Examen 2 Tomás Ricardo Basile Álvarez MAF

1) Para un parámetro $w>0$ se considera la EDO:

$$y''(x) + w^2 y(x) = f(x) \qquad x \in [0, \tfrac{3\pi}{2w}]$$
$$\alpha_1 y(0) + \beta_1 y'(0) = 0$$
$$\alpha_2 y\left(\tfrac{3\pi}{2w}\right) + \beta_2 y'\left(\tfrac{3\pi}{2w}\right) = 0$$

a) Determinar las condiciones de los parámetros $\alpha_1, \beta_1, \alpha_2, \beta_2, w$ para que exista la Función de Green y dar una expresión general de la solución.

Primero hay que buscar dos soluciones a la ecuación homogénea $y''(x) + w^2 y(x) = 0$ que cada una satisfaga cada una de las condiciones de frontera.

Es fácil ver que $y''(x) + w^2 y(x) = 0$ tiene como soluciones a $\cos(wx)$, $\text{sen}(wx)$ y por tanto, su solución general es:

$$y(x) = A\cos(wx) + B\,\text{sen}(wx)$$

$$\Rightarrow y'(x) = -Aw\,\text{sen}(wx) + Bw\cos(wx)$$

Ahora necesitamos una $y_1(x)$ de esta forma que satisfaga la primera condición:

$$\alpha_1 y_1(0) + \beta_1 y_1'(0) = 0$$

$$\rightarrow \alpha_1 [A\cos(0) + B\,\text{sen}(0)] + \beta_1[-Aw\,\text{sen}(0) + Bw\cos(0)] = 0$$

$$\rightarrow \alpha_1[A] + \beta_1[Bw] = 0 \qquad \rightarrow \alpha_1 A + \beta_1 w B = 0$$

$$\Rightarrow B = \frac{-\alpha_1}{\beta_1 w} A \qquad \leftarrow \text{Dado que } \beta_1 \neq 0$$

Podemos encontrar una solución en particular si $A=1 \Rightarrow B = \frac{-\alpha_1}{\beta_1 w}$

y entonces: $$y_1(x) = \cos(wx) - \frac{\alpha_1}{\beta_1 w}\,\text{sen}(wx)$$

Ahora necesitamos una $y_2(x)$ tal que $\alpha_2 y_2\left(\frac{3\pi}{2w}\right) + \beta_2 y'\left(\frac{3\pi}{2w}\right) = 0$

$$\Rightarrow \alpha_2\left[A\cos\left(w\left(\tfrac{3\pi}{2w}\right)\right) + B\,\text{sen}\left(w\left(\tfrac{3\pi}{2w}\right)\right)\right] + \beta_2\left[-Aw\,\text{sen}\left(w\left(\tfrac{3\pi}{2w}\right)\right) + Bw\cos\left(w\left(\tfrac{3\pi}{2w}\right)\right)\right] = 0$$

$$\Rightarrow \alpha_2\left[A\cos\left(\tfrac{3\pi}{2}\right) + B\,\text{sen}\left(\tfrac{3\pi}{2}\right)\right] + \beta_2\left[-Aw\,\text{sen}\left(\tfrac{3\pi}{2}\right) + Bw\cos\left(\tfrac{3\pi}{2}\right)\right] = 0$$

$$\Rightarrow \alpha_2[-B] + \beta_2[Aw] = 0 \qquad \rightarrow -\alpha_2 B + \beta_2 A w = 0$$

$$\Rightarrow B = \frac{\beta_2 w}{\alpha_2} A \qquad \leftarrow \text{Siempre y cuando } \alpha_2 \neq 0$$

Si $A=1$, nos queda una solución particular:

$$y_2(x) = \cos(wx) + \frac{\beta_2 w}{\alpha_2}\,\text{sen}(wx)$$

Ahora, para obtener la función de Green, hay que calcular antes el Wronskiano.

$$y_1(x) = \cos(wx) - \frac{\alpha_1}{\beta_1 w}\,\text{sen}(wx) \qquad y_2(x) = \cos(wx) + \frac{\beta_2 w}{\alpha_2}\,\text{sen}(wx)$$

$$y_1'(x) = -w\,\text{sen}(wx) - \frac{\alpha_1}{\beta_1}\cos(wx) \qquad y_2'(x) = -w\,\text{sen}(wx) + \frac{\beta_2 w^2}{\alpha_2}\cos(wx)$$

$$\Rightarrow W(x) = y_1 y_2' - y_2 y_1'$$

$$= \left[\cos wx - \frac{\alpha_1}{\beta_1 w}\,\text{sen}\,wx\right]\left[-w\,\text{sen}(wx) + \frac{\beta_2 w^2}{\alpha_2}\cos(wx)\right] - \left[\cos(wx) + \frac{\beta_2 w}{\alpha_2}\,\text{sen}(wx)\right]\left[-w\,\text{sen}(wx) - \frac{\alpha_1}{\beta_1}\cos(wx)\right]$$

$$= \cancel{-w\cos(wx)\,\text{sen}(wx)} + \frac{\beta_2 w^2}{\alpha_2}\cos^2(wx) + \frac{\alpha_1}{\beta_1}\text{sen}^2(wx) - \frac{\alpha_1\beta_2 w}{\beta_1\alpha_2}\text{sen}(wx)\cos(wx) + \cancel{w\cos(wx)\,\text{sen}(wx)} + \frac{\alpha_1}{\beta_1}\cos^2(wx)$$
$$+ \frac{\beta_2 w^2}{\alpha_2}\text{sen}^2(wx) + \frac{\beta_2\alpha_1 w}{\beta_1\alpha_2}\cos(wx)\,\text{sen}(wx)$$

$$= \frac{\beta_2 w^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}$$

Para que las soluciones sean L.I. debemos de tener que $W(x) \neq 0$
es decir, $\frac{\beta_2 w^2}{\alpha_2} + \frac{\alpha_1}{\beta_1} \neq 0$ ← Condición de los parámetros

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Entonces ya podemos calcular la función de Green:

$$\frac{y_2(x')\,y_1(x)}{W(x)} = \frac{\left[\cos(wx') + \frac{\beta_2 w}{\alpha_2}\text{sen}(wx')\right]\left[\cos(wx) - \frac{\alpha_1}{\beta_1 w}\text{sen}(wx)\right]}{\frac{\beta_2 w^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}$$

$$\frac{y_1(x')\,y_2(x)}{W(x)} = \frac{\left[\cos(wx') - \frac{\alpha_1}{\beta_1 w}\text{sen}(wx')\right]\left[\cos(wx) + \frac{\beta_2 w}{\alpha_2}\text{sen}(wx)\right]}{\frac{\beta_2 w^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}$$

