Quantum Computing

Tomás Ricardo Basile Álvarez
316617194

September 14, 2021

An introduction to Information Theory

Entropy and Shannon

We can calculate the amount of information in a message in a first method by simply counting the amount of bits necessary to transmit the message.

Shannon found another way of finding the amount of information in a message by thinking on the probability of getting said message.
He caracterizes the amount of information of a message using the probability of it happening. If it is a high probability event, the message has low information, it is not telling us anything interesting.

If p is the probability of a given message occurring, we define its information content I as:

$$I = -\log_2 p$$

So low probability events have a high information (example: $p = 0.005$ has a information of 7.6439).

We develop now a better definition:
Let X be a random variable characterized by a probability distribution $\vec{p}$, and suppose that it can assume one of the values $x_1, x_2, \cdots, x_n$ with probabilities $p_1, \cdots, p_n$ such that $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$.
The Shannon entropy of X is defined as:

$$H(X) = -\sum_i p_i \log_2 p_i$$

If the probability of a given x_j is 0, we use $0 \log 0 = 0$.
So this is a measurement of the amount of randomness or uncertainty in a signal. $\log_2 p_i$ is the information of the state x_i and we multiply it by the probability of that state happening.

Examples:

- Suppose a signal transmits $222222222222\cdots$, we want the entropy. The probability of obtaining a 2 is 1, so the entropy of the signal is:

$$H = -\log_2 1 = 0$$

- The signal is 12122122112112 with 0.5 probability of a 1 or a 2. Then, the entropy is:

$$H = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2} = 1$$

- 12332112231231231232 with 0.5 probability of a 1,2,3. Then the entropy is:

$$H = -\frac{1}{3}\log_2\frac{1}{3} - \frac{1}{3}\log_2\frac{1}{3} - \frac{1}{3}\log_2\frac{1}{3} = \cdots = 1.585$$

Shannon entropy measures the amount of uncertainty in the signal. If the signal is certain, Shannon gives 0.

Now suppose we require l_i bits to represent each x_i, then the **average bit rate** required to encode X is:

$$R_X = \sum_{l=1}^{n} l_i p_i$$

The Shannon entropy is less than the bit rate:

$$H(X) \leq R_X$$

The worst-case scenario in which we have the least information is when there is a chance $p_i = 1/n$ for each x_i.
In this case, $H(X) = -\sum \frac{1}{n}\log_2\frac{1}{n} = \log n$.
So in general:

$$0 \leq H(X) \leq \log_2(n)$$

The **relative entropy** of two variables X, Y characterized by probability distributions p, q is:

$$H(X||Y) = \sum p \log_2\frac{p}{q} = -H(X) - \sum p \log_2 q$$

Conditional Entropy: Suppose that we take a fixed value y_i from Y. From this we can get a conditional probability distribution $p(X|y_i)$ which are the probabilities of X given that y_i is certain. Then:

$$H(X|Y) = -\sum_j p(x_j|y_i)\log_2(p(x_j|y_i))$$

And it satisfies:

$$H(X|Y) \leq H(X)$$

With equality only when X, Y are independent:

$$H(X,Y) = H(Y) + H(X|Y)$$

Mutual information of X, Y: Is the difference in entropy of X and the entropy of X given Y:

$$I(X|Y) := H(X) - H(X|Y) = H(X) + H(Y) - H(X,Y)$$

Qubits and Quantum States

Qubit

Like a bit, a qubit can also be in one of two states, we label these states as $|0\rangle, |1\rangle$.
A qubit can be in a superposition of these two states, the true state of a qubit is $|\psi\rangle$ and is of the form:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Where $\alpha, \beta \in \mathbb{C}$.

Nevertheless, whenever we measure a qubit, we find it in one of its states

- $|\alpha|^2$ is the probability of finding $|\psi\rangle$ in $|0\rangle$
- $|\beta|^2$ is the probability of finding $|\psi\rangle$ in $|1\rangle$

Of course, we must have that:

$$|\alpha|^2 + |\beta|^2 = 1$$

We can write the qubit as:

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

And we can take linear combinations of these types of vectors.
The base vectors are of course:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Inner product

The inner product of two vectors $|u\rangle$ and $|v\rangle$ is written as:

$$\langle u|v\rangle$$

And it satisfies:

- $\langle u|v\rangle^* = \langle v|u\rangle$
- $\langle u|u\rangle \geq 0$
- $\langle u|\alpha v + \beta w\rangle = \alpha\langle u|v\rangle + \beta\langle u|w\rangle$

To calculate the inner product of two vectros, we need to calculate the **Hermitian conjugate** or dual or bra of $|u\rangle$ written as:

$$(|u\rangle)^\dagger = \langle u|$$

And defined as:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}^\dagger = \begin{pmatrix} a_1^* & a_2^* & \cdots & a_n^* \end{pmatrix}$$

That way, the inner product of two vectors is:

$$\langle a|b\rangle = \begin{pmatrix} a_1^* & a_2^* & \cdots & a_n^* \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \sum_{i=1}^{n} a_i^* b_i$$

Orthonormality

A group of vector $\{|u_i\rangle\}$ are orthonormal if:

$$\langle u_i|u_j\rangle = \delta_{ij}$$

Gram-Schmidt Orthogonalization

If $\{|v_i\rangle\}$ is a basis, then we define:

$$\begin{aligned} |w_1\rangle &= |v_1\rangle \\ |w_2\rangle &= |v_2\rangle - \frac{\langle w_1|v_2\rangle}{\langle w_1|w_1\rangle}|w_1\rangle \\ &\vdots \\ |w_n\rangle &= |v_n\rangle - \frac{\langle w_1|v_n\rangle}{\langle w_1|w_1\rangle}|w_1\rangle - \cdots - \frac{\langle w_{n-1}|v_n\rangle}{\langle w_{n-1}|w_{n-1}\rangle}|w_{n-1}\rangle \end{aligned}$$

If we now normalize this set, it becomes an orthonormal basis.

Bra-Kets

If $\{|u_i\rangle\}$ is an orthonormal set, then we can write for any state $|\psi\rangle$ we can write:

$$|\psi\rangle = \sum a_i|u_i\rangle$$

And the probability of finding the particle in a state $|u_i\rangle$ is given by:

$$p_i = |\langle u_i|\psi\rangle|^2 = a_i^2$$

Inequalities

- **Cauchy Schwarz:**

$$|\langle\psi|\phi\rangle|^2 \leq \langle\psi|\psi\rangle\langle\phi|\phi\rangle$$

- **Triangle Inequality**:

$$|\psi + \phi| \leq |\psi| + |\phi|$$

Matrices And Operators

An operator is a mathematical rule that can be applied to a function to convert it in another one.
An operator $\widehat{A}$ is a mathematical rule that transforms a ket $|\psi\rangle$ into another ket $|\phi\rangle$ like:

$$\widehat{A}|\psi\rangle = |\phi\rangle$$

Operators can also act on bras:

$$\langle\mu|\widehat{A} = \langle v|$$

An operator is **linear** is:

$$\widehat{A}(\alpha|\psi_1\rangle + \beta|\psi_2\rangle) = \langle\widehat{A}|\psi_1\rangle + \beta\widehat{A}|\psi_2\rangle$$

Observables:
In QM, dynamical variables like position, momentum, angular momentum, and energy are called observables.
An important postulate in QM is that there is a corresponding operator for every physical observable.

Pauli Operators

Operators can act on qubits.
An important set of operators are the **Pauli Operators**.

The Pauli operators are defined by their effects on qubits:

- $\sigma_0|0\rangle = |0\rangle \quad , \quad \sigma_0|1\rangle = |1\rangle$
- $\sigma_1|0\rangle = |1\rangle \quad , \quad \sigma_1|1\rangle = |0\rangle$
- $\sigma_2|0\rangle = -i|1\rangle \quad , \quad \sigma_2|1\rangle = i|0\rangle$
- $\sigma_3|0\rangle = |0\rangle \quad , \quad \sigma_3|1\rangle = -|1\rangle$

Outer Products

The product of a ket $|\psi\rangle$ and a bra $\langle\phi|$, written as $|\psi\rangle\langle\phi|$ is called the outer product. This quantity is an operator, which we can apply to a ket $|X\rangle$:

$$(|\psi\rangle\langle\phi|)|X\rangle = |\psi\rangle\langle\phi|X\rangle = (\langle\phi|X\rangle)|\Psi\rangle$$

So the result of this operator transforms the ket $|X\rangle$ into an operator proportional to $|\psi\rangle$.

For example, $\widehat{A} = |0\rangle\langle 0| + |1\rangle\langle 1|$ is the outer representation of the identity.

Closure relation

Given a basis set (orthonormal) $\{|u_i\rangle\}$ in n dimensions, we have:

$$I = \sum_{i=1}^{n} |u_i\rangle\langle u_i|$$

This is simply saying that:

$$|\psi\rangle = \widehat{I}|\psi\rangle = \left(\sum_{i=1}^{n} |u_i\rangle\langle u_i|\right)|\psi\rangle = \sum \langle u_i|\psi\rangle\ |u_i\rangle$$

Representations of operators using Matrices

If we know the action of $\widehat{A}$ on an orthonotmal basis $|u_i\rangle$, we can see that:

$$\widehat{A} = \widehat{I}\widehat{A}\widehat{I} = \left(\sum_i |u_i\rangle\langle u_i|\right)\widehat{A}\left(\sum_j |u_j\rangle\langle u_j|\right) = \sum_{ij}\langle u_i|\widehat{A}|u_j\rangle|u_i\rangle\langle u_j|$$

The quantity $\langle u_i|\widehat{A}|u_j\rangle$ is a complex number we denote by A_{ij}. Then:

$$\widehat{A} = \begin{pmatrix} \langle u_1|\widehat{A}|u_1\rangle & \langle u_1|\widehat{A}|u_2\rangle & \cdots & \langle u_1|\widehat{A}|u_n\rangle \\ \langle u_2|\widehat{A}|u_1\rangle & \langle u_2|\widehat{A}|u_2\rangle & \cdots & \langle u_2|\widehat{A}|u_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_n|\widehat{A}|u_1\rangle & \langle u_n|\widehat{A}|u_2\rangle & \cdots & \langle u_n|\widehat{A}|u_n\rangle \end{pmatrix}$$

Of course, this is with respect to the basis we defines.

In two dimensions that is:

$$A = \begin{pmatrix} \langle 0|A|0\rangle & \langle 0|A|1\rangle \\ \langle 1|A|0\rangle & \langle 1|A|1\rangle \end{pmatrix}$$

Therefore, the Pauli Matrices are:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Outer Products and Matrix Representations

The outer product of $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ and $|\phi\rangle = \begin{pmatrix} c \\ d \end{pmatrix}$ can be represented by:

$$|\psi\rangle\langle\phi| = \begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c^* & d^* \end{pmatrix} = \begin{pmatrix} ac^* & ad^* \\ bc^* & bd^* \end{pmatrix}$$

Hermitian, Unitary and Normal

The Hermitian of an operator $\widehat{A}$ is denoted by $\widehat{A}^\dagger$ and defined as:

$$\langle a|\widehat{A}^\dagger|b\rangle = \langle b|\widehat{A}|a\rangle^*$$

We have the following rules:

- $(\alpha\widehat{A})^T = \alpha^* A^T$
- $|\psi\rangle^T = \langle\psi|$
- $\langle\psi|^T = |\psi\rangle$
- $(AB)^T = B^T A^T$
- $(A|\psi\rangle)^T = \langle\psi|A^T$
- $(AB|\psi\rangle)^T = \langle\psi|B^T A^T$
- $(A+B)^T = A^T + B^T$

If $A = |\psi\rangle\langle\phi|$, then $A^T 2 = |\phi\rangle\langle\psi|$.
For a general matrix, the hermitian is the transpose conjugate.

Hermitian

Is an operator such that:

$$\widehat{A} = \widehat{A}^T$$

Pauli operators are hermitian.
The eigenvalues of a Hermitian operator are real.
The eigenvectors of different eigenvalues are orthogonal.

Unitary

It is an operator U such that:

$$UU^T = U^T U = I$$

Pauli matrices are also unitary.
The eigenvalues of a unitary operator are complex numbers of modulus 1.
A unitary operator with nondegenerate eigenvalues has mutually orthogonal eigenvectors.

Normal

It is such that:

$$AA^T = A^T A$$

Hermitian and unitary are normal.

Eigen

A vector $|\psi\rangle$ is an eigenvector of A with eigenvalue λ if:

$$A|\psi\rangle = \lambda|\psi\rangle$$

Spectral Decomposition

For every normal operator A, we can find a basis such that the matrix of A is diagonal. That is:

$$A = \sum_{i=1}^{n} a_i |u_i\rangle\langle u_i|$$

Where a_i are the eigenvalues.

To find the diagonal matrix, we simply calculate the eigen stuff of A. Then, the basis is the one formed by the eigenvectors.

Trace

The trace of an operator in a basis $|u_i\rangle$ is:

$$Tr(A) = \sum_{i=1}^{n} \langle u_i|A|u_i\rangle$$

Important properties:

- $Tr(ABC) = Tr(CAB) = Tr(BCA)$
- $Tr(|\phi\rangle\langle\psi|) = \langle\phi|\psi\rangle$
- $Tr(A|\psi\rangle\langle\phi|) = \langle\phi|A|\psi\rangle$
- Trace is base -independent. In particular, trace is equal to the sum of eigenvalues.
- Trace is linear.

Expectation Value of an operator

The expectation value of an operator A given that the particle is in state $|\psi\rangle$ is given by:

$$\langle A\rangle = \langle\psi|A|\psi\rangle$$

We can compute the expectation of A^2 like:

$$\langle A^2\rangle = \langle\psi|A^2|\psi\rangle$$

And the uncertainty or variance of an operator is:

$$\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$$

Functions of operators

The function of an operator can be written in terms of a Taylor series:

$$f(A) = \sum_{n=0}^{\infty} a_n A^n$$

For example:

$$e^{aA} = I + aA + \frac{a^2}{2!}A^2 + \cdots + \frac{a^n}{n!}A^n + \cdots$$

If an operator A is normal and has a spectral expansion given by $A = \sum_i a_i |u_i\rangle\langle u_i|$, then:

$$\boxed{f(A) = \sum_i f(a_i)|u_i\rangle\langle u_i|}$$

If H is a Hermitian operator, then:

$$U = e^{i\epsilon H}$$

For ϵ a constant is a unitary operator.

Thus, if $H = \sum_i \psi_i |u_i\rangle\langle u_i|$ we can write:

$$U = e^{i\epsilon H} = \sum_i e^{i\epsilon\phi_i}|u_i\rangle\langle u_i|$$

Or, if we take only the first term of the Taylor expansion, we get:

$$U = I + i\epsilon H$$

Unitary Transformations

A method known as a unitary transformation can be used to transform the matrix representation of an operator in one basis to a representation in another basis.
The change of basis from a basis $|u_i\rangle$ to a basis $|v_i\rangle$ is given by:

$$U = \begin{pmatrix} \langle v_1|u_1\rangle & \langle v_1|u_2\rangle \\ \langle v_2|u_1\rangle & \langle v_2|u_2\rangle \end{pmatrix}$$

If we have $|\psi\rangle$ in the $|u_i\rangle$ basis, then in the $|v_i\rangle$ basis it is written as:

$$|\psi'\rangle = U|\psi\rangle$$

Where $|\psi'\rangle$ is the same vector but written in the v basis.
If we have A in a basis $|u_i\rangle$, we can write it in $|v_i\rangle$ as:

$$A' = UAU^T$$

Projection Operators

A projection operator is an operator that can be formed by an outer product of a single ket:

$$P = |\psi\rangle\langle\psi|$$

A projection operator is Hermitian.
If $|\psi\rangle$ is normalized, then:

$$P^2 = P$$

We can define $P_i = |i\rangle\langle i|$. Then, any operator A can be written as:

$$A = \sum_{i=1}^{n} a_i |u_i\rangle\langle u_i| = \sum_{i=1}^{n} a_i P_i$$

And we have:

$$\sum_i P_i = I$$

If a particle is in state $|\psi\rangle$, the probability of finding the particle in state $|u_i\rangle$ is:

$$Pr(i) = |P_i|\psi\rangle|^2 = \langle\psi|P_i^T P_i|\psi\rangle = \langle\psi|P_i^2|\psi\rangle = \langle P_i|\psi\rangle$$

Positive operator

An operator A is positive semidefinite if:

$$\langle\psi|A|\psi\rangle \geq 0$$

A **POVM (Positive operator valued measure)** is a set of operators $\{E_1, E_2, \cdots, E_n\}$ such that E_i is positive semidefinite and fulfills:

$$\sum_i E_i = I$$

Commutator Algebra

The commutator of A, B is defined by:

$$[A, B] = AB - BA$$

When $[A, B] = 0$, we say that the operators commute. If not, we say they are incompatible.
Properties:

- $[A, B] = -[B, A]$

- $[A, B + C] = [A, B] + [A, C]$
- $[A, BC] = [A, B]C + B[A, C]$

For example, the commutator of position x and momentum p is:

$$[x, p] = i\hbar I$$

We can redefine a normal operator as one that:

$$[A, A^T] = 0$$

If two operators commute, they possess a set of common eigenvectors.
Let A, B be operators that $[A, B] = 0$, suppose $|u_i\rangle$ is a nondegenerate eigenvector of A with eigenvalue a_n, that is, $A|u_i\rangle = a_i|u_i\rangle$. Assume they are Hermitian. Then:

$$\langle u_i|[A, B]|u_j\rangle = \langle u_i|(AB - BA)|u_j\rangle = (a_i - a_j)\langle u_i|B|u_j\rangle$$

Since $[A, B] = 0$, then the first term is 0. So $(a_i - a_j)\langle u_i|B|u_j\rangle = 0$.
Then, if $i \neq j$:

$$\langle u_i|B|u_j\rangle \propto \delta_{ij}$$

This means that the eigenvectors of A, denoted by $|u_i\rangle$, are also eigenvectors of B.

Heisenberg uncertainty

It says:

$$\Delta A \Delta B \geq \frac{1}{2}|\langle [A, B]\rangle|$$

Polar Decomposition and singular Values

If the matrix of an operator A is nonsingular, then we can decompose A into a unitary operator U and a positive semidefinite Hermitian operator P in the following way:

$$A = UP$$

The operator P is given by:

$$P = \sqrt{A^T A}$$

And the poerator U is:

$$U = AP^{-1} = A(\sqrt{A^T A})^{-1}$$

And we have:

$$\det A = \det U \det P = re^{i\theta}$$

The Postulates of QM

- **State of a System:**
  The state of a quantum system is a vector $|\psi(t)\rangle$ in Hilbert space. The state contains all the information we need. It must be normalized $\langle\psi|\psi\rangle = 1$

- **Observable quantities Represented by operators:**
  Every dynamical variable A is represented by a Hermitian operator.

- **Measurements**
  The possible results of measurement of dynamical variable A are its eigenvalues a_n. Using spectral decomposition, we can write the operator A in terms of it's eigenvalues and corresponding projections operators $P_n = |u_n\rangle\langle u_n|$ as $A = \sum_n a_n P_n$.
  The probability of obtaining a_n in the measurement is:

  $$Pr(a_n) = \langle\psi|P_n|\psi\rangle = Tr(P_n|\psi\rangle\langle\psi|)$$

  The probability amplitude $c_n = \langle u_n|\psi\rangle$ gives us the probability of obtaining measurement result a_n as:

  $$Pr(a_n) = |c|^2$$

  A measurement result a_n causes the collapse of the wavefunction, meaning the system is left in state $|u_n\rangle$. So, in terms of $P_n = |u_n\rangle\langle u_n|$:

  $$|\psi\rangle \quad \Rightarrow \quad \text{(measurement)} \quad \Rightarrow \quad \frac{P_n|\psi\rangle}{\sqrt{\psi|P_n|\psi\rangle}}$$

- **Time evolution of the System**
  The time evolution is governed by Shrodinger's equation as:

  $$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$$

  Where H is the Hamiltonian operator.
  Given the initial state $|\psi(0)\rangle$, the time evolution is given by:

  $$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$$

  Therefore, the time evolution of Quantum state is given by the unitary operator:

  $$U = e^{-iHt/\hbar}$$

l

Summary Important

Dirac Notation

Operator: An operator $\widehat{A}$ is a mathematical rule that transforms a ket $|\psi\rangle$ into another ket $|\phi\rangle$ like:

$$\widehat{A}|\psi\rangle = |\phi\rangle$$

Ket: A ket is a vector representing the state of a system. In the case of spin 1/2 for example, it is a 2 term vector, in other cases it might be a function.
Bra: A bra is the conjugate transpose of the ket.

Inner product: The inner product of two vectors $|u\rangle$ and $|v\rangle$ is written as:

$$\langle u|v\rangle$$

And it satisfies:

- $\langle u|v\rangle^* = \langle v|u\rangle$
- $\langle u|u\rangle \geq 0$
- $\langle u|av + bw\rangle = \alpha\langle u|v\rangle + \beta\langle u|w\rangle$

For finite dimension vectors, the inner product is simply a matrix product of $\langle u|$ and $|v\rangle$.
For functions, the inner product is $\langle f|g\rangle = \int_{\mathbb{R}} f^* g dx$

Orthonormality: A group of vector $\{|u_i\rangle\}$ are orthonormal if:

$$\langle u_i|u_j\rangle = \delta_{ij}$$

Any spanning set of vectors can be orthogonalized with Gram Schmidt.

Outer Product: The product of a ket $|\psi\rangle$ and a bra $\langle\phi|$, written as $|\psi\rangle\langle\phi|$ is called the outer product. When applied to a ket $|X\rangle$, it gives:

$$(|\psi\rangle\langle\phi|)|X\rangle = \langle\phi|X\rangle \;\; |\psi\rangle$$

Projection:
If we have a normalized vector $|u_i\rangle$, then the operator given by:

$$P_i = |u_i\rangle\langle u_i|$$

gives the projection of a vector in the direction $|u_i\rangle$. That is:

$$P_i(|\psi\rangle) = |u_i\rangle\langle u_i|\psi\rangle \; = \; \langle u_i|\psi\rangle \; |u_i\rangle$$

Closure Relation: Given an orthonormal basis set $\{|u_i\rangle\}$ in n dimensions, we have:

$$I = \sum_{i=1}^{n} |u_i\rangle\langle u_i|$$

This is simply saying that the basis is complete:

$$|\psi\rangle = \widehat{I}|\psi\rangle = \left(\sum_{i=1}^{n} |u_i\rangle\langle u_i|\right)|\psi\rangle = \sum_{i=1}^{n}\langle u_i|\psi\rangle\ |u_i\rangle$$

Representation of an Operator using a matrix
Given an operator $\widehat{A}$ on a orthonormal basis $|u_i\rangle$, we can define the representation matrix as:

$$\widehat{A} = \begin{pmatrix} \langle u_1|\widehat{A}|u_1\rangle & \langle u_1|\widehat{A}|u_2\rangle & \cdots & \langle u_1|\widehat{A}|u_n\rangle \\ \langle u_2|\widehat{A}|u_1\rangle & \langle u_2|\widehat{A}|u_2\rangle & \cdots & \langle u_2|\widehat{A}|u_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_n|\widehat{A}|u_1\rangle & \langle u_n|\widehat{A}|u_2\rangle & \cdots & \langle u_n|\widehat{A}|u_n\rangle \end{pmatrix}$$

.

Types of operators:
Hermitian Conjugate: The hermitian A^T of an operator A is the operator such that:

$$\langle a|A^T|b\rangle = \langle b|A|a\rangle^*$$

Its matrix is obtained by transposing and conjugating.

- **Normal:** It is an operator such that:

  $$AA^T = A^T A$$

  The operator has the following properties:

  - $|A(|\psi\rangle)| = |A^T(|\psi\rangle)|$
  - If λ is an e.val of A then $\overline{\lambda}$ is an eigenvalue of A^T
  - If λ_1, λ_2 are different eigenvalues of A, then their eigenvectors are orthogonal.
  - A is normal in a finite dimensional space iff there is an orthonormal basis of eigenvectors of A

- **Self Adjoint:** An operator with $A = A^T$.
  It has the properties:

  - The eigenvalues are real.
  - It is normal, so it has the other properties.

- Its matrix representation is the same when we transpose and conjugate.

- **Unitary:** An operator with $AA^T = I$
  Properties:
  - The eigenvalues are complex numbers of norm 1.
  - It is normal, so it has all those properties.
  - The columns of its matrix representation are a orthonormal set, that way $AA^T = I$

Spectral Decomposition: For every normal operator A, we can find a basis of the space formed of eigenvectors $\{|u_i\rangle\}$. Therefore, the operator can be written as:

$$\begin{aligned} A &= \sum_{i=1}^{n} \lambda_i |u_i\rangle\langle u_i| \\ &= \sum_{i=1}^{n} \lambda_i P_i \end{aligned}$$

Where the eigenvectors are $A|u_i\rangle = \lambda_i |u_i\rangle$.
So the normal operator A acts only on n orthonormal directions and it ' enlarges ' each one by a factor λ_i.
If A is hermitian, the enlargment is real.
If A is unitary, the ' enlargement 'is actually a rotation with a complex number of norm 1.
And we have $\sum_i P_i = I$

Observables: In QM, dynamical variables are represented by a Hermitian operator.

Expectation Value of an Operator: Given an operator A and a particle in state $|\Psi\rangle$, the expectation value of A is:

$$\langle A\rangle = \langle\Psi|A|\Psi\rangle$$

And $\langle A^2\rangle = \langle\Psi|A^2|\Psi\rangle$.
The variance is $\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$

After measuring a variable with operator A, the particle will take a state such that the variance of A is 0 (because we already know A).
This determinate states are the eigenvectors of A, so the particle must transform into a state that is an eigenvector $|u_i\rangle$ of A.
And the value of the variable is the corresponding eigenvalue because the expectation value is now $\langle A\rangle = \langle u_i|A|u_i\rangle = \langle u_i|\lambda_i u_i\rangle = \lambda_i$.

Function of Operators: Given an operator written as $A = \sum_i \lambda_i |u_i\rangle\langle u_i|$ (where $|u_i\rangle$ are the eigenvectors of A).

Then we have that:

$$A^n = \sum_i \lambda_i^n |u_i\rangle\langle u_i|$$

Which can be proved by induction.
If we define a function f on an operator by its Taylor series, then:

$$f(A) = \sum_i f(\lambda_i)|u_i\rangle\langle u_i|$$

In particular, if H is a Hermitian operator, then:

$$U = e^{iaH}$$

for any real a is a unitary operator, because:

$$U = e^{iaH} = \sum_i e^{ia\lambda_i}|u_i\rangle\langle u_i|$$

So, we see that taking the conjugate is the same as taking the inverse, then $U^* = U^{-1}$, so U is unitary.

Measurement: When we measure an operator $A = \sum_n \lambda_n |u_n\rangle\langle u_n|$, then we are bound to get an eigenvalue of λ_n upon measurement.
The probability of getting this value is:

$$Pr(\lambda_n) = |\langle u_n|\Psi\rangle|^2$$

Time evolution of the system:
We can write Schrodinger's equation as:

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = H|\Psi\rangle$$

Where H is the Hamiltonian operator. Then, given the initial state $|\Psi(0)\rangle$ of the particle, the time evolution of the state is given by solving the equation:

$$|\Psi(t)\rangle = e^{-iHt/\hbar}|\Psi(0)\rangle$$

If the eigenvalues of H are h_i and eigenvectors are $|u_i\rangle$, then $H = h_i|u_i\rangle\langle u_i|$, and
$e^{-iHt/\hbar} = \sum_i e^{-ih_i t/\hbar}|u_i\rangle\langle u_i|$
So the time evolution is:

$$\begin{aligned} |\Psi(t)\rangle &= \sum_i e^{-ih_i t/\hbar}|u_i\rangle\langle u_i| \; |\Psi(0)\rangle \\ &= \sum_i e^{-ih_i t/\hbar}\langle u_i|\Psi(0)\rangle \;\; |u_i\rangle \end{aligned}$$

Tensor Product

Tensor products is useful to work with many particle systems.
We consider the two-particle case.

Suppose that H_1 and H_2 are two Hilbert spaces of dimension N_1, N_2. We can put these two Hilbert spaces together to construct a larger Hilbert space. We denote it by:

$$H = H_1 \otimes H_2$$

And we have that:

$$\dim(H) = \dim(H_1)\dim(H_2)$$

Representing composite states in QM

A state vector belonging to H is the tensor product of state vectors belonging to H_1 and H_2. Let $|\psi\rangle \in H_1$ and $|\chi\rangle \in H_2$ be two vectors that belong to the Hilbert spaces used to construct H. We can construct a vector $|\psi\rangle \in H$ using the tensor product:

$$|\psi\rangle = |\phi\rangle \otimes |\chi\rangle$$

The tensor product is bilinear, that is:

$$\begin{aligned} |\phi\rangle \otimes [\alpha|\chi_1\rangle + |\chi_2\rangle] &= \alpha|\phi\rangle \otimes |\chi_1\rangle + |\phi\rangle \otimes |\chi_2\rangle \\ [\alpha|\phi_1\rangle + |\phi_2\rangle] \otimes |\chi\rangle &= \alpha|\phi_1\rangle \otimes |\chi\rangle + |\phi_2\rangle \otimes |\chi\rangle \\ |\phi\rangle \otimes |\chi\rangle &= |\chi\rangle \otimes |\phi\rangle \end{aligned}$$

To construct a basis for H, we simply form the tensor products of basis vectors from the spaces H_1, H_2. Let $|u_i\rangle$ be the basis of H_1 and $|v_j\rangle$ be the basis of H_2. Then, a basis for $H = H_1 \otimes H_2$ can be given by:

$$|w_{ij}\rangle = |u_i\rangle \otimes |v_j\rangle$$

Example 4.1 Let H_1, H_2 be two Hilbert spaces for qubits, describe the basis of $H = H_1 \otimes H_2$.

