1$, we get the Minkowsky metric for spherical coordinates $ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$

Schwarzschild Geodesics

Circular Orbits

We are going to check if a particular curve is a geodesic. We try a circunferencial solution with radius R around an object of mass M with constant angular velocity ω with respect to our reference system out in infinity.
The trajectory is in the plane $\theta = \pi/2$. We can parametrize as:

$$\tau(t) = k\tau \quad , \quad r(\tau) = R_0 \quad , \quad \theta(\tau) = \frac{\pi}{2} \quad , \quad \phi(\tau) = \omega\tau k$$

Therefore $\phi = \omega t$. Here $k \neq 0$ is a normalization factor that can be determined from asking that $g_{\mu\nu}u^\mu u^\nu = -1$
We substitute into the geodesic equation:

$$\frac{d^2x^\nu}{d\tau^2} + \frac{dx^\mu}{d\tau}\frac{dx^\sigma}{d\tau}\Gamma^\nu_{\sigma\mu} = 0$$

Along with the metric, we obtain that the r component of the equation leads to:

$$-k^2\frac{(C(M)+R_0)(C(M)+2R_0^3\omega^2)}{2R_0^3} = 0$$

For this to be 0, we need that:

$$C(M) + 2R_0^3\omega^2 = 0$$

Or that:

$$\boxed{\omega^2 = \left(-\frac{C(M)}{2}\right)\frac{1}{R_0^3}}$$

This is the condition for circular geodesics and it is the **Kepler's third law**.
This allows us to determine $C(M)$, using the Kepler equation $\omega^2 = \dfrac{M}{R_0^3}$.
We get that:

$$C(M) = -2M$$

Conserved Quantities

We study now more general geodesics, of a body of mass m and due to an object of mass M. In this coordinates, the trajectory is parametrized by:

$$x^\mu(\tau) = \{t(\tau), r(\tau), \theta(\tau), \phi(\tau)\}$$

Such that they satisfy:

$$\ddot{x}^\nu + \dot{x}^\mu \dot{x}^\sigma \Gamma^\nu_{\sigma\mu} = 0$$

Where the dot denotes derivative with respect to the proper time.
The angular equations are:

$$\ddot{\theta} + 2\frac{\dot{r}\dot{\theta}}{r} = 0 \quad , \quad \ddot{\phi} + 2\frac{\dot{r}\dot{\phi}}{r} = 0$$

Which give:

$$r^2\dot{\theta} = C_\theta \quad , \quad r^2\dot{\phi} = C_\phi$$

With C_ϕ, C_θ are constants.
These conditions express the fact that the movement takes place in a plane, say $\theta = \pi/2$. This satisfies the θ equation, and therefore, the **total angular momentum** is:

$$L = mr^2\dot{\phi}$$

.

On the other hand, the temporal component is:

$$\ddot{t} - \frac{2M\dot{r}\dot{t}}{2Mr - r^2} = 0$$

Which we integrate to get:

$$C_t = -\left(1 - \frac{2M}{r}\right)\dot{t} = cte$$

To interpret it, we consider the 4-momentum, which is $p^\mu = mu^\mu$ and total energy is $E = p^t = m\dot{t}$
Therefore, the conserved quantity is:

$$E = \left(1 - \frac{2M}{r}\right)m\dot{t}$$

Which we interpret as the total energy of the particle, with a correction due to gravitational field.

We can now substitute this into the radial geodesic equation, and the result is:

$$\frac{M}{r^2}(E^2 - m^2\dot{r}^2)\left(1 - \frac{2M}{r}\right)^{-1} - \frac{L^2}{r^3}\left(1 - \frac{2M}{r}\right) + m^2\ddot{r} = 0$$

Which can be integrated to get:

$$(m^2\dot{r}^2 - E^2)\left(1 - \frac{2M}{r}\right)^{-1} + \frac{L^2}{r^2} = C_r = cte$$

Substituting E and L and after some algebra, we can identify that $C_r = -m^2$
And the radial equation is:

$$-m^2 = -\frac{E^2}{1 - \dfrac{2M}{r}} + \frac{m^2\dot{r}^2}{1 - \dfrac{2M}{r}} + \frac{L^2}{m^2} \qquad 50$$

.

