Mini - tarea 9 Tomás Ricardo Basile Álvarez.

1. La exponencial de una matriz $\mathbb{A}$ de $n \times n$ es: $\exp(\mathbb{A}t) = \mathbb{1} + \mathbb{A}t + \frac{1}{2!}\mathbb{A}^2 t^2 + \frac{1}{3!}\mathbb{A}^3 t^3 + \dots t.$

Calcula los primeros tres términos de la exponencial. Encuentre también los eigenvalores y eigenvectores

a) $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Eigenvalores: $\det(A - \lambda \mathbb{I}) = 0 \rightarrow \det \begin{pmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} = 0 \rightarrow (1-\lambda)(2-\lambda) = 0$

$\rightarrow \lambda_1 = 1 \quad , \lambda_2 = 2$

Eigenvectores: $\cdot) \ \underline{\lambda_1 = 1}$: $\quad Nuc(A - 1\mathbb{I}) = Nuc\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow y = 0 \qquad \rightarrow Nuc\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \left\langle \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \right\rangle$

$\therefore$ Su eigenvector es $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$\cdot\cdot) \ \underline{\lambda_2 = 2}$ $\quad Nuc(A - 2\mathbb{I}) = Nuc\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = \left\langle \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \right\rangle$

$\therefore$ Su eigenvector es $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

$\mathbb{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad \mathbb{A}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \qquad \mathbb{A}^3 = \mathbb{A}^2 \mathbb{A} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}$

$$\therefore e^{\mathbb{A}t} \approx \mathbb{1} + \mathbb{A}t + \frac{1}{2}\mathbb{A}^2 t^2 + \frac{1}{6}\mathbb{A}^3 t^3$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} t^2 + \frac{1}{6}\begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} t^3$$

$$= \begin{pmatrix} 1 + t + \frac{t^2}{2} + \frac{t^3}{6} & 0 \\ 0 & 1 + 2t + 2t^2 + \frac{4}{3}t^3 \end{pmatrix}$$

b) $A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$

Eigenvalores: $\det(\mathbb{A} - \lambda \mathbb{I}) = \det\begin{pmatrix} -\lambda & 1 \\ 2 & -\lambda \end{pmatrix} = \lambda^2 - 2 = 0 \rightarrow \underline{\lambda_1 = \sqrt{2}, \ \lambda_2 = -\sqrt{2}}$

Eigenvectores: •) $\lambda_1 = \sqrt{2}$

• $\text{Nuc}(\mathbb{A} - \lambda_1 \mathbb{I}) = \text{Nuc}\begin{pmatrix} -\sqrt{2} & 1 \\ 2 & -\sqrt{2} \end{pmatrix} \rightarrow \begin{matrix} -\sqrt{2}x + y = 0 \\ 2x - \sqrt{2}y = 0 \end{matrix} \rightarrow y = \sqrt{2}x$

$\therefore$ Eigenvector es: $\underline{\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}}$

••) $\lambda_2 = -\sqrt{2}$

$\text{Nuc}(\mathbb{A} - \lambda_2 \mathbb{I}) = \text{Nuc}\begin{pmatrix} \sqrt{2} & 1 \\ 2 & \sqrt{2} \end{pmatrix} \rightarrow \begin{matrix} \sqrt{2}x + y = 0 \\ 2x + \sqrt{2}y = 0 \end{matrix} \rightarrow y = -\sqrt{2}x$

$\therefore$ Eigenvector es: $\underline{\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}}$

exponencial:

$$\mathbb{A} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \rightarrow \mathbb{A}^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \rightarrow \mathbb{A}^3 = \mathbb{A}^2\mathbb{A} = \begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix}$$

$$\rightarrow e^{\mathbb{A}t} \approx \mathbb{1} + \mathbb{A}t + \frac{1}{2}\mathbb{A}^2 t^2 + \frac{1}{6}\mathbb{A}^3 t^3$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} t^2 + \frac{1}{6}\begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix} t^3$$

$$= \underline{\begin{pmatrix} 1 + t^2 & t + \frac{1}{3}t^3 \\ 2t + \frac{2}{3}t^3 & 1 + t^2 \end{pmatrix}}$$

c) $A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$

Eigenvalores: $\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -2 & -\lambda \end{pmatrix} = \lambda^2 + 2 = 0 \rightarrow \lambda_1 = \sqrt{2}\, i \quad \lambda_2 = -\sqrt{2}\, i$

Eigenvectores: ·) $\lambda_1 = \sqrt{2}\, i$

$\text{Nuc}(A - \lambda_1 I) = \text{Nuc}\begin{pmatrix} -\sqrt{2}i & 1 \\ -2 & -\sqrt{2}i \end{pmatrix} \Rightarrow \begin{array}{l} -\sqrt{2}i\,x + y = 0 \\ -2x - \sqrt{2}i\,y = 0 \end{array} \rightarrow y = \sqrt{2}i\,x \rightarrow$ Eigenvector: $\begin{pmatrix} 1 \\ \sqrt{2}i \end{pmatrix}$

··) $\lambda_2 = -\sqrt{2}\, i$

$\text{Nuc}(A - \lambda_2 I) = \text{Nuc}\begin{pmatrix} \sqrt{2}i & 1 \\ -2 & \sqrt{2}i \end{pmatrix} \rightarrow \begin{array}{l} \sqrt{2}i\,x + y = 0 \\ -2x + \sqrt{2}i\,y = 0 \end{array} \rightarrow y = -\sqrt{2}i\,x \rightarrow$ eigenvector: $\begin{pmatrix} 1 \\ -\sqrt{2}i \end{pmatrix}$

$A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \rightarrow A^2 = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \rightarrow A^3 = A^2 A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 4 & 0 \end{pmatrix}$

$\rightarrow e^{At} \approx I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} t^2 + \frac{1}{6}\begin{pmatrix} 0 & -2 \\ 4 & 0 \end{pmatrix} t^3$

$$= \begin{pmatrix} 1 - t^2 & t - \frac{1}{3}t^3 \\ -2t + \frac{2}{3}t^3 & 1 - t^2 \end{pmatrix}$$

d) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Eigenvalores: $\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} \to (1-\lambda)^2 = 0 \to \underline{\lambda_1 = 1}$

$\underline{\lambda_1 = 1}$, Eigenvectores:

$Nuc(A-\lambda_1 I) = Nuc\begin{pmatrix} 1-1 & 1 \\ 0 & 1-1 \end{pmatrix} = Nuc\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to y = 0$

$\to \underline{\bar{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$ es un eigenvector. (pero no hay dos eigenvectores L.i)

Buscamos otro vector $\bar{u}$ con $(A-\lambda I)^2 \bar{u} = 0$

$\to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 \bar{u} = 0 \to \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \bar{u} = 0$ ∴ ¡cualquier $\bar{u}$ sirve

pero para que sea l.i con $\bar{v}$, escogemos $\underline{\bar{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

Exponencial

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $\quad A^3 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

$$\to e^{At} \approx I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} t^2 + \frac{1}{6}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} t^3$$

$$= \begin{pmatrix} 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 & t + t^2 + \frac{1}{2}t^3 \\ 0 & 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 \end{pmatrix} \quad //$$