Y entonces, la función de Green es:

$$G(x,x') = \begin{cases} \dfrac{\left[\cos(wx') + \frac{\beta_2 w}{\alpha_2}\text{sen}(wx')\right]\left[\cos(wx) - \frac{\alpha_1}{\beta_1 w}\text{sen}(wx)\right]}{\frac{\beta_2 w^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}, & 0 < x < x' < \frac{3\pi}{2w} \\[2ex] \dfrac{\left[\cos(wx') - \frac{\alpha_1}{\beta_1 w}\text{sen}(wx')\right]\left[\cos(wx) + \frac{\beta_2 w}{\alpha_2}\text{sen}(wx)\right]}{\frac{\beta_2 w^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}, & 0 < x' < x < \frac{3\pi}{2w} \end{cases}$$

Y la solución a la ecuación diferencial será entonces $y(x) = \int_0^{\frac{3\pi}{2w}} G(x,x')\,f(x')\,dx'$

b) Resolver en el caso $f(x) = \operatorname{sen}(2\omega x)$

$$y(x) = \int_0^{\frac{3\pi}{2\omega}} G(x,x')\, f(x')\, dx' = \int_0^{x} G(x,x')\, \operatorname{sen}(2\omega x')\, dx' + \int_x^{\frac{3\pi}{2\omega}} G(x,x')\, \operatorname{sen}(2\omega x')\, dx'$$

$$= \int_0^x \frac{\left[\cos(\omega x') - \frac{\alpha_1}{\beta_1 \omega}\operatorname{sen}(\omega x')\right]\left[\cos(\omega x) + \frac{\beta_2 \omega}{\alpha_2}\operatorname{sen}(\omega x)\right]}{W} \operatorname{sen}(2\omega x')\, dx'$$

usamos la def de $G(x,x')$ en cada intervalo ✓

$$+ \int_x^{\frac{3\pi}{2\omega}} \frac{\left[\cos(\omega x') + \frac{\beta_2 \omega}{\alpha_2}\operatorname{sen}(\omega x')\right]\left[\cos(\omega x) - \frac{\alpha_1}{\beta_1 \omega}\operatorname{sen}(\omega x)\right]}{W} \operatorname{sen}(2\omega x')\, dx'$$

$$W \equiv \frac{\beta_2 \omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1} = cte$$

$$= \frac{\cos(\omega x) + \frac{\beta_2 \omega}{\alpha_2}\operatorname{sen}(\omega x)}{W}\left[\int_0^x \cos(\omega x')\operatorname{sen}(2\omega x')\, dx' - \frac{\alpha_1}{\beta_1 \omega}\int_0^x \operatorname{sen}(\omega x')\operatorname{sen}(2\omega x')\, dx'\right]$$

$$+ \frac{\cos(\omega x) - \frac{\alpha_1}{\beta_1 \omega}\operatorname{sen}(\omega x)}{W}\left[\int_x^{\frac{3\pi}{2\omega}} \cos(\omega x')\operatorname{sen}(2\omega x')\, dx' + \frac{\beta_2 \omega}{\alpha_2}\int_x^{\frac{3\pi}{2\omega}} \operatorname{sen}(\omega x')\operatorname{sen}(2\omega x')\, dx'\right]$$

* En la página siguiente calculamos estas antiderivadas explícitamente

$$= \frac{\cos(\omega x) + \frac{\beta_2 \omega}{\alpha_2}\operatorname{sen}(\omega x)}{W}\left[-\frac{2}{3\omega}\cos^3(\omega x')\Big|_0^x - \frac{\alpha_1}{\beta_1 \omega}\frac{2}{3\omega}\operatorname{sen}^3(\omega x')\Big|_0^x\right]$$

$$+ \frac{\cos(\omega x) - \frac{\alpha_1}{\beta_1 \omega}\operatorname{sen}(\omega x)}{W}\left[-\frac{2}{3\omega}\cos^3(\omega x')\Big|_x^{\frac{3\pi}{2\omega}} + \frac{\beta_2 \omega}{\alpha_2}\frac{2}{3\omega}\operatorname{sen}^3(\omega x')\Big|_x^{\frac{3\pi}{2\omega}}\right]$$

$$= \frac{\cos(\omega x) + \frac{\beta_2 \omega}{\alpha_2}\operatorname{sen}(\omega x)}{W}\left[-\frac{2}{3\omega}\cos^3(\omega x) + \frac{2}{3\omega} - \frac{2\alpha_1}{3\beta_1 \omega^2}\operatorname{sen}^3(\omega x)\right]$$

$$+ \frac{\cos(\omega x) - \frac{\alpha_1}{\beta_1 \omega}\operatorname{sen}(\omega x)}{W}\left[\frac{2}{3\omega}\cos^3(\omega x) - \frac{2\beta_2}{3\alpha_2} - \frac{2\beta_2}{3\alpha_2}\operatorname{sen}^3(\omega x)\right]$$

$$= \frac{-\frac{2}{3\omega}\cos^4(\omega x) - \frac{2\beta_2}{3\alpha_2}\operatorname{sen}(\omega x)\cos^3(\omega x) + \frac{2}{3\omega}\cos(\omega x) + \frac{2\beta_2}{3\alpha_2}\operatorname{sen}(\omega x) - \frac{2\alpha_1}{3\beta_1\omega^2}\cos(\omega x)\operatorname{sen}^3(\omega x) - \frac{2\alpha_1\beta_2}{3\alpha_2\beta_1\omega}\operatorname{sen}^4(\omega x)}{W}$$

$$+ \frac{\frac{2}{3\omega}\cos^4(\omega x) - \frac{2\alpha_1}{3\beta_1\omega^2}\operatorname{sen}(\omega x)\cos^3(\omega x) - \frac{2\beta_2}{3\alpha_2}\cos(\omega x) + \frac{2\beta_2\alpha_1}{3\alpha_2\beta_1\omega}\operatorname{sen}(\omega x) - \frac{2\beta_2}{3\alpha_2}\cos(\omega x)\operatorname{sen}^3(\omega x) + \frac{2\alpha_1\beta_2}{3\alpha_2\beta_1\omega}\operatorname{sen}^4(\omega x)}{W}$$