Effective Potential

We can write equation 50 as:

$$\epsilon = \frac{1}{2}m\dot{r}^2 + V_E(r)$$

Our problem turns into finding the non relativistic motion of a particle with mass m and total energy:

$$\epsilon = \frac{1}{2m}(E^2 - m^2)$$

With an effective potential of:

$$V_E(r) = -\frac{Mm}{r} + \frac{L^2}{2mr^2} - \frac{L^2M}{mr^3}$$

- **Newtonian case:** In this case $\epsilon = \frac{1}{2}m\dot{r}^2 + V_N(r)$ with $V_N(r) = -\frac{Mm}{r} + \frac{L^2}{2mr^2}$

  This potential has an only solution $R_0 = \frac{L^2}{2M}$ and an only extrema $R_c = \frac{L^2}{M}$ that turns out to be a minimum.
  Also, it has that $\lim_{r\to 0} V_N = \infty$ and $\lim_{r\to\infty} V_N = 0$
  Being $V_N < 0$ for $r > R_0$ and $V_N > 0$ for $r < R_0$.

  Therefore, this potential admits many types of orbits, closed and open dependiento de la energía ϵ de la partícula. En la Fig. (1) se muestra el potencial Newtoniano para el caso $L = 5M$.

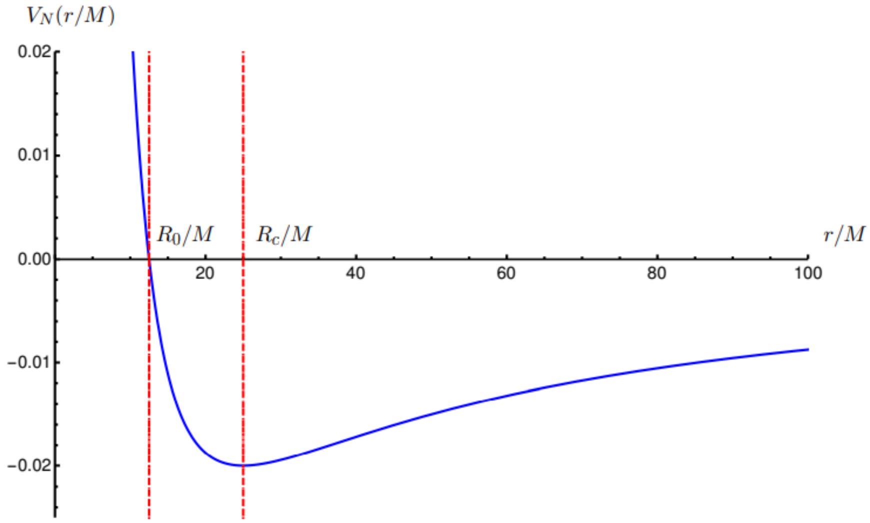


Figura 1: Potencial efectivo en el caso Newtoniano para $L = 5M$.

- **Schwarzschild:** Here the potential is $V_E(r) = V_N(r) - \dfrac{L^2 M}{r^3}$

  The potential has now two zeroes, $R_0^{\pm} = \dfrac{R_0}{2} \pm \dfrac{\sqrt{L^4 - 16L^2M^2}}{4M}$ so long as $L > 4M$

  Also, if $L > 2\sqrt{3}M$, it also has two extremma: $R_{\pm} = \dfrac{L^2}{2m} \pm \dfrac{\sqrt{L^4 - 12L^2M^2}}{2M}$
  It is easy to prove that if both conditions are met ($L > 4M$), then R_+ is a minimum and R_- a maximum.

  For big radius, $r > R_0^+$, V_E is similar to V_N and if gives very similar solutions. In small radii ($r < R_0^+$) there are differences. The Einstein potential admits unstable circular orbits when $r = R_-$.

