Tarea 3 Cálculo

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Haaser p.618

2e. Evalúese: $\displaystyle\int_0^3 x^2\sqrt{1+x}\ dx$

Sustitución: Sea $u=\sqrt{1+x}$, Entonces $du=\frac{1}{2\sqrt{1+x}}=\frac{dx}{2u}\to dx=2u\ du$
Además, $u^2=1+x$, luego: $x^2=(u^2-1)^2=u^4-2u^2+1$

$$\int_0^3 x^2\sqrt{1+x}\ dx=\int_0^3(u^4-2u^2+1)u(2u\ du)\ =\int_0^3(u^5-2u^3+u)(2u\ du)=$$
$$\int_0^3(2u^6-4u^4+2u^2)\ du=2\frac{1}{7}u^7-4\frac{1}{5}u^5+2\frac{1}{3}u^3\Big|_0^3=\frac{2}{7}\sqrt{1+x}^7-\frac{4}{5}\sqrt{1+x}^5+\frac{2}{3}\sqrt{1+x}^3\Big|_0^3$$
$$=\frac{2}{7}2^7-\frac{4}{5}2^5+\frac{2}{3}2^3-\frac{2}{7}+\frac{4}{5}-\frac{2}{3}=\frac{1096}{105}\approx 16{,}15$$

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3b. Demuéstrese que: $\displaystyle\int \sec^n x\ dx$

Por partes: $u=\sec^{n-2}x\to du=(n-2)\sec^{n-2}x\ \tan x\ ,\ dv=\sec^2 x\ dx\to v=\tan x$
$$\int\sec^n x\ dx=\sec^{n-2}x\ \tan x-\int(n-2)\sec^{n-2}x\ \tan^2 x\ dx$$
$$\int\sec^n x\ dx=\sec^{n-2}x\ \tan x-(n-2)\int\sec^{n-2}x(\sec^2 x-1)\ dx$$
$$\int\sec^n x\ dx=\sec^{n-2}x\ \tan x-(n-2)\int\sec^n x\ dx+(n-2)\int\sec^{n-2}x\ dx$$
$$\Rightarrow\int\sec^n x\ dx+(n-2)\int\sec^n x\ dx=\sec^{n-2}\ \tan x+(n-2)\int\sec^{n-2}x\ dx$$
$$\Rightarrow\int\sec^n x\ dx=\frac{1}{n-1}\sec^{n-2}x\tan x+\frac{n-2}{n-1}\int\sec^{n-2}x\ dx$$
$$=\frac{\operatorname{sen}x}{(n-1)\cos^{n-1}x}+\frac{n-2}{n-1}\int\sec^{n-2}x\ dx$$

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8a. Demuéstrese que el área de un un círculo de radio r es πr^2

Consideramos la cuarta parte de la circunferencia, ubicada en el primer cuadrante. De la ecuación de una circunferencia: $x^2 + y^2 = r^2 \rightarrow y^2 = r^2 - x^2$
Como $y \geq 0$ en este sector, $y = \sqrt{r^2 - x^2}$
Entonces $f(x) = \sqrt{r^2 - x^2}$ es la función a integrar, de 0 a r.
$\frac{1}{4}A(C) = \int_0^r \sqrt{r^2 - x^2}\, dx$
Sustitución trigonométrica: $x = r\operatorname{sen}(u) \rightarrow u = arcsen(\frac{x}{r}), dx = r\cos(u)$

$$\int_0^r \sqrt{r^2 - x^2} = \int_0^{\pi/2} \sqrt{r^2 - r^2\operatorname{sen}^2(u)}(r\cos(u))\, du = \int_0^{\pi/2} \sqrt{r^2(1 - \operatorname{sen}^2(u))}(r\cos(u))\, du =$$

$$\int_0^{\pi/2} r^2\sqrt{\cos^2(u)}\cos(u)\, du = \int_0^{\pi/2} r^2|\cos(u)|\cos(u)\, du$$

Como $\cos u > 0$ cuando $u \in [0, \frac{\pi}{2}]$ Entonces $\frac{1}{4}A(C) = \int_0^{\pi/2} r^2\cos^2 u\, du$

$$\frac{1}{4}A(C) = r^2\int_0^{\pi/2} \frac{1}{2} + \frac{\cos(2u)}{2}\, du = r^2\left(\frac{1}{2}u + \frac{\operatorname{sen}(2u)}{4}\right)\Bigg|_0^{\pi/2} = r^2(\frac{\pi}{4}) = \frac{1}{4}\pi r^2$$

$\therefore\ A(C) = \pi r^2$

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8b. Demuéstrese que el el área de un sector circular de ángulo α y radio r es $\frac{1}{2}r^2\alpha$.

