Weyl Quantum Channels

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Abstract

Many processes like dissipation and decoherence can be described using quantum channels. For systems of qubits, a particularly important set of channels is known as Pauli diagonal channels. In this work, we generalize this concept to d dimensional systems (qudits) and systems of many particles. We provide the set of conditions these channels must fulfill to be physically valid, find algebraic connections to group theory and understand the geometry of this set of channels. Finally, we consider Weyl component erasing (WCE) channels as a generalization of previous work on Pauli component erasing (PCE) channels.

Motivation: Pauli Diagonal Maps

Pauli diagonal maps define an important set of transformations of a density matrix representing one qubit. In this work we propose a generalization of them to systems of many particles with d dimensions.

Pauli Diagonal Maps

The density matrix for a single qubit can be written as

$$\rho = \frac{1}{2}\sum_{i=0}^{3} \alpha_i \sigma_i, \tag{1}$$

with $\sigma_0 = I$ and $\sigma_{1,2,3}$ the Pauli matrices. Normalization of ρ requires that $\alpha_0 = 1$. The other three components can be arranged into a vector that has to be inside a unit sphere called "Bloch sphere", shown in figure 1.

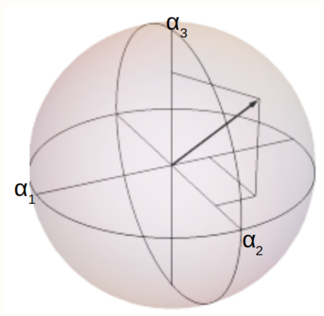

Figure 1: The Bloch sphere. Each point $(\alpha_1, \alpha_2, \alpha_3)$ corresponds to a density matrix of a qubit.

Pauli Diagonal Map: Map defined on a density matrix of one qubit as:

$$\alpha_i \to \tau_i \alpha_i. \tag{2}$$

That is, it applies the multiplier τ_i to each component α_i.

Generalization (Weyl Diagonal Maps)

We seek to generalize these channels for systems of d dimensions (qudits). For these systems, we can write a density matrix as

$$\rho = \frac{1}{d}\sum_{mn=0}^{d-1} \alpha_{mn} U_{mn}, \tag{3}$$

where U_{mn} are the Weyl matrices, a generalization of Pauli matrices:

$$U_{mn} = \sum_{j=0}^{d-1} \omega_d^{jm} |j\rangle\langle j \oplus n|, \quad \omega_d = e^{\frac{2\pi i}{d}}. \tag{4}$$

Weyl Diagonal Map: We define Weyl diagonal maps as a generalization of Pauli diagonal maps, given by

$$\alpha_{mn} \to \tau_{mn}\alpha_{mn}, \quad \tau_{mn} \in \mathbb{C}. \tag{5}$$

Composite System

For a system composed of N parts, each with arbitrary local dimension d_j, the density matrix can be written as

$$\rho = \frac{1}{D}\sum_{\vec{m}\vec{n}} \alpha_{\vec{m}\vec{n}} U_{\vec{m}\vec{n}}, \tag{6}$$

where $\vec{m}$ denotes a multi-index variable $(m_1, \cdots, m_N)$ with $m_j \in \{0, \cdots, d_j - 1\}$, and

$$U_{\vec{m}\vec{n}} = \bigotimes_{j=1}^{N} U_{m_j n_j}, \quad D = \prod_{j=1}^{N} d_j. \tag{7}$$

Then, we define a Weyl map on this system as

$$\alpha_{\vec{m}\vec{n}} \to \tau_{\vec{m}\vec{n}}\alpha_{\vec{m}\vec{n}}, \quad \tau_{\vec{m}\vec{n}} \in \mathbb{C}. \tag{8}$$

Conditions for Complete Positivity

For a map to be physically valid and be called a channel, it has to be completely positive (CP). A map is CP if and only if all the eigenvalues of its Choi matrix are real and non negative; this matrix for a Weyl map is found to be

$$\mathcal{D} = \frac{1}{D}\sum_{\vec{m}\vec{n}} \tau_{\vec{m}\vec{n}} U_{\vec{m}\vec{n}} \otimes U^*_{\vec{m}\vec{n}}. \tag{9}$$