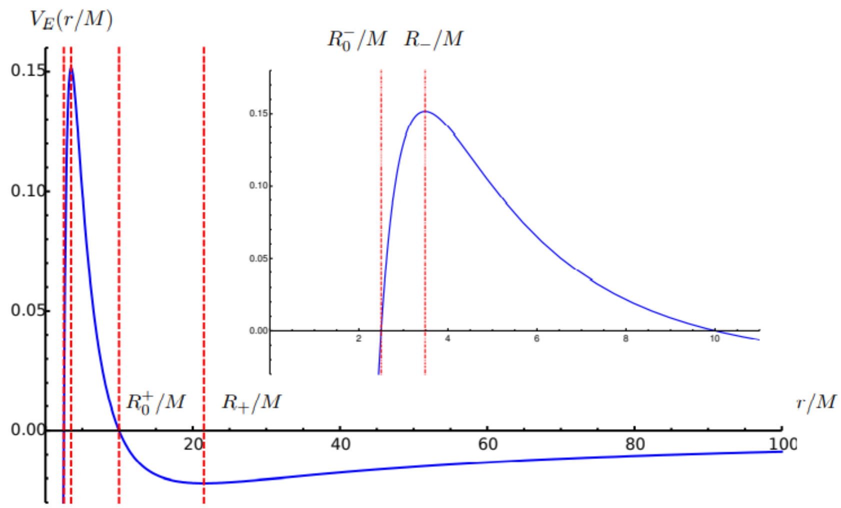


Figura 2: Potencial efectivo en el caso de Einstein para $L = 5M$.

Analysis of Schwarzschild metric

Dependency on r:

We return to the metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

We can see that the metric has two **singularities:**

- $r = 0$: We have $ds^2 = \infty dt^2$
- For $r = 2M$ (**Schwarzchild radius**)) we have that $ds^2 = \infty dr^2$

Does this mean the solution is not valid????
No.

First of all, the solution was taken for $r > R$. Definitively $R > 0$, so we avoid the first singularity. We avoid the second one if $R > 2M$ (which is usually a very small radius for any reasonable object).
Restoring the G and c factors, the radius is:

$$r_s = \frac{2GM}{c^2}$$

For the sun, $r_s = 3000m$, which is much less than the real radius, so the solution we gave remains true.
Only some stars end their life cycle with a radius less than r_s.

Nevertheless, there is the possibility that the singularity is due to the coordinates. To check if it is, we need to find some coordinates where the singularity disappears (which exist). To check a singularity is true, we need to find a **curvature scalar** (a number independent of coordinates) and see that it has a singularity at that point).

Radial Geodesics

Lest see what happens at r_s.
Notice that the equation $r = r_s$ specifies a 2+1 dimensional surface in spacetime that divides space in two regions:

- Interior region $r < r_s$
- Exterior region $r > r_s$

Consider a simple experiment:

Consider a particle with mass $m = 1$ in this metric, such that it follows a rdial trajectory from $r_i > r_s$ to r_s. It has no angular momentum, so it moves only radially, so we take $L = 0$. Then, the expression $-m^2 = -\frac{E^2}{1-2M/r} + \frac{m^2\dot{r}^2}{1-2M/r} + \frac{L^2}{r^2}$ we derived earlier leads to $E^2 = \dot{r}^2 + 1 - \frac{r_s}{r}$, so that:

$$\frac{dr}{d\tau} = -\sqrt{E^2 - 1 + \frac{r_s}{r}}$$

The minus sign means the particle moves towards the r_s.