$$= \frac{\left[\frac{2}{3\omega} - \frac{2\beta_2}{3\alpha_2}\right]\cos(\omega x) + \left[\frac{2\beta_2}{3\alpha_2} + \frac{2\beta_2\alpha_1}{3\alpha_2\beta_1\omega}\right]\operatorname{sen}(\omega x) + \left[-\frac{2\beta_2}{3\alpha_2} - \frac{2\alpha_1}{3\beta_1\omega^2}\right]\operatorname{sen}(\omega x)\cos^3(\omega x) + \left[-\frac{2\alpha_1}{3\beta_1\omega^2} - \frac{2\beta_2}{3\alpha_2}\right]\cos(\omega x)\operatorname{sen}^3(\omega x)}{W}$$

$$= \frac{\left[\frac{2}{3\omega} - \frac{2\beta_2}{3\alpha_2}\right]\cos(\omega x) + \left[\frac{2\beta_2}{3\alpha_2} + \frac{2\beta_2\alpha_1}{3\alpha_2\beta_1\omega}\right]\operatorname{sen}(\omega x) - \frac{2}{3\omega^2}\cancel{\left[\frac{\beta_2\omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}\right]}\operatorname{sen}(\omega x)\cos(\omega x)\cancel{\left[\cos^2(\omega x) + \operatorname{sen}^2(\omega x)\right]}}{\cancel{\frac{\beta_2\omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}}$$

$$\boxed{= \frac{\frac{2}{3\omega} - \frac{2\beta_2}{3\alpha_2}}{\frac{\beta_2\omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}\cos(\omega x) + \frac{\frac{2\beta_2}{3\alpha_2} + \frac{2}{3}\frac{\beta_2\alpha_1}{\alpha_2\beta_1\omega}}{\frac{\beta_2\omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}\operatorname{sen}(\omega x) - \frac{1}{3\omega^2}\operatorname{sen}(2\omega x)}$$

* Cálculo de los antiderivados:

i) $\int \cos(wx')\, sen(2wx')\, dx' = \int \cos(wx')\left(2\cos(wx')\, sen(wx')\right) dx'$

$= 2\int \cos^2(wx')\, sen(wx')\, dx'$

Sea $u = \cos(wx') \rightarrow du = -w\, sen(wx')\, dx'$

$= 2\int u^2 \frac{du}{-w} = -\frac{2}{w}\int u^2 du = \frac{-2}{3w} u^3 = -\frac{2}{3w}\cos^3(wx')$

ii) $\int sen(wx')\, sen(2wx')\, dx' = \int sen(wx')\left(2\cos(wx')\, sen(wx')\right) dx'$

$= 2\int sen^2(wx')\cos(wx')\, dx'$

Sea $u = sen(wx') \rightarrow du = w\cos(wx')\, dx'$

$\rightarrow = 2\int u^2 \frac{du}{w} = \frac{2}{w}\int u^2 du = \frac{2u^3}{3w} = \frac{2\, sen^3(wx')}{3w}$

Comprobación: $y(x) = \dfrac{\frac{2}{3w} - \frac{2\beta_2}{3\alpha_2}}{\frac{\beta_2 w^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}\cos(wx) + \dfrac{\frac{2\beta_2}{3\alpha_2} + \frac{2}{3}\frac{\beta_2\alpha_1}{\alpha_2\beta_1 w}}{\frac{\beta_2 w^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}} sen(wx) - \frac{2}{3w^2} sen(wx)\cos(wx)$

$\rightarrow y'(x) = -\dfrac{\frac{2}{3w} - \frac{2\beta_2}{3\alpha_2}}{\frac{\beta_2 w^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}} w\, sen(wx) + \dfrac{\frac{2\beta_2}{3\alpha_2} + \frac{2}{3}\frac{\beta_2\alpha_1}{\alpha_2\beta_1 w}}{\frac{\beta_2 w^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}} w\cos(wx) - \frac{2}{3w}\cos^2(wx) + \frac{2}{3w} sen^2(wx)$

$\rightarrow y''(x) = -\dfrac{\frac{2}{3w} - \frac{2\beta_2}{3\alpha_2}}{\frac{\beta_2 w^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}} w^2\cos(wx) - \dfrac{\frac{2\beta_2}{3\alpha_2} + \frac{2}{3}\frac{\beta_2\alpha_1}{\alpha_2\beta_1 w}}{\frac{\beta_2 w^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}} w^2 sen(wx) + \frac{8}{3}\cos(wx)\, sen(wx)$

Vemos que $y''(x) + w^2 y(x) = \frac{8}{3}\cos(wx)\, sen(wx) - \frac{2}{3w^2} w^2 sen(wx)\cos(wx)$

$= 2\cos(wx)\, sen(wx) = sen(2wx)$

Cumple la ec. dif.

i) $\alpha_1 y(0) + \beta_1 y'(0) = \alpha_1\left[\dfrac{\frac{2}{3w} - \frac{2\beta_2}{3\alpha_2}}{\frac{\beta_2 w^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}\right] + \beta_1\left[\dfrac{\frac{2\beta_2}{3\alpha_2} + \frac{2}{3}\frac{\beta_2\alpha_1}{\alpha_2\beta_1 w}}{\frac{\beta_2 w^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}} w - \frac{2}{3w}\right] = \ldots = 0$

ii) $\alpha_2 y\left(\frac{3\pi}{2w}\right) + \beta_2 y'\left(\frac{3\pi}{2w}\right) = \alpha_2\left[-\dfrac{\frac{3\beta_2}{3\alpha_2} + \frac{2}{3}\frac{\beta_2\alpha_1}{\alpha_2\beta_1 w}}{\frac{\beta_2 w^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}\right] + \beta_2\left[\dfrac{\frac{2}{3w} - \frac{2\beta_2}{3\alpha_2}}{\frac{\beta_2 w^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}} w + \frac{2}{3w}\right] = \ldots = 0$

Cumple las condiciones de Frontera.