Solution: The basis for both spaces is $\{|0\rangle\ ,\ |1\rangle\}$. The basis is:

$$\begin{aligned} |w_{11}\rangle &= |0\rangle \otimes |0\rangle \\ |w_{12}\rangle &= |0\rangle \otimes |1\rangle \\ |w_{21}\rangle &= |1\rangle \otimes |0\rangle \\ |w_{22}\rangle &= |1\rangle \otimes |1\rangle \end{aligned}$$

Now we consider the expansion of an arbitrary vector from H_1 and an arbitrary vector from H_2, that is:

$$|\phi\rangle = \sum_i \alpha_i |u_i\rangle$$
$$|\chi\rangle = \sum_i \beta |v_i\rangle$$

So, the vector $|\psi\rangle$ is simply:

$$|\psi\rangle = \sum_{ij} \alpha_i \beta_j |u_i\rangle \otimes |v_j\rangle$$

So the components of $|\psi\rangle$ are found by multiplying the components of the vector being tensor-multiplied.

Example 4.2
Let $|\phi\rangle \in H_1$ with basis vectors $|x\rangle, |y\rangle$ and expansion:

$$|\phi\rangle = a_x|x\rangle + a_y|y\rangle$$

And let $|\chi\rangle \in H_2$ with basis vector $|u\rangle$, $|v\rangle$,and expansion:

$$|\chi\rangle = b_u|u\rangle + b_v|v\rangle$$

Then, the tensor product is:

$$|\psi\rangle = a_x b_u|x\rangle \otimes |u\rangle + a_x b_v|x\rangle \otimes |v\rangle + a_y b_u|y\rangle \otimes |u\rangle + a_y b_v|y\rangle \otimes |v\rangle$$

Computing Inner Products

Suppose:

$$|\psi_1\rangle = |\phi_1\rangle \otimes |\chi_1\rangle$$
$$|\psi_2\rangle = |\phi_2\rangle \otimes |\chi_2\rangle$$

Then, we can calculate their inner product as:

$$\langle\psi_1|\psi_2\rangle = \langle\phi_1| \otimes \langle\chi_1|)(|\phi_2\rangle \otimes |\chi_2\rangle) = \langle\phi_1|\phi_2\rangle\langle\chi_1|\chi_2\rangle$$

Example: We have $|+\rangle, |-\rangle$ a basis of H_1, H_2, then, a basis of $H = H_1 \otimes H_2$ is:

$$|w_1\rangle = |+\rangle|+\rangle$$
$$|w_2\rangle = |+\rangle|-\rangle$$
$$|w_3\rangle = |-\rangle|+\rangle$$
$$|w_4\rangle = |-\rangle|-\rangle$$

We can check they are orthonormal by calculating the inner product, for example:

$$\langle w_2|w_2\rangle = (\langle+|\langle-|)(|+\rangle|-\rangle) = \langle+|+\rangle\langle-|-\rangle = (1)(1) = 1$$

Tensor Products of Column vectors

Let:

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \quad , \quad |\chi\rangle = \begin{pmatrix} c \\ d \end{pmatrix}$$

Where the base of the first vector is $|u_1\rangle, |u_2\rangle$ and of the second is $|v_1\rangle$, $|v_2\rangle$.
Then, the tensor product is:

$$\begin{aligned} |\psi\rangle \otimes |\chi\rangle &= (a|u_1\rangle + b|u_2\rangle) \otimes (c|v_1\rangle + d|v_2\rangle) \\ &= ac|u_1\rangle \otimes |v_1\rangle + ad|u_1\rangle \otimes |v_2\rangle + bc|u_2\rangle \otimes |v_1\rangle + bd|u_2\rangle \otimes |v_2\rangle \end{aligned}$$

Therefore, the component representation of $|\phi\rangle \otimes |\chi\rangle$ is:

$$|\phi\rangle \otimes |\chi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a\begin{pmatrix} c \\ d \end{pmatrix} \\ b\begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$$

Operators and Tensor products

Let $|\phi\rangle \in H_1$ and $|\chi\rangle \in H_2$ be two vectors.
Now let A be an operator that acts on $|\phi\rangle \in H_1$ and let B be an operator that acts on $|\chi\rangle \in H_2$.
That is, $A : H_1 \to H_1 \quad , \quad B : H_2 \to H_2$
Then $A \otimes B : H_1 \otimes H_2 \to H_2 \otimes H_2$ defined by:

$$(A \otimes B)(|\phi\rangle \otimes |\chi\rangle) = (A|\phi\rangle) \otimes (B|\chi\rangle)$$

Example: Let $|\psi\rangle = |a\rangle \otimes |b\rangle$ and $A|a\rangle = a|a\rangle$ and $B|b\rangle = b|b\rangle$. What is $A \otimes B|\psi\rangle$.
Then:

$$\begin{aligned} A \otimes B|\psi\rangle &= (A \otimes B)(|a\rangle \otimes |b\rangle) \\ &= A|a\rangle \otimes B|b\rangle \\ &= a|a\rangle \otimes b|b\rangle \\ &= ab(|a\rangle \otimes |b\rangle) \\ &= ab|\psi\rangle \end{aligned}$$

Example 4.9: Suppose that A is a projection operator in H_1 where $A = |0\rangle\langle 0|$ and B is a projection operator in H_2 where $B = |1\rangle\langle 1|$. Find $A \otimes B|\psi\rangle$ where $|\psi\rangle = \dfrac{|01\rangle + |10\rangle}{\sqrt{2}}$.

Solution:

$$\begin{aligned} A \otimes B|\psi\rangle &= A \otimes B\left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}[(A|0\rangle)(B|1\rangle) + (A|1\rangle)(B|0\rangle)] \\ &= \frac{1}{\sqrt{2}}[(|0\rangle)(|1\rangle) + (0)(0)] \\ &= \frac{1}{\sqrt{2}}|0\rangle|1\rangle \end{aligned}$$

Theorem: If A, B are Hermitian, then $A \otimes B$ is Hermitian.

Tensor Product of Matrices

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$
Be the representations of operators A, B.
Then the representation of $A \otimes B$ (which acts on tenors of the form $|\phi\rangle \otimes |\psi\rangle$) is:

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$$

It can be proven that:

$$(A \otimes B)^T = A^T \otimes B^T$$

Density Operator

An ensemble is a collection of systems 8. Members of the ensemble can be found in one of two or more different quantum states.

Consider a 2D Hilbert space with basis vectors $\{|a\rangle, |y\rangle\}$. We prepare a large number N of systems, where each member of the system can be in one of the two state vectors:

$$|a\rangle = \alpha|x\rangle + \beta|y\rangle$$
$$|b\rangle = \gamma|x\rangle + \delta|y\rangle$$

These states are normalized $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2 = 1$

Now we prepare n_a of these N systems in state $|a\rangle$ and n_b in state $|b\rangle$. Then $n_a + n_b = N$
And therefore:

$$\frac{n_a}{N} + \frac{n_b}{N} = 1$$

If we take any member of the ensemble and measure it, the probability of having a member in state $|a\rangle$ is $p = n_a/N$ and of having a member in state $|b\rangle$ is $1 - p$.

At the ensemble level, probability is acting in a classical way. While at the level of a single quantum system, the Born rule gives us the probabilities.

Density Operator for a pure state

If there exists some basis $|u_i\rangle$, and the system is in some known state $|\psi\rangle$, then we have:

$$|\psi\rangle = c_1|u_1\rangle + c_2|u_2\rangle + \cdots + c_n|u_n\rangle$$

Pure state: When a system is in a definite state like this.

But some states may be a mixture of any pure states. To describe them, we need to define the **density operator** ρ.
The density operator is an average operator that will allow us to describe a statistical mixture.

Density operator for a pure state
We begin by writing the average of some operator A:

$$\langle A\rangle = \langle\psi|A|\psi\rangle$$

We can expand it in some orthonormal basis $|\psi\rangle = c_1|u_1\rangle + c_2|u_2\rangle + \cdots + c_n|u_n\rangle$, this becomes:

$$\begin{aligned}\langle A\rangle = \langle\psi|A|\psi\rangle = (c_1^*\langle u_1| + c_2^*\langle u_2| + \cdots + c_n^*\langle u_n|)A(c_1|u_1\rangle + c_2|u_2\rangle + \cdots + c_n|u_n\rangle)\\ = \sum_{k,l=1}^{n} c_k^* c_l \langle u_k|A|u_l\rangle\\ = \sum_{k,l}^{n} c_k^* c_l A_{kl}\end{aligned}$$

Where we have that $c_m = \langle u_m|\Psi\rangle$, so $c_m^* = \langle\psi|u_m\rangle$. Therefore

$$c_k^* c_l = \langle\psi|u_k\rangle\langle u_l|\psi\rangle = \langle u_l|\psi\rangle\langle\psi|u_k\rangle$$

We call this the **density operator** and denote it by $\rho = |\psi\rangle\langle\psi|$.

So the expression for the average value of an operator A in a state $|\psi\rangle$ can be written as:

$$\langle A\rangle = \sum_{k,l=1}^{n} c_k^* c_l A_{kl} = \sum_{k,l=1}^{n} \langle u_l|(|\psi\rangle\langle\psi|)|u_k\rangle A_{kl} = \sum_{k,l=1}^{n} \langle u_l|\rho|u_k\rangle A_{kl}$$

Definition of density operator for a pure state

The density operator for a state $|\psi\rangle$ is defined by:

$$\rho = |\psi\rangle\langle\psi|$$

Given this definition, the average of an operator A in state $|\psi\rangle$ can be found as:

$$\begin{aligned}\langle A\rangle &= \sum_{k,l}^{n} c_k^* c_l \langle u_k|A|u_l\rangle\\ &= \sum_{k,l=1}^{n} \langle u_l|\psi\rangle\langle\psi|u_k\rangle\langle u_k|A|u_l\rangle\\ &= \sum_{k,l=1}^{n} \langle u_l|\rho|u_k\rangle\langle u_k|A|u_l\rangle\\ &= \sum_{l=1}^{n} \langle u_l|\rho\left(\sum_{k=1}^{n} |u_k\rangle\langle u_k|\right)A|u_l\rangle\\ &= \sum_{l=1}^{n} \langle u_l|\rho A|u_l\rangle\\ &= \sum_{l=1}^{n} \langle u_l|\rho A|u_l\rangle\\ &= Tr(\rho A)\end{aligned}$$

So, given a state $|\psi\rangle$, we define the density operator as

$$\rho = |\psi\rangle\langle\psi|$$

And with this definition, the expected value of the operator is given by:

$$\langle A\rangle = \sum_{l=1}^{n} \langle u_l|\rho A|u_l\rangle = Tr(\rho A)$$

Property:
An important property of the density operator for a pure state is that:

$$Tr(\rho) = 1$$

Another important property is that:

$$\rho^2 = (|\psi\rangle\langle\psi|)(|\psi\rangle\langle\psi|) = |\psi\rangle(\langle\psi|\psi\rangle)\langle\psi = |\psi\rangle\langle\psi| = \rho$$

Then, if a system is in a pure state $|\psi\rangle$, we have that:

$$Tr(\rho^2) = 1$$

Example 5.1

A system is in the state $|\psi\rangle = \frac{1}{\sqrt{3}}|u_1\rangle + i\sqrt{\frac{2}{3}}|u_2\rangle$, where the $|u_k\rangle$ constitute an orthonormal basis. Write down the density operator, and show it has unit trace.

Solution

First we write down the bra corresponding to the given state. Remember, we must take the conjugate of any complex numbers. We obtain

$$\langle\psi| = \frac{1}{\sqrt{3}}\langle u_1| - i\sqrt{\frac{2}{3}}\langle u_2|$$

Then, using (5.1), we have

$$\begin{aligned}\rho = |\psi\rangle\langle\psi| &= \left(\frac{1}{\sqrt{3}}|u_1\rangle + i\sqrt{\frac{2}{3}}|u_2\rangle\right)\left(\frac{1}{\sqrt{3}}\langle u_1| - i\sqrt{\frac{2}{3}}\langle u_2|\right)\\ &= \frac{1}{3}|u_1\rangle\langle u_1| - i\frac{\sqrt{2}}{3}|u_1\rangle\langle u_2| + i\frac{\sqrt{2}}{3}|u_2\rangle\langle u_1| + \frac{2}{3}|u_2\rangle\langle u_2|\end{aligned}$$

The trace is

$$\begin{aligned}Tr(\rho) &= \sum_{i=1}^{2}\langle u_i|\rho|u_i\rangle = \langle u_1|\rho|u_1\rangle + \langle u_2|\rho|u_2\rangle\\ &= \frac{1}{3}\langle u_1|u_1\rangle\langle u_1|u_1\rangle - i\frac{\sqrt{2}}{3}\langle u_1|u_1\rangle\langle u_2|u_1\rangle + i\frac{\sqrt{2}}{3}\langle u_1|u_2\rangle\langle u_1|u_1\rangle + \frac{2}{3}\langle u_1|u_2\rangle\langle u_2|u_1\rangle\\ &\quad + \frac{1}{3}\langle u_2|u_1\rangle\langle u_1|u_2\rangle - i\frac{\sqrt{2}}{3}\langle u_2|u_1\rangle\langle u_2|u_2\rangle + i\frac{\sqrt{2}}{3}\langle u_2|u_2\rangle\langle u_1|u_2\rangle + \frac{2}{3}\langle u_2|u_2\rangle\langle u_2|u_2\rangle\\ &= \frac{1}{3} + \frac{2}{3} = 1\end{aligned}$$

Mixed terms like $\langle u_1|u_2\rangle$ drop out because the basis is orthonormal, so $\langle u_i|u_j\rangle = \delta_{ij}$.

The evolution of the Density operator

The time evolution can be found by:

$$i\hbar\frac{d}{dt}|\psi\rangle = H|\psi\rangle$$

Then, since $H = H^*$, we can also write:

$$-i\hbar\frac{d}{dt}\langle\psi| = \langle\psi|H$$

The derivative of the density operator is:

$$
\begin{aligned}
\frac{d\rho}{dt} &= \frac{d}{dt}(|\psi\rangle\langle\psi|) = \left(\frac{d}{dt}|\psi\rangle\right)\langle\psi| + |\psi\rangle\left(\frac{d}{dt}\langle\psi|\right) \\
&= \left(\frac{H}{i\hbar}|\psi\rangle\right)\langle\psi| + |\psi\rangle\left(\langle\psi|\frac{H}{-i\hbar}\right) \\
&= \frac{H}{i\hbar}\rho - \rho\frac{H}{i\hbar} = \frac{1}{i\hbar}[H,\rho]
\end{aligned}
$$

Definition: Time evolution of the Density Operator:

$$
i\hbar\frac{d\rho}{dt} = [H,\rho]
$$

Density Operator for a Mixed State

WE need a density matrix for an ensemble. For this we need the following three steps:

- Construct a density for each individual state that can be found in the ensemble
- Weight it by the probability of finding that state in the ensemble
- Sum up the possibilities

Suppose we have an ensemble of two states:

$$
\begin{aligned}
|a\rangle &= \alpha|x\rangle + \beta|y\rangle \\
|b\rangle &= \gamma|x\rangle + \delta|y\rangle
\end{aligned}
$$

The density for each of these states are:

$$
\begin{aligned}
\rho_a &= |a\rangle\langle a| \\
\rho_b &= |b\rangle\langle b|
\end{aligned}
$$

Or, in terms of the basis states:

$$
\begin{aligned}
\rho_a &= |\alpha|^2|x\rangle\langle x| + \alpha\beta^*|x\rangle\langle y| + \alpha^*\beta|y\rangle\langle x| + |\beta|^2|y\rangle\langle y| \\
\rho_b &= |\gamma|^2|x\rangle\langle x| + \gamma\delta^*|x\rangle\langle y| + \gamma^*\delta|y\rangle\langle x| + |\delta|^2|y\rangle\langle y|
\end{aligned}
$$

Let's say the probability of a member of the ensemble being in state $|a\rangle$ is p. So, the density operator for the mixed state is:

$$
\rho = p\rho_a + (1-p)\rho_b = p|a\rangle\langle a| + (1-p)|b\rangle\langle b|
$$

We could expand it in terms of $|x\rangle, |y\rangle$.

In general, if there are n states $|\psi_i\rangle$ in the ensemble. And a member of the ensemble has a probability p_i of being $|\psi_i\rangle$, then we have:

$$\rho = \sum_{i=1}^{n} p_i \rho_i = \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i|$$

Key Properties:
An operator ρ is a density operator if and only if it satisfies the following requirements:

- It is hermitian $\rho = \rho^*$
- $Tr(\rho) = 1$
- ρ is a opsitive operator, meaning $\langle u|\rho u\rangle \geq 0$ (or equivalently, it is Hermitian and has nonnegative eigenvalues).

Example 5.2

Consider the state

$$|a\rangle = \begin{pmatrix} e^{-i\phi}\sin\theta \\ \cos\theta \end{pmatrix}$$

Is $\rho = |a\rangle\langle a|$ a density operator?

Solution

In the $\{|0\rangle, |1\rangle\}$ basis, the state is written as

$$|a\rangle = \begin{pmatrix} e^{-i\phi}\sin\theta \\ \cos\theta \end{pmatrix} = e^{-i\phi}\sin\theta\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos\theta\begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{-i\phi}\sin\theta|0\rangle + \cos\theta|1\rangle$$

The dual vector is

$$\langle a| = e^{i\phi}\sin\theta\langle 0| + \cos\theta\langle 1|$$

So we have

$$\begin{aligned}\rho &= |a\rangle\langle a| = (e^{-i\phi}\sin\theta|0\rangle + \cos\theta|1\rangle)(e^{i\phi}\sin\theta\langle 0| + \cos\theta\langle 1|) \\ &= \sin^2\theta|0\rangle\langle 0| + e^{-i\phi}\sin\theta\cos\theta|0\rangle\langle 1| + e^{i\phi}\sin\theta\cos\theta|1\rangle\langle 0| + \cos^2\theta|1\rangle\langle 1|\end{aligned}$$

The matrix representation of this density operator is

$$\begin{pmatrix} \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle \\ \langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle \end{pmatrix} = \begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}$$

First, we check to see if the matrix is Hermitian. The transpose of the matrix is

$$\rho^T = \begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}^T = \begin{pmatrix} \sin^2\theta & e^{i\phi}\sin\theta\cos\theta \\ e^{-i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}$$

Taking the complex conjugate gives

$$\rho^\dagger = (\rho^T)^* = \begin{pmatrix} \sin^2\theta & e^{i\phi}\sin\theta\cos\theta \\ e^{-i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}^* = \begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}$$

Since $\rho = \rho^\dagger$, the matrix is Hermitian. Second, we see that $Tr(\rho) = \sin^2\theta + \cos^2\theta = 1$. The trace of a density matrix is always unity. Finally, we consider an arbitrary state

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

Then

$$\langle\psi|\rho|\psi\rangle = |a|^2 \sin^2\theta + ab^* e^{i\phi} \sin\theta\cos\theta + a^* b e^{-i\phi} \sin\theta\cos\theta + |b|^2 \cos^2\theta$$

For complex numbers z and w, we can write

$$(z + w)(z^* + w^*) = zz^* + wz^* + w^*z + ww^* = |z + w|^2$$

It is also true that the modulus of any complex number satisfies $|\varsigma|^2 \geq 0$, and also $|z + w|^2 \geq 0$. So we make the following definitions:

$$z = ae^{-i\phi}\sin\theta, \quad \Rightarrow zz^* = |a|^2 \sin^2\theta$$
$$w = b\cos\theta, \quad \Rightarrow ww^* = |b|^2 \cos^2\theta$$

We see that we can identify

$$ab^* e^{i\phi} \sin\theta\cos\theta = wz^*$$
$$a^* b e^{-i\phi} \sin\theta\cos\theta = zw^*$$

Now recall that for any complex number z, $|z|^2 \geq 0$. So

$$\langle\psi|\rho|\psi\rangle = |ae^{-i\phi}\sin\theta + b\cos\theta|^2 \geq 0$$

since $|z + w|^2 \geq 0$. Hence the operator is positive. Since ρ is Hermitian, has unit trace, and is a positive operator, it qualifies as a density operator. We can also verify that the density operator is positive by examining its eigenvalues. It can be shown that the eigenvalues of

$$\rho = \begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}$$

are given by $\lambda_{1,2} = \{1, 0\}$. Since both eigenvalues are nonnegative and the matrix is Hermitian, we conclude that the operator is positive.

Proof that the trace of a mixed density operator is 1

As we saw earlier, the mixed density operator is $\rho = p_i|\psi_i\rangle\langle\psi_i|$, so its trace is:

$$Tr(\rho) = Tr\left(\sum_{i=1}^{n} p_i|\psi_i\rangle\langle\psi_i|\right) = \sum_{i=1}^{n} p_i Tr(|\psi_i\rangle\langle\psi_i|) = \sum_{i=1}^{n} p_i\langle\psi_i|\psi_i\rangle = \sum_{i=1}^{n} p_i = 1$$

Proof that the density operator of a mixed state is positive

Consider an arbitrary vector $|\phi\rangle$, then:

$$\langle\phi|\rho|\phi\rangle = \sum_{i=1}^{n} p_i\langle\phi|\psi_i\rangle\langle\psi_i|\phi\rangle = \sum_{i=1}^{n} p_i|\langle\phi|\psi_i\rangle|^2 \geq 0$$

Expectation Values

The same results stay for mixed states. The value of an operator with respect to a statistical mixture of states is:

$$\langle A\rangle = Tr(\rho A)$$

Proof: Let's say $\rho = p_i|\psi_i\rangle\langle\psi_i|$. Then, it makes sense that $\langle A\rangle = p_i\langle\psi_i|A|\psi_i\rangle$, therefore:

$$\begin{aligned}\langle A\rangle &= p_i\langle\psi_i|A|\psi_i\rangle\\ &= p_i\langle\psi_i|u_j\rangle\langle u_j|A\langle u_k|\psi_i\rangle|u_k\rangle\\ &= p_i\langle\psi_i|u_j\rangle\langle u_k|\psi_i\rangle\langle u_j|A|u_k\rangle\\ &= p_i\langle u_k|\psi_i\rangle\langle\psi_i|u_j\rangle\langle u_j|A|u_k\rangle\\ &= p_i\langle u_k|\psi_i\rangle\langle\psi_i A|u_k\rangle\\ &= \langle u_k|(p_i|\psi_i\rangle\langle\psi_i|)A|u_k\rangle\\ &= \langle u_k|\rho A|u_j\rangle\\ &= Tr(\rho A)\end{aligned}$$

Probability of Obtaining a Given Measurement Result

Given a projection operator $P_n = |u_n\rangle\langle u_n|$ corresponding to measurement result a_n, the probability of obtaining a_n is:

$$p(a_n) = \langle u_n|\rho|u_n\rangle = Tr(|u_n\rangle\langle u_n|\rho) = Tr(P_n\rho)$$

Proof:
So, given a basis $|u_i\rangle$, if the mix has a probability p_i of being in $|\psi_i\rangle = \sum c_{ij}|u_j\rangle$. Then, when we measure the observable with eigenvalues $|u_j\rangle$, the probability of getting the nth state (with eigenvalue a_n) as a result is:

$$\begin{aligned}p(a_n) &= \sum_i p_i|c_{in}|^2\\ &= \sum_i p_i|\langle u_n|\psi_i\rangle|^2 = \sum_i p_i\langle\psi_i|u_n\rangle\langle u_n|\psi_i\rangle\\ &= \sum_i\langle u_n|\psi_i\rangle\langle\psi_i|u_n\rangle\\ &= \langle u_n|\rho|u_n\rangle = \sum_j\langle u_j|u_n\rangle\langle u_n|\rho|u_j\rangle \quad \text{por ortogonalidad, se cancela casi toda la suma excepto j=n}\\ &= Tr(P_n\rho)\end{aligned}$$

Using the more general measurement operator formalism, the probability of obtaining measurement result m associated with measurement operator M_m is:

$$P(m) = Tr(M_m^* M_m\rho)$$

After a measurement described by the projection operator, the system is in the state:

$$\rho \to \frac{P_n \rho P_n}{Tr(P_n \rho)}$$

Or in terms of the general measurement operators, the state after measurement is:

$$\frac{M_m \rho M_m^*}{Tr(M_m^* M_m \rho)}$$

Example 5.4

A system is found to be in the state

$$|\psi\rangle = \frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle$$

(a) Write down the density operator for this state.
(b) Write down the matrix representation of the density operator in the $\{|0\rangle, |1\rangle\}$ basis. Verify that $Tr(\rho) = 1$, and show this is a pure state.
(c) A measurement of Z is made. Calculate the probability that the system is found in the state $|0\rangle$ and the probability that the system is found in the state $|1\rangle$.
(d) Find $\langle X \rangle$.

Solution

(a) To write down the density operator, first we construct the dual vector $\langle\psi|$. This can be done by inspection

$$\langle\psi| = \frac{1}{\sqrt{5}}\langle 0| + \frac{2}{\sqrt{5}}\langle 1|$$

The density operator is

$$\begin{aligned}\rho = |\psi\rangle\langle\psi| &= \left(\frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle\right)\left(\frac{1}{\sqrt{5}}\langle 0| + \frac{2}{\sqrt{5}}\langle 1|\right) \\ &= \frac{1}{5}|0\rangle\langle 0| + \frac{2}{5}|0\rangle\langle 1| + \frac{2}{5}|1\rangle\langle 0| + \frac{4}{5}|1\rangle\langle 1|\end{aligned}$$

(b) The matrix representation of the density operator in the $\{|0\rangle, |1\rangle\}$ basis is found by writing

$$[\rho] = \begin{pmatrix} \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle \\ \langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle \end{pmatrix}$$

For the state given in this problem, we have

$$\langle 0|\rho|0\rangle = \frac{1}{5}$$

$$\langle 0|\rho|1\rangle = \frac{2}{5} = \langle 1|\rho|0\rangle$$

$$\langle 1|\rho|1\rangle = \frac{4}{5}$$

So in the $\{|0\rangle, |1\rangle\}$ basis the density matrix is

$$\rho = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}$$

The trace is just the sum of the diagonal elements. In this case

$$Tr(\rho) = \frac{1}{5} + \frac{4}{5} = 1$$

To determine whether or not this is a pure state, we need to determine if $Tr(\rho^2) = 1$. Now

$$\begin{aligned}\rho^2 &= \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} = \begin{pmatrix} \frac{1}{25} + \frac{4}{25} & \frac{2}{25} + \frac{8}{25} \\ \frac{2}{25} + \frac{8}{25} & \frac{4}{25} + \frac{16}{25} \end{pmatrix} \\ &= \begin{pmatrix} \frac{5}{25} & \frac{10}{25} \\ \frac{10}{25} & \frac{20}{25} \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} = \rho\end{aligned}$$

Since $\rho^2 = \rho$, it follows that $Tr(\rho^2) = 1$ and this is a pure state.

(c) In this simple example we can see by inspection that the probability that the system is in state $|0\rangle$ is 1/5 while the probability that the system is found in state $|1\rangle$ is 4/5. Let's see if we can verify this using the density operator formalism. First, we write down the projection operators in matrix form. The measurement operator that corresponds to the measurement result $|0\rangle$ is $P_0 = |0\rangle\langle 0|$ while the measurement operator $P_1 = |1\rangle\langle 1|$. The matrix representation in the given basis is

$$P_0 = \begin{pmatrix} \langle 0|P_0|0\rangle & \langle 0|P_0|1\rangle \\ \langle 1|P_0|0\rangle & \langle 1|P_0|1\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P_1 = \begin{pmatrix} \langle 0|P_1|0\rangle & \langle 0|P_1|1\rangle \\ \langle 1|P_1|0\rangle & \langle 1|P_1|1\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The probability of finding the system in state $|0\rangle$ is

$$p(0) = Tr(P_0\rho) = Tr\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}\right] = Tr\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ 0 & 0 \end{pmatrix} = \frac{1}{5}$$

The probability of finding the system in state $|1\rangle$ is

$$p(1) = Tr(P_1\rho) = Tr\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}\right] = Tr\begin{pmatrix} 0 & 0 \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} = \frac{4}{5}$$

(d) We can find the expectation value of X by calculating $Tr(X\rho)$. First, let's do the matrix multiplication:

$$X\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & \frac{4}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

The trace is the sum of the diagonal elements of this matrix

$$\langle X\rangle = Tr(X\rho) = Tr\begin{pmatrix} \frac{2}{5} & \frac{4}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} = \frac{2}{5} + \frac{2}{5} = \frac{4}{5}$$

Characterizing Mixed States

Coherence is the capability of different components of a state to interfere with one another. In a statistical mixture there will be no coherences, but in a pure state or a state in a linear superposition there will be coherences:

- A mixed state is a classical statistical mixture of two or more states. It has no coherences, so ρ_{mn} is diagonal.
- A pure state will have nonzero off diagonal terms.