We want to calculate the time it takes according to a person out in infinity (coordinate time). For that, we use the chain rule along with $E = \left(1 - \frac{2M}{r}\right) m\dot{t}$, so we find:

$$\frac{dr}{dt} = -\frac{1}{E}(1 - \frac{r_s}{r})\sqrt{E^2 - 1 + \frac{r_s}{r}}$$

We can integrate this to find $t(r)$ as:

$$\boxed{t(r) = -E\int_{r_i}^{r} \left(1 - \frac{r_s}{s}\right)^{-1} \left(E^2 - 1 + \frac{r_s}{s}\right)^{-1/2} ds}$$

The function $t(r)$ gives us the coordinate time it takes for the particle to move from r_i to an arbitrary r. If $E = 1$, this integral can be calculated explicitly and gives:

$$t(r) = 2\left(r_s arctanh\left(\sqrt{\frac{s}{r_s}}\right) - \frac{1}{3}(s + 3r_s)\sqrt{\frac{s}{r_s}}\right)\Bigg|_{r_i}^{r}$$

WE observe that t diverges as r gets closer to r_s.
The particle takes an infinite coordinate time to get to r_s. The particle never reaches r_s

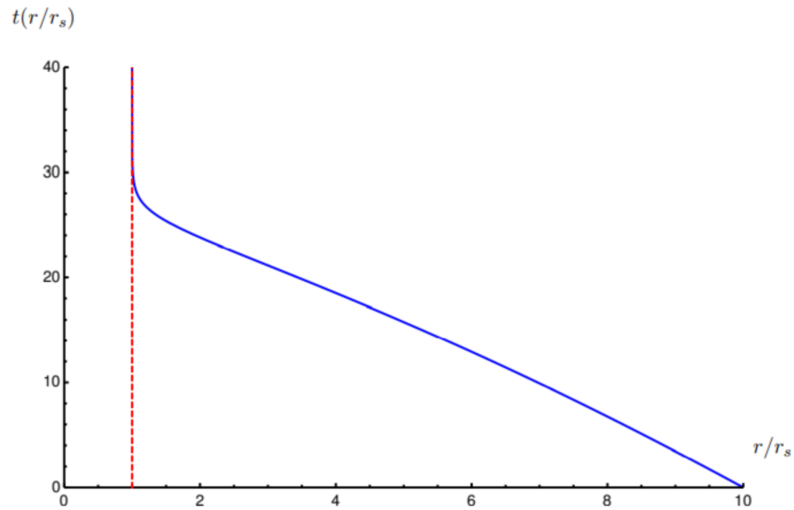


Figura 3: Tiempo coordenado que le toma a la partícula moverse desde el exterior en $r_i = 10r_s$ hasta r_s

.

This is due to a the singularity obviously.
To determine the type of singularity, we calculate the proper time of the particle, for this, we integrate $\frac{dr}{d\tau} = -\sqrt{E^2 - 1 + \frac{r_s}{r}}$ to obtain $\tau(r)$:

$$\boxed{\tau(r) = -\int_{r_i}^{r} \left(E^2 - 1 + \frac{r_s}{r}\right)^{-1/2} ds}$$

We can analize once again the case for $E = 1$ and we see that:

$$\tau(r) = -\frac{2}{3} s \sqrt{\frac{s}{r_s}}\Big|_{r_i}^{r}$$

We see that it is a finite time.
So the particle takes a finite proper time to reach r_s.
The particle reaches r_s with no problem and it actually continues getting closer to $r = 0$.

The natural conclusion is that this is a coordinate singularity.