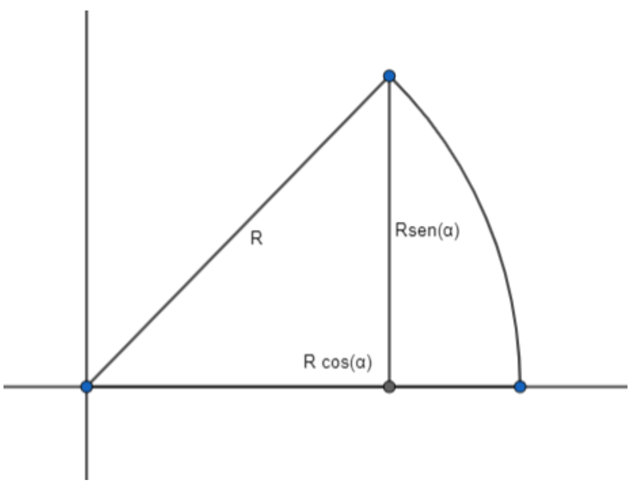

Figura 1: Sector circular

El área de un sector circular puede dividirse en dos partes, un triángulo de base $r\cos\theta$ y altura $r\operatorname{sen}\theta$, y un área curva. El área total, será la suma de ambas.

Para hallar el área del tríangulo, sabemos que:
$A_t = \frac{b\times h}{2} = \frac{r\cos\theta \times r\operatorname{sen}\theta}{2}$

Ahora hallaremos el área curva.
Por la ecuación de una circunferencia, sabemos que $r^2 = x^2 + y^2 \Rightarrow y = \sqrt{r^2 - x^2}$
Integramos: $\int_{r\cos\theta}^{r} \sqrt{r^2 - x^2}\, dx$

Resolvemos por cambio de variable trigonométrica.
Sea $\cos t = \frac{x}{r} \Rightarrow x = r\cos t \Rightarrow dx = -r \operatorname{sen} t \; dt$ y cambiamos de intervalo: para $x = r \Rightarrow \cos t = 1 \Rightarrow 0 = t$ y para $x = r\cos\theta \Rightarrow \cos\theta = \cos t \Rightarrow \theta = t$

$$\Rightarrow \int_{r\cos\theta}^{r} \sqrt{r^2 - x^2}\; dx = \int_{\theta}^{0} \sqrt{r^2 - r^2\cos^2 t}\; (-r\operatorname{sen} t)dt = \int_{0}^{\theta} \sqrt{r^2 - r^2\cos^2 t}\; (r\operatorname{sen} t)dt = \int_{0}^{\theta} \sqrt{r^2(1-\cos^2 t)}\; (r\operatorname{sen} t)dt$$
$$= \int_{0}^{\theta} \sqrt{r^2\operatorname{sen}^2 t}\; (r\operatorname{sen} t)dt = \int_{0}^{\theta} r\sqrt{\operatorname{sen}^2 t}\; (r\operatorname{sen} t)dt = \int_{0}^{\theta} r|\operatorname{sen} t| r\operatorname{sen} t\; dt = r^2\int_{0}^{\theta} \operatorname{sen}^2 t\; dt = r^2\int_{0}^{\theta}\left(\frac{1}{2} - \frac{1}{2}\cos 2t\right)dt$$
$$= r^2\left(\frac{1}{2}t - \frac{1}{4}\operatorname{sen} 2t\right)\Big|_0^\theta = r^2\left(\frac{1}{2}\theta - \frac{1}{4}\operatorname{sen} 2\theta\right) = r^2\left(\frac{1}{2}\theta - \frac{1}{4}2\operatorname{sen}\theta\cos\theta\right) = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\operatorname{sen}\theta\cos\theta$$
$$\therefore\; A = \frac{1}{2}r^2\operatorname{sen}\theta\cos\theta + \frac{1}{2}r^2\theta - \frac{1}{2}r^2\operatorname{sen}\theta\cos\theta$$
$$\therefore\; A = \frac{1}{2}r^2\theta$$