In a pure state, $\rho^2 = \rho$, so $Tr(\rho^2) = 1$. Therefore:

- $Tr(\rho^2) < 1$ for a mixed state
- $Tr(\rho^2) = 1$ for a pure state.

Probability of finding an element of the ensemble in a given state

The density matrix in a basis $\{|0\rangle, |1\rangle\}$ is given by:

$$\rho = \begin{pmatrix} \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle \\ \langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle \end{pmatrix}$$

Where $\langle 0|\rho|0\rangle$ is the probability of a member of the ensemble in state $|0\rangle$. $\langle 1|\rho|1\rangle$ is the probability of a member of the ensemble found in state $|1\rangle$

Example 5.10

Consider an ensemble in which 40% of the systems are known to be prepared in the state

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$$

and 60% of the systems are prepared in the state

$$|\phi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

(a) Find the density operators for each of these states, and show they are pure states. If measurements are made on systems in each of these states, what are the probabilities they are found to be in states $|0\rangle$ and state $|1\rangle$, respectively?

(b) Determine the density operator for the ensemble.

(c) Show that $Tr(\rho) = 1$.

(d) A measurement of Z is made on a member drawn from the ensemble. What are the probabilities it is found to be in state $|0\rangle$ and state $|1\rangle$, respectively?

Solution

(a) By looking at $|\psi\rangle$ and using the Born rule, we see that the probability of finding the system in state $|0\rangle$ is 1/3, while the probability of finding the system in the state $|1\rangle$ is 2/3. If the system is prepared in state $|\phi\rangle$, the probability that measurement finds the system in state $|0\rangle$ is 1/4, and the probability that the system is found in state $|1\rangle$ is 3/4.

The density operators for each individual state are given by $\rho_\psi = |\psi\rangle\langle\psi|$ and $\rho_\phi = |\phi\rangle\langle\phi|$. We obtain

$$\rho_\psi = |\psi\rangle\langle\psi| = \left(\frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle\right)\left(\frac{1}{\sqrt{3}}\langle 0| + \sqrt{\frac{2}{3}}\langle 1|\right)$$

$$= \frac{1}{3}|0\rangle\langle 0| + \frac{\sqrt{2}}{3}|0\rangle\langle 1| + \frac{\sqrt{2}}{3}|1\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1|$$

$$\rho_\phi = |\phi\rangle\langle\phi| = \left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right)\left(\frac{1}{2}\langle 0| + \frac{\sqrt{3}}{2}\langle 1|\right)$$

$$= \frac{1}{4}|0\rangle\langle 0| + \frac{\sqrt{3}}{4}|0\rangle\langle 1| + \frac{\sqrt{3}}{4}|1\rangle\langle 0| + \frac{3}{4}|1\rangle\langle 1|$$

The matrix representations are given by

$$\rho_\psi = \begin{pmatrix} \langle 0|\rho_\psi|0\rangle & \langle 0|\rho_\psi|1\rangle \\ \langle 1|\rho_\psi|0\rangle & \langle 1|\rho_\psi|1\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} \end{pmatrix}$$

and

$$\rho_\phi = \begin{pmatrix} \langle 0|\rho_\phi|0\rangle & \langle 0|\rho_\phi|1\rangle \\ \langle 1|\rho_\phi|0\rangle & \langle 1|\rho_\phi|1\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix}$$

To determine whether or not these matrices are pure states, we square each density matrix and then compute the trace. We calculate the first explicitly:

$$\rho_\psi^2 = \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} \end{pmatrix}$$

Since $\rho_\psi = \rho_\psi^2$ and $Tr(\rho_\psi) = 1$, this is a pure state. It is easy to show that ρ_ϕ is also a pure state.

(b) We use (5.7), which we restate here:

$$\rho = \sum_{i=1}^{n} p_i \rho_i = \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i|$$

The probabilities given in the problem are

$$p_\psi = 40\% = \frac{2}{5}$$

$$p_\phi = 60\% = \frac{3}{5}$$

The density matrix for the ensemble is

$$\rho = p_\psi \rho_\psi + p_\phi \rho = \frac{2}{5}\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} \end{pmatrix} + \frac{3}{5}\begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{17}{60} & \frac{(8\sqrt{2}+9\sqrt{3})}{60} \\ \frac{(8\sqrt{2}+9\sqrt{3})}{60} & \frac{43}{60} \end{pmatrix}$$

(c) Notice that $Tr(\rho) = 17/60 + 43/60 = 60/60 = 1$, which must be the case for a density matrix.

(d) To find the respective probabilities, recall that we can write the matrix representation in the $\{|0\rangle, |1\rangle\}$ basis:

$$\rho = \begin{pmatrix} \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle \\ \langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle \end{pmatrix}$$

The probability of finding a member of the ensemble in the state $|0\rangle$ is given by $\langle 0|\rho|0\rangle$. Looking at the density matrix calculated in part (c), we see that the probability is $17/60 \approx 0.28$.

Similarly, the probability of finding a member of the ensemble in the state $|1\rangle$ is given by $\langle 1|\rho|1\rangle$. Looking at the density matrix calculated in part (c), we see that the probability is $43/60 \approx 0.72$.

Partial Trace and Reduced Density

Imagine that Alice and Bob each share one member of the entangled EPR pair:

$$|\beta_{10}\rangle = \frac{|0_A\rangle|0_B\rangle - |1_A\rangle|1_B\rangle}{\sqrt{2}}$$

We construct the density operator:

$$
\begin{aligned}
\rho &= |\beta_{10}\rangle\langle\beta_{10}| \\
&= \frac{|0_A\rangle|0_B\rangle\langle 0_A|\langle 0_B| - |0_A\rangle|0_B\rangle\langle 1_A|\langle 1_B| - |1_A\rangle|1_B\rangle\langle 0_A|\langle 0_B| + |1_A\rangle|1_B\rangle\langle 1_A|\langle 1_B|}{2}
\end{aligned}
$$

The idea for the partial trace is to obtain the density operator for one of the composite systems alone. Suppose we are Bob, to get the density operator for Bob, we need to trace over Alice's basis states. We have:

$$
\begin{aligned}
\rho_B &= Tr_A(\rho) = Tr_A(|\beta_{10}\rangle\langle\beta_{10}|) \\
&= \langle 0_A|(|\beta_{10}\rangle\langle\beta_{10}|)|0_A\rangle + \langle 1_A|(|\beta_{10}\rangle\langle\beta_{10}|)|1_A\rangle
\end{aligned}
$$

Using our expression for ρ, we have

$$
\begin{aligned}
&\langle 0_A|(|\beta_{10}\rangle\langle\beta_{10}|)|0_A\rangle \\
&= \langle 0_A|\left(\frac{|0_A\rangle|0_B\rangle\langle 0_A|\langle 0_B| - |0_A\rangle|0_B\rangle\langle 1_A|\langle 1_B| - |1_A\rangle|1_B\rangle\langle 0_A|\langle 0_B| + |1_A\rangle|1_B\rangle\langle 1_A|\langle 1_B|}{2}\right)|0_A\rangle \\
&= \frac{1}{2}\left(\frac{\begin{array}{c}\langle 0_A|0_A\rangle|0_B\rangle\langle 0_B|\langle 0_A|0_A\rangle - \langle 0_A|0_A\rangle|0_B\rangle\langle 1_B|\langle 0_A|1_A\rangle - \langle 0_A|1_A\rangle|1_B\rangle\langle 0_B|\langle 0_A|0_A\rangle \\ + \langle 0_A|1_A\rangle|1_B\rangle\langle 1_B|\langle 1_A|0_A\rangle\end{array}}{2}\right) \\
&= \frac{|0_B\rangle\langle 0_B|}{2}
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle 1_A|(|\beta_{10}\rangle\langle\beta_{10}|)|1_A\rangle \\
&= \langle 1_A|\left(\frac{|0_A\rangle|0_B\rangle\langle 0_A|\langle 0_B| - |0_A\rangle|0_B\rangle\langle 1_A|\langle 1_B| - |1_A\rangle|1_B\rangle\langle 0_A|\langle 0_B| + |1_A\rangle|1_B\rangle\langle 1_A|\langle 1_B|}{2}\right)|1_A\rangle \\
&= \frac{1}{2}\left(\frac{\begin{array}{c}\langle 1_A|0_A\rangle|0_B\rangle\langle 0_B|\langle 0_A|1_A\rangle - \langle 1_A|0_A\rangle|0_B\rangle\langle 1_B|\langle 0_A|1_A\rangle - \langle 1_A|1_A\rangle|1_B\rangle\langle 0_B|\langle 0_A|1_A\rangle \\ + \langle 1_A|1_A\rangle|1_B\rangle\langle 1_B|\langle 1_A|1_A\rangle\end{array}}{2}\right) \\
&= \frac{|1_B\rangle\langle 1_B|}{2}
\end{aligned}
$$

Therefore:

$$\rho_B = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}$$

This is Bob's state alone. The matrix representation is:

$$\rho_B = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

But $Tr(\rho_B^2) < 1$, so it is a mixed state.

possession.

What about the state of the joint system? The matrix representation of ρ as given in (15) is

$$[\rho]=\begin{pmatrix}\langle 00|\rho|00\rangle & \langle 00|\rho|01\rangle & \langle 00|\rho|10\rangle & \langle 00|\rho|11\rangle\\ \langle 01|\rho|00\rangle & \langle 01|\rho|01\rangle & \langle 01|\rho|10\rangle & \langle 01|\rho|11\rangle\\ \langle 10|\rho|00\rangle & \langle 10|\rho|01\rangle & \langle 10|\rho|10\rangle & \langle 10|\rho|11\rangle\\ \langle 11|\rho|00\rangle & \langle 11|\rho|01\rangle & \langle 11|\rho|10\rangle & \langle 11|\rho|11\rangle\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{2} & 0 & 0 & \frac{-1}{2}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \frac{-1}{2} & 0 & 0 & \frac{1}{2}\end{pmatrix}$$

It can be easily verified that

$$\rho^2=\begin{pmatrix}\frac{1}{2} & 0 & 0 & \frac{-1}{2}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \frac{-1}{2} & 0 & 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0 & 0 & \frac{-1}{2}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \frac{-1}{2} & 0 & 0 & \frac{1}{2}\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & 0 & 0 & \frac{-1}{2}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \frac{-1}{2} & 0 & 0 & \frac{1}{2}\end{pmatrix}$$

So we have $Tr(\rho^2)=1$. That is, the joint system described by the state $|\beta_{10}\rangle$ is a pure state, while Alice and Bob alone see completely mixed states. We will see more of joint systems when we study entanglement in detail.

Example 5.11

Suppose that

$$|A\rangle=\frac{|0\rangle-i|1\rangle}{\sqrt{2}},\quad |B\rangle=\sqrt{\frac{2}{3}}|0\rangle+\frac{1}{\sqrt{3}}|1\rangle$$

(a) Write down the product state $|A\rangle|B\rangle$.

(b) Compute the density operator. Is this a pure state?

Solution

(a) The product state is

$$|A\rangle \otimes |B\rangle = \left(\frac{|0\rangle - i|1\rangle}{\sqrt{2}}\right) \otimes \left(\sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle\right)$$

$$= \frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{6}}|01\rangle - \frac{i}{\sqrt{3}}|10\rangle - \frac{i}{\sqrt{6}}|11\rangle$$

(b) The density operator is

$$\rho = \left(\frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{6}}|01\rangle - \frac{i}{\sqrt{3}}|10\rangle - \frac{i}{\sqrt{6}}|11\rangle\right)\left(\frac{1}{\sqrt{3}}\langle 00| + \frac{1}{\sqrt{6}}\langle 01| + \frac{i}{\sqrt{3}}\langle 10| + \frac{i}{\sqrt{6}}\langle 11|\right)$$

$$= \frac{1}{3}|00\rangle\langle 00| + \frac{1}{\sqrt{18}}|00\rangle\langle 01| + \frac{i}{3}|00\rangle\langle 10| + \frac{i}{\sqrt{18}}|00\rangle\langle 11| + \frac{1}{\sqrt{18}}|01\rangle\langle 00|$$

$$+ \frac{1}{6}|01\rangle\langle 01| + \frac{i}{\sqrt{18}}|01\rangle\langle 10| + \frac{i}{6}|01\rangle\langle 11|$$

$$- \frac{i}{3}|10\rangle\langle 00| - \frac{i}{\sqrt{18}}|10\rangle\langle 01| + \frac{1}{3}|10\rangle\langle 10| + \frac{1}{\sqrt{18}}|10\rangle\langle 11| - \frac{i}{\sqrt{18}}|11\rangle\langle 00|$$

$$- \frac{i}{6}|11\rangle\langle 01| + \frac{1}{\sqrt{18}}|11\rangle\langle 10| + \frac{1}{6}|11\rangle\langle 11|$$

The matrix representation is

$$\rho = \begin{pmatrix} \frac{1}{3} & \frac{1}{\sqrt{18}} & \frac{i}{3} & \frac{i}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} & \frac{1}{6} & \frac{i}{\sqrt{18}} & \frac{i}{6} \\ \frac{-i}{3} & \frac{-i}{\sqrt{18}} & \frac{1}{3} & \frac{1}{\sqrt{18}} \\ \frac{-i}{\sqrt{18}} & \frac{-i}{6} & \frac{1}{\sqrt{18}} & \frac{1}{6} \end{pmatrix}$$

Notice that the trace is unity

$$Tr(\rho) = \frac{1}{3} + \frac{1}{6} + \frac{1}{3} + \frac{1}{6} = 1$$

Squaring, we have

$$\rho^2 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{i}{3} & \frac{i}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} & \frac{1}{6} & \frac{i}{3\sqrt{2}} & \frac{i}{6} \\ \frac{-i}{3} & \frac{-i}{3\sqrt{2}} & \frac{1}{3} & \frac{1}{3\sqrt{2}} \\ \frac{-i}{3\sqrt{2}} & \frac{-i}{6} & \frac{1}{3\sqrt{2}} & \frac{1}{6} \end{pmatrix}$$

Of course, this matrix is just the original density matrix, so

$$Tr(\rho^2) = \frac{1}{3} + \frac{1}{6} + \frac{1}{3} + \frac{1}{6} = 1$$

We see that the product state is a pure state.

Density Operator and the Bloch Vector

The density operator of a system in a 2D Hilbert space can be decomposed in the following way:

$$\vec{\sigma} = \sigma_x \widehat{x} + \sigma_y \widehat{y} + \sigma_z \widehat{z}$$

Then, a density operator can be decomposed as:

$$\rho = \frac{1}{2}(I + \vec{S} \cdot \vec{\sigma})$$

$\vec{S}$ is called the Bloch vector. If satisfies $|\vec{S}| \leq 1$ with equality for pure states. where:

$$\vec{S} = S_x \widehat{x} + S_y \widehat{y} + S_z \widehat{z} = \langle X \rangle \widehat{x} + \langle Y \rangle \widehat{y} + \langle Z \rangle \widehat{z}$$

Where also:

$$S_x = Tr(\rho X) \quad , \quad S_y = Tr(\rho Y) \quad , \quad S_z = Tr(\rho Z)$$

Example 5.12

Consider the following matrix:

$$\rho = \begin{pmatrix} \frac{5}{8} & \frac{i}{4} \\ \frac{-i}{4} & \frac{3}{8} \end{pmatrix}$$

(a) Is this a valid density operator?

(b) Does this represent a pure state or a mixed state?

Solution

(a) First it is easy to verify that the matrix is Hermitian. The transpose is

$$\rho^T = \begin{pmatrix} \frac{5}{8} & \frac{-i}{4} \\ \frac{i}{4} & \frac{3}{8} \end{pmatrix}$$

Taking the complex conjugate, we observe that $\rho = \rho^{dag}$, so the matrix is Hermitian. Next we see that

$$Tr(\rho) = \frac{5}{8} + \frac{3}{8} = 1$$

as required for density operators. Finally, we check the eigenvalues. A simple calculation shows that the eigenvalues of the matrix are

$$\lambda_{1,2} = \frac{4 \pm \sqrt{5}}{8}$$

Both eigenvalues satisfy $0 < \lambda_{1,2} < 1$, so the matrix represents a positive operator. We conclude that this is a valid density matrix.

(b) We compute the components of the Bloch vector:

$$S_x = Tr(X\rho) = Tr\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{5}{8} & \frac{i}{4} \\ \frac{-i}{4} & \frac{3}{8} \end{pmatrix}\right] = Tr\begin{pmatrix} \frac{-i}{4} & \frac{3}{8} \\ \frac{5}{8} & \frac{i}{4} \end{pmatrix} = 0$$

$$S_y = Tr(Y\rho) = Tr\left[\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \frac{5}{8} & \frac{i}{4} \\ \frac{-i}{4} & \frac{3}{8} \end{pmatrix}\right] = Tr\begin{pmatrix} \frac{-1}{4} & \frac{-i3}{8} \\ \frac{i5}{8} & \frac{-1}{4} \end{pmatrix} = \frac{-1}{2}$$

$$S_z = Tr(Z\rho) = Tr\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{5}{8} & \frac{i}{4} \\ \frac{-i}{4} & \frac{3}{8} \end{pmatrix}\right] = Tr\begin{pmatrix} \frac{5}{8} & \frac{i}{4} \\ \frac{i}{4} & \frac{-3}{8} \end{pmatrix} = \frac{1}{4}$$

The magnitude of the Bloch vector is

$$\begin{aligned}
|\vec{S}| &= \sqrt{S_x^2 + S_y^2 + S_z^2} \\
&= \sqrt{\left(\frac{-1+}{2}\right)^2 + \left(\frac{1}{4}\right)^2} \\
&= \sqrt{\frac{1}{4} + \frac{1}{16}} \\
&= \sqrt{\frac{5}{16}} \\
&= \frac{\sqrt{5}}{4} \approx 0.56 < 1
\end{aligned}$$

Since $|\vec{S}| < 1$, we conclude that this density matrix represents a mixed state.

Resumen

For a pure state $|\psi\rangle = c_1|u_1\rangle + \cdots + c_n|u_n\rangle$ (pure state means that all the members of the ensemble have this state). Then, we define the density operator as:

$$\rho = |\psi\rangle\langle\psi|$$

Which helps us calculate **expectation values:**

$$\langle A\rangle = Tr(\rho A)$$

And it has the **properties:**

- $\rho = \rho^*$

- $Tr(\rho) = 1$
- $Tr(\rho^2) = 1$
- $\rho^2 = \rho$

Evolution of the density operator:

$$i\hbar \frac{d\rho}{dt} = [H, \rho]$$

Density Operator for a mixed state:

We have an ensemble with p_i probability of finding a member in state $|\psi_i\rangle$. Then the **density operator is:**

$$\rho = \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i|$$

Key Properties:

- $\rho = \rho^*$
- $Tr(\rho) = 1$
- $Tr(\rho^2) < 1$
- $\langle u|\rho u\rangle \geq 0$

For an operator A, the **expectation value is:**

$$\langle A\rangle = Tr(\rho A)$$

Probability of obtaining a given measurement result:
If we measure an observable, the probability of getting the value associated with the eigenvector $|u_n\rangle$ (say the eigenvalue a_n) is:

$$p(a_n) = Tr(P_n \rho)$$

Or using the measurement operator M_m formalism, the probability is:

$$P(m) = Tr(M_m^* M_m \rho)$$

After measurement, the density matrix becomes:

$$\begin{aligned} \rho &\rightarrow \frac{P_n \rho P_n}{Tr(P_n \rho)} \\ &\rightarrow \frac{M_m \rho M_m^*}{Tr(M_m^* M_m \rho)} \end{aligned}$$

Lecture Notes

Suppose we have an ensemble of pure states $\{q_i, |\psi_i\rangle\}_{i=1}^{N}$ (with q_i the probabilities of each pure state).
Then, let's say we make a measurement with operatores $\{M_k\}$, then each state transforms into:

$$|\psi_i\rangle \to \frac{M_k|\psi_i\rangle}{\sqrt{p_{k|i}}} = |\psi_i^k\rangle$$

Where $p_{k|i} = \langle\psi_i|M_k^\dagger M_k|\psi_i\rangle$ is the probability of outcome k given state $|\psi_i\rangle$
If we choose any state of the ensemble, the probability of obtaining a value of p_k is:

$$\begin{aligned} p_k &= \sum_i p_{k|i} q_i \\ &= \sum_i q_i \langle\psi_i|M_k^\dagger M_k|\psi_i\rangle \\ &= Tr\left[M_k^\dagger M_k \left(\sum_i q_i|\psi_i\rangle\langle\psi_i|\right)\right] \end{aligned}$$

So we define $\rho = \sum_i q_i|\psi_i\rangle\langle\psi_i|$

Then, we find that the probability of obtaining result p_k is:

$$p_k = Tr(E_k\rho)$$

And the resulting density operator is:

$$\begin{aligned} \rho_k &= \sum_i p_{i|k}|\psi_i^k\rangle\langle\psi_i^k| \\ &= \sum_i p_{i|k}\frac{M_k|\psi_i\rangle\langle\psi_i|M_k^\dagger}{p_{k|i}} \\ &= \sum_i \frac{q_i}{p_k} M_k|\psi_i\rangle\langle\psi_i|M_k^\dagger \\ &= \frac{M_k\rho M_k^\dagger}{p_k} \\ &= \frac{M_k\rho M_k^\dagger}{Tr[\rho M_k^\dagger M_k]} \end{aligned}$$

Quantum Measurement Theory

Consider the general qubit:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

When the measurement is made, the qubit is forces into $|\psi\rangle \rightarrow |0\rangle$ or $|\psi\rangle \rightarrow |1\rangle$. It isn't possible to determine α or β.

The measurement of a quantum system involves some type of interaction or coupling of the system with a measuring device (ancilla), the complete system + device is called open system.
The time evolution of a closed system is described by:

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$$

If the initial state is $|\psi(0)\rangle$, then the solution is:

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$$

Where

$$U = e^{-iHt/\hbar}$$

is the **unitary evolution operator.**

If the system is initially described by some density operator ρ_0, then the state of the system at time t will be:

$$\rho_t = U\rho_0 U^*$$

The dynamics of a quantum system is trace preserving. So $Tr(\rho_t) = Tr(\rho_0) = 1$

A quantum operation involving measurement, described by a measurement operator M_m, transforms a density operator ρ into:

$$\rho' = M_m \rho M_m^*$$

Projective Measurements

Also called Von Neumann measurements. Mutually exclusive measurements like knowing if a qubit is in $|0\rangle$ or $|1\rangle$ is described by an operator P.
This operator is Hermitian:

$$P = P^*$$

And equal to its own square:

$$P^2 = P$$

The projection operators are **orthogonal** if $P_1 P_2|\psi\rangle = 0$. Or in general:

$$P_i P_j = \delta_{ij} P_i$$

A **complete** set of orthogonal projection operators is one for which:

$$\sum_i P_i = I$$

Every complete set of orthogonal projectors specifies a measurement that can be realized. If the dimension of the Hilbert space is d, and there are m projection operators, it must be true that $m \leq d$.
For example, for measuring qubits, the projection operators would be:

$$P_0 = |0\rangle\langle 0| \quad , \quad P_1 = |1\rangle\langle 1|$$

Consider a set of mutually orthogonal projection operators $\{P_1, P_2, \cdots, P_n\}$. And let the system be in a state $|\psi\rangle$. The probability of finding the ith outcome is:

$$Pr(i) = |P_i|\psi\rangle|^2 = (P_i|\psi\rangle)^*(P_i|\psi\rangle) = \langle\psi|P_i^2|\psi\rangle = \langle\psi|P_i|\psi\rangle = Tr(P_i|\psi\rangle\langle\psi|)$$

Which makes sense because it is equal to $\langle\psi|u_i\rangle\langle u_i|\psi\rangle = |\langle\psi|u_i\rangle^2$

Now we consider some observable A, with eigenvectors $|u_i\rangle$ each with eigenvalue a_i. The spectral decomposition of A allows us to write the operator as:

$$A = \sum_{i=1}^{n} a_i |u_i\rangle\langle u_i| = \sum_{i=1}^{n} a_i P_i$$

Where $P_i = |u_i\rangle\langle u_i|$. We can expand the state $|\psi\rangle$ as:

$$|\psi\rangle = \sum_{i=1}^{n} \langle u_i|\psi\rangle|u_i\rangle = \sum_{i=1}^{n} c_i|u_i\rangle$$

The probability for measuring the eigenvalue a_i is given by:

$$Pr(i) = |\langle u_i|\psi\rangle|^2$$

After measurement, the state changes to:

$$|\psi'\rangle = \frac{P_i|\psi\rangle}{\sqrt{\langle\psi|P_i|\psi\rangle}}$$

The expectation value of an observable A is given by:

$$\langle A\rangle = \sum_i a_i \langle\psi|P_i|\psi\rangle$$

Example 6.1

A system is in the state

$$|\psi\rangle = \frac{2}{\sqrt{19}}|u_1\rangle + \frac{2}{\sqrt{19}}|u_2\rangle + \frac{1}{\sqrt{19}}|u_3\rangle + \frac{2}{\sqrt{19}}|u_4\rangle + \sqrt{\frac{6}{19}}|u_5\rangle$$

where $\{|u_1\rangle, |u_2\rangle, |u_3\rangle, |u_4\rangle, |u_5\rangle\}$ are a complete and orthonormal set of vectors. Each $|u_i\rangle$ is an eigenstate of the system's Hamiltonian corresponding to the possible measurement result $H|u_n\rangle = n\varepsilon|u_n\rangle$, where $n = 1, 2, 3, 4, 5$.

(a) Describe the set of projection operators corresponding to the possible measurement results.

(b) Determine the probability of obtaining each measurement result. What is the state of the system after measurement if we measure the energy to be 3ε?

(c) What is the average energy of the system?