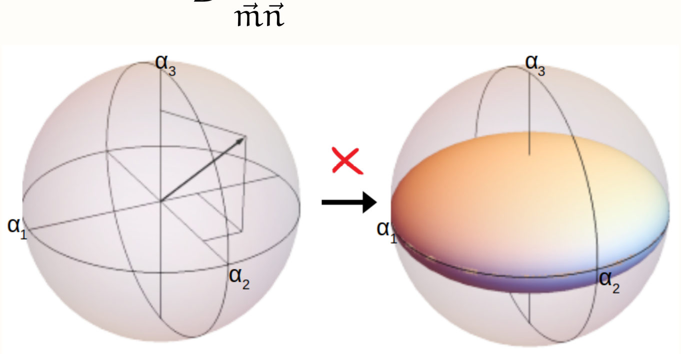

Figure 2: Example of a Pauli diagonal channel that is not completely positive.

Diagonalization

To test if the Choi matrix has real and non negative eigenvalues, we first diagonalize it and find that its eigenvalues are a linear function of the multipliers $\tau_{\vec{m}\vec{n}}$ given by:

$$\lambda_{\vec{\mu}\vec{\nu}} = \frac{1}{D}\sum_{\vec{m}\vec{n}} A_{\substack{\vec{\mu}\vec{\nu}\\ \vec{m}\vec{n}}} \tau_{\vec{m}\vec{n}}, \quad A_{\substack{\vec{\mu}\vec{\nu}\\ \vec{m}\vec{n}}} := \prod_{j=1}^{N} \omega_{d_j}^{\mu_j n_j - m_j \nu_j}. \tag{10}$$

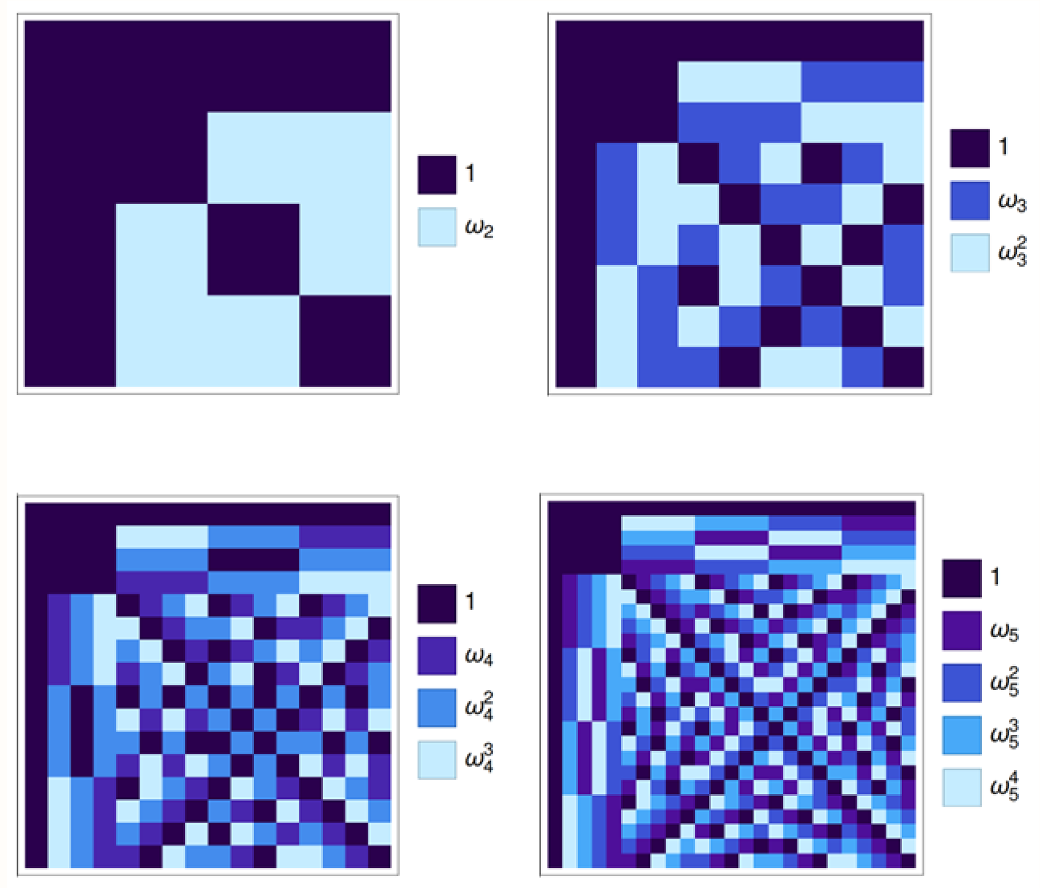

Figure 3: Plots of $A_{\substack{\mu\nu\\ mn}} = \omega_d^{\mu n - m\nu}$ for $d = 2, 3, 4, 5$. For a d dimensional system of one particle, this matrix transforms the multipliers τ_{mn} into the eigenvalues $\lambda_{\mu\nu}$.