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8c. Demuéstrese que el área de una elipse con semiejes a y b es de πab

Sabemos que la ecuación de una elipse es $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Gracias a la simetría de la elipse, podemos calcular el área de uno de los cuadrantes de la elipse y multiplicarla por 4, por lo que sólo tomaremos la ecuación para $x \geq 0,\; y \geq 0$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \;\Rightarrow\; \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \;\Rightarrow\; y = b\sqrt{1 - \frac{x^2}{a^2}} \;=\; f(x)$
Entonces ahora integramos:
$\frac{A}{4} = \int_0^a b\sqrt{1 - \frac{x^2}{a^2}}dx$ Y usando el cambio de variable, sea $x = a\;\cos u \;\Rightarrow\; u = \arccos\frac{x}{a}$ y $dx = -a\;\operatorname{sen} u du$, donde $x : 0 \Rightarrow a$ y $u : \frac{\pi}{2} \Rightarrow 0$

$$\Rightarrow \int_0^a b\sqrt{1-\frac{x^2}{a^2}}\; dx = \int_{\frac{\pi}{2}}^{0} b\sqrt{1 - \frac{a^2cos^2u}{a^2}}\;(-a)\operatorname{sen} u\; du = -ab\int_{\frac{\pi}{2}}^{0}\sqrt{1 - \frac{a^2cos^2u}{a^2}}\;\operatorname{sen} u\; du = ab\int_0^{\frac{\pi}{2}}\sqrt{1 - \frac{a^2cos^2u}{a^2}}\;\operatorname{sen} u\; du$$
$$= \int_0^{\frac{\pi}{2}} \frac{1-\cos 2u}{2}\; du = ab\left[\frac{u}{2} - \frac{sen2u}{4}\right]_0^{\frac{\pi}{2}} = \frac{ab\pi}{4}$$
Ahora este resultado debemos multiplicarlo por 4 y entonces nos queda finalmente que: $A = \frac{ab\pi}{4}\;(4)$
$\therefore\; A = ab\pi$

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Haaser p.724

2c. Encúentrese la siguiente integral: $\int \sqrt{x} \ln x \ dx$

Por partes:

$u = \ln x \to du = \frac{1}{x}, dv = \sqrt{x} \to v = \frac{2}{3}x^{3/2}$
$= \frac{2}{3}x^{3/2}\ln x - \int \frac{2}{3}x^{3/2}\frac{1}{x} \ dx = \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int \sqrt{x} \ dx = \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C$

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2d. Encúentrese la siguiente integral: $\int x^2 \arctan x \ dx$

Por partes: $u = \arctan x \to du = \frac{1}{1+x^2}dx, dv = x^2 \to v = \frac{x^3}{3}$
$= \frac{x^3}{3}\arctan x - \frac{1}{3}\int \frac{x^3}{1+x^2} \ dx = \frac{1}{3}x^3 \arctan x - \frac{1}{3}\int \frac{x^3 + x - x}{1+x^2} \ dx = \frac{1}{3}x^3 \arctan x - \frac{1}{3}\int \frac{x(1+x^2) - x}{1+x^2} \ dx = \frac{1}{3}x^3 \arctan x - \frac{1}{3}\int x - \frac{x}{1+x^2} \ dx = \frac{1}{3}x^3 \arctan - \frac{1}{3}\left(\frac{1}{2}x^2\right) + \frac{1}{3}\int \frac{x}{1+x^2}dx = \frac{1}{3}x^3 \arctan x - \frac{1}{6}x^2 + \frac{1}{6}\int \frac{2x}{1+x^2}dx$
$= \frac{1}{3}x^3 \arctan x - \frac{1}{6}x^2 + \frac{1}{6}\ln(1+x^2) + C$