Solution

(a) The possible measurement results are $\varepsilon, 2\varepsilon, 3\varepsilon, 4\varepsilon$, and 5ε. These measurement results correspond to the basis states $|u_1\rangle$, $|u_2\rangle$, $|u_3\rangle$, $|u_4\rangle$, and $|u_5\rangle$, respectively. Hence the projection operators corresponding to each measurement result are

$$P_1 = |u_1\rangle\langle u_1|$$

$$P_2 = |u_2\rangle\langle u_2|$$

$$P_3 = |u_3\rangle\langle u_3|$$

$$P_4 = |u_4\rangle\langle u_4|$$

$$P_5 = |u_5\rangle\langle u_5|$$

Since the $|u_i\rangle$'s are a set of orthnormal basis vectors, the completeness relation is satisfied and

$$\sum_i P_i = I$$

(b) We can calculate the probability of obtaining each measurement result using (6.14) or (6.17). Let's apply (6.17) to calculate the probability of finding ε or 2ε. First we need to check and see if the state is normalized. This is done by calculating

$$\sum_{i=1}^{5} |c_i|^2$$

and seeing if the result is 1. We have

$$\begin{aligned}\sum_{i=1}^{5} |c_i|^2 &= \left|\frac{2}{\sqrt{19}}\right|^2 + \left|\frac{2}{\sqrt{19}}\right|^2 + \left|\frac{1}{\sqrt{19}}\right|^2 + \left|\frac{2}{\sqrt{19}}\right|^2 + \left|\sqrt{\frac{6}{19}}\right|^2 \\ &= \frac{4}{19} + \frac{4}{19} + \frac{1}{19} + \frac{4}{19} + \frac{6}{19} \\ &= \frac{19}{19} = 1\end{aligned}$$

The state is normalized, so we can proceed. Before doing so, recall that the fact that the basis states are orthonormal means that

$$\langle u_i | u_j \rangle = \delta_{ij}$$

So, in the first case, applying the Born rule we have

$$\begin{aligned}
\Pr(\varepsilon) = |\langle u_2|\psi\rangle|^2 &= \left| \langle u_1| \left(\frac{2}{\sqrt{19}}|u_1\rangle + \frac{2}{\sqrt{19}}|u_2\rangle + \frac{1}{\sqrt{19}}|u_3\rangle + \frac{2}{\sqrt{19}}|u_4\rangle + \sqrt{\frac{6}{19}}|u_5\rangle \right) \right|^2 \\
&= \left| \frac{2}{\sqrt{19}}\langle u_1|u_1\rangle + \frac{2}{\sqrt{19}}\langle u_1|u_2\rangle + \frac{1}{\sqrt{19}}\langle u_1|u_3\rangle + \frac{2}{\sqrt{19}}\langle u_1|u_4\rangle + \sqrt{\frac{6}{19}}\langle u_1|u_5\rangle \right|^2 \\
&= \left| \frac{2}{\sqrt{19}}(1) + \frac{2}{\sqrt{19}}(0) + \frac{1}{\sqrt{19}}(0) + \frac{2}{\sqrt{19}}(0) + \sqrt{\frac{6}{19}}(0) \right|^2 \\
&= \left| \frac{2}{\sqrt{19}} \right|^2 \\
&= \frac{4}{19}
\end{aligned}$$

The probability of obtaining the second measurement result is

$$\begin{aligned}
\Pr(\varepsilon) = |\langle u_2|\psi\rangle|^2 &= \left| \langle u_2| \left(\frac{2}{\sqrt{19}}|u_1\rangle + \frac{2}{\sqrt{19}}|u_2\rangle + \frac{1}{\sqrt{19}}|u_3\rangle + \frac{2}{\sqrt{19}}|u_4\rangle + \sqrt{\frac{6}{19}}|u_5\rangle \right) \right|^2 \\
&= \left| \frac{2}{\sqrt{19}}\langle u_2|u_2\rangle \right|^2 \\
&= \left| \frac{2}{\sqrt{19}} \right|^2 \\
&= \frac{4}{19}
\end{aligned}$$

To calculate the remaining probabilities, let's use the projection operators and apply (6.14). We find that

$$\begin{aligned}
P_3|\psi\rangle &= (|u_3\rangle\langle u_3|) \left(|\psi\rangle = \frac{2}{\sqrt{19}}|u_1\rangle + \frac{2}{\sqrt{19}}|u_2\rangle + \frac{1}{\sqrt{19}}|u_3\rangle + \frac{2}{\sqrt{19}}|u_4\rangle + \sqrt{\frac{6}{19}}|u_5\rangle \right) \\
&= |u_3\rangle \left(\frac{1}{\sqrt{19}}\langle u_3|u_3\rangle \right) = \frac{1}{\sqrt{19}}|u_3\rangle
\end{aligned}$$

Therefore

$$\Pr(3\varepsilon) = \langle \psi | P_3 | \psi \rangle$$

$$= \left(\frac{2}{\sqrt{19}} \langle u_1| + \frac{2}{\sqrt{19}} \langle u_2| + \frac{1}{\sqrt{19}} \langle u_3| + \frac{2}{\sqrt{19}} \langle u_4| + \sqrt{\frac{6}{19}} \langle u_5| \right) \left(\frac{1}{\sqrt{19}} |u_3\rangle \right)$$

$$= \left(\frac{2}{\sqrt{19}} \right) \left(\frac{1}{\sqrt{19}} \right) \langle u_1|u_3\rangle + \left(\frac{2}{\sqrt{19}} \right) \left(\frac{1}{\sqrt{19}} \right) \langle u_2|u_3\rangle + \frac{1}{19} \langle u_3|u_3\rangle$$

$$+ \left(\frac{2}{\sqrt{19}} \right) \left(\frac{1}{\sqrt{19}} \right) \langle u_4|u_3\rangle + \left(\sqrt{\frac{6}{19}} \right) \left(\frac{1}{\sqrt{19}} \right) \langle u_5|u_3\rangle$$

$$= \frac{1}{19}$$

Similarly we find that

$$P_4|\psi\rangle = (|u_4\rangle\langle u_4|)|\psi\rangle = \frac{2}{\sqrt{19}}|u_4\rangle$$

$$P_5|\psi\rangle = (|u_5\rangle\langle u_5|)|\psi\rangle = \sqrt{\frac{6}{19}}|u_4\rangle$$

So we write

$$\Pr(4\varepsilon) = \langle \psi|P_4|\psi\rangle = \frac{4}{19}$$

$$\Pr(5\varepsilon) = \langle \psi|P_5|\psi\rangle = \frac{6}{19}$$

If a measurement is made and we find the energy to be 3ε, we apply (6.20). The state of the system after measurement is

$$|\psi'\rangle = \frac{P_3|\psi\rangle}{\sqrt{\langle \psi|P_3|\psi\rangle}} = \frac{1/\sqrt{19}[|\mu_3\rangle}{\sqrt{1/19}} = |\mu_3\rangle$$

(c) The average energy of the system is found using (6.21). We find that

$$\langle H\rangle = \sum_{i=1}^{5} E_i \langle \psi|P_i|\psi\rangle = \varepsilon\langle \psi|P_1|\psi\rangle + 2\varepsilon\langle \psi|P_2|\psi\rangle + 3\varepsilon\langle \psi|P_3|\psi\rangle$$

$$+ 4\varepsilon\langle \psi|P_4|\psi\rangle + 5\varepsilon\langle \psi|P_5|\psi\rangle$$

$$= \varepsilon\frac{4}{19} + 2\varepsilon\frac{4}{19} + 3\varepsilon\frac{1}{19} + 4\varepsilon\frac{4}{19} + 5\varepsilon\frac{6}{19}$$

$$= \frac{61}{19}\varepsilon$$

Example 6.2

A qubit is in the state

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle$$

A measurement with respect to Y is made. Given that the eigenvalues of the Y matrix are ± 1, determine the probability that the measurement result is $+1$ and the probability that the measurement result is -1.

Solution

First we verify that the state is normalized

$$
\begin{aligned}
\langle\psi|\psi\rangle &= \left(\frac{\sqrt{3}}{2}\langle 0| - \frac{1}{2}\langle 1|\right)\left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\right) \\
&= \frac{3}{4}\langle 0|0\rangle - \frac{\sqrt{3}}{4}\langle 1|0\rangle - \frac{\sqrt{3}}{4}\langle 0|1\rangle + \frac{1}{4}\langle 1|1\rangle \\
&= \frac{3}{4} + \frac{1}{4} = 1
\end{aligned}
$$

Since $\langle\psi|\psi\rangle = 1$ the state is normalized. Recall that $Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. You need to show that the eigenvectors of the Y matrix are

$$
|u_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |u_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}
$$

corresponding to the eigenvalues ± 1, respectively. The dual vectors in each case, found by computing the transpose of each vector can taking the complex conjugate of each element, are

$$
\langle u_1| = (|u_1\rangle)^\dagger = \frac{1}{\sqrt{2}}(1 \quad -i), \quad \langle u_2| = (|u_2\rangle)^\dagger = \frac{1}{\sqrt{2}}(1 \quad i)
$$

The projection operators corresponding to each possible measurement result are

$$
\begin{aligned}
P_{+1} &= |u_1\rangle\langle u_1| = \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}(1 \quad -i) = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \\
P_{-1} &= |u_2\rangle\langle u_2| = \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix}(1 \quad i) = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}
\end{aligned}
$$

Writing the state $|\psi\rangle$ as a column vector, we have

$$
|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle = \frac{\sqrt{3}}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix}
$$

Hence

$$
\begin{aligned}
P_{+1}|\psi\rangle &= \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \sqrt{3}+i \\ -1+i\sqrt{3} \end{pmatrix} \\
P_{-1}|\psi\rangle &= \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \sqrt{3}-i \\ -1-i\sqrt{3} \end{pmatrix}
\end{aligned}
$$

Now, if a measurement is made of the Y observable, the probability of finding $+1$ is

$$
\begin{aligned}
\Pr(+1) &= \langle\psi|P_{+1}|\psi\rangle = \frac{1}{2}(\sqrt{3} \quad -1)\frac{1}{4}\begin{pmatrix} \sqrt{3}+i \\ -1+i\sqrt{3} \end{pmatrix} \\
&= \frac{1}{8}(3 + i\sqrt{3} + 1 - i\sqrt{3}) = \frac{1}{8}(3+1) = \frac{1}{2}
\end{aligned}
$$

Similarly find

$$
\begin{aligned}
\Pr(-1) &= \langle\psi|P_{-1}|\psi\rangle = \frac{1}{2}(\sqrt{3} \quad -1)\frac{1}{4}\begin{pmatrix} \sqrt{3}-i \\ -1-i\sqrt{3} \end{pmatrix} \\
&= \frac{1}{8}(3 - i\sqrt{3} + 1 + i\sqrt{3}) = \frac{1}{8}(3+1) = \frac{1}{2}
\end{aligned}
$$

Example 6.4

Describe the action of the operators $P_0 \otimes I$ and $I \otimes P_1$ on the state

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

Solution

The first operator, $P_0 \otimes I$, tells us to apply the projection operator, $P_0 = |0\rangle\langle 0|$, to the *first qubit* and to leave the second qubit alone. The result is

$$P_0 \otimes I|\psi\rangle = \frac{1}{\sqrt{2}}[(|0\rangle\langle 0|0\rangle) \otimes |1\rangle - (|0\rangle\langle 0|1\rangle \otimes |0\rangle)] = \frac{|01\rangle}{\sqrt{2}}$$

Interestingly, applying a projective measurement to the first qubit causes the second qubit to assume a definite state. As we will see in the next chapter, this is a property of entangled systems. Apparently it doesn't matter if the qubits are spatially separated for a collapse of the system to occur.

To find the properly normalized state of the system after measurement we use (6.20). We have

$$\langle\psi|P_0 \otimes I|\psi\rangle = \left(\frac{\langle 01| - \langle 10|}{\sqrt{2}}\right)\frac{|01\rangle}{\sqrt{2}} = \frac{\langle 0|0\rangle\langle 1|1\rangle - \langle 1|0\rangle\langle 0|1\rangle}{2} = \frac{1}{2}$$

The state after measurement is

$$|\psi'\rangle = \frac{P_0 \otimes I|\psi\rangle}{\sqrt{\langle\psi|P_0 \otimes I|\psi\rangle}} = \frac{|01\rangle/\sqrt{2}}{(1/\sqrt{2})} = |01\rangle$$

As can be seen, while applying (6.20) in the single qubit case can seem like overkill, in this case this allows us to quickly write down the properly normalized state after measurement.

The second operator, $I \otimes P_1$, tells us to leave the *first qubit alone* and to apply the projection operator $P_1 = |1\rangle\langle 1|$ to the second qubit. This gives

$$I \otimes P_1|\psi\rangle = \frac{1}{\sqrt{2}}[|0\rangle \otimes (|1\rangle\langle 1|1\rangle) - |1\rangle \otimes (|1\rangle\langle 1|0\rangle)] = \frac{|01\rangle}{\sqrt{2}}$$

We have therefore the same state, but this time doing the projective measurement represented by $P_1 = |1\rangle\langle 1|$ the second qubit has forced the first qubit into the state $|0\rangle$. Let's redo the calculation using matrices. The operator is

$$I \otimes P_1 = \begin{pmatrix} 1 \cdot P_1 & 0 \cdot P_1 \\ 0 \cdot P_1 & 1 \cdot P_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then we have

$$|01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|10\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

So the state of the system prior to measurement is

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

The action of the operator $I \otimes P_1$ is then computed as follows:

$$I \otimes P_1|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}|01\rangle$$

Example 6.5

A system is in the state

$$|\psi\rangle = \frac{1}{\sqrt{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

(a) What is the probability that measurement finds the system in the state $|\phi\rangle = |01\rangle$?

(b) What is the probability that measurement finds the first qubit in the state $|0\rangle$? What is the state of the system after measurement?

Solution

(a) Given that the system is in the state $|\psi\rangle$, the probability of finding it in the state $|\phi\rangle = |01\rangle$ is calculated using the Born rule, which is $\Pr = |\langle\phi|\psi\rangle|^2$. Since $\langle 0|1\rangle = \langle 1|0\rangle = 0$, we have

$$\begin{aligned}
\langle\phi|\psi\rangle &= \langle 01| \left(\frac{1}{\sqrt{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle \right) \\
&= \frac{1}{\sqrt{8}}\langle 0|0\rangle\langle 1|0\rangle + \sqrt{\frac{3}{8}}\langle 0|0\rangle\langle 1|1\rangle + \frac{1}{2}\langle 0|1\rangle\langle 1|0\rangle + \frac{1}{2}\langle 0|1\rangle\langle 1|1\rangle \\
&= \sqrt{\frac{3}{8}}
\end{aligned}$$

Therefore the probability is

$$\Pr = |\langle\phi|\psi\rangle|^2 = \frac{3}{8}$$

(b) To find the probability that measurement finds the first qubit in the state $|0\rangle$, we can apply $P_0 \otimes I = |0\rangle\langle 0| \otimes I$ to the state. So the projection operator P_0 is applied to the first qubit and the identity operator to the second qubit, leaving the second qubit unchanged. This obtains

$$\begin{aligned}
P_0 \otimes I|\psi\rangle &= (|0\rangle\langle 0| \otimes I) \left(\frac{1}{\sqrt{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle \right) \\
&= \frac{1}{\sqrt{8}}|0\rangle\langle 0|0\rangle \otimes |0\rangle + \sqrt{\frac{3}{8}}|0\rangle\langle 0|0\rangle \otimes |1\rangle + \frac{1}{2}|0\rangle\langle 0|1\rangle \otimes |0\rangle + \frac{1}{2}|0\rangle\langle 0|1\rangle \otimes |1\rangle \\
&= \frac{1}{\sqrt{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle
\end{aligned}$$

The probability of obtaining this result is

$$\begin{aligned}
\Pr &= \langle\psi|P_0 \otimes I|\psi\rangle \\
&= \left(\frac{1}{\sqrt{8}}\langle 00| + \sqrt{\frac{3}{8}}\langle 01| + \frac{1}{2}\langle 10| + \frac{1}{2}\langle 11| \right) \left(\frac{1}{\sqrt{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle \right) \\
&= \frac{1}{8} + \frac{3}{8} = \frac{1}{2}
\end{aligned}$$

The state of the system after measurement using (6.20) is found to be

$$\begin{aligned}
|\psi'\rangle &= \frac{\frac{1}{\sqrt{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle}{\sqrt{\langle\psi|P_0 \otimes I|\psi\rangle}} = \sqrt{2}\left(\frac{1}{\sqrt{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle \right) \\
&= \frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}|01\rangle
\end{aligned}$$

Example 6.6

A three-qubit system is in the state

$$|\psi\rangle = \left(\frac{\sqrt{2}+i}{\sqrt{20}}\right)|000\rangle + \frac{1}{\sqrt{2}}|001\rangle + \frac{1}{\sqrt{10}}|011\rangle + \frac{i}{2}|111\rangle$$

(a) Is the state normalized? What is the probability that the system is found in the state $|000\rangle$ if all 3 qubits are measured?

(b) What is the probability that a measurement on the first qubit only gives 0? What is the postmeasurement state of the system?

Solution

(a) To determine if the state is normalized, we compute the sum of the squares of the coefficients:

$$\begin{aligned}\sum_i |c_i|^2 &= \left(\frac{\sqrt{2}+i}{\sqrt{20}}\right)\left(\frac{\sqrt{2}-i}{\sqrt{20}}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{10}}\right)\left(\frac{1}{\sqrt{10}}\right) \\ &\quad + \left(\frac{i}{2}\right)\left(-\frac{i}{2}\right) \\ &= \frac{3}{20} + \frac{1}{2} + \frac{1}{10} + \frac{1}{4} = \frac{20}{20} = 1\end{aligned}$$

So the state is normalized. The probability the system is found in state $|000\rangle$ if all three qubits are measured is

$$\Pr(000) = \left(\frac{\sqrt{2}+i}{\sqrt{20}}\right)\left(\frac{\sqrt{2}-i}{\sqrt{20}}\right) = \frac{3}{20} = 0.15$$

(b) The probability that a measurement on the first qubit is zero can be found by acting on the state with the operator $P_0 \otimes I \otimes I$ and computing $\langle\psi|P_0 \otimes I \otimes I|\psi\rangle$. This will project onto the $|0\rangle$ state for the first qubit while leaving the second and third qubits alone. We find that

$$
\begin{aligned}
P_0 \otimes I \otimes I|\psi\rangle &= \left(\frac{\sqrt{2}+i}{\sqrt{20}}\right)(|0\rangle\langle 0| \otimes I \otimes I)|000\rangle + \frac{1}{\sqrt{2}}(|0\rangle\langle 0| \otimes I \otimes I)|001\rangle \\
&\quad + \frac{1}{\sqrt{10}}(|0\rangle\langle 0| \otimes I \otimes I)|011\rangle + \frac{i}{2}(|0\rangle\langle 0| \otimes I \otimes I)|111\rangle \\
&= \left(\frac{\sqrt{2}+i}{\sqrt{20}}\right)|000\rangle + \frac{1}{\sqrt{2}}|001\rangle + \frac{1}{\sqrt{10}}|011\rangle
\end{aligned}
$$

The last term vanishes, since $\langle 0|1\rangle = 0$ So

$$\frac{i}{2}(|0\rangle\langle 0| \otimes I \otimes I)|111\rangle = \frac{i}{2}(|0\rangle\langle 0|1\rangle) \otimes |1\rangle \otimes |1\rangle = 0$$

Hence the probability that measurement on the first qubit finds 0 is

$$\langle\psi|P_0 \otimes I \otimes I|\psi\rangle = \left|\frac{\sqrt{2}+i}{\sqrt{20}}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{1}{\sqrt{10}}\right|^2 = \frac{3}{20} + \frac{1}{2} + \frac{1}{10} = \frac{3}{4}$$

The postmeasurement state is

$$
\begin{aligned}
|\psi'\rangle &= \frac{P_0 \otimes I \otimes I|\psi\rangle}{\sqrt{\langle\psi|P_0 \otimes I \otimes I|\psi\rangle}} = \sqrt{\frac{4}{3}}\left(\left(\frac{\sqrt{2}+i}{\sqrt{20}}\right)|000\rangle + \frac{1}{\sqrt{2}}|001\rangle + \frac{1}{\sqrt{10}}|011\rangle\right) \\
&= \left(\frac{\sqrt{2}+i}{\sqrt{15}}\right)|000\rangle + \sqrt{\frac{2}{3}}|001\rangle + \sqrt{\frac{2}{15}}|011\rangle
\end{aligned}
$$

Example 6.8

A two qubit system is in the state

$$|\phi\rangle = \frac{\sqrt{3}}{2}|00\rangle + \frac{1}{2}|11\rangle$$

A Y gate is applied to the first qubit. After this is done, what are the possible measurement results if both qubits are measured, and what are the respective probabilities of each measurement result?

Solution

The action of the Y gate on the computational basis states is

$$Y|0\rangle = i|1\rangle, \quad Y|1\rangle = -i|0\rangle$$

Hence

$$Y \otimes I|\phi\rangle = \frac{\sqrt{3}}{2}(Y \otimes I)|00\rangle + \frac{1}{2}(Y \otimes I)|11\rangle = i\frac{\sqrt{3}}{2}|10\rangle - \frac{i}{2}|01\rangle$$

If both qubits are measured, the possible measurement results are 10 and 01. The probability of finding 10 is

$$\left|i\frac{\sqrt{3}}{2}\right|^2 = \left(i\frac{\sqrt{3}}{2}\right)\left(-i\frac{\sqrt{3}}{2}\right) = \frac{3}{4}$$

The probability of finding 01 is

$$\left|\frac{i}{2}\right|^2 = \left(\frac{i}{2}\right)\left(-\frac{i}{2}\right) = \frac{1}{4}$$

These probabilities sum to one, as they should.

Generalized Measurements

We denote a measurement operator by M_m, where m is an index that denotes a possible measurement result. Given a state $|\psi\rangle$, the probability that we find the result m is:

$$Pr(m) = \langle\psi|M_m^* M_m|\psi\rangle$$

And after the measurement, the state is:

$$|\psi'\rangle = \frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^* M_m|\psi\rangle}}$$

Completeness relation:

$$\sum_m M_m^* M_m = I$$

Now, if the system is described by a density operator ρ, the probability of finding measurement result m is:

$$Pr(m) = Tr(M_m^* M_m \rho)$$

And the density changes to:

$$\begin{aligned}
\rho' &= \frac{M_m \rho M_m^*}{Tr(M_m^* M_m \rho)} \\
&= \frac{|u_i\rangle\langle u_i|\rho|u_i\rangle\langle u_i|}{\langle u_i|\rho|u_i\rangle}
\end{aligned}$$

Example 6.9

A quantum system has a density matrix given by

$$\rho = \frac{5}{6}|0\rangle\langle 0| + \frac{1}{6}|1\rangle\langle 1|$$

What is the probability that the system is in the state $|0\rangle$?

Solution

Of course, one can read off that the probability the system is in the state $|0\rangle$ is 5/6. But let's see how we can apply (6.30). It's easy enough:

$$\begin{aligned}
\Pr(0) &= Tr(|0\rangle\langle 0|\rho) = \langle 0|\rho|0\rangle \\
&= \langle 0|\left(\frac{5}{6}|0\rangle\langle 0| + \frac{1}{6}|1\rangle\langle 1|\right)|0\rangle \\
&= \frac{5}{6}\langle 0|0\rangle\langle 0|0\rangle + \frac{1}{6}\langle 0|1\rangle\langle 1|0\rangle = \frac{5}{6}
\end{aligned}$$

Example 6.10

A system has the density operator

$$\rho = \frac{1}{3}|u_1\rangle\langle u_1| - i\frac{\sqrt{2}}{3}|u_1\rangle\langle u_2| + i\frac{\sqrt{2}}{3}|u_2\rangle\langle u_1| + \frac{2}{3}|u_2\rangle\langle u_2|$$

where the $|u_k\rangle$ constitute an orthonormal basis. What is the probability that a measurement finds the system in the state $|u_2\rangle$?

Solution

The projection operator corresponding to this measurement result is

$$P_2 = |u_2\rangle\langle u_2|$$

The probability is

$$\Pr(|u_2\rangle) = Tr(|u_2\rangle\langle u_2|\rho) = \langle u_2|\rho|u_2\rangle = \frac{2}{3}$$

Positive Opeartor Values Measures

A POVM consists of a set of positive operators commonly denoted by E_m. The probability of obtaining measurement result m is:

$$Pr(m) = \langle\psi|E_m|\psi\rangle$$

When the system is mixed and there is a density operator, we have:

$$Pr(m) = Tr(E_m\rho)$$

In addition, it must have:

$$\sum_m E_m = I$$

The measurement operators in a POVM can be constructed from an arbitrary measurement operator by taking:

$$E_m = M_m^* M_m$$

The E_m do not have to be projection operators, it can be more general. The POVM allows us to construct a more general type of measurement operator to describe measurements where projective measurements do not apply in the real world. A POVM is applicable in this case because it allows us to describe measurements on the system without regard to the postmeasurement state.

Example 6.11

A system is in the state

$$|\psi\rangle = \frac{2}{\sqrt{5}}|0\rangle + \frac{1}{\sqrt{5}}|1\rangle$$

Describe the probabilities of measuring 0 and 1 for this state in the POVM formalism.

Solution

In the simple case of a single qubit, we actually have a POVM using the projection operators. In this example we denote

$$E_0 = |0\rangle\langle 0|, \quad E_1 = |1\rangle\langle 1|$$

Notice that $\sum_m E_m = E_0 + E_1 = |0\rangle\langle 0| + |1\rangle\langle 1| = I$. The matrix representations of these operators in the computational basis are

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Each matrix has two eigenvalues, namely $\{1, 0\}$ indicating that these operators are positive semidefinite.

By (6.33), the respective probabilities are

$$\begin{aligned}
\Pr(0) = \langle\psi|E_0|\psi\rangle &= \left(\frac{2}{\sqrt{5}}\langle 0| + \frac{1}{\sqrt{5}}\langle 1|\right)(|0\rangle\langle 0|)\left(\frac{2}{\sqrt{5}}|0\rangle + \frac{1}{\sqrt{5}}|1\rangle\right) \\
&= \left(\frac{2}{\sqrt{5}}\langle 0| + \frac{1}{\sqrt{5}}\langle 1|\right)\left(\frac{2}{\sqrt{5}}|0\rangle\langle 0|0\rangle + \frac{1}{\sqrt{5}}|1\rangle\langle 0|1\rangle\right) \\
&= \left(\frac{2}{\sqrt{5}}\langle 0| + \frac{1}{\sqrt{5}}\langle 1|\right)\frac{2}{\sqrt{5}}|0\rangle \\
&= \frac{4}{5}\langle 0|0\rangle + \frac{2}{5}\langle 1|0\rangle = \frac{4}{5} \\
\Pr(1) = \langle\psi|E_1|\psi\rangle &= \left(\frac{2}{\sqrt{5}}\langle 0| + \frac{1}{\sqrt{5}}\langle 1|\right)(|1\rangle\langle 1|)\left(\frac{2}{\sqrt{5}}|0\rangle + \frac{1}{\sqrt{5}}|1\rangle\right) \\
&= \left(\frac{2}{\sqrt{5}}\langle 0| + \frac{1}{\sqrt{5}}\langle 1|\right)\left(\frac{2}{\sqrt{5}}|0\rangle\langle 1|0\rangle + \frac{1}{\sqrt{5}}|1\rangle\langle 1|1\rangle\right) \\
&= \left(\frac{2}{\sqrt{5}}\langle 0| + \frac{1}{\sqrt{5}}\langle 1|\right)\frac{1}{\sqrt{5}}|1\rangle \\
&= \frac{2}{5}\langle 0|1\rangle + \frac{1}{5}\langle 1|1\rangle = \frac{1}{5}
\end{aligned}$$

POVM's can be useful when projective measurements are not. For example, POVM's provide the ability to distinguish between nonorthogonal states.

Example 6.12

A system can be in one of two states $|\psi\rangle$ or $|\phi\rangle$. The states are not orthogonal; in fact $|\langle\psi|\phi\rangle| = \cos\theta$. Describe a POVM that can distinguish between the two states. Assume that the states are normalized.

Solution

Consider the POVM consisting of the following measurement operators:

$$E_1 = \frac{I - |\phi\rangle\langle\phi|}{1+\cos\theta}, \quad E_2 = \frac{I - |\psi\rangle\langle\psi|}{1+\cos\theta}, \quad E_3 = I - E_1 - E_2$$

Each of these operators corresponds to a different measurement outcome. These operators satisfy the completeness relation, since $\sum_m E_m = E_1 + E_2 + E_3 = I$. Now consider the first measurement outcome associated with E_1. The probabilities associated with each state are $\langle\psi|E_1|\psi\rangle$ and $\langle\phi|E_1|\phi\rangle$. In the first case,

$$\begin{aligned}
\langle\psi|E_1|\psi\rangle &= \langle\psi|\frac{I - |\phi\rangle\langle\phi|}{1+\cos\theta}|\psi\rangle \\
&= \frac{\langle\psi\psi\rangle - \langle\psi\phi\rangle\langle\phi\psi\rangle}{1+\cos\theta} = \frac{1-|\langle\psi\phi\rangle|^2}{1+\cos\theta} \\
&= \frac{1-\cos\theta^2}{1+\cos\theta} = \frac{(1-\cos\theta)(1+\cos\theta)}{1+\cos\theta} = 1-\cos\theta
\end{aligned}$$

Meanwhile, in the second case,

$$\begin{aligned}
\langle\phi|E_1|\phi\rangle &= \langle\phi|\frac{I - |\phi\rangle\langle\phi|}{1+\cos\theta}|\phi\rangle \\
&= \frac{\langle\phi\phi\rangle - \langle\phi\phi\rangle\langle\phi\phi\rangle}{1+\cos\theta} = \frac{1-1}{1+\cos\theta} = 0
\end{aligned}$$

Hence the operator E_1 allows us to identify the state $|\psi\rangle$ with probability $1-\cos\theta$. If the system is in the state $|\phi\rangle$, the probability is zero—this measurement never identifies the state $|\phi\rangle$. A similar exercise shows that the operator

$$E_2 = \frac{1 - |\psi\rangle\langle\psi|}{1+\cos\theta}$$

never identifies the state $|\psi\rangle$, but it identifies the state $|\phi\rangle$ with probability $1-\cos\theta$. These operators have provided a means of imperfectly distinguishing between two nonorthogonal quantum states.

If the measurement outcome E_3 is obtained, no information about the state is available.

Entanglement

Two systems are entangled when the state of one is correlated with the state of the other.

Suppose we have the following state:

$$|\psi\rangle = \frac{|0\rangle|1\rangle - |1\rangle|0\rangle}{\sqrt{2}}$$

If we measure $Z \times I$ (to measure the state of the first particle), if we get $|0\rangle$, then the state of the second particle must be $|1\rangle$. And if we measure 1 for the first particle, the state of the second must be $|0\rangle$.

An interesting property is that we can write this same state in terms of the X basis and it is:

$$|\psi\rangle = -\frac{|+-\rangle - |-+\rangle}{\sqrt{2}}$$

Once again, if we measure the state of the first particle, the state of the second is defined. This shows that the states are correlated in all directions. So we could not have definite states prior to measurement, because we would need a definite state of Z and X at the same time.

When a system is entangled, this means that the individual components systems are linked together as a single entity. Any measurement that measures a part of the system is really a measurement of the entire system. The wavefunction then collapses and both particles assume definite states.

Bell's Theorem

We consider spin measurements in three directions $\vec{a}, \vec{b}, \vec{c}$. And imagine a large ensemble of systems prepared so that Alice and Bob can measure spin along the three directions $\vec{a}, \vec{b}, \vec{c}$. With three different directions to consider and 2 possible states ($\pm$) along each directions, there will be $2^3 = 8$ different populations in total .

Local Realistic Theory:
Here we consider the local realist position, based only on conservation of angular momentum. We only make the simple assumption that if Alice measures $+$ along a given direction, Bob will measure $-$ along that same directions.