Geometry of Weyl Maps

Not every operator defined by equation (8) is a quantum channel, since some violate CP conditions. For example, for Pauli channels on one qubit, it is known that the conditions are

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} \geqslant 0, \tag{11}$$

which implies that the multipliers τ_1, τ_2, τ_3 must be inside the tetrahedron shown in figure 4.

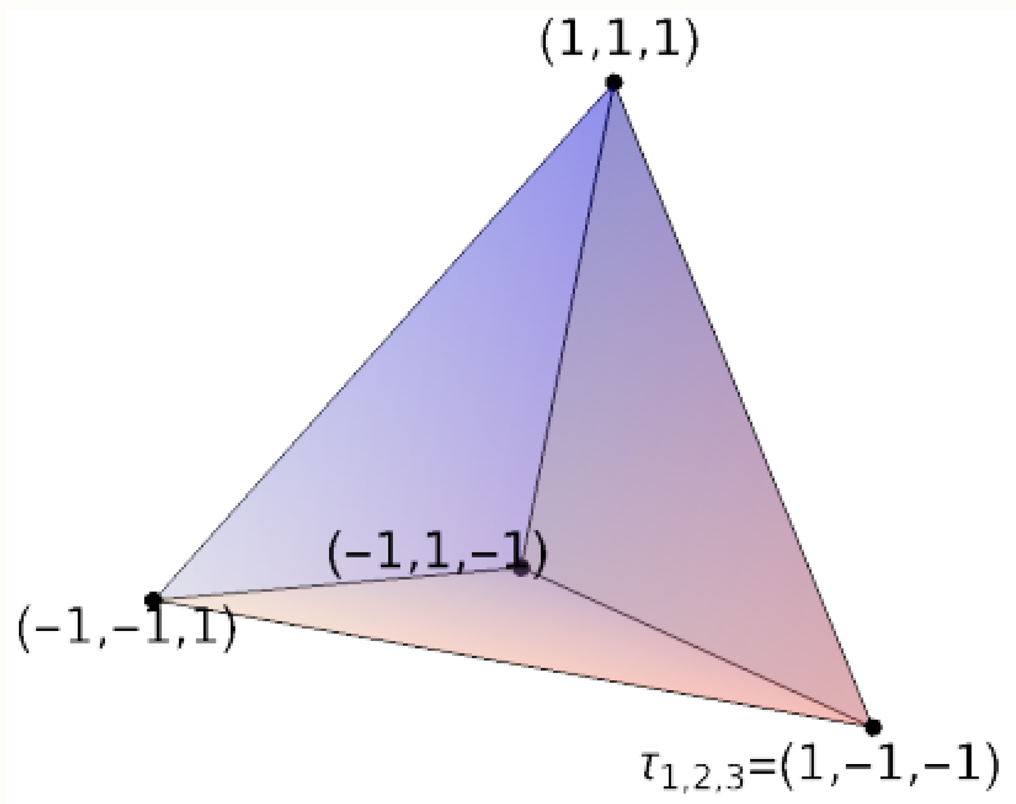

Figure 4: Only the values of the multipliers τ_1, τ_2, τ_3 inside the tetrahedron correspond to maps that satisfy conditions 11 and are therefore CP. The vertices are unitary channels that apply the matrices I, σ_1, σ_2 or σ_3 to the qubit.

For Weyl maps, the CP conditions take the form:

$$\sum_{\vec{m}\vec{n}} \left[\mathrm{Re}\left(A_{\substack{\vec{\mu}\vec{\nu}\\ \vec{m}\vec{n}}} \right) - \mathrm{Im}\left(A_{\substack{\vec{\mu}\vec{\nu}\\ \vec{m}\vec{n}}} \right) \right] \sigma_{\vec{m}\vec{n}} \geqslant 0 \tag{12}$$

where $\sigma_{\vec{m}\vec{n}} := \mathrm{Re}(\tau_{\vec{m}\vec{n}}) + \mathrm{Im}(\tau_{\vec{m}\vec{n}})$. These conditions define a polytope in the space of variables $\sigma_{\vec{m}\vec{n}}$ that generalizes the tetrahedron of figure 4.