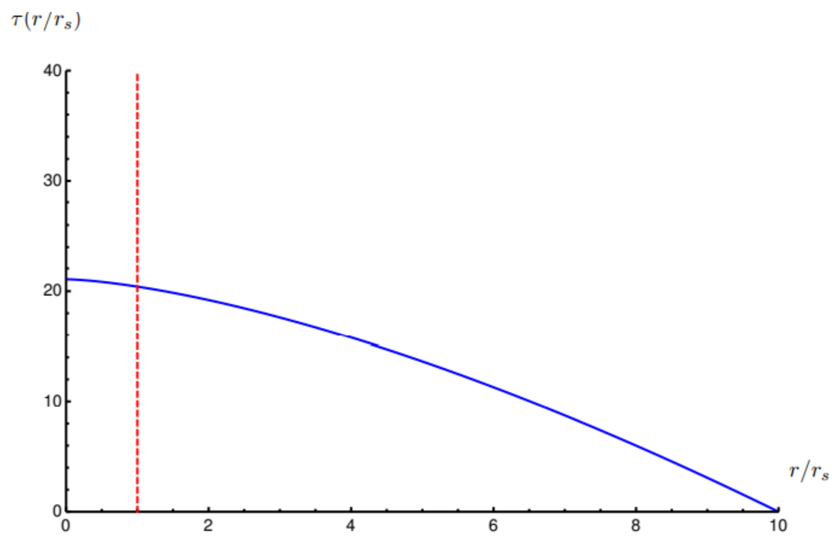


Figura 4: Tiempo propio que le toma a la partícula moverse desde el exterior en $r_i = 10r_s$ hasta algún $r_s > r > 0$ en el interior.

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Lets say that the particle emits light rays from $r_{ii} < r_s$ in the radial direction towards r_i, to tell the observers that he managed to cross the singularity.
The rays follow a radial trajectory, so that it is parametrized as:

$$x^\mu(\lambda) = (t(\lambda), r(\lambda), \theta_0, \phi_0)$$

Then, imposing that the tangent vector be null, we get that:

$$0 = -\left(1 - \frac{r_s}{r}\right)\dot{t}^2 + \left(1 - \frac{r_s}{r}\right)^{-1}\dot{r}^2$$

The dot represents derivative respect to λ. Then, we can find that:

$$\frac{dt}{dr} = \pm\left(1 - \frac{r_s}{r}\right)^{-1}$$

The sign is determined by the direction of the ray. We need it positive for it to go outwards. Then, we can find:

$$\begin{aligned} t(r) &= -\int_{r_{ii}}^{r}\left(1 - \frac{r_s}{r}\right)^{-1} ds \\ &= -r_s\left(\frac{s}{r_s} + \log\left(1 - \frac{r_s}{s}\right) + \log\frac{s}{r_s}\right)\Bigg|_{r_{ii}}^{r} \end{aligned}$$

We see that the light rays take an infinite time to get out of r_s

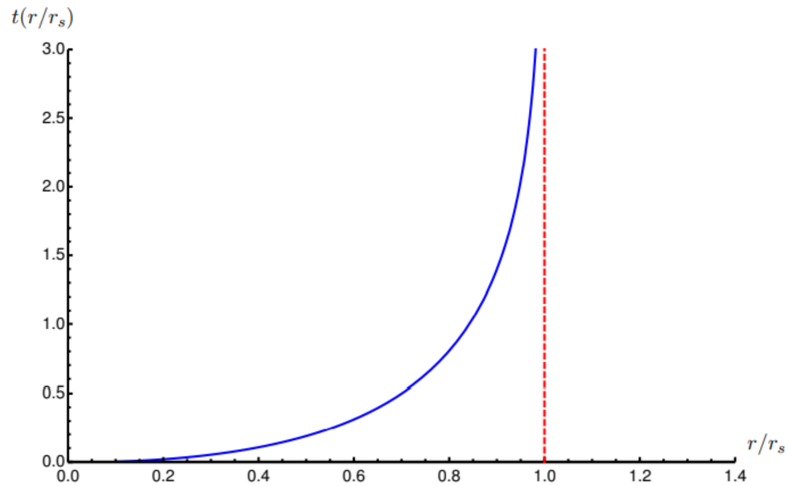


Figura 5: Tiempo que le toma a la luz moverse desde el interior en $r_{ii} = r_s/10$ hasta r_s.