Resolver para $f(x) = \cos(2\omega x)$:

$$y(x) = \int_0^{\frac{3\pi}{2\omega}} G(x,x')\,f(x')\,dx' = \int_0^x G(x,x')\cos(2\omega x')\,dx' + \int_x^{\frac{3\pi}{2\omega}} G(x,x')\cos(2\omega x')\,dx'$$

$$= \int_0^x \frac{\left[\cos(\omega x') - \frac{\alpha_1}{\beta_1\omega}\,\text{sen}(\omega x')\right]\left[\cos(\omega x) + \frac{\beta_2\omega}{\alpha_2}\,\text{sen}(\omega x)\right]}{W}\cos(2\omega x')\,dx'$$

con $W \equiv \frac{\beta_2\omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}$

$$+ \int_x^{\frac{3\pi}{2\omega}} \frac{\left[\cos(\omega x') + \frac{\beta_2\omega}{\alpha_2}\,\text{sen}(\omega x')\right]\left[\cos(\omega x) - \frac{\alpha_1}{\beta_1\omega}\,\text{sen}(\omega x)\right]}{W}\cos(2\omega x')\,dx'$$

$$= \frac{\cos(\omega x) + \frac{\beta_2\omega}{\alpha_2}\,\text{sen}(\omega x)}{W}\left[\int_0^x \cos(\omega x')\cos(2\omega x')\,dx' - \frac{\alpha_1}{\beta_1\omega}\int_0^x \text{sen}(\omega x')\cos(2\omega x')\,dx'\right]$$

$$+ \frac{\cos(\omega x) - \frac{\alpha_1}{\beta_1\omega}\,\text{sen}(\omega x)}{W}\left[\int_x^{\frac{3\pi}{2\omega}} \cos(\omega x')\cos(2\omega x')\,dx' + \frac{\beta_2\omega}{\alpha_2}\int_x^{\frac{3\pi}{2\omega}} \text{sen}(\omega x')\cos(2\omega x')\,dx'\right]$$

* Las antiderivadas las resolvemos en la siguiente hoja

$$= \frac{\cos(\omega x) + \frac{\beta_2\omega}{\alpha_2}\,\text{sen}(\omega x)}{W}\left[\frac{1}{\omega}\,\text{sen}(\omega x') - \frac{2}{3\omega}\,\text{sen}^3(\omega x')\Big|_0^x - \frac{\alpha_1}{\beta_1\omega}\left(\frac{1}{\omega}\cos(\omega x') - \frac{2}{3\omega}\cos^3(\omega x')\right)\Big|_0^x\right]$$

$$+ \frac{\cos(\omega x) - \frac{\alpha_1}{\beta_1\omega}\,\text{sen}(\omega x)}{W}\left[\frac{1}{\omega}\,\text{sen}(\omega x') - \frac{2}{3\omega}\,\text{sen}^3(\omega x')\Big|_x^{\frac{3\pi}{2\omega}} + \frac{\beta_2\omega}{\alpha_2}\left(\frac{1}{\omega}\cos(\omega x') - \frac{2}{3\omega}\cos^3(\omega x')\right)\Big|_x^{\frac{3\pi}{2\omega}}\right]$$

$$= \frac{\cos(\omega x) + \frac{\beta_2\omega}{\alpha_2}\,\text{sen}(\omega x)}{W}\left[\frac{1}{\omega}\,\text{sen}(\omega x) - \frac{2}{3\omega}\,\text{sen}^3(\omega x) - \frac{\alpha_1}{\beta_1\omega^2}\cos(\omega x) + \frac{2\alpha_1}{3\beta_1\omega^2}\cos^3(\omega x) + \frac{\alpha_1}{3\beta_1\omega^2}\right]$$

$$+ \frac{\cos(\omega x) - \frac{\alpha_1}{\beta_1\omega}\,\text{sen}(\omega x)}{W}\left[-\frac{1}{3\omega} - \frac{1}{\omega}\,\text{sen}(\omega x) + \frac{2}{3\omega}\,\text{sen}^3(\omega x) - \frac{\beta_2}{\alpha_2}\cos(\omega x) + \frac{2\beta_2}{3\alpha_2}\cos^3(\omega x)\right]$$

$$= \Big[\frac{1}{\omega}\cos(\omega x)\,\text{sen}(\omega x) + \frac{\beta_2}{\alpha_2}\,\text{sen}^2(\omega x) - \frac{2}{3\omega}\cos(\omega x)\,\text{sen}^3(\omega x) - \frac{2\beta_2}{3\alpha_2}\,\text{sen}^4(\omega x) - \frac{\alpha_1}{\beta_1\omega^2}\cos^2\omega x - \frac{\alpha_1\beta_2}{\alpha_2\beta_1\omega}\cos\omega x\,\text{sen}\,\omega x + \frac{2\alpha_1}{3\beta_1\omega^2}\cos^4\omega x$$
$$+ \frac{2\alpha_1\beta_2}{3\alpha_2\beta_1\omega}\,\text{sen}(\omega x)\cos^3(\omega x) + \frac{\alpha_1}{3\beta_1\omega^2}\cos(\omega x) + \frac{\alpha_1\beta_2}{3\beta_1\alpha_2\omega}\,\text{sen}(\omega x)\Big] / W$$

$$+ \Big[-\frac{1}{3\omega}\cos\omega x + \frac{\alpha_1}{3\beta_1\omega^2}\,\text{sen}(\omega x) - \frac{1}{\omega}\cos\omega x\,\text{sen}\,\omega x + \frac{\alpha_1}{\beta_1\omega^2}\,\text{sen}^2(\omega x) + \frac{2}{3\omega}\cos\omega x\,\text{sen}^3\omega x - \frac{2\alpha_1}{3\beta_1\omega^2}\,\text{sen}^4(\omega x) - \frac{\beta_2}{\alpha_2}\cos^2\omega x + \frac{\alpha_1\beta_2}{\beta_1\alpha_2\omega}\cos\omega x\,\text{sen}\,\omega x$$
$$+ \frac{2\beta_2}{3\alpha_2}\cos^4(\omega x) - \frac{2\alpha_1\beta_2}{3\beta_1\alpha_2\omega}\cos^3(\omega x)\,\text{sen}(\omega x)\Big] / W$$

$$= \Big(\left[\frac{\beta_2}{\alpha_2} + \frac{\alpha_1}{\beta_1\omega^2}\right]\text{sen}^2(\omega x) + \left[-\frac{2\beta_2}{3\alpha_2} - \frac{2\alpha_1}{3\beta_1\omega^2}\right]\text{sen}^4(\omega x) + \left[-\frac{\alpha_1}{\beta_1\omega^2} - \frac{\beta_2}{\alpha_2}\right]\cos^2\omega x + \left[\frac{2\alpha_1}{3\beta_1\omega^2} + \frac{2\beta_2}{3\alpha_2}\right]\cos^4(\omega x)$$
$$+ \left[\frac{\alpha_1}{3\beta_1\omega^2} - \frac{1}{3\omega}\right]\cos\omega x + \left[\frac{\alpha_1\beta_2}{3\beta_1\alpha_2\omega} + \frac{\alpha_1}{3\beta_1\omega^2}\right]\text{sen}(\omega x)\Big) / W$$