TABLE 7.2 Measurements for particles Alice and Bob

	Alice			Bob		
Population	a	b	c	a	b	c
N_1	+	+	+	−	−	−
N_2	+	+	−	−	−	+
N_3	+	−	+	−	+	−
N_4	+	−	−	−	+	+
N_5	−	+	+	+	−	−
N_6	−	−	+	+	+	−
N_7	−	+	−	+	−	+
N_8	−	−	−	+	+	+

The total number of particles is N, where N_i is the number of particles in the state i, so that:

$$N = N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7 + N_8 + N_9$$

The directions along which Alice and Bob decide to measure is random. We see that:

$$\frac{N_3 + N_4}{N} = Pr(+\vec{a}; +\vec{b})$$

Because if Alice measures $\vec{a}$ and Bob measures $\vec{b}$, the probability of getting $+, +$ is only possible in states N_3, N_4.
Similarly:

$$\frac{N_2 + N_4}{N} = Pr(+\vec{a}; +\vec{c})$$

$$\frac{N_3 + N_7}{N} = Pr(+\vec{c}; +\vec{b})$$

Then, we can use that obviously $\frac{N_3 + N_4}{N} \leq \frac{N_2 + N_4}{N} + \frac{N_3 + N_7}{N}$.

So that we get **Bell's inequality:**

$$Pr(+\vec{a}; +\vec{b}) \leq Pr(+\vec{a}; +\vec{c}) + Pr(+\vec{c}; +\vec{b})$$

Local realistic theories satisfy Bell's inequality, but quantum mechanics doesn't.

Quantum Mechanics:

We need to consider a qubit oriented in an arbitrary direction. Consider a vector $\vec{n} = \sin\theta\cos\phi\widehat{x} + \sin\theta\sin\phi\widehat{y} + \cos\theta\widehat{z}$. The eigenvectors of $\vec{\sigma}\cdot\vec{n}$ are:

$$|+_n\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

$$|-_n\rangle = \cos\frac{\theta}{2}|0\rangle - e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

To get the eigenvectors of $\vec{\sigma} \cdot \vec{x}$ for example, we set $\theta = \pi/2$ and $\phi = 0$.
If the system is in state $|+_n\rangle$, then we have:

$$\langle 0|+_n\rangle = \cos\frac{\theta}{2}$$

Therefore, the probability that measurement finds $|0\rangle$ given that the system is in state $|+_n\rangle$ is:

$$|\langle 0|+_n\rangle|^2 = \cos^2\frac{\theta}{2}$$

And the probability of finding the state $|-_n\rangle$ in state $|1\rangle$ is:

$$|\langle 1|+_n\rangle|^2 = \sin^2\frac{\theta}{2}$$

We prepare the system in the singlet state (which is invariant under rotations), so we consider along the $\vec{a}$ axis:

$$|\psi\rangle = \frac{|+_a\rangle|-_a\rangle - |-_a\rangle|+_a\rangle}{\sqrt{2}}$$

Then we calculate $Pr(+\vec{a};+\vec{c})$ by looking at the inner product:

$$\langle +_a +_c|\psi\rangle = \cdots = \frac{\langle +_c|-_a\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\sin\frac{\theta_{ac}}{2}$$

With θ_{ac} the angle between the directions $\vec{a}$ and $\vec{c}$.
Hence the probability is:

$$Pr(+\vec{a};+\vec{c}) = \frac{1}{2}\sin^2\frac{\theta_{ac}}{2}$$
$$Pr(+\vec{a};+\vec{b}) = \frac{1}{2}\sin^2\frac{\theta_{ab}}{2}$$
$$Pr(+\vec{c};+\vec{b}) = \frac{1}{2}\sin^2\frac{\theta_{cb}}{2}$$

So Bell's inequality would be:

$$\sin^2\frac{\theta_{ab}}{2} \leq \sin^2\frac{\theta_{ac}}{2} + \sin^2\frac{\theta_{cb}}{2}$$

We take $\vec{a}, \vec{b}$ and $\vec{c}$ to lie in a plane with $\vec{c}$ bisecting the angle θ_{ab}. So that $\theta_{ac} = \theta_{cb} = \theta$ and $\theta_{ab} = 2\theta$. And Bells inequality becomes $\sin^2\theta \leq 2\sin^2\theta/2$ which is not true if $0 < \theta < \pi/2$.

So Quantum mechanics predicts a violation of Bell's inequality which was derived under local realism. Experiment agrees with quantum mechanics.

Bipartite Systems and the Bell Basis

When a system consists of two subsystems we say it is a bipartite system. The Hilbert space of the composite system is the tensor product of the Hilbert space that describes Alice's system and the Hilbert space that describes Bob's. Then the Hilbert space of the composite system is:

$$H = H_A \otimes H_B$$

If the basis states for Alice is $|a_i\rangle$ and the basis states for Bob is $|b_j\rangle$, then the basis for the bipartite system is:

$$|\alpha_{ij}\rangle = |a_i\rangle \otimes |b_j\rangle = |a_i\rangle|b_j\rangle = |a_i b_j\rangle$$

This basis is orthonormal:

$$\langle\alpha_{ij}|\alpha_{kl}\rangle = \delta_{ik}\delta_{jl}$$

A general state $|\psi\rangle$ can be written as:

$$|\psi\rangle = \sum_{i,j} c_{ij}|\alpha_{ij}\rangle = \sum_{ij} |a_i b_j\rangle\langle a_i b_j|\psi\rangle$$

And the probability of being in state $|a_i b_j\rangle$ is:

$$Pr(a_i b_j) = |\langle a_i b_j|\psi\rangle|^2$$

And we can represent a matrix as:

$$A = \sum_{i,j,k,l} = |a_i b_j\rangle A_{ijkl}\langle a_k b_l|$$

Where $A_{ijkl} = \langle a_i b_j|A|a_k b_l\rangle$

For example, the bell states form a basis for the bipartite space of two qubits. Remember that the Bell basis is:

$$|\beta_{xy}\rangle = \frac{|0y\rangle + (-1)^x|1\bar{y}\rangle}{\sqrt{2}}$$

Where $\bar{y}$ denotes 'not' y. x is called the phase bit and y the parity bit.

Example 7.1

Show that the operator $Z \otimes Z$ acts on (7.31) via the parity bit as $Z \otimes Z|\beta_{xy}\rangle = (-1)^y|\beta_{xy}\rangle$.

Solution

Recall the action of the Z operator

$$Z|0\rangle = |0\rangle, \quad Z|1\rangle = -|1\rangle$$

This can be written more abstractly as $Z|a\rangle = (-1)^a|a\rangle$. Look at the first term in (7.31). The operator on the first qubit does nothing because it's $|0\rangle$. So we obtain

$$Z \otimes Z|0y\rangle = (-1)^y|0y\rangle$$

In the second case, we have

$$(Z \otimes Z)(-1)^x|1\overline{y}\rangle = (-1)^x(Z|1\rangle) \otimes (Z|\overline{y}\rangle) = (-1)^x(-1)(-1)^{\overline{y}}|1\overline{y}\rangle$$

Now, if $\overline{y} = 0$, then $(-1)(-1)^0 = (-1)(+1) = -1$. But, if $\overline{y} = 0$, then obviously $y = 1$, and this is the same as $(-1)^y$. If y = 1, then $(-1)(-1)^1 = (-1)(-1) = +1 = (-1)^y$, since $y = 0$ in that case. So we've found that $(Z \otimes Z)(-1)^x|1\overline{y}\rangle = (-1)^x(-1)^y|1\overline{y}\rangle$. Putting everything together we have

$$Z \otimes Z|\beta_{xy}\rangle = \frac{(-1)^y|0y\rangle + (-1)^x(-1)^y|1\overline{y}\rangle}{\sqrt{2}} = (-1)^y|\beta_{xy}\rangle$$

When is a State Entangles

Not all states $|\psi\rangle \in H_A \otimes H_B$ are entangled, the state of each composite system can only be described with reference to the other state. If the two states are not entangled, we say that they are a **product state** or **separable**. If $|\psi\rangle \in H_A$ and $|\phi\rangle \in H_B$, then $|\chi\rangle = |\psi\rangle \otimes |\phi\rangle$ is a prodcut state.

For a 2 qubit state, the state described by $|\psi\rangle = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ is separable if and only if $ad = bc$.

Example 7.2

Are the Bell states given in (7.27) through (7.30) entangled?

Solution

The Bell states are clearly entangled (in fact they could be said to be the quintessential entangled state), but let's apply criterion (7.32) to show they are not separable. Writing each state as a column vector, we have

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \quad |\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

$$|\beta_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \quad |\beta_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

For $|\beta_{00}\rangle$, we have $a = d = 1/\sqrt{2}, b = c = 0$, so $ad = 1/2 \neq bc$. So $|\beta_{00}\rangle$ is not a product state and must be entangled. For $|\beta_{01}\rangle, a = d = 0, b = c = 1/\sqrt{2}, \Rightarrow ad = 0 \neq bc = 1/2$. We conclude that $|\beta_{01}\rangle$ is also entangled. For $|\beta_{10}\rangle$, we find that $ad = -1/2 \neq bc = 0$, and for $|\beta_{11}\rangle$, we have $ad = 0 \neq bc = -1/2$, so these states are also entangled by (7.32).

Example 7.3

A system of two qubits is in the state $|00\rangle$. We operate on this state with $H \otimes H$, where H is the Hadamard matrix. Is the state $H \otimes H|00\rangle$ entangled?

Solution

First let's write down the matrix representation of $H \otimes H$ in the computational basis. The Hadamard matrix is given by

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{7.33}$$

So we find that

$$H \otimes H = \frac{1}{\sqrt{2}}\begin{pmatrix} H & H \\ H & -H \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

The state in question has a column vector representation

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

In this case $ad = (1)(0) = 0 = bc$ so $|00\rangle$ is clearly a product state according to (7.32).

Now let's calculate $H \otimes H|00\rangle$:

$$H \otimes H|00\rangle = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Using (7.32), we see that this is also a product state, since

$$ad = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$
$$bc = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$
$$\Rightarrow ad = bc$$

In fact this state is the tensor product

$$\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

Example 7.4

Alice and Bob each possess one member of a pair of interacting magnetic dipoles (spin-1/2 particles). The interaction Hamiltonian for two interacting magnetic dipoles separated by a distance r is given by

$$H_I = \frac{\mu^2}{r^3}(\vec{\sigma}_A \cdot \vec{\sigma}_B - 3Z_A Z_B)$$

where $\vec{\sigma}_A = X_A\,\hat{x} + Y_A\,\hat{y} + Z_A\,\hat{z}$ and similarly for Bob. Find the allowed energies of the system and show that the eigenvectors of the Hamiltonian include entangled states. Then rewrite the Hamiltonian using it's eigenvectors.

Solution

First we express the Hamiltonian in matrix form. We have

$$\vec{\sigma}_A \cdot \vec{\sigma}_B = X_A \otimes X_B + Y_A \otimes Y_B + Z_A \otimes Z_B$$

The first term is

$$X_A \otimes X_B = \begin{pmatrix} 0 \cdot X_B & 1 \cdot X_B \\ 1 \cdot X_B & 0 \cdot X_B \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Similarly we find that

$$Y_A \otimes Y_B = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Z_A \otimes Z_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So

$$\begin{aligned}
\vec{\sigma}_A \cdot \vec{\sigma}_B &= X_A \otimes X_B + Y_A \otimes Y_B + Z_A \otimes Z_B \\
&= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Therefore the matrix representation of the Hamiltonian is

$$H_I = \frac{\mu^2}{r^3}(\vec{\sigma}_A \cdot \vec{\sigma}_B - 3Z_AZ_B) = \frac{\mu^2}{r^3}\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \frac{\mu^2}{r^3}\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

Keep in mind that the matrix is written in the following way:

$$H_I = \begin{pmatrix} \langle 00|H_I|00\rangle & \langle 00|H_I|01\rangle & \langle 00|H_I|10\rangle & \langle 00|H_I|11\rangle \\ \langle 01|H_I|00\rangle & \langle 01|H_I|01\rangle & \langle 01|H_I|10\rangle & \langle 01|H_I|11\rangle \\ \langle 10|H_I|00\rangle & \langle 10|H_I|01\rangle & \langle 10|H_I|10\rangle & \langle 10|H_I|11\rangle \\ \langle 11|H_I|00\rangle & \langle 11|H_I|01\rangle & \langle 11|H_I|10\rangle & \langle 11|H_I|11\rangle \end{pmatrix}$$

And a state vector $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$is given by

$$|\psi\rangle = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

The eigenvalues of this matrix are the energies that the system can assume. With a 4×4matrix, it's easiest to find the eigenvalues and eigenvectors using a computer. Using Mathematica® we find that the eigenvalues of H_I are $\mu^2/r^3\{4, -2, -2{,}0\}$. The eigenvectors corresponding to each of these eigenvalues are in turn

$$|\phi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$|\phi_2\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |11\rangle, \quad |\phi_3\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle$$

and

$$|\phi_4\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Comparison with (7.28) and (7.30) shows that two of the eigenvectors are Bell states—and hence represent states in which the particles in possession of Alice and Bob are entangled. Specifically, $|\phi_1\rangle = |\beta_{01}\rangle$ and $|\phi_4\rangle = |\beta_{11}\rangle$.

In the basis of its eigenstates, H_I is diagonal with the entries along the diagonal given by its eigenvalues

$$H_I = \begin{pmatrix} \langle\phi_1|H_I|\phi_1\rangle & \langle\phi_1|H_I|\phi_2\rangle & \langle\phi_1|H_I|\phi_3\rangle & \langle\phi_1|H_I|\phi_4\rangle \\ \langle\phi_2|H_I|\phi_1\rangle & \langle\phi_2|H_I|\phi_2\rangle & \langle\phi_2|H_I|\phi_3\rangle & \langle\phi_2|H_I|\phi_4\rangle \\ \langle\phi_3|H_I|\phi_1\rangle & \langle\phi_3|H_I|\phi_2\rangle & \langle\phi_3|H_I|\phi_3\rangle & \langle\phi_3|H_I|\phi_4\rangle \\ \langle\phi_4|H_I|\phi_1\rangle & \langle\phi_4|H_I|\phi_2\rangle & \langle\phi_4|H_I|\phi_3\rangle & \langle\phi_4|H_I|\phi_4\rangle \end{pmatrix}$$
$$= \frac{\mu^2}{r^3} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The Pauli Representation

The Pauli representation provides a means to write down a density operator of a single qubit or two qubit systems in terms of Pauli matrices. For a single quibit, the **pauli representation is given by:**

$$\rho = \frac{1}{2}\sum_{i=0}^{3} c_i \sigma_i$$

The coefficients are $c_i = Tr(\rho\sigma_i) = \langle\sigma_i\rangle$

Example 7.5

A certain density matrix is given by

$$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$$

Find its Pauli representation.

Solution

Proceeding by brute force, and recalling that $\sigma_0 = I$ the identity operator, we have

$$c_0 = Tr(\rho\sigma_0) = Tr\begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} = \frac{3}{4} + \frac{1}{4} = 1$$

Next we find

$$c_1 = Tr(\rho\sigma_1) = Tr\begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = Tr\begin{pmatrix} 0 & \frac{3}{4} \\ \frac{1}{4} & 0 \end{pmatrix} = 0$$

Similarly $c_2 = Tr(\rho\sigma_2) = 0$. Finally,

$$c_3 = Tr(\rho\sigma_3) = Tr\begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = tr\begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{-1}{4} \end{pmatrix} = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

The Pauli representation is

$$\rho = \sigma_0 - \frac{1}{2}\sigma_3 = I - \frac{1}{2}Z$$

The Pauli representation for a system of two qubits is given by

$$\rho = \frac{1}{4}\sum_{i,j} c_{ij}\sigma_i \otimes \sigma_j \tag{7.35}$$

where $c_{ij} = \langle \sigma_i \otimes \sigma_j \rangle = Tr(\rho\, \sigma_i \otimes \sigma_j)$. If the density operator ρ is a separable state, then

$$|c_{11}| + |c_{22}| + |c_{33}| \leq 1 \tag{7.36}$$

Example 7.6

Show that $H \otimes H|00\rangle$ is a separable state using the criterion (7.36), and show that $|\beta_{00}\rangle$ is entangled.

Solution

First let's find the density operator for $H \otimes H\ |00\rangle$. This can be written down using

$$\rho = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)\frac{1}{2}(\langle 00| + \langle 01| + \langle 10| + \langle 11|)$$

The density matrix turns out to be

$$\rho = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

The first term is

$$c_{11} = \langle \sigma_1 \otimes \sigma_1 \rangle = \langle X \otimes X \rangle = Tr(\rho X \otimes X)$$

Now

$$X \otimes X = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\rho X \otimes X = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Hence

$$c_{11} = Tr(\rho X \otimes X) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$$

Using

$$Y \otimes Y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

we find that

$$\begin{aligned} c_{22} &= Tr(\rho\sigma_2 \otimes \sigma_2) = Tr(\rho Y \otimes Y) \\ &= Tr\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\ &= Tr\frac{1}{4}\begin{pmatrix} -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix} = \frac{1}{4}(-1+1+1-1) = 0 \end{aligned}$$

Finally, we have

$$Z \otimes Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\rho Z\otimes Z=\frac{1}{4}\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1&-1&-1&1\\1&-1&-1&1\\1&-1&-1&1\\1&-1&-1&1\end{pmatrix}$$

The trace of this matrix, which is the sum of the diagonal elements, also vanishes, so we have

$$|c_{11}|+|c_{22}|+|c_{33}|=1+0+0=1$$

Hence (7.36) is satisfied, and this is a separable state. Now let's check $|\beta_{00}\rangle$. The density operator for this state is

$$\rho=\frac{1}{2}(|00\rangle+|11\rangle)(\langle 00|+\langle 11|)=\frac{1}{2}(|00\rangle\langle 00|+|00\rangle\langle 11|+|11\rangle\langle 00|+|11\rangle\langle 11|)$$

The density matrix is thus

$$\rho=\begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix}$$

We find that

$$c_{11}=Tr(\rho X\otimes X)=Tr\frac{1}{2}\begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$$

$$=Tr\frac{1}{2}\begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix}=\frac{1}{2}+0+0+\frac{1}{2}=1$$

Next we find that

$$c_{22}=Tr(\rho Y\otimes Y)=Tr\frac{1}{2}\begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix}\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}$$

$$=Tr\frac{1}{2}\begin{pmatrix}-1&0&0&-1\\0&0&0&0\\0&0&0&0\\-1&0&0&-1\end{pmatrix}=-\frac{1}{2}+0+0-\frac{1}{2}=-1$$

The last coefficient we need is

$$c_{33} = Tr(\rho Z \otimes Z) = Tr\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= Tr\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2} + 0 + 0 + \frac{1}{2} = 1$$

So in this case

$$|c_{11}| + |c_{22}| + |c_{33}| = |1| + |-1| + |1| = 1 + 1 + 1 = 3$$

Hence by (7.36) the state is entangled.

Entanglement Fidelity

Consider a density operator for a single qubit that is diagonal with respect to the computational basis

$$\rho = f|0\rangle\langle 0| + (1-f)|1\rangle\langle 1|$$

The parameter f is known as the **entanglement fidelity**. For example, if:

$$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$$

The entanglement fidelity is 3/4.

Using Bell State for Density Operator Representation

The density operator of a 2-qubit system that is diagonal with respect to the Bell basis can be represented in terms of the Bell states expanding as:

$$\begin{aligned}\rho &= \sum_{i,j} c_{ij}|\beta_{ij}\rangle\langle\beta_{ij}| \\ &= c_{00}|\beta_{00}\rangle\langle\beta_{00}| + c_{01}|\beta_{01}\rangle\langle\beta_{01}| + c_{10}|\beta_{10}\rangle\langle\beta_{10}| + c_{11}|\beta_{11}\rangle\langle\beta_{11}|\end{aligned}$$

This type of expansion is possible because we can write outer products of Bell states using Pauli operators.

When written like this, a density operator ρ is separable iff $c_{00} \leq \dfrac{1}{2}$

Example 7.7

A density matrix for a certain two-qubit system in the $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ basis is

$$\rho = \begin{pmatrix} \frac{1}{8} & 0 & 0 & \frac{1}{8} \\ 0 & \frac{3}{8} & \frac{-3}{8} & 0 \\ 0 & \frac{-3}{8} & \frac{3}{8} & 0 \\ \frac{1}{8} & 0 & 0 & \frac{1}{8} \end{pmatrix}$$

Can this state be written in a diagonal form with respect to the Bell basis? Is this a separable state?

Solution

This matrix represents the density operator

$$\rho = \frac{1}{8}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|) + \frac{3}{8}(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|)$$

In terms of the Bell basis, this operater is rewritten as

$$\rho = \frac{1}{4}|\beta_{00}\rangle\langle\beta_{00}| + \frac{3}{4}|\beta_{11}\rangle\langle\beta_{11}|$$

Since $c_{00} = 1/4 < 1/2$, by (7.43) this is a separable state.

Schmidt Decomposition

Consider a composite space $H_A \otimes H_B$ and let $|\psi\rangle \in H_A \otimes H_B$ be a pure state. Then there exists an expansion of $|\psi\rangle$ of the form:

$$|\psi\rangle = \sum_i \lambda_i |a_i\rangle|b_i\rangle$$

Where $|a_i\rangle$ are orthonormal states in system A and $|b_i\rangle$ are orthonormal states belonging to system B. The expansion coefficients λ_i are such that $\lambda_i \geq 0$ and $\sum_i \lambda_i^2 = 1$.
The Schmidt coefficients are calculated from the matrix:

$$Tr_B(|\psi\rangle\langle\psi|)$$

This matrix has eigenvalues λ_i^2. The Schmidt number is the number of nonzero eigenvalues λ_i.

- If the Schmidt number is 1, the state is separable
- If the Schmidt number is > 1, the state is entangled

Example 7.8

Consider the state $|\psi\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$. Is this state separable? What is the Schmidt number?

Solution

The density operator for this state is

$$\begin{aligned}\rho = |\psi\rangle\langle\psi| = \frac{1}{4}(&|00\rangle\langle 00| - |00\rangle\langle 01| - |00\rangle\langle 10| + |00\rangle\langle 11| - |01\rangle\langle 00| + |01\rangle\langle 01| \\ &+ |01\rangle\langle 10| - |01\rangle\langle 11| - |10\rangle\langle 00| + |10\rangle\langle 01| + |10\rangle\langle 10| - |10\rangle\langle 11| + |11\rangle\langle 00| \\ &- |11\rangle\langle 01| - |11\rangle\langle 10| + |11\rangle\langle 11|)\end{aligned}$$

We can trace out system B immediately giving

$$\begin{aligned}\rho_A &= Tr_B(|\psi\rangle\langle\psi|) = \langle 0|\psi\rangle\langle\psi|0\rangle + \langle 1|\psi\rangle\langle\psi|1\rangle \\ &= \frac{1}{4}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|) + \frac{1}{4}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|) \\ &= \frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|)\end{aligned}$$

The matrix representation is

$$\rho_A = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

The eigenvalues of this matrix are $\lambda_1 = 1$, $\lambda_2 = 0$. The Schmidt number is the number of nonzero eigenvalues, and since Sch $= 1$, this is a separable state. In fact this is the product state given by

$$|\psi\rangle = \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

Example 7.9

Show that the singlet state

$$|S\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

is entangled by computing its Schmidt number.

Solution

The density operator in this case is

$$\rho = |S\rangle\langle S| = \left(\frac{|01\rangle - |10\rangle}{\sqrt{2}}\right)\left(\frac{\langle 01| - \langle 10|}{\sqrt{2}}\right)$$
$$= \frac{1}{2}(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|)$$

Tracing out system B, we obtain

$$\rho_A = Tr_B\left(\frac{1}{2}(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|)\right)$$
$$= \langle 0|\frac{1}{2}(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|)|0\rangle$$
$$+ \langle 1|\frac{1}{2}(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|)|1\rangle$$
$$= \frac{1}{2}(|1\rangle\langle 1| + |0\rangle\langle 0|) = \frac{1}{2}I$$

This matrix has two nonzero eigenvalues, namely $\lambda_1 = \lambda_2 = 1/2$. Since the Schmidt number is $Sch = 2 > 1$, this is an entangled state.

Purification

Purification is the process by which we create a reference system B such that, given the system A, the state $|\phi_A\phi_B\rangle$ is a pure state.
The starting point is the density matrix of the mixed state ρ_A. The state $|\phi_B\rangle$ is a purification of ρ_A if:

$$\rho_A = Tr_B(|\phi_B\rangle\langle\phi_B|)$$

If ρ_A is a mixed state, we can use purificatoin to analyze the system as a pure state by expanding the Hilbert space to the larger space defined by $|\phi_A\phi_B\rangle$. Suppose:

$$\rho_A = \sum_i p_i|a_i\rangle\langle a_i|$$

Let $|b_i\rangle$ be an orthonormal basis for system B. The purification is then:

$$|\phi_B\rangle = \sum_i \sqrt{p_i}|a_i\rangle \otimes |b_i\rangle$$

		Pure States	General States
Postulate 1	**State space**	Hilbert space $\mathscr{H}$	Trace-class operator space $\mathscr{D}$
	State	ket vector $\lvert\psi\rangle \in \mathscr{H}$ s.t. $\langle\psi\vert\psi\rangle = 1$	density operator ρ s.t. $\begin{cases} \mathrm{Tr}[\rho] = 1 \\ \rho > 0 \end{cases}$
	Inner product	$f(\lvert\mu\rangle, \lvert\omega\rangle) \equiv \langle\mu\vert\omega\rangle$, $\forall \lvert\mu\rangle, \lvert\omega\rangle \in \mathscr{H}$	$f(A, B) \equiv \mathrm{Tr}\left[A^\dagger B\right]$, $\forall A, B \in \mathscr{D}$ Hilbert-Schmidt inner product
Postulate 2	**Expansion**	tensor product $\otimes$	tensor product $\otimes$
Postulate 3	**Dynamics** w/ Hamiltonian H	Schrödinger equation: $\frac{d\lvert\psi(t)\rangle}{dt} = -iH\lvert\psi(t)\rangle$	Liouville-von Neumann equation: $\frac{d\rho(t)}{dt} = -i[H, \rho(t)]$
Postulate 4	**Measurement** w/ meas. ops. $\{M_k\}_{k\in K}$	outcome $k \in K$ w.p. $p_k = \langle\psi\rvert M_k^\dagger M_k \lvert\psi\rangle$ $\lvert\psi\rangle \mapsto \frac{M_k\lvert\psi\rangle}{\sqrt{p_k}}$	outcome $k \in K$ w.p. $p_k = \mathrm{Tr}\left[M_k \rho M_k^\dagger\right]$ $\rho \mapsto \frac{M_k \rho M_k^\dagger}{p_k}$

Quantum Gates and Circuits

Classical Logic Gates

Here are some of the gates:

Input	NOT
0	1
1	0

ore complicated operation
ıput gates, including the (
:s. Let's examine each on
› bits as input, which we l
3 is 1, and is 0 otherwise.

A	B	A OR B
0	0	0
0	1	1
1	0	1
1	1	1

A	B	A NAND B
0	0	1
0	1	1
1	0	1
1	1	0

ng property of being *universal*
ed using only NAND gates. In t
hing but NAND gates, or the c
ample of how other logic operat
suppose that happens if we sup
The truth table is reduced to th

A	A	A OR B
0	0	1
1	1	0

Single Qubit Gates

The gates are unitary operations. That is, operators U such that $UU^* = U^*U = I$.
A quantum gate with n inputs and outputs can be represented by a matrix of dimensions 2^n.