Algebraic Property

From the CP condition, it can be shown that given any two multipliers $\tau_{\vec{m}\vec{n}}, \tau_{\vec{m}'\vec{n}'}$ such that $|\tau_{\vec{m}\vec{n}}| = |\tau_{\vec{m}'\vec{n}'}| = 1$, it follows that

$$\tau_{\vec{m}\oplus\vec{m}', \vec{n}\oplus\vec{n}'} = \tau_{\vec{m}\vec{n}}\tau_{\vec{m}'\vec{n}'}, \tag{13}$$

where $\vec{m} \oplus \vec{m}' = (m_1 + m_1' \pmod{d_1}, \cdots, m_N + m_N' \pmod{d_N})$.

MNU Channels

We define a subset of Weyl channels called MNU channels as those in which $|\tau_{\vec{m}\vec{n}}| = 1$ for all $(\vec{m}, \vec{n})$. That is, the channels act by only giving a phase to each component $\alpha_{\vec{m}\vec{n}}$. Using property 13, it is shown that MNU channels are exactly those in the vertices of the polytope of Weyl diagonal channels and therefore every Weyl channel is a convex sum of MNU channels.

WCE channels

We generalize PCE channels to d level systems as those where

$$\tau_{\vec{m}\vec{n}} = 0, 1. \tag{14}$$

That is, channels that preserve or delete components of the density matrix. Property 13 implies that

$$\text{if } \tau_{\vec{m}\vec{n}} = 1 \text{ and } \tau_{\vec{m}'\vec{n}'} = 1 \Rightarrow \tau_{\vec{m}\oplus\vec{m}', \vec{n}\oplus\vec{n}'} = 1. \tag{15}$$

This means that the set of indices with multiplier equal to 1 is closed under addition, that is, it is a subgroup of the group of all indices. Therefore, the indices with multipliers equal to 1 cannot be chosen arbitrarily. Figure 5 shows the set of WCE channels for one particle with $d = 6$.