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2m. Encúentrese la siguiente integral: $\int_0^\infty e^{-x}\cos x \ dx$

$$\int_0^\infty e^{-x}\cos x \ dx = -\cos x e^{-x} - \int_0^\infty (-\operatorname{sen} x)(-e^{-x}) \ dx$$
$$= -\cos x e^{-x} - \int_0^\infty \operatorname{sen} x e^{-x} \ dx$$
$$= -\cos x e^{-x} - (-e^{-x}\operatorname{sen} x - \int_0^\infty (-e^{-x})(\cos x) \ dx)$$
$$= -\cos x e^{-x} + e^{-x}\operatorname{sen} x - \int_0^\infty e^{-x}(\cos x) \ dx$$
$$\Rightarrow \ 2\int_0^\infty e^{-x}(\cos x) \ dx = -\cos x e^{-x} + e^{-x}\operatorname{sen} x$$
$$\Rightarrow \int_0^\infty e^{-x}(\cos x) \ dx = e^{-x}\left(\frac{\operatorname{sen} x - \cos x}{2}\right)\Bigg|_0^\infty$$
$$= \lim_{a\to\infty} e^{-x}\left(\frac{\operatorname{sen} x - \cos x}{2}\right)\Bigg|_0^a = \lim_{a\to\infty} e^{-a}\left(\frac{\operatorname{sen} a - \cos a}{2}\right) - (-\frac{1}{2})$$
$$= 0 + \frac{1}{2} = \frac{1}{2}$$

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3d. Obténganse la siguiente fórmulas de reducción (n es un entero positivo) $\int \cos^n x\ dx$

por partes: $u = cos^{n-1} x \rightarrow du = -(n-1)\cos^{n-2} x \operatorname{sen} x, dv = \cos x \rightarrow v = \operatorname{sen} x$

$$= \cos^{n-1} x \operatorname{sen} x + \int (n-1)\cos^{n-2} x \operatorname{sen}^2 x\ dx$$

$$= \cos^{n-1} x \operatorname{sen} x + (n-1)\int \cos^{n-2} x (1-\cos^2 x)\ dx$$

$$= \cos^{n-1} x \operatorname{sen} x + (n-1)\int \cos^{n-2} x\ dx - (n-1)\int \cos^n x\ dx$$

$$\Rightarrow n \int \cos^n x\ dx = \cos^{n-1} x \operatorname{sen} x + (n-1)\int \cos^{n-2} x\ dx$$

$$\Rightarrow \int \cos^n x\ dx = \frac{1}{n}\cos^{n-1} x \operatorname{sen} x + \frac{n-1}{n}\int \cos^{n-2} x\ dx$$

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Haaser p.738

5o. Encuéntrese la siguiente integral: $\int_0^1 \frac{5x^2+6x+17}{(x^2+x+1)^2(2-x)}$

Por fracciones parciales: $\dfrac{5x^2+6x+17}{(x^2+x+1)^2(2-x)} = \dfrac{Ax+B}{x^2+x+1} + \dfrac{Cx+D}{(x^2+x+1)^2} + \dfrac{E}{2-x} =$

$\dfrac{(Ax+b)(x^2+x+1)(2-x)+(Cx+D)(2-x)+E(x^2+x+1)^2}{(x^2+x+1)^2(2-x)} =$

$= \frac{2Ax^3+2Ax^2+2Ax+2Bx^2+2Bx+2B-Ax^4-Ax^3-Ax^2-Bx^3-Bx^2-Bx-Cx^2+2Cx-Dx+2D+Ex^4+2Ex^3+3Ex^2+2Ex+E}{(x^2+x+1)^2(2-x)}$

$= \frac{(E-X)x^2+(A-B+2E)x^3+(A+B-C+2E)x^2+(2A+B+2C-D+2E)x+(2B+2D+E)}{(x^2+x+1)^2(2-x)}$

Por lo tanto:

$E - A = 0$

$A - B + 2E = 0$

$A + B - C + 3E = 5$

$2A + B + 2C - D + 2E = 6$

$2B + 2D + E = 17$

Por la primera ecuación: A=E, lo cual al sustituir en la segunda nos da, B=3A. Al sustituir todo esto en la tercera, obtenemos C=7A-5. Sustituimos ahora estos resultados en la última ecuación para obtener: D=(17-7A)/2.

Y finalmente por al cuarta ecuación se obtiene que A=1.