The standard computational basis is given by:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The Pauli X matrix can do the operation of a Not gate. This matrix is:

$$X = U_{not} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Hence we have:

$$U_{not}|0\rangle = |1\rangle$$
$$U_{not}|1\rangle = |0\rangle$$

The action of a Not gate on an arbitrary state $|j\rangle$ can be written as:

$$X|j\rangle = |j \oplus 1\rangle$$

Example 8.1

The NOT operator takes $|0\rangle \mapsto |1\rangle$ and $|1\rangle \mapsto |0\rangle$. Describe the unitary operator that will implement the NOT operation in outer product form, and find its matrix representation with respect to the basis

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Solution

In the standard or computational basis, the matrix representation of the NOT operator is given by (8.3). We can also write this as

$$X = \begin{pmatrix} \langle 0|X|0\rangle & \langle 0|X|1\rangle \\ \langle 1|X|0\rangle & \langle 1|X|1\rangle \end{pmatrix}$$

You can see that this will work if

$$X = |0\rangle\langle 1| + |1\rangle\langle 0| \tag{8.7}$$

(check it). Then the action of the operator on the standard or computational basis states is

$$\begin{aligned} X|0\rangle &= (|0\rangle\langle 1| + |1\rangle\langle 0|)|0\rangle \\ &= |0\rangle\langle 1|0\rangle + |1\rangle\langle 0|0\rangle \\ &= |1\rangle \end{aligned}$$

and

$$\begin{aligned} X|1\rangle &= (|0\rangle\langle 1| + |1\rangle\langle 0|)|1\rangle \\ &= |0\rangle\langle 1|1\rangle + |1\rangle\langle 0|1\rangle \\ &= |0\rangle \end{aligned}$$

where we have used the orthonormality of the basis states. To find the representation of the NOT operator in the $\{|+\rangle, |-\rangle\}$ basis, we need to find the unitary transformation connecting this basis with the standard or computational basis. That is, we need to find a matrix with components given by

$$U_{trans} = \begin{pmatrix} \langle +|0\rangle & \langle +|1\rangle \\ \langle -|0\rangle & \langle -|1\rangle \end{pmatrix}$$

We know already the representation of the $\{|+\rangle, |-\rangle\}$ states in the standard or computational basis:

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So it's a simple matter to compute each matrix component:

$$\langle +|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}$$
$$\langle +|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}$$
$$\langle -|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}$$
$$\langle -|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = -\frac{1}{\sqrt{2}}$$

Hence the transformation matrix between the two bases is given by

$$U_{trans} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} = H \tag{8.8}$$

This matrix is nothing other than the Hadamard matrix H. For this reason the $\{|+\rangle, |-\rangle\}$ basis is sometimes called the *Hadamard basis*. Now we can apply this unitary transformation to the matrix representation of the NOT gate to find its representation with respect to the Hadamard basis. It is easy to verify that $H = H^\dagger = H^{-1}$, so the unitary transformation that takes NOT from the standard or computational basis to the Hadamard basis is just

$$U_{NOT}^H = HU_{NOT}H = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}$$
$$= \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

Example 8.2

A rotation matrix by an angle γ is given by

$$R(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}$$

Describe the action of this operator on a qubit $|\psi\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$.

Solution

The rotation matrix acts on the state as follows:

$$R(\gamma)|\psi\rangle = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\gamma\cos\theta - \sin\gamma\sin\theta \\ \sin\gamma\cos\theta + \cos\gamma\sin\theta \end{pmatrix}$$

Now recall some basic trig identities:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

So the rotated state can be written as

$$|\psi'\rangle = \begin{pmatrix} \cos(\gamma+\theta) \\ \sin(\gamma+\theta) \end{pmatrix} = \cos(\gamma+\theta)|0\rangle + \sin(\gamma+\theta)|1\rangle$$

Consider the Bloch sphere picture. The rotation operator has rotated the state vector relative to the z axis by the angle γ. More specifically for those who like a more concrete interpretation, the rotation operator has altered the relative length of each probability amplitude. If the original qubit were measured, the probability that we find the system in the state $|0\rangle$ is given by $\cos^2\theta$, while the probability that we find the system in the state $|1\rangle$ is given by $\sin^2\theta$. If we rotate the state before measurement, then these probabilities are changed to $\cos^2(\gamma+\theta)$ and $\sin^2(\gamma+\theta)$, respectively.

The other Pauli matrices, being unitary 2x2 matrices are also valid single qubit gates. The Z operator is sometimes called the **phase flip** because:

$$Z(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle - \beta|1\rangle$$

Example 8.3

Describe the action of the phase shift gate when considering the Bloch sphere representation of a qubit.

Solution

We write the qubit as

$$|\psi\rangle = \cos\theta|0\rangle + e^{i\phi}\sin\theta|1\rangle$$

The phase shift operator (using an angle γ) can be written in outer product notation as follows:

$$P = |0\rangle\langle 0| + e^{i\gamma}|1\rangle\langle 1| \tag{8.12}$$

Hence

$$\begin{aligned} P|\psi\rangle &= (|0\rangle\langle 0| + e^{i\gamma}|1\rangle\langle 1|)(\cos\theta|0\rangle + e^{i\phi}\sin\theta|1\rangle) \\ &= \cos\theta|0\rangle + e^{i(\gamma+\phi)}\sin\theta|1\rangle \end{aligned}$$

Therefore we see that the phase shift operator takes the azimuthal angle $\phi \to \phi + \gamma$.

We have seen that the Z gate is a special case of the phase shift operator where we take the angle to be π. There are other special cases of interest. The first of these is when we take $\theta = \pi/2$. By Euler's identity, $e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = i$. The resulting gate is called the S gate, which has the matrix representation in the standard or computational basis given by

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \tag{8.13}$$

If we let $\theta = \pi/4$, then we have the $\pi/8$ or T gate:

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = e^{i\pi/8}\begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix} \tag{8.14}$$

Of course, we have already seen the Hadamard matrix:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{8.15}$$

Example 8.4

Write the Hadamard matrix in outer product form (using the standard or computational basis) and describe its action on the basis states $\{|0\rangle, |1\rangle\}$.

Solution

The matrix representation given in (8.15) can be rewritten as

$$H \doteq \begin{pmatrix} \langle 0|H|0\rangle & \langle 0|H|1\rangle \\ \langle 1|H|0\rangle & \langle 1|H|1\rangle \end{pmatrix}$$

Comparing this to (8.15), we see that the outer product representation of the Hadamard operator must be

$$H = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|) \tag{8.16}$$

Now let's see how the Hadamard operator acts on $|0\rangle$:

$$\begin{aligned} H|0\rangle &= \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)|0\rangle \\ &= \frac{1}{\sqrt{2}}(|0\rangle\langle 0|0\rangle + |0\rangle\langle 1|0\rangle + |1\rangle\langle 0|0\rangle - |1\rangle\langle 1|0\rangle) \\ &= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \end{aligned}$$

Similarly we find that

$$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Therefore the action of the Hadamard gate on the standard or computational basis states is to map the $\{|0\rangle, |1\rangle\}$ states into the superposition states

$$\left\{\frac{|0\rangle + |1\rangle}{\sqrt{2}}, \frac{|0\rangle - |1\rangle}{\sqrt{2}}\right\}$$

In general, the Hadamard gate takes the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ into the state

$$H|\psi\rangle = \left(\frac{\alpha+\beta}{\sqrt{2}}\right)|0\rangle + \left(\frac{\alpha-\beta}{\sqrt{2}}\right)|1\rangle \tag{8.17}$$

This means that the probability of finding the qubit in the state $|0\rangle$ is changed from

$$|\alpha|^2 \text{to} \left|\frac{\alpha+\beta}{\sqrt{2}}\right|^2 = \left(\frac{\alpha^*+\beta^*}{\sqrt{2}}\right)\left(\frac{\alpha+\beta}{\sqrt{2}}\right) = \frac{1}{2}(|\alpha|^2 + |\beta|^2 + \mathrm{Re}(\alpha\beta^*))$$

and similarly for the probability of finding the system in the $|1\rangle$ state. We can regroup the terms in (8.17) to give another interpretation of the output of a Hadamard gate:

$$H|\psi\rangle = \alpha\frac{|0\rangle + |1\rangle}{\sqrt{2}} + \beta\frac{|0\rangle - |1\rangle}{\sqrt{2}} = \alpha|+\rangle + \beta|-\rangle \tag{8.18}$$

That is, the Hadamard gate has turned a state that, with respect to the standard or computational basis, had the probability $|\alpha|^2$ of finding the system in the state $|0\rangle$ and the probability $|\beta|^2$ of finding the system in the state $|1\rangle$ into a state that has the probability $|\alpha|^2$ of finding the system in the state $|+\rangle$ and the probability $|\beta|^2$ of finding the system in the state $|-\rangle$.

Example 8.5

Prove that if an operator U is unitary and Hermitian, then $\exp(-i\theta U) = \cos\theta I - i\sin\theta U$.

Solution

If a matrix or operator U is unitary, then

$$UU^\dagger = U^\dagger U = I$$

If the operator is also Hermitian, then

$$U = U^\dagger$$

Combining these two relations yields

$$U^2 = UU = UU^\dagger = I$$

So we have

$$\exp(-i\theta U) = I - i\theta U + (-i)^2\frac{\theta^2}{2!}U^2 + (-i)^3\frac{\theta^3}{3!}U^3 + (-i)^4\frac{\theta^4}{4!}U^4 + (-i)^5\frac{\theta^5}{5!}U^5 + \cdots$$

Since $U^2 = I$ and $i^2 = -1$, this relation becomes

$$\begin{aligned}\exp(-i\theta U) &= \left(I - \frac{\theta^2}{2!}I + \frac{\theta^4}{4!}I - \cdots\right) - i\theta U + i\frac{\theta^3}{3!}U - i\frac{\theta^5}{5!}U + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right)I - i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)U \\ &= \cos\theta I - \sin\theta U\end{aligned}$$

By exponentiating a given matrix, we can come up with more gates. In fact we can create rotation operators to represent rotation about the x, y, and z axes on the Bloch sphere by exponentiating the Pauli matrices. These are given by

$$R_x(\gamma) = e^{-i\gamma X/2} = \begin{pmatrix} \cos\left(\frac{\gamma}{2}\right) & -i\sin\left(\frac{\gamma}{2}\right) \\ -i\sin\left(\frac{\gamma}{2}\right) & \cos\left(\frac{\gamma}{2}\right) \end{pmatrix} \tag{8.20}$$

$$R_y(\gamma) = e^{-i\gamma Y/2} = \begin{pmatrix} \cos\left(\frac{\gamma}{2}\right) & -\sin\left(\frac{\gamma}{2}\right) \\ \sin\left(\frac{\gamma}{2}\right) & \cos\left(\frac{\gamma}{2}\right) \end{pmatrix} \tag{8.21}$$

$$R_z(\gamma) = e^{-i\gamma Z/2} = \begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix} \tag{8.22}$$

The Z-Y Decomposition

Given a single qubit operator U, we can find real numbers a, b, c, d such that:

$$U = e^{ia} R_z(b) R_y(c) R_z(d)$$

Basic Quantum Circuit Diagrams

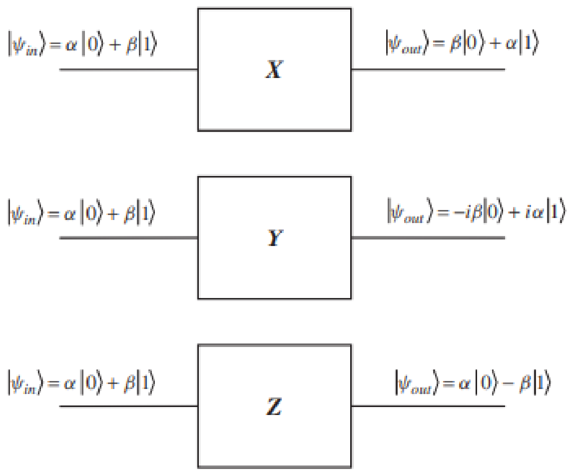


Figure 8.1 Circuit diagram representations of the Pauli operators and their actions on a single arbitrary qubit

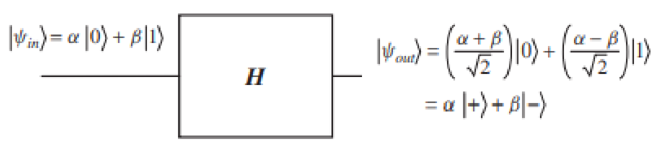


Figure 8.2 The Hadamard gate

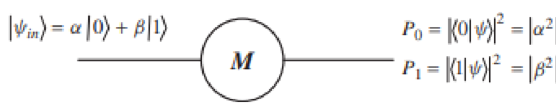


Figure 8.3 Representation of measurement in a quantum circuit

Controlled Gates

These types of gates allow us to include if else logic to the circuit. We call the control bit C. If $C = 0$ we will do nothing, if $C = 1$, then the gate performs some action.
Recall that given an operator on two qubits, we can find its matrix representation as:

$$U \doteq \begin{pmatrix} \langle 00|U|00\rangle & \langle 00|U|01\rangle & \langle 00|U|10\rangle & \langle 00|U|11\rangle \\ \langle 01|U|00\rangle & \langle 01|U|01\rangle & \langle 01|U|10\rangle & \langle 01|U|11\rangle \\ \langle 10|U|00\rangle & \langle 10|U|01\rangle & \langle 10|U|10\rangle & \langle 10|U|11\rangle \\ \langle 11|U|00\rangle & \langle 11|U|01\rangle & \langle 11|U|10\rangle & \langle 11|U|11\rangle \end{pmatrix}$$

CNOT gate:

The CNOT gate does the following:

$$|a, b\rangle \to |a, b \oplus a\rangle$$

So if $a = |0\rangle$, it will do nothing and if $a = |1\rangle$, it will do a not to the second qubit. The action on all pair of qubits is:

$$\begin{aligned} |00\rangle &\to |00\rangle \\ |01\rangle &\to |01\rangle \\ |10\rangle &\to |11\rangle \\ |11\rangle &\to |10\rangle \end{aligned}$$

The matrix representation is:

$$CN = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|$$

$$CN = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Example 8.6

Using Dirac notation, find the action of the controlled NOT gate when the control bit is $|1\rangle$ and the target qubit is given by $|0\rangle, |1\rangle$ and $\alpha|0\rangle + \beta|1\rangle$.

Solution

In the first case, CN will act on the state $|10\rangle$. Using (8.29), we have

$$\begin{aligned} CN|10\rangle &= (|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|)|10\rangle \\ &= |00\rangle\langle 00||10\rangle + |01\rangle\langle 01||10\rangle + |10\rangle\langle 11||10\rangle + |11\rangle\langle 10||10\rangle \end{aligned}$$

To calculate each of the inner products, we use the rule for calculating inner products given in (4.8):

$$\langle ab|cd = \langle a|c\rangle\langle b|d\rangle$$

Hence

$$\begin{aligned} \langle 00|10\rangle &= \langle 0|1\rangle\langle 0|0\rangle = 0 \\ \langle 01|10\rangle &= \langle 0|1\rangle\langle 1|0\rangle = 0 \\ \langle 11|10\rangle &= \langle 1|1\rangle\langle 1|0\rangle = 0 \\ \langle 10|10\rangle &= \langle 1|1\rangle\langle 0|0\rangle = 1 \end{aligned}$$

We conclude that

$$CN|10\rangle = |11\rangle$$

When the target qubit is $|1\rangle$, we have

$$\begin{aligned} CN|11\rangle &= (|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|)|11\rangle \\ &= |00\rangle\langle 00||11\rangle + |01\rangle\langle 01||11\rangle + |10\rangle\langle 11||11\rangle + |11\rangle\langle 10||11\rangle \\ &= |10\rangle \end{aligned}$$

So we've confirmed that the controlled NOT gate flips the target qubit when the control bit is $|1\rangle$. Now we can use what we've learned to find the action on the target qubit when it's in the state $\alpha|0\rangle + \beta|1\rangle$. In this case

$$CN(\alpha|10\rangle + \beta|11\rangle) = \alpha CN|10\rangle + \beta CN|11\rangle = \alpha|11\rangle + \beta|10\rangle$$

Therefore the CN takes $\alpha|0\rangle + \beta|1\rangle$ to $\beta|0\rangle + \alpha|1\rangle$ when the control bit is $|1\rangle$.

Example 8.7

Describe a circuit that will generate the Bell states.

Solution

In Chapter 7 we learned that the Bell states are given by

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \tag{8.30}$$

$$|\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \tag{8.31}$$

$$|\beta_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} \tag{8.32}$$

$$|\beta_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \tag{8.33}$$

To see how we can draw a circuit to generate these states, consider the action of the CNOT gate when the *control qubit* is in a superposition state. For example, let's take it to be $|c\rangle = |0\rangle + |1\rangle$, and let the target qubit be $|0\rangle$. Then the CNOT gate will act on the sum $|00\rangle + |10\rangle$. From (8.29), we see that the action of the CN gate on this state is

$$\begin{aligned} CN(|00\rangle + |10\rangle) &= (|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|)(|00\rangle + |10\rangle) \\ &= |00\rangle\langle 00||00\rangle + |01\rangle\langle 01||00\rangle + |10\rangle\langle 11||00\rangle + |11\rangle\langle 10||00\rangle \\ &\quad + |00\rangle\langle 00||10\rangle + |01\rangle\langle 01||10\rangle + |10\rangle\langle 11||10\rangle + |11\rangle\langle 10||10\rangle \\ &= |00\rangle + |11\rangle \end{aligned}$$

This is almost the Bell state given by (8.30). All we are missing is the normalization constant $1/\sqrt{2}$. Of course, if we look back at (8.16) and the action of the Hadamard gate on $|0\rangle$, we see that this gives us the factor we need. So we will start with a qubit given by $|0\rangle$ and act on it with a Hadamard gate. That should give us the state

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

The resulting qubit output from the Hadamard gate is then passed as the control qubit for the *CN* gate. The result will be the state $|\beta_{00}\rangle$. We can use a similar thought process to see how to generate the other Bell states. The circuit required to do this, in general, is shown in Figure 8.5. One thing to notice about this circuit is that moving from left to right indicates the passage of *time*. So a wire in a quantum

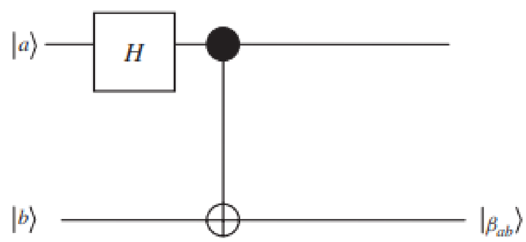


Figure 8.5 Diagram for a quantum circuit that creates Bell states. Time moves from left to right, with wires used to represent the passage of time where the state is left alone. First we apply a Hadamard gate to the qubit —$a\rangle$ to generate a superposition state. This is then used as the control bit in a *CN* gate. The result is the bell state —$\beta_{ab}\rangle$

circuit is simply the time evolution of a quantum state. The operation of the circuit goes as follows, with the steps applied in order:

- Take a qubit $|a\rangle$ where $|a=0\rangle$ or $|a=1\rangle$, and act on it with a Hadamard gate.
- Use the resulting output as the control bit that is passed to a *CN* gate. If the target qubit is denoted by $|b\rangle$, where $b=\{0,1\}$, the target output of the *CN* gate will be the Bell state $|\beta_{ab}\rangle$.

The Bell state $|\beta_{ab}\rangle$ can be written as

$$|\beta_{ab}\rangle = \frac{|0,b\rangle + (-1)^a|1,\overline{a}\rangle}{\sqrt{2}} \tag{8.34}$$

where $\overline{a}$ represents *NOT a*.

As we mentioned earlier, it is possible to generate any type of controlled U gate we wish. For example, we can have a *controlled-Hadamard* gate. Let's denote this by *CH*. The action of the controlled Hadamard gate is as follows: If the control qubit is $|0\rangle$, nothing happens to the target qubit. If the control qubit is $|1\rangle$, then we apply a Hadamard gate to the target qubit. The matrix representation of the controlled Hadamard gate is

$$CH = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{8.35}$$

The Dirac notation representation of this operator is

$$CH = |00\rangle\langle 00| + |01\rangle\langle 01| + \frac{1}{\sqrt{2}}(|10\rangle\langle 10| + |10\rangle\langle 11| + |11\rangle\langle 10| - |11\rangle\langle 11|) \tag{8.36}$$

Example 8.8

Find the action of the CH gate using (8.35) and (8.36) when the input states are $|01\rangle$and$|11\rangle$.

Solution

Using the matrix representation of the CH gate, we need to write out the states $|01\rangle$ and $|11\rangle$. From (4.9) the column vector representation of these states are

$$|01\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|11\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

We should find that applied to the state $|01\rangle$, the CH gate does nothing, since the control qubit is set to 0. We have

$$CH|01\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle$$

In the second case, we have

$$CH|11\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

Hence the target qubit has been taken into the state $|0\rangle - |1\rangle/\sqrt{2}$. Now let's redo the calculations using Dirac notation. For the first state, we have

$$
\begin{aligned}
CH|01\rangle &= \left[|00\rangle\langle 00| + |01\rangle\langle 01| + \frac{1}{\sqrt{2}}(|10\rangle\langle 10| + |10\rangle\langle 11| + |11\rangle\langle 10| - |11\rangle\langle 11|)\right]|01\rangle \\
&= |00\rangle\langle 00||01\rangle + |01\rangle\langle 01||01\rangle + \frac{1}{\sqrt{2}}(|10\rangle\langle 10||01\rangle + |10\rangle\langle 11||01\rangle + |11\rangle\langle 10||01\rangle \\
&\quad - |11\rangle\langle 11||01\rangle) = |01\rangle
\end{aligned}
$$

For the second state, we find that

$$
\begin{aligned}
CH|11\rangle &= \left[|00\rangle\langle 00| + |01\rangle\langle 01| + \frac{1}{\sqrt{2}}(|10\rangle\langle 10| + |10\rangle\langle 11| + |11\rangle\langle 10| - |11\rangle\langle 11|)\right]|01\rangle \\
&= |00\rangle\langle 00||11\rangle + |01\rangle\langle 01||11\rangle + \frac{1}{\sqrt{2}}(|10\rangle\langle 10||11\rangle + |10\rangle\langle 11||11\rangle + |11\rangle\langle 10||01\rangle \\
&\quad - |11\rangle\langle 11||11\rangle) = \frac{|10\rangle - |11\rangle}{\sqrt{2}}
\end{aligned}
$$

Example 8.9

We wish to investigate the use of the controlled NOT gate as a cloning machine. Specifically can it clone the state $a|0\rangle - b|1\rangle$? Begin by supposing that the superposition state given by $a|0\rangle - b|1\rangle$ is used as the control qubit and that the target qubit is given by $|1\rangle$. Then consider the case where the target qubit is $|0\rangle$.

Solution

First let's write down what the output state would be if the gate could clone the state. If it could, then it would make a copy of $a|0\rangle - b|1\rangle$, and the output state would be the product state given by

$$a|0\rangle - b|1\rangle \otimes a|0\rangle - b|1\rangle = a^2|00\rangle - ab|01\rangle - ba|10\rangle + b^2|11\rangle$$

The input state is

$$a|0\rangle - b|1\rangle \otimes |1\rangle = a|01\rangle - b|11\rangle$$

The controlled NOT gate acts on this state as follows:

$$CN(a|01\rangle - b|11\rangle) = aCN|01\rangle - bCN|11\rangle = a|01\rangle - b|10\rangle$$

You can see that $a|01\rangle - b|10\rangle \neq a|0\rangle - b|1\rangle \otimes a|0\rangle - b|1\rangle$. If the target qubit is $|0\rangle$, then we have

$$CN(a|00\rangle - b|10\rangle) = aCN|00\rangle - bCN|10\rangle = a|00\rangle - b|11\rangle$$

This state is also not equal to the product state $a|0\rangle - b|1\rangle \otimes a|0\rangle - b|1\rangle$, so we aren't any closer to cloning the state. This example shows that the CN gate can't clone, in general. Can you think of any specific states that the CN gate might clone?

Gate Decomposition

A large part of working with quantum circuits is decomposing an arbitrary controlled unitary operation U into a series of single qubit. For example. the figure shows some arbitrary controlled U operation An equivalent circuit, consisting of two controlled NOT gates and the single qubit gates A, B, C will result in the same output.

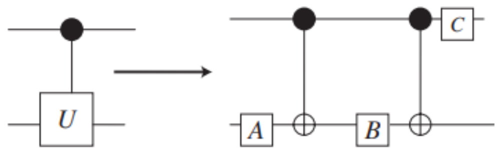


Figure 8.6 We replaced a controlled-U operation by an equivalent circuit consisting of controlled NOT gates and single-qubit gates

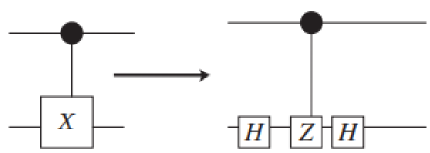


Figure 8.7 A controlled NOT gate is equivalent to a circuit consisting of two Hadamard gates and a controlled Z gate

Quantum Algorithms

A qubit can exist in a superposition of states, giving a quantum computer a hidden realm where exponential computations are possible. So it allows us to perform parallel computations that cannot be done by a classical computer. However, since measurement finds a qubit in one state or the other, we end with a determined result.

Nature allows us to get around this to a certain extent and extract information by using quantum interference.

Hadamard Gates

Recall that a Hadamard gate acts as follows:

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad ; \quad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Or, on a general $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ qubit:

$$H|\psi\rangle = \left(\frac{\alpha+\beta}{\sqrt{2}}\right)|0\rangle + \left(\frac{\alpha-\beta}{\sqrt{2}}\right)|1\rangle$$

Also recall that $H^2 = I$

Hadamard gates in series
For example:

$$\begin{aligned}(H \otimes H)|1\rangle|1\rangle = (H|1\rangle)(H|1\rangle) &= \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \\ &= \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)\end{aligned}$$

When we apply n Hadamard gates in parallel like this, we call it a **Hadamard transform**.

For $n = 3$ we have for example:

$$(H \otimes H \otimes H)|0\rangle|0\rangle|0\rangle = \frac{1}{\sqrt{2^3}}(|000\rangle + |001\rangle + |010\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle)$$

.

There is a general formula for the **Hadamard** transform on $|0\rangle^{\otimes n}$:

$$H^{\otimes n}|0\rangle^{\otimes n}) = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$$

Where $x \in \{0,1\}^n$ runs over all n long qubit strings.

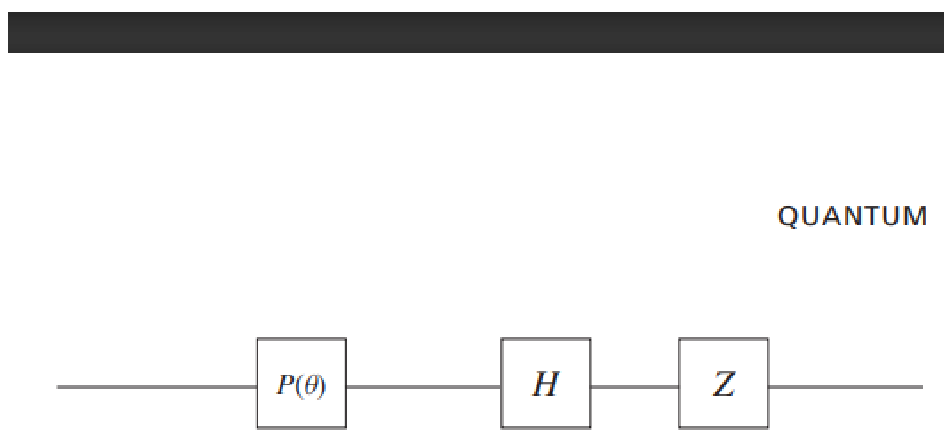
Phase Gate

Another useful gate is the **discrete phase gate**, defined as follows:

$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{(2\pi i/2^k)} \end{pmatrix}$$

Matrix Representation of Serial and Parallel Operations

$$ZHP(\theta)$$



Explicitly, we have:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{i\theta} \\ -1 & e^{i\theta} \end{pmatrix}$$

When quantum operations are performed in parallel, we perform the tensor product:

$$H \otimes H = \frac{1}{\sqrt{2}} \begin{pmatrix} H & H \\ H & -H \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Quantum Parallelism and function evaluation

We can evaluate a function $f(x)$ at many points simultaneously. Consider a simple function that takes a single bit as input and produces a single bit as output. That is $x \in \{0, 1\}$. There are only 4 possible functions like this:

$$f(x) = x \quad ; \quad f(x) = x \oplus 1 \quad ; \quad f(x) = 0 \quad ; \quad f(x) = 1$$

The identity and bit flip functions are called **balanced** because the outputs are opposite for half the inputs. While the others are constant.

Deutsch's Algorithm

It is an algorithm that allows us to know if a function is balanced or constant.

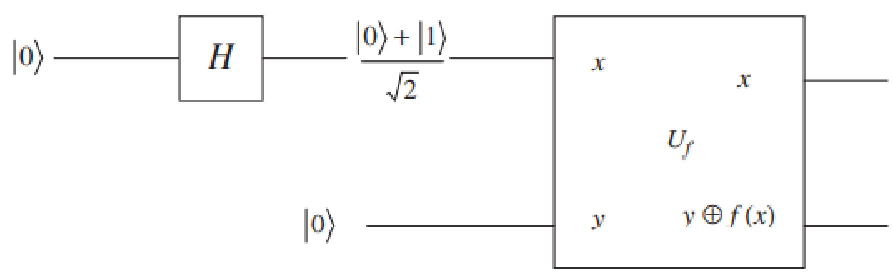
The first step is to imagine a unitary operation denoted U_f that acts on two qubits.
It leaves the first qubit alone and produces the exclusive or ($\oplus$) of the second qubit with the function f evaluated with the first qubit as argument. That is:

$$U_f|x, y\rangle = |x, y \oplus f(x)\rangle$$

Suppose $|x\rangle$ is in a superposition state, and let's see the effect of U_f on $|xy\rangle$ for $y = 0$

$$U_f\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|0\rangle = \frac{1}{\sqrt{2}}(U_f|00\rangle + U_f|10\rangle) = \frac{|0, 0 \pm f(0)\rangle + |1, 0 \pm f(1)\rangle}{\sqrt{2}}$$

The circuit creates a superposition state that has information about every possible value of $f(x)$ in a single step.