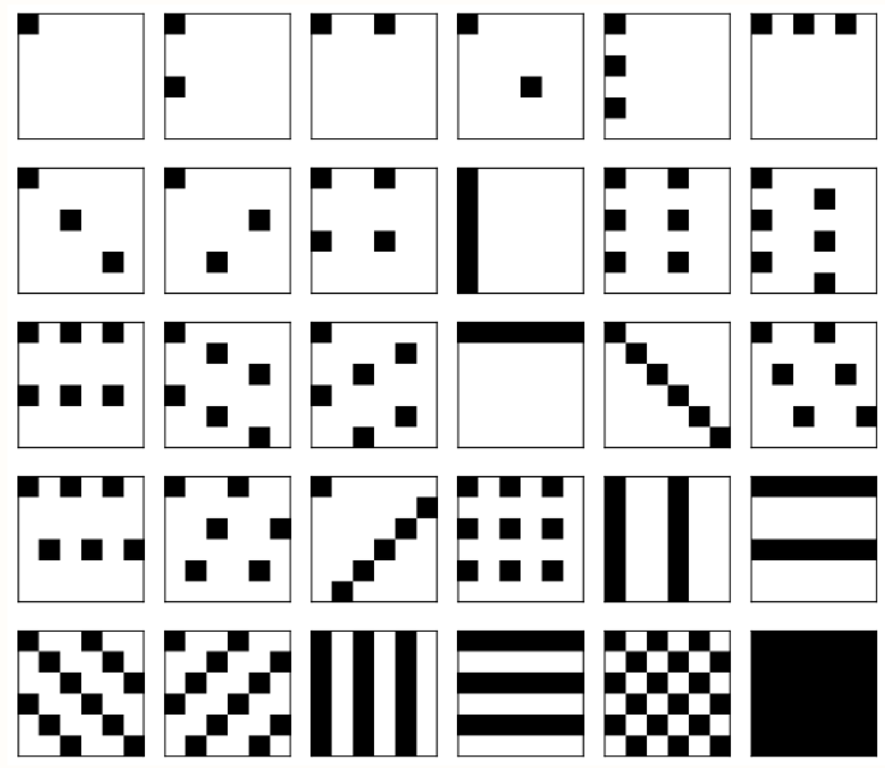

Figure 5: Each of the 30 squares represents one of the 30 possible WCE channels for a particle with $d = 6$. A WCE channel is represented by a 6×6 grid, where a black square in position (m, n) indicates that $\tau_{mn} = 1$ and a white one indicates that $\tau_{mn} = 0$. Out of all possible WCE maps, only these ones satisfy equation 15 and are therefore CP. That is, they are the subgroups of the set of all indices (m, n) for $m, n \in \{0, 1, \cdots, 5\}$.

Generators

Some of these channels can be constructed as the composition of other WCE channels. We seek to find the set of WCE channels whose compositions generate all other WCE channels, which we call generators.

Theorem of generators: A WCE channel is a generator if and only if the set of indices for which $\tau_{\vec{m}\vec{n}} = 1$ has the form $S_{\vec{\mu}\vec{\nu}} := \{(\vec{p}, \vec{q}) \,|\, \vec{p}\cdot\vec{\nu} - \vec{q}\cdot\vec{\mu} = 0 \pmod{D}\}$ for some $(\vec{\mu}, \vec{\nu})$ and $D^2/|S_{\vec{\mu}\vec{\nu}}|$ is a prime power (where $\vec{p}\cdot\vec{v} := p_1 v_1 + p_2 v_2 + \cdots + p_N v_N$).

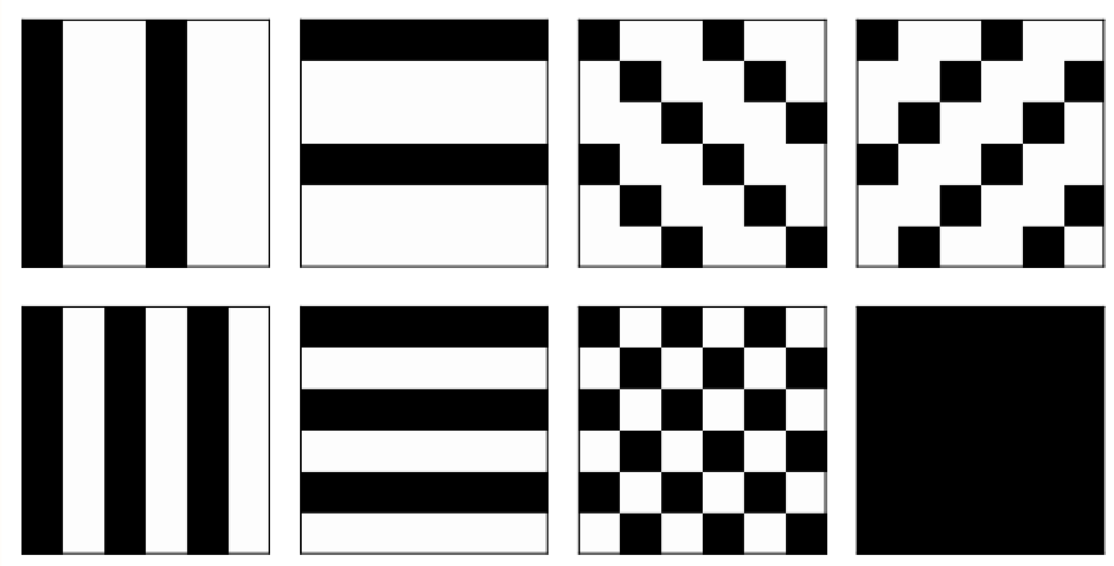

Figure 6: Out of the 30 WCE channels for a qudit with $d = 6$ in figure 5, these are the ones that satisfy the conditions of the theorem of generators. That is, they are the generators and therefore any WCE channel for a qudit with $d = 6$ can be constructed by composing these channels.

Conclusions

- We introduced a new type of channels that generalize Pauli diagonal channels to systems of multiple qudits.
- We diagonalized analitically the Choi matrix of those channels and found the conditions for complete positivity.
- We found the extreme points of the set of these channels.
- We defined WCE channels as a generalization of PCE and studied their algebraic structure.

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