Y entonces las soluciones son: $A = 1, B = 3, C = 2, D = 5, E = 1$

Por lo tanto la integral es:

$$\int \frac{x+3}{x^2+x+1}dx \ \ (1) + \int \frac{2x+5}{(x^2+x+1)^2}dx \ \ (2) + \int \frac{1}{2-x}dx \ \ (3)$$

$(1) = \displaystyle\int \frac{x+3}{x^2+x+1}dx = \int \frac{2x+2}{2(x^2+x+1)}dx + \frac{5}{2}\int \frac{dx}{x^2+x+1} = \frac{1}{2}\ln|x^2+x+1| + \frac{5}{2}\int \frac{dx}{(x+1/2)^2+3/4} = \frac{1}{2}\ln|x^2+x+1| + \frac{5}{\sqrt{3}}\arctan(\frac{2x+1}{\sqrt{3}})$

(2) $= \dfrac{2x+5}{(x^2+x+1)^2}dx = \displaystyle\int \frac{2x+1}{(x^2+x+1)^2} + 4\int \frac{dx}{(x^2+x+1)^2} = \int \frac{2x+1}{(x^2+x+1)^2}dx +$ $4\displaystyle\int \frac{dx}{((x+1/2)^2+3/4)^2} = -\frac{1}{x^2+x+1} + \frac{16\arctan(\frac{2x+1}{\sqrt{3}}}{3\sqrt{3}} + \frac{16(2x+1)}{3((2x+1)^2+3)}$ (Por la fórmula de reducción)

(3) $= \displaystyle\int \frac{dx}{2-x} = -\ln|2-x|$

$\therefore$ La integral es igual a:

$\dfrac{1}{2}\ln|x^2+x+1| + \dfrac{5}{\sqrt{3}}\arctan\left(\dfrac{2x+1}{\sqrt{3}}\right) - \dfrac{1}{x^2+x+1} + \dfrac{16}{3\sqrt{3}}\arctan\left(\dfrac{2x+1}{\sqrt{3}}\right) + \dfrac{16(2x+1)}{3((2x+1)^2+3)} -$ $\ln|2-x|\Big|_0^1$

$= \dfrac{1}{2}\ln|3| + \dfrac{5}{\sqrt{3}}\arctan\left(\dfrac{3}{\sqrt{3}}\right) - \dfrac{1}{3} + \dfrac{16}{3\sqrt{3}}\arctan\left(\dfrac{3}{\sqrt{3}}\right) + \dfrac{4}{3} - \ln|1| -$

$\left(\dfrac{1}{2}\ln|1| + \dfrac{31}{3\sqrt{3}}\arctan\left(\dfrac{1}{\sqrt{3}}\right) - 1 + \dfrac{4}{3} - \ln|2|\right)$

$= 7{,}7968 - 2{,}7639 \approx 5{,}0328$

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Haaser P.743

5c. Use sustitución estereográfica para encontrar: $\int \frac{dx}{1+\operatorname{sen}x+\cos x}$

Sustitución estereográfica: $t = \tan(\frac{x}{2}), \cos x = \frac{1-t^2}{1+t^2}, \operatorname{sen}x = \frac{2t}{1+t^2}, dx = \frac{2}{1+t^2}dt$

$= \displaystyle\int \frac{\frac{2}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}dt = 2\int \frac{\frac{1}{1+t^2}}{\frac{t^2+1+2t-t^2+1}{1+t^2}} = 2\int \frac{1}{2t+2} = \int \frac{1}{t+1} = \ln|t+1| + C =$ $\ln|\tan(\dfrac{x}{2})+1| + C$

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Haaser p.747

1w. Encuentre la siguiente integral: $\int \frac{\sqrt{3+x^2}}{\sqrt{2+x^2}}x\ dx$

Sustitución: Sea $u = \sqrt{x^2+3} \to x = \sqrt{u^2-3}, dx = \frac{u}{\sqrt{u^2-3}}du$

$\displaystyle\int \frac{u}{\sqrt{2+(u^2-3)}}\sqrt{u^2-3}\left(\frac{u}{\sqrt{u^2-3}}\right)\ du = \int \frac{u^2}{\sqrt{u^2-1}}\ du$

Sea $u = \sec(t) \to t = arcsec(u), du = \sec t\tan t\ dt$

$= \displaystyle\int \frac{\sec^2 t}{\sqrt{\sec^2 t-1}}\sec t\tan t\ dt = \int \sec^3 t\ dt = \frac{1}{2}(\sec t\tan t + \ln|\sec t+\tan t|) + C$