Quantum Noise and Quantum Information

Classical Noise and Markov Processes

Imagine a classical bit stored on a hard disk. There will be a probability p for it to flip and a probability $1 - p$ for it to stay the same

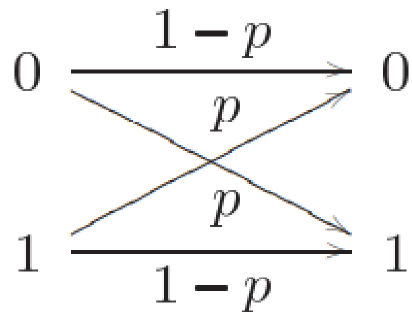


Figure 8.1. After a long time a bit on a hard disk drive may flip with probability p.

These are the ' transition probabilities '. If there was a probability p_0 of the bit being at 0 and p_1 of being at 1, then the new probabilities after the possible flip are:

$$\begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \vec{q} = E\vec{p}$$

Quantum Operations

Similar to how classical states transform, quantum states transform as:

$$\rho' = \epsilon(\rho)$$

Where ρ is the density matrix before the event and ρ' after.
For example, a unitary transformation makes the density matrix evolve as:

$$\epsilon(\rho) = U\rho U^\dagger$$

And a measurement operator makes it evolve as:

$$\epsilon_m(\rho) = M_m \rho M_m^\dagger$$

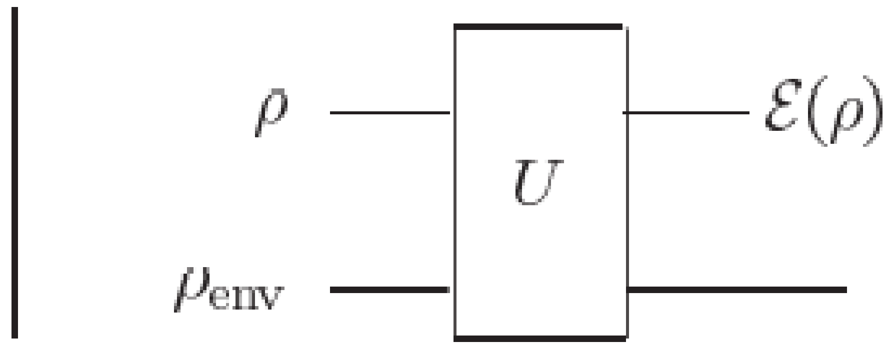
Environments and Quantum Operations

A natural way to describe the dynamics of an open system is to regard it as arising from an interaction between the system of interest (principal system) and an environment, which together form a closed quantum system.
We assume that the complete system input state is $\rho \otimes \rho_{env}$. Then, after a transformation U, the complete system evolves as:

$$\rho \otimes \rho_{env} \rightarrow U(\rho \otimes \rho_{env})U^\dagger$$

Then, we take the partial trace, to know the evolution of the principal system's state:

$$\epsilon(\rho) = Tr_{env}\left[U(\rho \otimes \rho_{env})U^{\dagger}\right]$$



In particular, we sometimes think of the environment as starting in the state $|0\rangle$ (which for some reason is a valid supposition). Then, the evolution of ρ (the principal system's density matrix) is given by:

$$\epsilon(\rho) = \rho' = tr_{env}(U(\rho \otimes |0\rangle\langle 0|)U^{\dagger})$$

.

Generalization:

An important generalization of this is that at the end of the evolution, we may be interested in the density matrix of the environment instead of that for the principal system. Or on any part of the complete system we may so choose. To keep that part of the complete system, we leave out the rest by tracing it out.

$$\epsilon(\rho) = \rho' = Tr_{Rest}(U(\rho \otimes |0\rangle\langle 0|)U^{\dagger})$$

Operator Sum Representation

Quantum operations can be represented in an elegant form known as **operator sum representation.**

Specific example: Let $\{|e_k\rangle\}$ be an orthonormal basis for the state space of the environment, which starts at $|e_0\rangle\langle e_0|$.
Then, the evolution of the principal system can be written as:

$$\begin{aligned}\epsilon(\rho) &= tr_{env}(U[\rho \otimes |e_0\rangle\langle e_0|]U^{\dagger}) \\ &= \sum_k \langle e_k|\, U[\rho \otimes |e_0\rangle\langle e_0|\,]U^{\dagger}\, |e_k\rangle \\ &= \sum_k E_k \rho E_k^{\dagger}\end{aligned}$$

Where $E_k := \langle e_k|U|e_0\rangle$

Notice that E_k is still an operator in the principal system's space, since $\langle e_k|U|e_0\rangle$ only contracts over the environment part.

Operation Elements: In general, operations elements $\{E_k\}$ are a set of operators (on the principal system's space I think [or more like on the part of the system of interest]) that are used to express the quantum operation ϵ.
They are used such that the evolution of ρ is given by:

$$\epsilon(\rho) = \sum_k E_k \rho E_k^\dagger$$

Theorem 1: The operation elements satisfy the relation:

$$\sum_k E_k^\dagger E_k = I$$

This is satisfied by quantum operations which are **trace preserving**.
There is a generalization for **non trace preserving** quantum operations which satisfy $\sum_k E_k^\dagger E_k \leq I$

Excercise 8.4

Exercise 8.4: (Measurement) Suppose we have a single qubit principal system, interacting with a single qubit environment through the transform

$$U = P_0 \otimes I + P_1 \otimes X\,, \tag{8.16}$$

where X is the usual Pauli matrix (acting on the environment), and $P_0 \equiv |0\rangle\langle 0|$, $P_1 \equiv |1\rangle\langle 1|$ are projectors (acting on the system). Give the quantum operation for this process, in the operator-sum representation, assuming the environment starts in the state $|0\rangle$.

$\mathcal{E}(\rho) = tr_{env}(U\rho U^\dagger)$

$= tr_{env}\left((P_0 \otimes I + P_1 \otimes X)(\rho \otimes |0\rangle\langle 0|)(P_0 \otimes I + P_1 \otimes X)\right)$

$= tr_{env}\left(|0\rangle\langle 0|\rho \otimes |0\rangle\langle 0| + |1\rangle\langle 1|\rho \otimes |1\rangle\langle 0|\right)(P_0 \otimes I + P_1 \otimes X)$

$= tr_{env}\big(|0\rangle\langle 0|\rho|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0|\rho|1\rangle\langle 1| \otimes |0\rangle\langle 1|$

$+ \cancel{|1\rangle\langle 1|\rho \otimes |1\rangle\langle 0|}$

$+ |1\rangle\langle 1|\rho|0\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1| \otimes |1\rangle\langle 1|\big)$

$= \langle 0|_{env} \sim |0\rangle_{env} + \langle 1|_{env} \sim |1\rangle_{env}$

$= |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|$

$E_k \equiv \langle e_k|U|e_0\rangle_{env} \longrightarrow E_0 = \langle e_0|_{env}U|e_0\rangle_{env} = \langle e_0|_{env} P_0 \otimes I + P_1 \otimes X|e_0\rangle_{env} = P_0$

$E_1 = \langle e_1|_{env}U|e_0\rangle_{env} = \langle e_1|_{env} P_0 \otimes I + P_1 \otimes X|e_0\rangle_{env} = P_1$

Vemos que $\sum E_k^\dagger E_k = I \Rightarrow P_0 + P_1 = I$ ✓

y efectivamente $\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger = P_0 \rho P_0 + P_1 \rho P_1$

Excercise 8.5 Just as in the previous exercise, but now let:

$$U = \frac{X}{\sqrt{2}} \otimes I + \frac{Y}{\sqrt{2}} \otimes X$$

Spin flips. $U = \frac{X}{\sqrt{2}} \otimes I + \frac{Y}{\sqrt{2}} \otimes X$

$E_k = \langle e_k|_{env} U |e_0\rangle_{env}$

$\Rightarrow E_0 = \langle e_0| \frac{X}{\sqrt{2}} \otimes I + \frac{Y}{\sqrt{2}} \otimes X |e_0\rangle_{env}$

$= \frac{X}{\sqrt{2}}$

$E_1 = \langle e_1|_{env} \frac{X}{\sqrt{2}} \otimes I + \frac{Y}{\sqrt{2}} \otimes X |e_0\rangle_{env}$

$= \frac{Y}{\sqrt{2}}$

$\Rightarrow$ Vemos que $\sum_k E_k^\dagger E_k = I$

so, $\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger = \frac{1}{2} X \rho X^\dagger + \frac{1}{2} Y \rho Y^\dagger$

$= \frac{1}{2}\left(X \rho X + Y \rho Y\right)$

.

Physical Interpretation

Imagine that a measurement of the environment is performed in the basis $|e_k\rangle$ after the unitary transformation U has been applied. According to the principle of implicit measurement, we see that such a measurement affects only the state of the environment, and does not change the principal system.
Let ρ_k be the state of the principal system given that outcome k occurs, so:

$$\rho_k \propto Tr_E(|e_k\rangle\langle e_k|\, U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger |e_k\rangle\langle e_k|\,) = \langle e_k|\, U(\rho \otimes |e_0\rangle\langle e_0|\,)U^\dagger |e_k\rangle$$
$$= E_k \rho E_k^\dagger$$

Normalizing we have:

$$\rho_k = \frac{E_k \rho E_k^\dagger}{tr(E_k \rho E_k^\dagger)}$$

the probability of outcome k is given by:

$$p(k) = Tr(|e_k\rangle\langle e_k|\; U(\rho \otimes |e_0\rangle\langle e_0|\;)U^\dagger |e_k\rangle\langle e_k|)$$

Therefore:

$$\epsilon(\rho) = \sum_k p(k)\rho_k = \sum_k E_k \rho E_k^\dagger$$

This gives us the interpretation for the operation elements $\{E_k\}$.
The action of the quantum operation is equivalent to taking the state ρ and randomly replacing it by $\dfrac{E_k\rho E_k^\dagger}{tr(E_k\rho E_k^\dagger)}$, with probability $tr(E_k\rho E_k^\dagger)$.

Then, it is similar to the concept of noisy communication channels used in classical information theory.

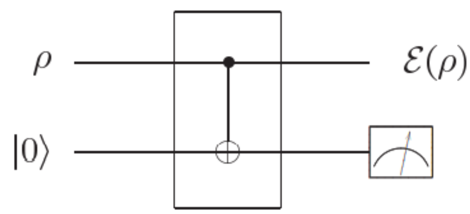


Figure 8.5. Controlled-NOT gate as an elementary model of single qubit measurement.

.

Example: Suppose we choose the state $|e_k\rangle = |0_E\rangle$ and $|1_E\rangle$.
This can be interpreted as doing a measurement in the computational basis of the environment qubit, as shown in Figure 8.5. Doing such a measurement does not change the state of the principal system.
The controlled Not may be expanded as:

$$U = |0_P 0_E\rangle\langle 0_P 0_E| + |0_P 1_E\rangle\langle 0_P 1_E| + |1_P 1_E\rangle\langle 1_P 0_E| + |1_P 0_E\rangle\langle 1_P 1_E|$$

Thus:

$$E_0 = \langle 0_E|U|0_E\rangle = |0_P\rangle\langle 0_P|$$
$$E_1 = \langle 1_E|U|0_E\rangle = |1_P\rangle\langle 1_P|$$

And therefore:

$$\epsilon(\rho) = E_0\rho E_0 + E_1\rho E_1$$

Measurements and the Operator Sum Representation

Given an open quantum system, how do we determine an operator sum representation?

One answer is given the unitary system environment transformation operator U, and a basis of state $|e_k\rangle$ for the environment, the operation elements are:

$$E_k := \langle e_k|U|e_0\rangle$$

We can extend it by allowing the possibility that a measurement is performed on the combined system-environment after the unitary interaction. This possibility is naturally connected to non trace preserving quantum operations, that is, maps that do the following: $\epsilon(\rho) = \sum_k E_k \rho E_k^\dagger$ such that $\sum_k^\dagger E_k \leq I$

Suppose the principal system is initially in state ρ. (We denote the principal by Q and environment by E). Q starts in a state ρ and E starts in a state σ, so the joint system starts at:

$$\rho^{QE} = \rho \otimes \sigma$$

The system evolves under some unitary interaction U
Then, a projective measurement is performed on the joint system, described by projectors P_m.
The case where no measurement is made corresponds to the special case where there is only a single measurement outcome, $m = 0$, which corresponds to the projector $P_0 \equiv I$.

The final state of QE after evolution and measurement (and normalization) given the measurement outcome m occurred, is:

$$\rho^{QE} \rightarrow \frac{P_m U(\rho \otimes \sigma) U^\dagger P_m}{tr(P_m U(\rho \otimes \sigma) U^\dagger P_m)}$$

Tracing out E, we see that the final state of Q alone is:

$$\rho^{Q} \rightarrow \frac{tr_E(P_m U(\rho \otimes \sigma) U^\dagger P_m)}{tr(P_m U(\rho \otimes \sigma) U^\dagger P_m)}$$

So, we define a map:

$$\epsilon_m(\rho) := tr_E(P_m U(\rho \otimes \sigma) U^\dagger P_m)$$

So the final state of Q alone is the normalization of this:

$$\rho \rightarrow \frac{\epsilon_m(\rho)}{tr(\epsilon_m(\rho))}$$

ρ Solita indica ρ^Q

Note that $tr(\epsilon_m(\rho))$ is the probability of outcome m of the measurement occurring.

Now let $\sigma = \sigma_j q_j |j\rangle\langle j|$ be an ensemble decomposition of σ. Introduce an orthonormal basis $|e_k\rangle$ for the system E. Note that:

$$\begin{aligned}\epsilon_m(\rho) &= \sum_{jk} q_j tr_E(\ |e_k\rangle\langle e_k|P_m U(\rho \otimes |j\rangle\langle j|\ )U^\dagger P_m|e_k\rangle\langle e_k|\ ) \\ &= \sum_{jk} E_{jk}\rho E_{jk}^\dagger\end{aligned}$$

Where

$$E_{jk} := \sqrt{q_j}\langle e_k|P_m U|j\rangle$$

This is a generalization of equation 8.10.
This gives an explicit means for calculating the operators appearing in an operator sum representation for ϵ_m, given that the initial state σ of E is known and the dynamics between Q and E are known.
This operations ϵ_m can be thought as defining a kind of measurement process generalizing the description of measurements.

System-Environment Models for any operator sum representation

Given a set of operators $\{E_k\}$ is there a reasonable way to model environmental system and dynamics which give rise to a quantum operation with those operation elements?
We will only see how to do this for a quantum operations mapping the input space to the same output space.

We show that for any quantum operation ϵ (trace preserving or not) with operation elements $\{E_k\}$, there exists a model environment, E, starting in a pure state $|e_0\rangle$, and model dynamics specified by a unitary operator U and projector P onto E such that:

$$\epsilon(\rho) = tr_E(PU(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger P)$$

To see this, suppose ϵ is trace preserving, so that it is described by operation elements $\{E_k\}$ with $\sum_k E_k^\dagger E_k = I$. Let $|e_k\rangle$ be an orthonormal basis for E. Define an operator U which has the following action on states of the form $|\psi\rangle|e_0\rangle$:

$$U|\psi\rangle|e_0\rangle := \sum_k E_k|\psi\rangle|e_k\rangle$$

Note that for arbitrary states $|\psi\rangle$ and $|\phi\rangle$ of the principal system,

$$\langle\psi|\langle e_0|U^\dagger U|\phi\rangle|e_0\rangle = \sum_k \langle\psi|E_k^\dagger E_k|\phi\rangle = \langle\psi|\phi\rangle$$

By the completeness relation.
Thus the operator U can be extended to a unitary operator acting on the entire state space of the joint system, with:

$$tr_E(U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger) = \sum_k E_k \rho E_k^\dagger$$

Box 8.1: Mocking up a quantum operation

Given a trace-preserving quantum operation expressed in the operator-sum representation, $\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$, we can construct a physical model for it in the following way. From (8.10), we want U to satisfy

$$E_k = \langle e_k|U|e_0\rangle \,, \tag{8.41}$$

where U is some unitary operator, and $|e_k\rangle$ are orthonormal basis vectors for the environment system. Such a U is conveniently represented as the block matrix

$$U = \begin{bmatrix} [E_1] & \cdot & \cdot & \cdot & \dots \\ [E_2] & \cdot & \cdot & \cdot & \dots \\ [E_3] & \cdot & \cdot & \cdot & \dots \\ [E_4] & \cdot & \cdot & \cdot & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \tag{8.42}$$

in the basis $|e_k\rangle$. Note that the operation elements E_k only determine the first block column of this matrix (unlike elsewhere, here it is convenient to have the first label of the states be the environment, and the second, the principal system). Determination of the rest of the matrix is left up to us; we simply choose the entries such that U is unitary. Note that by the results of Chapter 4, U can be implemented by a quantum circuit.

.
.

Summary

We have a complete state ρ, then it can evolve in many ways:

- **According to a unitary:** $\epsilon(\rho) = U\rho U^\dagger$
- **According to a measurement:** $\epsilon_m(\rho) = M_m \rho M_m^\dagger$
  The evolution depends on the actual result of the measurement.

- If the complete system starts in $\rho \otimes \rho_E$, it evolves, and we only care about the principal part, we have:

$$\epsilon(\rho) = Tr_E(U(\rho \otimes \rho_E)U^\dagger)$$

Operator Sum Representation: In general, a quantum operation can be written as:

$$\epsilon(\rho) = \sum_k E_k \rho E_k^\dagger$$

Where E_k are operators such that $\sum_k E_k^\dagger E_k = I$
In the special case where the initial environment state is $|e_0\rangle\langle e_0|$ and the environment basis is $\{|e_k\rangle\}$, the elements are $E_k = \langle e_k|U|e_0\rangle$.

- **With Measurement:** Suppose that a measurement is performed in the environment basis $|e_k\rangle$ after the unitary transformation U. Let ρ_k be the state if measurement k happens:

$$\rho_k = \frac{E_k \rho E_k^\dagger}{tr(E_k \rho E_k^\dagger}$$

  Where $Tr(E_k \rho E_k^\dagger)$ is the probability that we get this k result.

  If we don't know the result of measurement, then:

$$\epsilon(\rho) = \sum_k E_k \rho E_k^\dagger$$

- Suppose the initial system is $\rho^{QE} = \rho \otimes \sigma$.
  $\sigma = q_j |j\rangle\langle j|$ is the density of the initial environment.
  Then, let's say the system undergoes a complete unitary transformation U. And then we measure with operators P_m.
  If the result of measurement is m, then the resulting density operator of the principal system is:

$$\begin{aligned}\epsilon_m(\rho) &= \sum_{jk} q_j tr_E(|e_k\rangle\langle e_k| P_m U (\rho \otimes |j\rangle\langle j|) U^\dagger P_m |e_k\rangle\langle e_k|) \\ &= \sum_{jk} E_{jk} \rho E_{jk}^\dagger\end{aligned}$$

  Where

$$E_{jk} := \sqrt{q_j}\langle e_k| P_m U |j\rangle$$

  This result happens with probability $Tr(\epsilon_m(\rho))$

Axiomatic Approach to Quantum Operations

We define a quantum operation ϵ as a map from the set of density operators of the input space Q_1 to the set of density operators for the output space Q_2. For simplicity, we usually take $Q_1 = Q_2 = Q$.

A1 First, $tr[\epsilon(\rho)]$ is the probability that the process represented by ϵ occurs, when ρ is the initial state. Thus, $0 \leq tr[\epsilon(\rho)] \leq 1$ for any state ρ.

A2 Second, ϵ is a **convex linear map** on the set of density matrices, that is, for probabilities $\{p_i\}$,

$$\epsilon\left(\sum_i p_i \rho_i\right) = \sum_i p_i \epsilon(\rho_i)$$

A3 ϵ is **completely positive map.** That is, if ϵ maps density operators of system Q_1 to density operators of system Q_2, then $\epsilon(A)$ must be positive for any positive operator A.
Furthermore, if we introduce an extra system R of arbitrary dimensionality, it must be true that $(I \otimes \epsilon)(A)$ is positive for any positive operator A on the combined system RQ_1, where I denotes the identity map on system R.

First Property

For example, suppose that we are doing a projective measurement in the computational basis of a single qubit. Then two quantum operations are used to describe this process, defined by $\epsilon_0(\rho) = |0\rangle\langle 0|\rho|0\rangle\langle 0|$ and $\epsilon_1(\rho) = |1\rangle\langle 1|\rho|1\rangle\langle 1|$.
the probabilities of the respective outcomes are correctly given by $tr[\epsilon_0(\rho)]$ and $tr[\epsilon_1(\rho)]$.
With this convention, the correctly normalized final quantum state is

$$\frac{\epsilon(\rho)}{tr[\epsilon(\rho)]}$$

In the case where the process is deterministic (no measurement taking place), this reduces to the requirement that $tr[\epsilon(\rho)] = 1 = tr(\rho)$, so the quantum operation is **trace preserving**.

If there is a ρ such that $tr[\epsilon(\rho)] < 1$, then the quantum operation is non trace preserving.
A **physical** quantum operation is one that satisfies the requirement that probabilities never exceed 1, $tr[\epsilon(\rho)] \leq 1$.

Second Property:

Suppose the input ρ to the quantum operation is obtained by randomly selecting the state from an ensemble $\{p_i, \rho_i\}$ of quantum states, that is $\rho = \sum_i p_i \rho_i$.
Then we would expect that the resulting state, $\epsilon(\rho)/tr[\epsilon(\rho)] = \epsilon(\rho)/p(\epsilon)$ corresponds to a

random selection from the ensemble $\{p(i|\epsilon), \epsilon(\rho_i)/tr[\epsilon(\rho_i)]\}$, where $p(i|\epsilon)$ is the probability that the state prepared was ρ_i, given that the process represented by ϵ occurred. Thus, we demand that:

$$\epsilon(\rho) = p(\epsilon)\sum_i p(i|\epsilon)\frac{\epsilon(\rho_i)}{tr[\epsilon(\rho_i)]}$$

Where $p(\epsilon) = tr[\epsilon(\rho)]$ is the probability that the process described by ϵ occurs on input of ρ. By Bayes' rule,

$$p(i|\epsilon) = p(\epsilon|i)\frac{p_i}{p(\epsilon)} = \frac{tr[\epsilon(\rho_i)]p_i}{p(\epsilon)}$$

.

Third Property

Not only must $\epsilon(\rho)$ be a valid density matrix, so long as ρ is valid, but if $\rho = \rho_{RQ}$ is the density matrix of a joint system of R and Q, if ϵ acts only on Q, then $\epsilon(\rho_{RQ})$ must still result in a valid density matrix (up to normalization) of the joint system.

These three axioms are sufficient to define quantum operations.

Theorem 8.1: The map ϵ satisfies axioms $A1, A2, A3$ iff:

$$\epsilon(\rho) = \sum_i E_i \rho E_i^\dagger$$

For some set of operators $\{E_i\}$ which map the input Hilber space to the output Hilber space, and $\sum_i E_i^\dagger E_i \leq I$

Proof
Suppose $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$. $\mathcal{E}$ is obviously linear, so to check that $\mathcal{E}$ is a quantum operation we need only prove that it is completely positive. Let A be any positive operator acting on the state space of an extended system, RQ, and let $|\psi\rangle$ be some state of RQ. Defining $|\varphi_i\rangle \equiv (I_R \otimes E_i^\dagger)|\psi\rangle$, we have

$$\begin{aligned} \langle\psi|(I_R \otimes E_i)A(I_R \otimes E_i^\dagger)|\psi\rangle &= \langle\varphi_i|A|\varphi_i\rangle & (8.51)\\ &\geq 0, & (8.52)\end{aligned}$$

by the positivity of the operator A. It follows that

$$\langle\psi|(\mathcal{I} \otimes \mathcal{E})(A)|\psi\rangle = \sum_i \langle\varphi_i|A|\varphi_i\rangle \geq 0, \quad (8.53)$$

and thus for any positive operator A, the operator $(\mathcal{I} \otimes \mathcal{E})(A)$ is also positive, as required. The requirement $\sum_i E_i^\dagger E_i \leq I$ ensures that probabilities are less than or equal to 1. This completes the first part of the proof.

Suppose next that $\mathcal{E}$ satisfies axioms **A1**, **A2** and **A3**. Our aim will be to find an operator-sum representation for $\mathcal{E}$. Suppose we introduce a system, R, with the same dimension as the original quantum system, Q. Let $|i_R\rangle$ and $|i_Q\rangle$ be orthonormal bases for R and Q. It will be convenient to use the same index, i, for these two bases, and this can certainly be done as R and Q have the same dimensionality. Define a joint state $|\alpha\rangle$ of RQ by

$$|\alpha\rangle \equiv \sum_i |i_R\rangle|i_Q\rangle. \tag{8.54}$$

The state $|\alpha\rangle$ is, up to a normalization factor, a maximally entangled state of the systems R and Q. This interpretation of $|\alpha\rangle$ as a maximally entangled state may help in understanding the following construction. Next, we define an operator σ on the state space of RQ by

$$\sigma \equiv (\mathcal{I}_R \otimes \mathcal{E})(|\alpha\rangle\langle\alpha|). \tag{8.55}$$

We may think of this as the result of applying the quantum operation $\mathcal{E}$ to one half of a maximally entangled state of the system RQ. It is a truly remarkable fact, which we will now demonstrate, that the operator σ completely specifies the quantum operation $\mathcal{E}$. That is, to know how $\mathcal{E}$ acts on an arbitrary state of Q, it is sufficient to know how it acts on a single maximally entangled state of Q with another system!

The trick which allows us to recover $\mathcal{E}$ from σ is as follows. Let $|\psi\rangle = \sum_j \psi_j |j_Q\rangle$ be any state of system Q. Define a corresponding state $|\tilde{\psi}\rangle$ of system R by the equation

$$|\tilde{\psi}\rangle \equiv \sum_j \psi_j^* |j_R\rangle. \tag{8.56}$$

Notice that

$$\langle\tilde{\psi}|\sigma|\tilde{\psi}\rangle = \langle\tilde{\psi}| \left(\sum_{ij} |i_R\rangle\langle j_R| \otimes \mathcal{E}(|i_Q\rangle\langle j_Q|) \right) |\tilde{\psi}\rangle \tag{8.57}$$

$$= \sum_{ij} \psi_i \psi_j^* \mathcal{E}(|i_Q\rangle\langle j_Q|) \tag{8.58}$$

$$= \mathcal{E}(|\psi\rangle\langle\psi|). \tag{8.59}$$

Let $\sigma = \sum_i |s_i\rangle\langle s_i|$ be some decomposition of σ, where the vectors $|s_i\rangle$ need not be normalized. Define a map

$$E_i(|\psi\rangle) \equiv \langle\tilde{\psi}|s_i\rangle. \tag{8.60}$$

A little thought shows that this map is a linear map, so E_i is a linear operator on the state space of Q. Furthermore, we have

$$\sum_i E_i|\psi\rangle\langle\psi|E_i^\dagger = \sum_i \langle\tilde{\psi}|s_i\rangle\langle s_i|\tilde{\psi}\rangle \tag{8.61}$$

$$= \langle\tilde{\psi}|\sigma|\tilde{\psi}\rangle \tag{8.62}$$

$$= \mathcal{E}(|\psi\rangle\langle\psi|). \tag{8.63}$$

Thus

$$\mathcal{E}(|\psi\rangle\langle\psi|) = \sum_i E_i|\psi\rangle\langle\psi|E_i^\dagger, \tag{8.64}$$

for all pure states, $|\psi\rangle$, of Q. By convex-linearity it follows that

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger \tag{8.65}$$

in general. The condition $\sum_i E_i^\dagger E_i \leq I$ follows immediately from axiom **A1** identifying the trace of $\mathcal{E}(\rho)$ with a probability. $\square$

Freedom in the Operator Sum representation

The operator sum representation is not a unique description.
Consider quantum operators ϵ and F acting on a single qubit with the operator sum representations $\epsilon(\rho) = \sum_k E_k \rho E_k^\dagger$ and $F(\rho) = \sum_k F_k \rho F_k^\dagger$, where the operation elements are defined by:

$$E_1 = \frac{I}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad E_2 = \frac{Z}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

And

$$F_1 = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad F_2 = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

What is interesting is that they both give rise to the same quantum operation, that is $F(\rho) = \epsilon(\rho)$.

Theorem 8.2: Suppose $\{E_1, \cdots, E_m\}$ and $\{F_1, \cdots, F_n\}$ are operation elements giving rise to quantum operations ϵ and F. By appending zero operators to the shorter list we ensure $m = n$.
Then $\epsilon = F$ iff there exist complex numbers u_{ij} such that $E_i = \sum_j u_{ij} F_j$ and u_{ij} is a $m \times m$ unitary matrix.

Theorem 8.3: All quantum operations ϵ on a system of Hilbert space dimension d can be generated by an operator sum representation containing at most d^2 elements,

$$\epsilon(\rho) = \sum_{k=1}^{M} E_k \rho E_k^\dagger$$

Where $1 \leq M \leq d^2$

.

Examples of Quantum Noise and Quantum Operations

Trace and Partial Trace

The simplest operation is the trace map $\rho \to tr(\rho)$. Which can be thought of a quantum operation if we let H_Q be any input Hilbert space spanned by $|1\rangle, \cdots, |d\rangle$, and let H_Q' be a one dimensional output space, spanned by the state $|0\rangle$. We define:

$$\epsilon(\rho) = \sum_{i=1}^{d} |0\rangle\langle i|\rho|i\rangle\langle 0|$$

So that ϵ is a quantum operation, by theorem 8.1. Note that $\epsilon(\rho) = tr(\rho)|0\rangle\langle 0|$, so trace is a quantum operation.