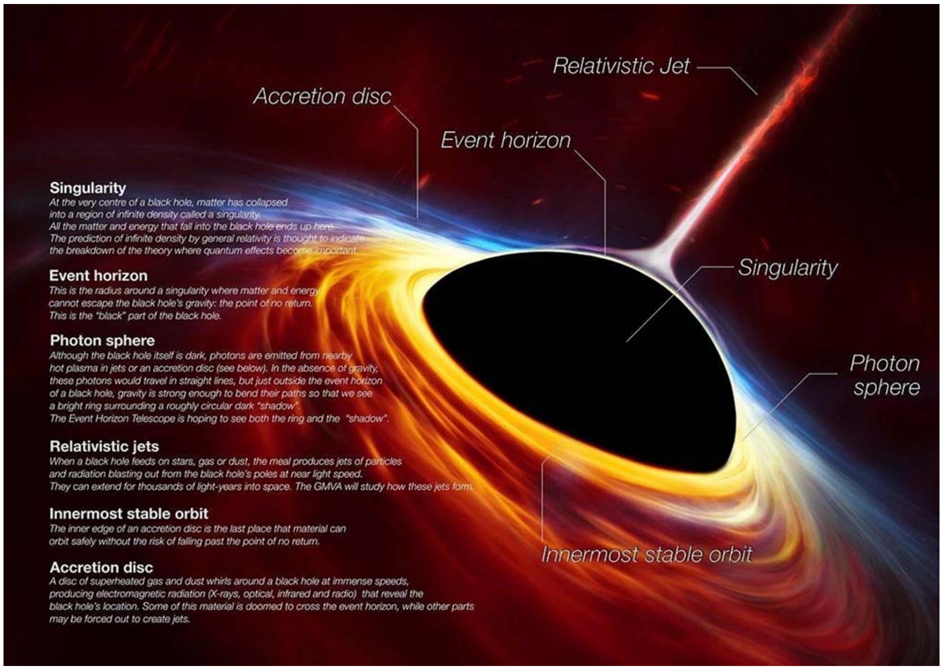
¿Qué es lo que el observador en infinito ve en $r = r_s$? Una esfera totalmente negra, pues el interior no puede emitir rayos de luz. Es por esta característica que a los objetos astronómicos que cumplen $R < r_s$ se les conoce como **agujeros negros**[1].

Black Hole

A black hole is not a hole (something 2 dimensional) in space that starts swallowing anything near. The metric that describe these objects is the same that describes the sun, moon, earth, etc. From far away, it is a completely dark sun. It is a sphere, not a plane hole.
All the study we did for Schwarzschild geodesics outside r_s is true for black holes, it doesn't swallow automatically everything around it.

Accretion disc: It is a disc of gas and dust surrounding the black hole and that orbits it.

Schwarzschild: The simplest idealization of a BH.



Event Horizon:

What distinguishes a BH is that its radius is less than r_s, that is $R < r_s$.

The surface $r = r_s$ is known as the **event horizon** (it is not really a surface, there is nothing there).

We see the metric:

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_s}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

We see that at r_s there is a change of signs in two components of the metric:

- For $r > r_s$, $g_{tt} < 0$ and $g_{rr} > 0$
- For $r < r_s$, $g_{tt} > 0$ and $g_{rr} < 0$

The important thing here is not if these components diverge on r_s, but the change in signs they have.
The general definition of a event horizon is that of a **null hypersurface**, such that its normal vector is null.
The unit normal vector to a surface with constant r has components:

$$n^\mu = \left(0, \frac{1}{\sqrt{|g_{rr}|}}, 0, 0\right)$$

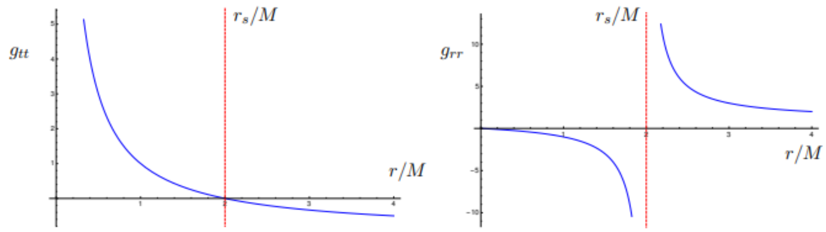


Figura 9: Componentes temporal y radial de la métrica de Schwarzschild. Ambas cambian de signo al cruzar por r_s

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So that:

$$n^\mu n_\mu = \begin{cases} -1 & , \ r > r_s \\ 0 & , \ r = r_s \\ 1 & , \ r < r_s \end{cases}$$

The case $r = r_s$ is the only that cannot be done using this coordinates.
What happens when crossing the event horizon. The response has to do with the change of signs of the norm or n.