$= \dfrac{1}{2}(u\sqrt{u^2-1} + \ln|u+\sqrt{u^2-1}|) + C$

$= \dfrac{\sqrt{x^2+3}\sqrt{x^2+2} + \ln|\sqrt{x^2+3}+\sqrt{2+x^2}|}{2} + C$

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Spivak p 524

1i. Resuelva la siguiente integral $\int \frac{\sqrt[5]{x^3}+\sqrt[6]{x}}{\sqrt{x}}dx$

$$= \int \frac{x^{1/5}+x^{1/6}}{x^{1/2}}dx = \int \frac{x^{3/5}}{x^{1/2}}\ dx + \int \frac{x^{1/6}}{x^{1/2}}dx = \int x^{3/5-1/2}\ dx + \int x^{1/6-1/2}\ dx =$$
$$\int x^{1/10}\ dx + \int x^{-1/3}\ dx = \frac{10}{11}x^{11/10} + \frac{3}{2}x^{2/3} + C$$

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1 ii. Resolver: $\int \frac{dx}{\sqrt{x-1}+\sqrt{x+1}}$

$$\int \frac{dx}{\sqrt{x-1}+\sqrt{x+1}}\left(\frac{\sqrt{x-1}-\sqrt{x+1}}{\sqrt{x-1}-\sqrt{x+1}}\right)dx = \int \frac{\sqrt{x-1}-\sqrt{x+1}}{(x-1)-(x+1)}dx = \int \frac{\sqrt{x-1}-\sqrt{x+1}}{-2}dx =$$
$$-\frac{1}{2}\int \sqrt{x-1}\ dx + \frac{1}{2}\int \sqrt{x+1}\ dx = -\frac{1}{3}(x-1)^{3/2} + \frac{1}{3}(x+1)^{3/2} + C$$

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3 vi. Resolver la siguiente integral: $\int \frac{\ln(\ln x)}{x}dx$

Por partes: Sea $u = \ln(\ln x), dv = \frac{1}{x}dx$, entonces $du = \frac{1}{\ln x}\frac{1}{x}dx = \frac{1}{x\ln x}dx, v = \ln|x|$
Como $\ln x$ está definida en $\mathbb{R}^+$, $\frac{\ln|x|}{\ln x} = 1 \forall x \in \mathbb{R}^+$
$$\Rightarrow \int \frac{\ln(\ln x)}{x}dx = \ln|x|\ln(\ln x) - \int \frac{1}{x}dx = \ln|x|\ \ln(\ln x) - \ln|x| = \ln|x|\ (\ln(\ln x) - 1)$$

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5 x. Resolver la siguiente integral: $\int \frac{\sqrt{x-1}}{\sqrt{x+1}}\frac{1}{x^2}\ dx$

$\int \sqrt{\frac{x-1}{x+1}}\frac{1}{x^2}dx = \int \frac{\sqrt{x-1}}{x^2\sqrt{x+1}}dx = \int \frac{2u^2}{(u^2+1)^2\sqrt{u^2+2}}du$ Con $u = \sqrt{x+1} \to dx = 2u\ du$
$= 2\int \frac{u^2}{(u^2+1)^2\sqrt{u^2+2}}du$
Resolviendo ahora:
$\int \frac{u^2}{(u^2+1)^2\sqrt{u^2+2}}du = \int \frac{2^{\frac{3}{2}}\sec^2(v)\tan^2(v)}{(2\tan^2(v)+1)^2\sqrt{2\tan^2(v)+2}}dv$ Con $u = \tan(v), du = \sec^2(v)dv$
$= 2\int \frac{\sec(v)\tan^2(v)}{(2\tan^2(v)+1)^2}dv$
Resolviendo ahora:
$\int \frac{sec(v)\tan^2(v)}{(2\tan^2(v)+1)^2}dv = \int \cos(v)\frac{\sin^2(v)}{(\sin^2(v)+1)^2}dv$
$= \int \frac{w^2}{(w^2+1)^2}dw$ con $w = \text{sen}(v), dw = cos(v)dv$
$= \int (\frac{w^2+1}{(w^2+1)^2} - \frac{1}{(w^2+1)^2})dw$
$= \int (\frac{1}{w^2+1} - \frac{1}{(w^2+1)^2})dw$