Observation:: Partial trace is a quantum operation

Suppose we have a joint system QR and wish to trace out R. Let $|j\rangle$ be a basis for system R. Define a linear operator $E_i : H_{QR} \to H_Q$ by:

$$E_i \left(\sum_j \lambda_j |q_j\rangle |j\rangle \right) := \lambda_i |q_i\rangle$$

Where λ_j are complex numbers, and $|q_j\rangle$ are arbitrary states of Q. Define ϵ to be the quantum operation with operation elements $\{E_i\}$, that is,

$$\epsilon(\rho) := \sum_i E_i \rho E_i^\dagger$$

By theorem 8.1, this is a quantum operation from system QR to system Q. Notices that:

$$\epsilon(\rho \otimes |j\rangle\langle j'|) = \rho\delta_{j,j'} = tr_R(\rho \otimes |j\rangle\langle j'|)$$

Where ρ is any Hermitian operator on the state space of W and $|j\rangle$ and $|j'\rangle$ are members of the orthonormal basis for system R. So $\epsilon = tr_R$

Therefore, the partial trace is a quantum operation.

Visualization

A qubit is represented by a 2x2 density matrix that has trace 1 and is positive (and therefore Hermitian). Then, it has the general form:

$$\rho = \begin{pmatrix} a & b \\ b^* & 1-a \end{pmatrix}$$

And positivity means that the eigenvalues are both positive:

$$\lambda_\pm = \frac{1}{2}(1 \pm \sqrt{1-4|\rho|}) \geq 0$$

In general, any qubit density matrix can be represented by:

$$\rho = \frac{1}{2}(I + \vec{v} \cdot \vec{\sigma})$$

Where $\vec{v} = (v_x, v_y, v_z)$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. In these terms, the density matrix is:

$$\rho = \frac{1}{2}\begin{pmatrix} 1+v_z & v_x - iv_y \\ v_x + iv_y & 1 - v_z \end{pmatrix}$$

Unit trace is guaranteed, and positivity can be made explicit by noting that $|\rho| = \frac{1}{4}(1-||\vec{v}||^2)$. So that:

$$\lambda_\pm = \frac{1}{2}(1 \pm ||\vec{v}||)$$

Thus, we require that:

$$\boxed{||\vec{v}|| \leq 1}$$

.

Also, we can see that we have:

$$\boxed{Tr(\rho^2) = \frac{1}{2}(1 + |\vec{v}|^2)}$$

So pure states have $|\vec{v}| = 1$ and mixed states have $|\vec{v}| < 1$.

Therefore, any state can be represented by a point in a unit sphere.

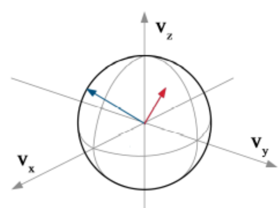


FIG. 1. The Bloch sphere is a geometric representation of the collection of all Bloch vectors $\vec{v}$ which describe valid qubit density operators. Thus, the sphere is of radius 1, its surface represents all pure states, and its interior represents all mixed states. In this diagram the blue vector lies on the surface of the sphere indicating a pure state, whereas the red vector lies in its interior indicating a mixed state.

- **Z poles** ($\vec{v} = (0, 0, \pm 1)$): the density matrix is:

$$\rho = \frac{I \pm \sigma_z}{2}$$

  Which is $|0\rangle\langle 0|$ for $v_z = 1$ and $|1\rangle\langle 1|$ for $v_z = -1$

- **X poles** ($\vec{v} = (\pm 1, 0, 0)$): The density matrix takes the form:

$$\rho = \frac{I \pm \sigma_x}{2} = \frac{1}{2}(|0\rangle \pm |1\rangle)(\langle 0| \pm \langle 1|)$$

  So $|+\rangle\langle +|$ for $v_x = 1$ and $|-\rangle\langle -|$ for $v_x = -1$

- **Y poles** $\vec{v} = (0, \pm 1, 0))$
  The density matrix is:

$$\rho = \frac{I \pm \sigma_y}{2} = \frac{1}{2}(|0\rangle \pm i|1\rangle)(\langle 0| \pm (-i)\langle 1|)$$

- **Center:** ($\vec{v} = (0, 0, 0)$: In this case $\rho = I/2$

In this representation, it turns out that an arbitrary trace preserving quantum operation is equivalent to a map of the form:

$$\vec{r} \rightarrow_{\epsilon} \vec{r'} = M\vec{r} + \vec{c}$$

Where M is a 3x3 real matrix and $\vec{c}$ a constant vector. This is an affine map of the Bloch sphere into itself.

Proof: Suppose the operators E_i generating the operator sum representation for ϵ are written in the form:

$$E_i = \alpha_i I + \sum_{k=1}^{3} a_{ik}\sigma_k$$

Then, it is easy to prove that:

$$M_{jk} = \sum_l \left[a_{lj}a_{lk}^* + a_{lj}^* a_{lk} + \left(|\alpha_l|^2 - \sum_p a_{lp}a_{lp}^* \right) \delta_{jk} + i \sum_p \epsilon_{jkp}(\alpha_l \alpha_{lp}^* - \alpha_l^* \alpha_{lp}) \right]$$

$$c_k = 2i \sum_l \sum_{jp} \epsilon_{jpk} a_{lj} a_{lp}^*$$

We can decompose the matrix in polar decomposition as $M = U|M|$ where U is unitary. Because M is real, it follows that $|M|$ is real and Hermitian (symmetric matrix). And because M is real, we may assume U is too. And thus an orthongonal matrix $U^T U = I$. Thus we may write:

$$M = OS$$

Where O is real orthogonal matrix with determinant 1, representing a proper rotation, and S is a real symmetric matrix.
So M is just a deformation of the Bloch sphere along principal axes determined by S, followd by a proper rotation due to O. And then a displacement due to $\vec{c}$

Bit flip and phase flip

- **Bit Flip**

  The bit flip channel flips the state of a qubit from $|0\rangle$ to $|1\rangle$ (and vice versa) with probability $1-p$. It has operation elements:

  $$E_0 = \sqrt{p} I = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad E_1 = \sqrt{1-p} X = \sqrt{1-p} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  Clearly these are the correct operators, since the first one has probability p of leaving all the same and the second one has probability $1-p$ of flipping. We can also see this by writing:
  $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

  $$\epsilon(\rho) = \sum_k E_k \rho E_k^\dagger = pI\rho I + (1-p) X\rho X$$
  $$= \cdots p\rho + (1-p) \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

So a probability p of leaving everything the same and a probability $1-p$ of flipping (because the original would be $\rho = a|0\rangle\langle 0| + b|1\rangle\langle 0| + c|0\rangle\langle 1| + d|1\rangle\langle 1|$ and therefore a flip change would result in $a|1\rangle\langle 1| + b|0\rangle\langle 1| + c|1\rangle\langle 0| + d|0\rangle\langle 0|$

So as, explained in the physical interpratation section, we have a probability $p = tr(E_0\rho E_0^\dagger)$ of getting a result $E_0\rho E_0^\dagger$. And a probability $1-p = tr(E_1\rho E_1^\dagger)$ of getting a result $E_1\rho E_1^\dagger$

We can represent this operation using the Bloch vector $\vec{r} \to M\vec{r} + \vec{c}$.
It results that the effect of the bit flip on the Bloch sphere, for $p = 0.3$ is given by

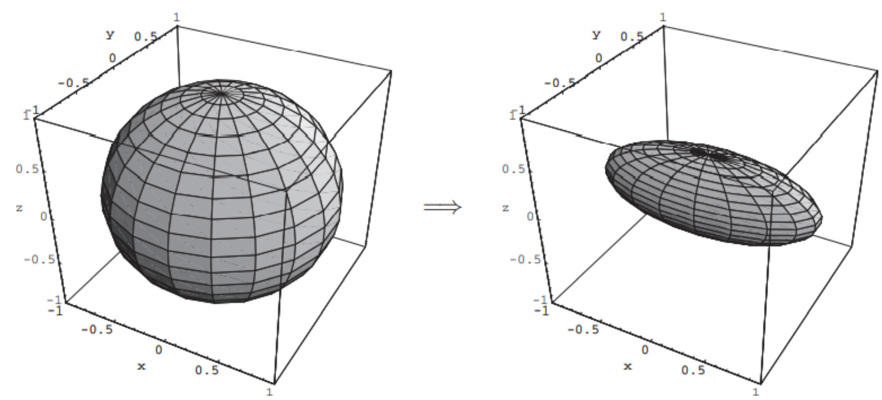


Figure 8.8. The effect of the bit flip channel on the Bloch sphere, for $p = 0.3$. The sphere on the left represents the set of all pure states, and the deformed sphere on the right represents the states after going through the channel. Note that the states on the $\hat{x}$ axis are left alone, while the $\hat{y}$-$\hat{z}$ plane is uniformly contracted by a factor of $1-2p$.

As we saw earlier, $tr(\rho^2)$ for a single qubit is equal to $(1+|r|^2)/2$, where $|r|$ is the norm of the Bloch vector. So, seeing the drawing, this operation only can reduce $Tr(\rho^2)$

- **Phase Flip:** This one has elements:

$$E_0 = \sqrt{p}I = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad E_1 = \sqrt{1-p}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So we have:

$$\epsilon(\rho) = p\rho + (1-p)Z\rho Z$$

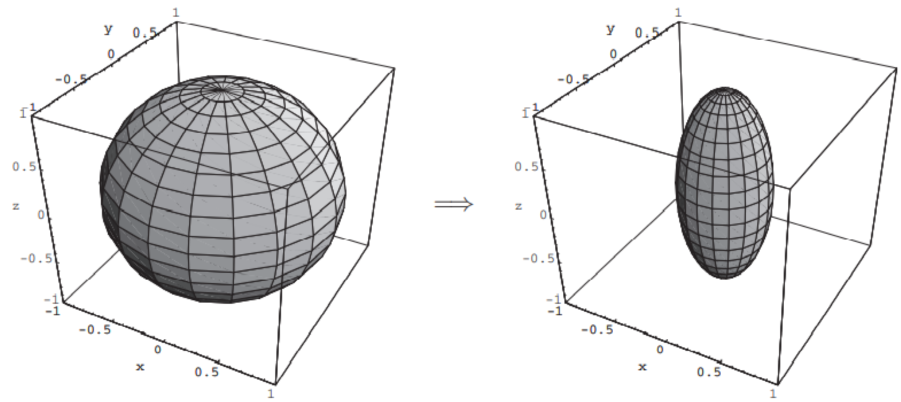


Figure 8.9. The effect of the phase flip channel on the Bloch sphere, for $p = 0.3$. Note that the states on the $\hat{z}$ axis are left alone, while the $\hat{x} - \hat{y}$ plane is uniformly contracted by a factor of $1 - 2p$.

When $p = 1/2$, it is completely contracted to the z axis.

- **Bit phase flip**: Elements:

$$E_0 = \sqrt{p} I = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad E_1 = \sqrt{1-p} Y = \sqrt{1-p} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

So we have:

$$\epsilon(\rho) = p\rho + (1-p) Y \rho Y$$

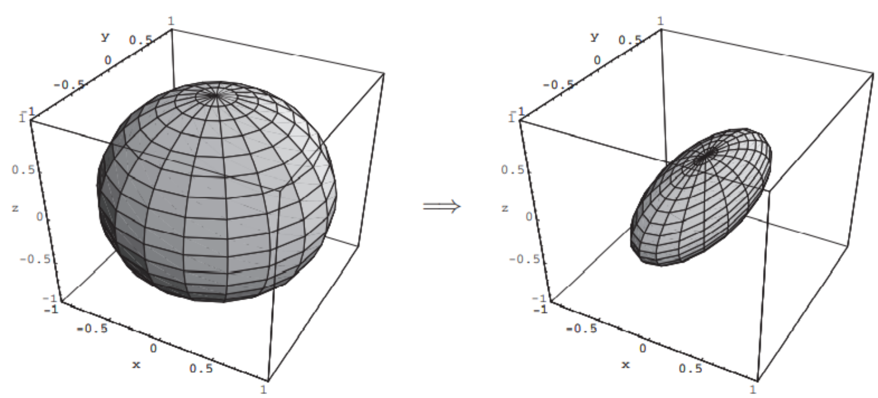


Figure 8.10. The effect of the bit–phase flip channel on the Bloch sphere, for $p = 0.3$. Note that the states on the $\hat{y}$ axis are left alone, while the $\hat{x}$-$\hat{z}$ plane is uniformly contracted by a factor of $1 - 2p$.

Depolarizing Channel

We take a single qubit, and with probability p that qubit is depolarized. That is, it is replaced by the completely mixed state $I/2$. With probability $1-p$ the qubit is left untouched. The state of the quantum system after the noise is:

$$\epsilon(\rho) = p\frac{I}{2} + (1-p)\rho$$

The effect is seen here:

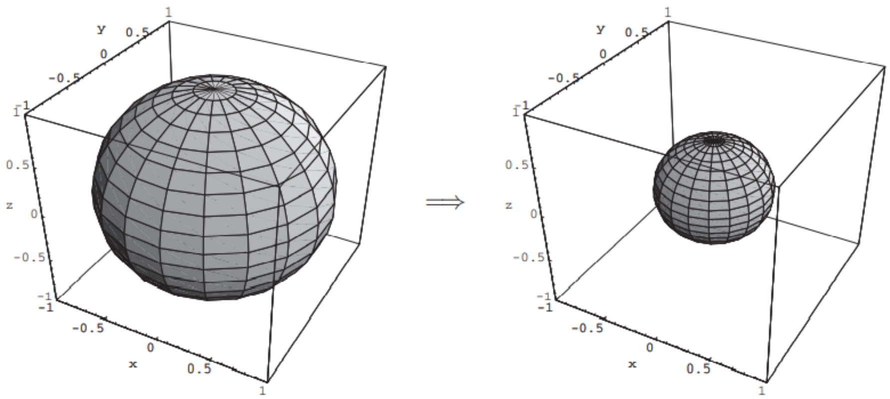


Figure 8.11. The effect of the depolarizing channel on the Bloch sphere, for $p = 0.5$. Note how the entire sphere contracts uniformly as a function of p.

.

Here we see a circuit simulating the depolarizing channel. The top of the circuit is the input to the depolarizing channel, while the bottom two lines are an 'environment'. We have used an environment with two mixed state inputs. The idea is that the third qubit, initially a mixture of the state $|0\rangle$ with probability $1-p$ and state $|1\rangle$ with probability p acts as a control for whether or not the completely mixed state $I/2$ in the second qubit is swapped into the first qubit.
To get the operator sum representation (which is not given by the last equation), we can write for arbitrary ρ:

$$\frac{I}{2} = \frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{4}$$

And then substitute this into $\epsilon(\rho)$ to write it as:

$$\epsilon(\rho) = \left(1 - \frac{3p}{4}\right)\rho + \frac{p}{4}(X\rho X + Y\rho Y + Z\rho Z)$$

Showing that the depolarizing channel has operation elements $\{\sqrt{1-3p/4}I, \sqrt{p}X/2, \sqrt{p}Y/2, \sqrt{p}Z/2\}$. It is more convinient to write it as:

$$\epsilon(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$$

Which has the interpretation that the state ρ is left alone with probability $1-p$ and each operator X, Y, Z acts with probability $p/3$.

This channel can be generalized for d-dimensional quantum systems as:

$$\epsilon(\rho) = \frac{pI}{d} + (1-p)\rho$$

Amplitude Damping

n important application of quantum operations is the description of energy dissipation – effects due to loss of energy from a quantum system.

Suppose we have a single optical mode containing the quantum state $a|0\rangle + b|1\rangle$, a superposition of zero or one photons. The scattering of a photon from this mode can be modeled by thinking of inserting a partially silvered mirror, a beamsplitter. This beamsplitter allows the photon to couple to another optical mode (the environment), according to the unitary tranformation $B = \exp\left[\theta(a^\dagger b - ab^\dagger)\right]$, where $a, a^\dagger, b, b^\dagger$ are the annihilation and creation operators for photons in the two modes.
The output after the beamsplitter, assuming the environment starts with no photons, is simply $B|0\rangle(a|0\rangle + b|1\rangle) = a|00\rangle + b(\cos\theta|01\rangle + \sin\theta|10\rangle)$. Tracing out the environment gives us the quantum operation:

$$\epsilon_{AD}(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$$

Where $E_k = \langle k|B|0\rangle$ are:

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}$$
$$E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

Where $\gamma = \sin^2\theta$ can be thought of as the probability of losing a photon.

No linear combination can be made of E_0 and E_1 to give an operation element proportional to the identity. The E_1 operation changes a $|1\rangle$ into a $|0\rangle$ state, corresponding to the physical process of losing a quantum of energy to the environment. E_0 leaves $|0\rangle$ unchanged, but reduces the amplitude of a $|1\rangle$ state; physically, this happens because a quantum of energy was not lost to the environment and thus the environment now perceives it to be more likely that the system is in the $|0\rangle$ state, rather than the $|1\rangle$ state.

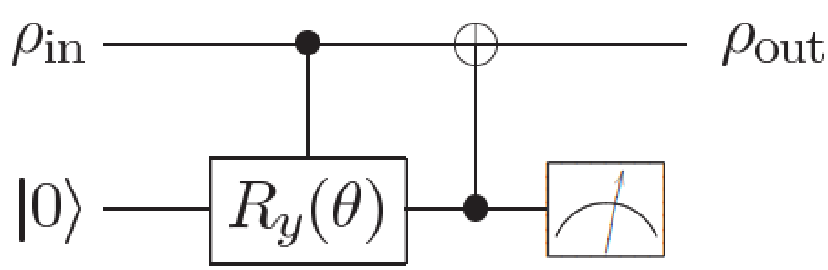


Figure 8.13. Circuit model for amplitude damping

A general characteristic of quantum operation is the set of states that are left invariant under the operation. For example, the phase flip leaves $\vec{z}$ axis of the Bloch sphere unchanged;

this corresponds to states of the form $p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ for arbitrary p. In the case of amplitude damping, only the ground state $|0\rangle$ is invariant.

Quantum operation for dissipation to an environment at finite temperature?
This process ϵ_{GAD}, called **generalized amplitude damping**, is defined for single qubits by the operation elements:

$$E_0 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$E_1 = \sqrt{p}\begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

$$E_2 = \sqrt{1-p}\begin{pmatrix} \sqrt{1-\gamma} & 0 \\ 0 & 1 \end{pmatrix}$$

$$E_3 = \sqrt{1-p}\begin{pmatrix} 0 & 0 \\ \sqrt{p} & 0 \end{pmatrix}$$

Where the stationary state

$$\rho_\infty = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}$$

satisfies $\epsilon_{GAD}(\rho_\infty) = \rho_\infty$.

We can visualize the effect of amplitude damping in the Bloch representation as the Bloch vector transformation:

$$(r_x, r_y, r_z) = (r_x\sqrt{1-\gamma}, r_y\sqrt{1-\gamma}, \gamma + r_z(1-\gamma))$$

When γ is replaced with a time varying function like $1 - e^{-t/T_1}$ (t is time and T_1 a constant parameter). This can be visualized as a flow of the Bloch sphere towards a fixed point at the north pole, where $|0\rangle$ is located.

While generalized damping performs the transformation:

$$(r_x, r_y, r_z) \rightarrow \left(r_x\sqrt{1-\gamma}, r_y\sqrt{1-\gamma}, \gamma(2p-1) + r_z(1-\gamma)\right)$$

Comparing to the usual amplitude damping, we see that they differ on the final fixed state.

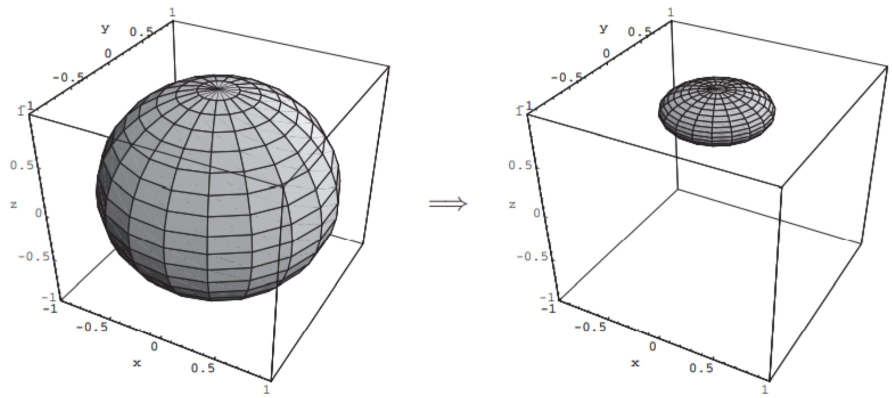


Figure 8.14. The effect of the amplitude damping channel on the Bloch sphere, for $p = 0.8$. Note how the entire sphere shrinks towards the north pole, the $|0\rangle$ state.

Phase Damping

A noise process that is uniquely quantum mechanical, which describes the loss of information without loss of energy is **phase damping**.
The energy eigenstates of a quantum system do not change as a function of time, but do accumulate a phase which is proportional to the eigenvalue. When a system evolves during an unknown amount of time, information about this relative phases is lost.

Suppose we have a qubit $|\psi\rangle a|0\rangle b|1\rangle$ upon which the rotation operation $R_z(\theta)$ is applied, there the angle of rotation θ is random. We shall call this random R_z operation a **phase kick**. Let us assume that the phase kick angle θ is well represented as a random variable which has a Gaussian distribution with mean 0 and variance 2λ.

The output state for this process is given by the density matrix obtained from averaging over θ,

$$\begin{aligned}\rho &= \frac{1}{\sqrt{4\pi\lambda}} \int_{-\infty}^{\infty} R_z(\theta)|\psi\rangle\langle\psi|R_z^\dagger(\theta) e^{-\theta^2/4\lambda} d\theta \\ &= \begin{pmatrix} |a|^2 & ab^* e^{-\lambda} \\ a^* b e^{-\lambda} & |b|^2 \end{pmatrix}\end{aligned}$$

So the random phase kicking causes the value of the off diagonal elements of the density matrix to decay exponentially to zero with time.

.

Applications of Quantum Operations

Master Equations

The main objective is to describe the time evolution of an open system with a differential equation which describes non unitary behaviour. This is provided by the **Lindblad form** of the **master equation:**

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] + \sum_j \left[2L_j\rho L_j^\dagger - \{L_j^\dagger L_j, \rho\}\right]$$

H is the Hamiltonian (Hermitian operator) and L_j are the **Lindblad operators**. It is also generally assumed that the system and environment begin in a product state.
To determine a master equation, one usually begins with a system-environment model Hamiltonian, and then makes the Born and Markov approximations to determine L_j. Note that $tr[\rho(t)] = 1$.

Example: Consider a two level atom coupled to the vacuum, undergoing spontaneous emission. The coherent part of the atom's evolution is described by Hamiltonian $H = -\hbar\omega\sigma_z/2$. $\hbar\omega$ is the difference in energy of the atomic levels.
Spontaneous emission causes an atom in the excited ($|1\rangle$) state to drop to the ground ($|0\rangle$) state, emitting a photon.
This is described by the Lindblad operator $\sqrt{\gamma}\sigma_-$, where $\sigma_- := |0\rangle\langle 1|$ is the atomic lowering operator, and γ is the rate of spontaneous emission. The master equation describing this process is:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] + \gamma\left[2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-\right]$$

Where $\sigma_+ := \sigma_-^\dagger$

To solve it, we make the change of variable:

$$\tilde{\rho}(t) := e^{iHt}\rho(t)e^{-iHt}$$

The equation becomes:

$$\frac{d\tilde{\rho}}{dt} = \gamma\left[2\tilde{\sigma}_-\tilde{\rho}\tilde{\sigma}_+ - \tilde{\sigma}_+\tilde{\sigma}_-\tilde{\rho} - \tilde{\rho}\tilde{\sigma}_+\tilde{\sigma}_-\right]$$

This is easily solved using the Bloch representation for $\tilde{\rho}$. The solution is:

$$\begin{aligned} \lambda_x &= \lambda_x(0)e^{-\gamma t} \\ \lambda_y &= \lambda_y(0)e^{-\gamma t} \\ \lambda_z &= \lambda_z(0)e^{-2\gamma t} + 1 - e^{-2\gamma t} \end{aligned}$$

Defining $\gamma' = 1 - \exp(-2t\gamma)$ we can easily check that this evolution is equivalent to:

$$\tilde{\rho}(t) = \epsilon(\tilde{\rho}(0)) := E_0\rho\widetilde{(0)}E_0^\dagger + E_1\tilde{\rho}(0)E_1^\dagger$$

Where:

$$E_0 := \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma'} \end{pmatrix} \quad , \quad E_1 := \begin{pmatrix} 0 & \sqrt{\gamma'} \\ 0 & 0 \end{pmatrix}$$

Are the operation elements.
The effect of ϵ is amplitude damping.

However, a quantum process described in terms of an operator-sum representation cannot necessarily be written down as a master equation. For example, quantum operations can describe non-Markovian dynamics, simply because they describe only state changes, not continuous time evolution. Nevertheless, each approach has its own place.

Quantum Process Tomography

State tomography is the procedure of experimentally determining an unknown quantum state. Suppose we are given an unknown state, ρ, of a single qubit. How can we determine what the state ρ is?

A quantum operation on a d-dimensional quantum system can be completely determined by experimentally measuring the output density matrices produced from d^2 pure state inputs

If we are given a single copy, it is impossible, since we cannot distinguish non ortogonal quantum states with certainty. But we can do it if we have a large number of copies of ρ. Then, the set $I/\sqrt{2}, X/\sqrt{2}, Y/\sqrt{2}, Z/\sqrt{2}$ forms an orthonormal set of matrices with respect to the Hilber Schmidt inner product $\langle A, B\rangle = Tr(A^\dagger B)$. so ρ can be expanded as:

$$\rho = \frac{tr(\rho)I + tr(X\rho)X + tr(Y\rho)Y + tr(Z\rho)Z}{2}$$

Recall however that $Tr(A\rho)$ has the interpretation as the average value of A when measuring a large number of times.
So we can estimate the coefficients making many measurements. This can be generalized for many qubit systems

Now that we know how to do quantum state tomography, how can we use it to do quantum process tomography? The experimental procedure may be outlined as follows. Suppose the state space of the system has d dimensions; for example, $d = 2$ for a single qubit. We choose d^2 pure quantum states $|\psi_1\rangle, \dots, |\psi_{d^2}\rangle$, chosen so that the corresponding density matrices $|\psi_1\rangle\langle\psi_1|, \dots, |\psi_{d^2}\rangle\langle\psi_{d^2}|$ form a *basis set* for the space of matrices. We explain in more detail how to choose such a set below. For each state $|\psi_j\rangle$ we prepare the quantum system in that state and then subject it to the process which we wish to characterize. After the process has run to completion we use quantum state tomography to determine the state $\mathcal{E}(|\psi_j\rangle\langle\psi_j|)$ output from the process. From a purist's point of view we are now done, since in principle the quantum operation $\mathcal{E}$ is now determined by a linear extension of $\mathcal{E}$ to all states.

In practice, of course, we would like to have a way of determining a useful representation of $\mathcal{E}$ from experimentally available data. We will explain a general procedure for doing so, worked out explicitly for the case of a single qubit. Our goal is to determine a set of operation elements $\{E_i\}$ for $\mathcal{E}$,

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger. \tag{8.150}$$

However, experimental results involve numbers, not operators, which are a theoretical concept. To determine the E_i from measurable parameters, it is convenient to consider an equivalent description of $\mathcal{E}$ using a *fixed* set of operators $\tilde{E}_i$, which form a basis for the set of operators on the state space, so that

$$E_i = \sum_m e_{im} \tilde{E}_m \tag{8.151}$$

for some set of complex numbers e_{im}. Equation (8.150) may thus be rewritten as

$$\mathcal{E}(\rho) = \sum_{mn} \tilde{E}_m \rho \tilde{E}_n^\dagger \chi_{mn}, \tag{8.152}$$

where $\chi_{mn} \equiv \sum_i e_{im} e_{in}^*$ are the entries of a matrix which is positive Hermitian by definition. This expression, known as the *chi matrix representation*, shows that $\mathcal{E}$ can be completely described by a complex number matrix, χ, once the set of operators E_i has been fixed.

8.2 From eqn (2.147) (on page 100),

$$\rho_m = \frac{M_m \rho M_m^\dagger}{Tr(M_m^\dagger M_m \rho)} = \frac{M_m \rho M_m^\dagger}{Tr(M_m \rho M_m^\dagger)} = \frac{\mathcal{E}_m(\rho)}{Tr\mathcal{E}_m(\rho)}.$$

And from eqn (2.143) (on page 99), $p(m) = Tr(M_m^\dagger M_m \rho) = Tr(M_m \rho M_m^\dagger) = Tr\mathcal{E}_m(\rho)$.