$$= \frac{\frac{1}{\omega^2}\left[\frac{\beta_2\omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}\right]\left[\text{sen}^2\omega x - \cos^2\omega x\right] + \frac{2}{3\omega^2}\left[\frac{\beta_2\omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}\right]\left[\cos^4(\omega x) - \text{sen}^4(\omega x)\right] + \left[\frac{\alpha_1}{3\beta_1\omega^2} - \frac{1}{3\omega}\right]\cos\omega x + \left[\frac{\alpha_1\beta_2}{3\beta_1\alpha_2\omega} + \frac{\alpha_1}{3\beta_1\omega^2}\right]\text{sen}(\omega x)}{\frac{\beta_2\omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}$$

porque $\text{sen}^2\omega x - \cos^2\omega x = -\cos(2\omega x)$
$\cos^4\omega x - \text{sen}^4\omega x = \cos(2\omega x)$

$$= -\frac{1}{\omega^2}\cos(2\omega x) + \frac{2}{3\omega^2}\cos(2\omega x) + \frac{\frac{\alpha_1}{3\beta_1\omega^2} - \frac{1}{3\omega}}{\frac{\beta_2\omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}\cos(\omega x) + \frac{\frac{\alpha_1\beta_2}{3\beta_1\alpha_2\omega} + \frac{\alpha_1}{3\beta_1\omega^2}}{\frac{\beta_2\omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}\,\text{sen}(\omega x)$$

$$\Rightarrow \boxed{y(x) = -\frac{1}{3\omega^2}\cos(2\omega x) + \frac{\frac{\alpha_1}{3\beta_1\omega^2} - \frac{1}{3\omega}}{\frac{\beta_2\omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}\cos(\omega x) + \frac{\frac{\alpha_1\beta_2}{3\beta_1\alpha_2\omega} + \frac{\alpha_1}{3\beta_1\omega^2}}{\frac{\beta_2\omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}\,\text{sen}(\omega x)}$$

* Cálculo de Antiderivadas

•) $\int \cos(\omega x') \cos(2\omega x') = \int \cos(\omega x') \left[1 - 2\,\text{sen}^2(\omega x') \right] dx'$

hacemos $u = \text{sen}(\omega x')$

$\rightarrow du = \omega \cos(\omega x')\, dx'$

$\hookrightarrow = \int 1 - 2u^2 \, \frac{du}{\omega} = u - \frac{2}{3}u^3 = \frac{1}{\omega}\,\text{sen}(\omega x') - \frac{2}{3\omega}\,\text{sen}^3(\omega x')$

••) $\int \text{sen}(\omega x') \cos(2\omega x')\, dx' = \int \text{sen}(\omega x') \left(-1 + 2\cos^2(\omega x')\right) dx'$

Sea $u = \cos(\omega x')$

$\rightarrow du = \omega\,\text{sen}(\omega x')\, dx'$

$\hookrightarrow = \int (-1 + 2u^2)\left(-\frac{1}{\omega}\, du\right) = \int \frac{1}{\omega}\, du - \int \frac{2u^2}{\omega} = \frac{1}{\omega} u - \frac{2u^3}{3\omega} = \frac{1}{\omega}\cos(\omega x') - \frac{2}{3\omega}\cos^3(\omega x')$

Comprobación

$$y(x) = -\frac{1}{3\omega^2}\cos(2\omega x) + \frac{\frac{\alpha_1}{3\beta_1\omega^2} - \frac{1}{3\omega}}{\frac{\beta_2\omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}\cos(\omega x) + \frac{\frac{\alpha_1\beta_2}{3\beta_1\alpha_2\omega} + \frac{\alpha_1}{3\beta_1\omega^2}}{\frac{\beta_2\omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}\operatorname{sen}(\omega x)$$

$$\Rightarrow y'(x) = \frac{2}{3\omega}\operatorname{sen}(2\omega x) - \frac{\frac{\alpha_1}{3\beta_1\omega^2} - \frac{1}{3\omega}}{\frac{\beta_2\omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}\,\omega\operatorname{sen}(\omega x) + \frac{\frac{\alpha_1\beta_2}{3\beta_1\alpha_2\omega} + \frac{\alpha_1}{3\beta_1\omega^2}}{\frac{\beta_2\omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}\,\omega\cos(\omega x)$$

$$\Rightarrow y''(x) = \frac{4}{3}\cos(2\omega x) - \frac{\frac{\alpha_1}{3\beta_1\omega^2} - \frac{1}{3\omega}}{\frac{\beta_2\omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}\,\omega^2\cos(\omega x) - \frac{\frac{\alpha_1\beta_2}{3\beta_1\alpha_2\omega} + \frac{\alpha_1}{3\beta_1\omega^2}}{\frac{\beta_2\omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}\,\omega^2\operatorname{sen}(\omega x)$$

$$\Rightarrow \cdot)\quad y''(x) + \omega^2 y(x) = \frac{4}{3}\cos(2\omega x) - \frac{1}{3}\frac{\omega^2}{\omega^2}\cos(2x\omega) = \underline{\cos(2\omega x)}\ \checkmark$$

← Cumple la ec. dif.

$$\cdot\cdot)\quad \alpha_1 y(0) + \beta_1 y'(0) = \alpha_1\left[-\frac{1}{3\omega^2} + \frac{\frac{\alpha_1}{3\beta_1\omega^2} - \frac{1}{3\omega}}{\frac{\beta_2\omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}\right] + \beta_1\left[\frac{\frac{\alpha_1\beta_2}{3\beta_1\alpha_2\omega_2} + \frac{\alpha_1}{3\beta_1\omega^2}}{\frac{\beta_2\omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}\,\omega\right] = \ldots = 0\ \checkmark$$

$$\cdot\cdot\cdot)\quad \alpha_2 y\left(\frac{3\pi}{2\omega}\right) + \beta_2 y'\left(\frac{3\pi}{2\omega}\right) = \alpha_2\left[-\frac{\frac{\alpha_1\beta_2}{3\beta_1\alpha_2\omega} + \frac{\alpha_1}{3\beta_1\omega^2}}{\frac{\beta_2\omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}\right] + \beta_2\left[\frac{\frac{\alpha_1}{3\beta_1\omega^2} - \frac{1}{3\omega}}{\frac{\beta_2\omega^2}{\alpha_2} + \frac{\alpha_1}{\beta_1}}\,\omega\right] = \ldots = 0\ \checkmark$$

↑

Cumple condiciones iniciales.