$= \int \frac{1}{w^2+1}dw - \int \frac{1}{(w^2+1)^2}dw$
Ahora bien:
$\int \frac{1}{w^2+1}dw = \arctan(w)$
y además:
$\int \frac{1}{(w^2+1)^2}dw = \frac{w}{2(w^2+1)^2} + \frac{1}{2}\int \frac{1}{w^2+1}dw$ (por fórmula de reducción)

Entonces:
$\frac{w}{2(w^2+1)^2} + \frac{1}{2}\int \frac{1}{w^2+1}dw = \frac{\arctan(w)}{2} + \frac{w}{2(w^2+1)}$

Remplazamos las integrales ya hechas:
$\int \frac{1}{w^2+1}dw - \int \frac{1}{(w^2+1)^2}dw = \frac{\arctan(w)}{2} - \frac{w}{2(w^2+1)} = \frac{\arctan(\sin(v))}{2} - \frac{\sin(v)}{2(\sin^2(v)+1)}$

Seguimos remplazamos las integrales ya hechas:
$2\int \frac{sec(v)\tan^2(v)}{(2\tan^2(v)+1)^2}dv = \arctan(\sin(v)) - \frac{\sin(v)}{\sin^2(v)+1} = \arctan(\frac{u}{\sqrt{2}\sqrt{\frac{u^2}{2}+1}}) - \frac{u}{\sqrt{2}\sqrt{\frac{u^2}{2}+1}(\frac{u^2}{2(\frac{u^2}{2}+1)}+1)}$

Remplazamos por última vez:
$2\int \frac{u^2}{(u^2+1)^2\sqrt{u^2+2}}du = 2\arctan(\frac{u}{\sqrt{2}\sqrt{\frac{u^2}{2}+1}}) - (\frac{\sqrt{2}u}{\sqrt{\frac{u^2}{2}+1}(\frac{u^2}{2(\frac{u^2}{2}+1)}+1)})$
$= 2\arctan(\frac{\sqrt{x-1}}{\sqrt{2}\sqrt{\frac{x-1}{2}+1}}) - (\frac{\sqrt{2}\sqrt{x-1}}{\sqrt{\frac{x-1}{2}+1}(\frac{x-1}{2(\frac{x-1}{2}+1)}+1)})$
El problema queda entonces resuelto:
$\int \frac{\sqrt{x-1}}{x^2\sqrt{x+1}}dx = 2\arctan(\frac{\sqrt{x-1}}{\sqrt{2}\sqrt{\frac{x-1}{2}+1}}) - (\frac{\sqrt{2}\sqrt{x-1}}{\sqrt{\frac{x-1}{2}+1}(\frac{x-1}{2(\frac{x-1}{2}+1)}+1)}) + C$

Lo cual se simplifica a:
$2\arctan\left(\frac{\sqrt{x-1}}{\sqrt{x+1}}\right) - \frac{\sqrt{x-1}\sqrt{x+1}}{x} + C$

$\blacksquare$

7 ix. Resuelva: $\int \sqrt{\tan x}\, dx$

Sea $t^2 = \tan x, x = \arctan(t^2), dx = \frac{2t}{1+t^4}$
$$= \int t\frac{2t}{1+t^4}dt = 2\int \frac{t^2}{1+t^4}dt = \int \frac{2}{t^2+\frac{1}{t^2}}dt = \int \frac{1+\frac{1}{t^2}+1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt = \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt + \int \frac{1-\frac{1}{t^2}}{(t+\frac{1}{t})^2-2}dt$$
Sea $u = t - \frac{1}{t} \rightarrow du = 1 - \frac{1}{t^2}, v = t - \frac{1}{t} \rightarrow dv = 1 - \frac{1}{t^2}$
$$= \int \frac{du}{u^2+2} + \int \frac{dv}{v^2-2} = \frac{1}{2}\int \frac{du}{(\frac{u}{\sqrt{2}})^2+1} + \int \frac{A}{v-\sqrt{2}} + \frac{B}{v+\sqrt{2}}dv$$
(Para las fracciones parciales: $\frac{Av+\sqrt{2}A+Bv-\sqrt{2}B}{v^2-2} \rightarrow A+B=0, \sqrt{2}A-\sqrt{2}B=1 \rightarrow$
$A = \frac{1}{2\sqrt{2}}, B = -\frac{1}{2\sqrt{2}}$)
$$\frac{\sqrt{2}}{2}\int \frac{\frac{1}{\sqrt{2}}}{(\frac{u}{\sqrt{2}})^2+1}du + \frac{1}{2\sqrt{2}}\int \frac{dv}{v-\sqrt{2}} - \frac{1}{2\sqrt{2}}\int \frac{dv}{v+\sqrt{2}}$$
$$= \frac{\sqrt{2}}{2}\arctan(\frac{u}{\sqrt{2}}) + \frac{1}{2\sqrt{2}}\ln|v-\sqrt{2}| - \frac{1}{2\sqrt{2}}\ln|v+\sqrt{2}| + C$$