Remember that in Minkowsky space time, the trajectories of light are solutions to $\dfrac{dt}{dr} = \pm 1$
So that the equation that describes a light ray is $t(r) = \pm r$

This is independent of the point in spacetime we designate as origin.
Because nothing can travel faster than light, what happens in any point in space can only affect its **future cone**.

In SCHD, things are different. The equation for a light ray is now:

$$\frac{dt}{dr} = \pm \left(1 - \frac{r_s}{r}\right)^{-1}$$

For $r_s << r$, this tens to the Minkowskian solution, so far from the black hole, the cones are similar to those in Minkowsky spacetime.
While $r \to r_s$, the cones get more vertical until they have an infinite slope in the event horizon.

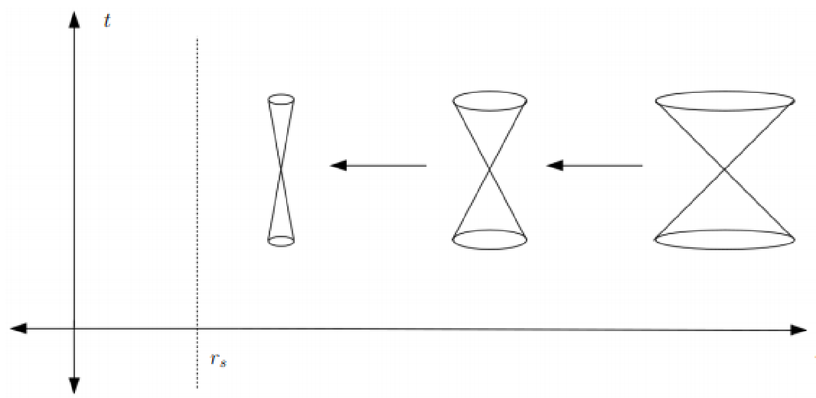

Figura 11: Conos de luz en el espacio-tiempo de Schwarzschild en el plano $r - t$

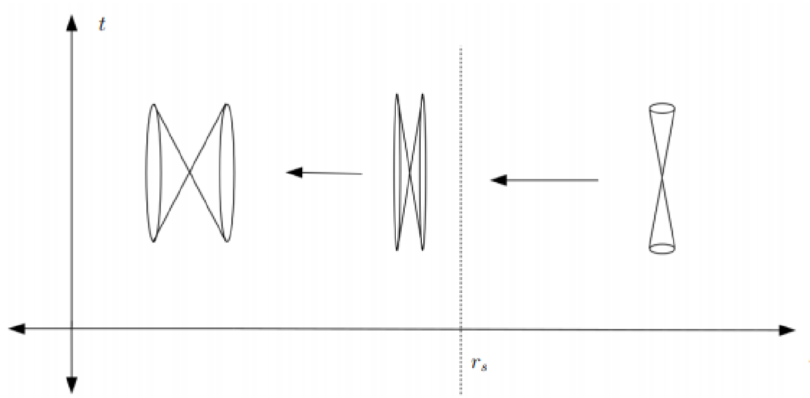

Figura 12: Conos de luz en el espacio-tiempo de Schwarzschild cruzando el horizonte de eventos.

This explains what we saw earlier.

Because the light cone closes up as we reach the horizon, an observer at infinity never sees the particle cross.

After crossing the horizon, the signs of the metric invert. So that r becomes a temporal coordinate and t a tmeporal one in the interior. So now the cones open horizontally.
Crossing the event horizon, advancing forward in time corresponds to to advancing towards $r = 0$, while trying to go to the outside corresponds to traveling to the past.
So nothing can escape the event horizon and the destiny of any object inside is to keep on advancing towards $r = 0$