2. Resolver la siguiente EDP con condiciones iniciales

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} = i\hbar \frac{\partial \psi}{\partial t} \quad , \quad (x,t) \in \Omega = [0, L] \times [0, \infty)$$

$$\psi(0,t) = \psi(L,t) = 0 \qquad t \geq 0$$

$$\psi(x,0) = \sin^2(\pi x / L) \quad , \quad x \in [0, L]$$

Proponemos una solución separable de la forma $\psi(x,t) = X(x)\, T(t)$ y sustituimos:

$$-\frac{\hbar^2}{2m}\frac{\partial^2 (X(x)\, T(t))}{\partial x^2} = i\hbar \frac{\partial (X(x)\, T(t))}{\partial t}$$

$$\Rightarrow -\frac{\hbar^2}{2m} T(t) \frac{d^2 X}{dx^2} = i\hbar\, X(x) \frac{dT(t)}{dt}$$

$$\Rightarrow \frac{1}{X(x)}\frac{d^2 X}{dx^2} = -\frac{2mi}{\hbar}\frac{1}{T(t)}\frac{dT(t)}{dt}$$

Ambos lados son iguales $\forall\ x, t$ pero dependen de distintas variables $\Rightarrow$ deben de ser constantes

$\Rightarrow$ Igualamos a una constante $-k^2$

$$\rightarrow \frac{1}{X(x)}\frac{d^2 X}{dx^2} = -k^2 \quad , \quad -\frac{2mi}{\hbar}\frac{1}{T(t)}\frac{dT(t)}{dt} = -k^2$$

$$\rightarrow \frac{d^2 X}{dx} = -k^2 X(x) \quad , \quad \frac{dT(t)}{dt} = \frac{\hbar k^2}{2mi} T(t) = -\frac{\hbar k^2 i}{2m} T(t)$$

Solución (por inspección)

Solución (por inspección)

$$T(t) = e^{-\frac{\hbar k^2 i}{2m} t}$$

← no le ponemos coeficiente C porque igual al multiplicar por $X(x)$, esta constante se "absorbería" con A y B

$$X(x) = A\cos(kx)$$
$$X(x) = B\,\mathrm{sen}(kx)$$

$$\Rightarrow X(x) = A\cos(kx) + B\,\mathrm{sen}(kx)$$

$\therefore$ Solución general: $\psi(x,t) = X(x)\, T(t) = e^{-\frac{\hbar k^2 i}{2m} t}\left[A\cos(kx) + B\,\mathrm{sen}(kx)\right]$

Aplicamos las condiciones de frontera:

◦) $\Psi(0,t) = 0$

$\rightarrow e^{-\frac{\hbar k^2 i}{2m}t}\left[A\cos(0) + B\,\text{sen}(0)\right] = 0 \rightarrow e^{-\frac{\hbar k^2 i}{2m}t}[A] = 0 \quad \forall t \quad \Rightarrow \quad A = 0$

◦◦) $\Psi(L,t) = 0$

$\rightarrow e^{-\frac{\hbar k^2 i}{2m}t}\left[\cancel{A^0\cos(kL)} + B\,\text{sen}(kL)\right] = 0 \quad \Rightarrow \quad B e^{-\frac{\hbar k^2 i t}{2m}}\,\text{sen}(kL) = 0 \quad \Rightarrow \quad k_n L = \pi n$ ← con $n \in \mathbb{N}$

$\rightarrow k_n = \frac{\pi n}{L}$

$\therefore \Psi(x,t) = e^{-\frac{\hbar k_n^2 i}{2m}t} B_n\,\text{sen}\left(\frac{\pi n}{L}x\right)$

Escribimos la solución general como combinación lineal de estas soluciones

$$\Psi(x,t) = \sum_{n=1}^{\infty} e^{-\frac{\hbar k_n^2 i t}{2m}} B_n\,\text{sen}\left(\frac{\pi n}{L}x\right)$$

Empezamos en $n=1$, porque en $n=0$ nos queda $e^{-\frac{\hbar k_0 i t}{2m}} B_0\,\text{sen}\left(\frac{\pi(0)}{L}x\right) = 0$

Queremos que cumpla con la última condición de frontera:

$\Psi(x,0) = \text{sen}^2(\pi x/L)$

$$\Rightarrow \sum_{n=1}^{\infty} e^{-\frac{\hbar k_n^2 i (0)}{2m}} B_n\,\text{sen}\left(\frac{\pi n}{L}x\right) = \text{sen}^2(\pi x/L)$$

$$\rightarrow \sum_{n=1}^{\infty} e^{0} B_n\,\text{sen}\left(\frac{\pi n}{L}x\right) = \text{sen}^2\left(\frac{\pi x}{L}\right) \quad \Rightarrow \quad \sum_{n=1}^{\infty} B_n\,\text{sen}\left(\frac{\pi n}{L}x\right) = \text{sen}^2\left(\frac{\pi x}{L}\right)$$

Buscamos los B_n apropiados tales que esta serie de senos sea correcta.

Se trata de una serie de senos en el intervalo $[0,L]$ y como hemos visto en clase y usado varias veces, los coeficientes para una serie de este tipo están dados por

$$B_n = \frac{2}{L}\int_0^L f(x)\,\text{sen}\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L \text{sen}^2\left(\frac{\pi x}{L}\right)\text{sen}\left(\frac{n\pi x}{L}\right)dx$$

Lo cual nos da los coeficientes para la serie de senos de la extensión impar en $[-L,L]$ de la función $\text{sen}^2\left(\frac{\pi x}{L}\right)$

$$\Rightarrow B_n = \frac{2}{L}\int_0^L \operatorname{sen}^2\left(\frac{\pi x}{L}\right)\operatorname{sen}\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_0^L \frac{1}{2}\left[1-\cos\left(\frac{2\pi x}{L}\right)\right]\operatorname{sen}\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\int_0^L \operatorname{sen}\left(\frac{n\pi x}{L}\right) - \cos\left(\frac{2\pi x}{L}\right)\operatorname{sen}\left(\frac{n\pi x}{L}\right)\, dx$$

← identidad de producto a suma de $\cos x \operatorname{sen} y = \frac{1}{2}\operatorname{sen}(x+y) - \frac{1}{2}\operatorname{sen}(x-y)$

$$= \frac{1}{L}\int_0^L \operatorname{sen}\left(\frac{n\pi x}{L}\right) - \frac{1}{2}\left[\operatorname{sen}\left(\frac{2\pi x}{L}+\frac{n\pi x}{L}\right) - \operatorname{sen}\left(\frac{2\pi x}{L}-\frac{n\pi x}{L}\right)\right] dx$$

$$= \frac{1}{L}\int_0^L \operatorname{sen}\left(\frac{n\pi x}{L}\right)dx - \frac{1}{2L}\int_0^L \operatorname{sen}\left((2+n)\frac{\pi}{L}x\right)dx + \frac{1}{2L}\int_0^L \operatorname{sen}\left((2-n)\frac{\pi x}{L}\right)dx$$

$$= -\frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{1}{2\pi(2+n)}\cos\left((2+n)\frac{\pi}{L}x\right)\Big|_0^L - \frac{1}{2\pi(2-n)}\cos\left((2-n)\frac{\pi x}{L}\right)\Big|_0^L$$

$$= -\frac{1}{n\pi}\cos(n\pi) + \frac{1}{n\pi} + \frac{1}{2\pi(2+n)}\cos\left((2+n)\pi\right) - \frac{1}{2\pi(2+n)} - \frac{1}{2\pi(2-n)}\cos\left((2-n)\pi\right) + \frac{1}{2\pi(2-n)}$$

$$= -\frac{1}{n\pi}\cos(n\pi) + \frac{1}{2\pi(2+n)}\cos((2+n)\pi) - \frac{1}{2\pi(2-n)}\cos((2-n)\pi) + \frac{4}{4n\pi - n^3\pi}$$

$$= -\frac{1}{n\pi}\cos(n\pi) + \frac{1}{2\pi(2+n)}\cos(n\pi+2\pi) - \frac{1}{2\pi(2-n)}\cos(n\pi-2\pi) + \frac{4}{4n\pi-n^3\pi}$$

$$= \left(-\frac{1}{n\pi} + \frac{1}{2\pi(2+n)} - \frac{1}{2\pi(2-n)}\right)\cos(n\pi) + \frac{4}{4n\pi - n^3\pi}$$

$$= \frac{4}{\pi(n^3-4n)}\cos(n\pi) - \frac{4}{\pi(n^3-4n)}$$

↳ Para n par: $\cos(n\pi) = 1 \rightarrow B_n = \frac{4}{\pi(n^3-4n)} - \frac{4}{\pi(n^3-4n)} = 0$

Para n impar: $\cos(n\pi) = -1 \rightarrow B_n = \frac{-4}{\pi(n^3-4n)} - \frac{4}{\pi(n^3-4n)} = \frac{-8}{\pi(n^3-4n)}$

Entonces, con estos B_n calculados, la solución completa queda como:

$$\Psi(x,t) = \sum_{n \text{ impar}} e^{-\frac{\hbar k_n^2 i}{2m}t}\left(-\frac{8}{\pi(n^3-4n)}\right)\operatorname{sen}\left(\frac{\pi n}{L}x\right)$$

← $k_n = \frac{\pi n}{L}$

$$= \sum_{j=0}^{\infty} e^{-\frac{\hbar\left(\frac{\pi(2j+1)}{L}\right)^2 i}{2m}t}\left(-\frac{8}{\pi[(2j+1)^3 - 4(2j+1)]}\right)\operatorname{sen}\left(\frac{\pi(2j+1)}{L}x\right)$$

← hacemos $n = 2j+1$ para que n corra en los impares.

$$= \frac{8}{\pi}\sum_{j=0}^{\infty} e^{-\frac{\hbar\left(\frac{\pi(2j+1)}{L}\right)^2 i}{2m}t}\left(\frac{1}{4(2j+1)-(2j+1)^3}\right)\operatorname{sen}\left(\frac{\pi(2j+1)}{L}x\right)$$

Extra: Usando la transformada de Fourier, dar una solución $u(x,t)$ al EDP

$$\frac{\partial u}{\partial t} = \frac{i\lambda}{4\pi}\frac{\partial^2 u}{\partial x^2} \qquad u(x,0) = f(x)$$

y calcular la solución explícita cuando $f(x) = \delta(x-a) + \delta(x+a)$, $a > 0$

Proponemos una solución de la forma $u(x,t) = X(x)\,T(t)$ y sustituimos

$$\frac{\partial (X(x)\,T(t))}{\partial t} = \frac{i\lambda}{4\pi}\frac{\partial^2 (X(x)\,T(t))}{\partial x^2}$$

$$\rightarrow X(x)\frac{dT(t)}{dt} = \frac{i\lambda}{4\pi}T(t)\frac{d^2X(x)}{dx^2} \quad \rightarrow \quad \frac{1}{T(t)}\frac{dT(t)}{dt} = \frac{i\lambda}{4\pi}\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}$$

$$\rightarrow -\frac{4\pi}{\lambda}i\,\frac{1}{T(t)}\frac{dT(t)}{dt} = \frac{1}{X(x)}\frac{d^2X(x)}{dx^2}$$

Ambos lados son iguales $\forall\, x,t$. Pero el lado izquierdo depende de t y el derecho de x $\Rightarrow$ deben de ser iguales a una constante $-k^2$

$$\rightarrow \frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -k^2 \quad , \quad -\frac{4\pi}{\lambda}i\,\frac{1}{T(t)}\frac{dT(t)}{dt} = -k^2$$

$$\rightarrow \frac{d^2X(x)}{dx^2} = -k^2X(x) \qquad \rightarrow \frac{dT(t)}{dt} = \frac{-\lambda k^2}{4\pi}i\,T(t)$$

Soluciones por inspección

$$X(x) = Ae^{ikx} + Be^{-ikx} \qquad T(t) = e^{-\frac{\lambda k^2}{4\pi}it}$$

no incluimos coeficiente C porque al multiplicar por $X(x)$ igual sería "absorbido"

Entonces: $u(x,t) = X(x)\,T(t) = \left(Ae^{ikx} + Be^{-ikx}\right)e^{-\frac{\lambda k^2}{4\pi}it}$

No tenemos condiciones de frontera, por lo que k es libre de tomar cualquier valor real.
Como ya tenemos el término Ae^{ikx}, no hace falta considerar el Be^{-ikx} que se conseguiría al cambiar k por $-k$.
Para cada k podemos poner un coeficiente $A(k)$ distito que dependa de k

Entonces, la solución es: $u(x,t) = A(k)\,e^{ikx}\,e^{-\frac{\lambda k^2}{4\pi}it}$

Y la solución más general se consigue al "sumar" sobre las k (integrar de hecho porque k corre en $\mathbb{R}$)

$$\Rightarrow u(x,t) = \int_{\mathbb{R}} A(k)\,e^{ikx}\,e^{-\frac{\lambda k^2}{4\pi}it}\,dk$$

Ahora bien, tenemos la condición inicial que $u(x,0) = f(x)$:

$$f(x) = u(x,0) = \int_{\mathbb{R}} A(k)\, e^{ikx}\, e^{-\frac{\lambda k^2}{4\pi}(0)i}\, dk$$

$$= \int_{\mathbb{R}} A(k)\, e^{ikx}\, dk$$

$$\rightarrow \quad f(x) = \int_{\mathbb{R}} A(k)\, e^{ikx}\, dk$$

Pero la integral ↑ es la definición de la transformada inversa de Fourier de $A(k)$

Por lo que f y A son una pareja de transformadas de Fourier. ← Teorema de inversión

$\Rightarrow$ A es la transformada de f

$$\Rightarrow \quad A(k) = \frac{1}{2\pi} \int_{\mathbb{R}} f(x)\, e^{-ikx}\, dx$$

Y por tanto, la solución general con condición $u(x,0) = f(x)$ es:

$$u(x,t) = \int_{\mathbb{R}} A(k)\, e^{ikx}\, e^{-\frac{\lambda k^2}{4\pi} it}\, dk$$ con $A(k)$ la transformada de f

Calcula cuando $f(x) = \delta(x-a) + \delta(x+a)$

Calculamos primero la transformada de $f(x)$

$$\rightarrow \quad A(k) = \frac{1}{2\pi} \int_{\mathbb{R}} (\delta(x-a) + \delta(x+a))\, e^{-ikx}\, dx$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \delta(x-a)\, e^{-ikx}\, dx + \frac{1}{2\pi} \int \delta(x+a)\, e^{-ikx}\, dx$$

$$= \frac{1}{2\pi} e^{-iak} + \frac{1}{2\pi} e^{iak} = \frac{e^{iak} + e^{-iak}}{2\pi} = \frac{1}{\pi} \cos(ak)$$

Por la expresión compleja de cos

Entonces, la solución a la ecuación diferencial es:

$$u(x,t) = \int_{\mathbb{R}} \frac{1}{\pi} \cos(ak)\, e^{ikx}\, e^{-\frac{\lambda k^2}{4\pi} it}\, dk$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \left(e^{iak} + e^{-iak}\right) e^{ikx}\, e^{-\frac{\lambda k^2}{4\pi} it}\, dk$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\left(ak + kx - \frac{\lambda}{4\pi} t k^2\right)} + e^{i\left(-ak + kx - \frac{\lambda}{4\pi} t k^2\right)}\, dk$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\left[(a+x)k - \frac{\lambda t}{4\pi} k^2\right]} + e^{i\left[(-a+x)k - \frac{\lambda t}{4\pi} k^2\right]}\, dk$$

Para resolver estas integrales, resolveré primero el caso más general de:

$$\int_{\mathbb{R}} e^{i(\alpha k + \beta k^2)}\,dk = \int_{\mathbb{R}} e^{\beta i\left(k^2 + \frac{\alpha}{\beta}k + \frac{\alpha^2}{4\beta^2}\right) - \frac{\alpha^2}{4\beta}i}\,dk$$

completar el cuadrado

$$= \int_{\mathbb{R}} e^{\beta i\left(k + \frac{\alpha}{2\beta}\right)^2} e^{-\frac{\alpha^2}{4\beta}i}\,dk$$

$$= e^{-\frac{\alpha^2}{4\beta}i} \int_{\mathbb{R}} e^{\beta i\left(k + \frac{\alpha}{2\beta}\right)^2}\,dk$$

$$= e^{-\frac{\alpha^2}{4\beta}i} \int_{\mathbb{R}} e^{\beta i u^2}\,du$$

hacemos la sustitución $u = k + \frac{\alpha}{2\beta}$ $\rightarrow du = dk$, $k \in (-\infty, \infty)$, $u \in (-\infty, \infty)$

$$= \frac{e^{-\frac{\alpha^2}{4\beta}i}}{\sqrt{\beta}} \int_{\mathbb{R}} e^{iv^2}\,dv$$

hacemos $v = \sqrt{\beta}u \rightarrow v^2 = \beta u^2$
$\rightarrow dv = \sqrt{\beta}\,du$

$$= \frac{e^{-\frac{\alpha^2}{4\beta}i}}{\sqrt{\beta}} \int_{\mathbb{R}} \cos(v^2)\,dv + i\frac{e^{-\frac{\alpha^2}{4\beta}i}}{\sqrt{\beta}} \int_{\mathbb{R}} \operatorname{sen}(v^2)\,dv$$

Éstas son las conocidas integrales de Fresnel que dan $\sqrt{\frac{\pi}{2}}$ cada una

$$= \frac{e^{-\frac{\alpha^2}{4\beta}i}}{\sqrt{\beta}} \left[\sqrt{\frac{\pi}{2}} + i\sqrt{\frac{\pi}{2}}\right]$$

$$= \sqrt{\frac{\pi}{2\beta}}\, e^{-\frac{\alpha^2}{4\beta}i}(1+i) \quad \ldots (1)$$

Entonces, siguiendo el desarrollo de antes,

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\left[(a+x)k - \frac{\lambda t}{4\pi}k^2\right]} + e^{i\left[(-a+x)k - \frac{\lambda t}{4\pi}k^2\right]}\,dk$$

Las dos integrales tienen la forma $\int_{\mathbb{R}} e^{i[\alpha k + \beta k^2]}\,dk$ con $\alpha = a+k$, $\beta = -\frac{\lambda t}{4\pi}$, $\alpha = -a+x$, $\beta = -\frac{\lambda t}{4\pi}$ Usamos el resultado (1)

$$= \frac{1}{2\pi}\sqrt{\frac{\pi}{\frac{-\lambda t}{2\pi}}}\, e^{-\frac{(a+x)^2}{-\lambda t/\pi}i}(1+i) + \frac{1}{2\pi}\sqrt{\frac{\pi}{\frac{-\lambda t}{2\pi}}}\, e^{-\frac{(-a+x)^2}{-\lambda t/\pi}i}(1+i)$$

$$= i\sqrt{\frac{1}{2\lambda t}}(1+i)\left[e^{\frac{\pi(a+x)^2}{\lambda t}i} + e^{\frac{\pi(-a+x)^2}{\lambda t}i}\right]$$

$$= (-1+i)\sqrt{\frac{1}{2\lambda t}}\left[e^{\frac{\pi(a+x)^2}{\lambda t}i} + e^{\frac{\pi(-a+x)^2}{\lambda t}i}\right]$$