$$= \frac{\sqrt{2}}{2}\arctan(\frac{t-\frac{1}{t}}{\sqrt{2}}) + \frac{1}{2\sqrt{2}}\ln\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right| + C$$
$$= \frac{\sqrt{2}}{2}\arctan(\frac{t^2-1}{\sqrt{2}t}) + \frac{1}{2\sqrt{2}}\ln\left|\frac{t^2+1-\sqrt{2}t}{t^2+1+\sqrt{2}t}\right| + C$$
$$= \frac{\sqrt{2}}{2}\arctan(\frac{\tan x-1}{\sqrt{2\tan x}}) + \frac{1}{2\sqrt{2}}\ln\left|\frac{\tan x+1-\sqrt{2\tan x}}{\tan x+1+\sqrt{2\tan x}}\right| + C$$

■

9 viii. Encuentra la integral: $\int \frac{dx}{x-x^{3/5}}$

$$= \int \frac{dx}{(x^{2/5}-1)x^{3/5}} = \frac{5}{2}\int \frac{1}{x^{2/5}-1}(\frac{2}{5}x^{-3/5})\ dx$$
Sea $u = x^{2/5}-1 \to du = \frac{2}{5}x^{-3/5}$
$$\frac{5}{2}\int \frac{du}{u} = \frac{5}{2}\ln|u| + C = \frac{5}{2}\ln|x^{2/5}-1| + C$$

■

11 iii. Resuelva la siguiente integral: $\int \frac{dx}{a\operatorname{sen} x + b\cos x}$

Sustitución por variable estereográfica. $\displaystyle\int \frac{\frac{2}{1+t^2}}{a\frac{2t}{1+t^2}+b\frac{1-t^2}{1+t^2}}dt = \int \frac{2dt}{a(2t)+b(1-t^2)} =$
$$2\int \frac{dt}{-bt^2+2at+b} = -\frac{2}{b}\int \frac{dt}{t^2-2\frac{a}{b}t-1} = -\frac{2}{b}\int \frac{dt}{t^2-\frac{2a}{b}t+\frac{a^2}{b^2}-\frac{a^2}{b^2}-1} =$$
$$-\frac{2}{b}\int \frac{dt}{(t-\frac{a}{b})^2-(\frac{a^2}{b^2}-1)} = -\frac{2}{b}\int \frac{dt}{(t-\frac{a}{b})^2-\frac{a^2-b^2}{b^2}} =$$
$$-\frac{2}{b}\frac{b^2}{a^2-b^2}\int \frac{dt}{\frac{b^2}{a^2-b^2}(t-\frac{a}{b})^2-1} = \frac{-2b}{a^2-b^2}\int \frac{dt}{\left(\frac{b}{\sqrt{a^2-b^2}}t-\frac{a}{\sqrt{a^2-b^2}}\right)^2-1}$$
$$= -\frac{2}{\sqrt{a^2-b^2}}\int \frac{\frac{b}{\sqrt{a^2-b^2}}}{\left(\frac{b}{\sqrt{a^2-b^2}}t-\frac{a}{\sqrt{a^2-b^2}}\right)^2-1}$$
Sea $u = \frac{b}{\sqrt{a^2-b^2}}t-\frac{a}{\sqrt{a^2-b^2}} \to du = \frac{b}{\sqrt{a^2-b^2}}$
$$= -\frac{2}{\sqrt{a^2-b^2}}\int \frac{du}{u^2-1} = -\frac{1}{\sqrt{a^2-b^2}}\int \frac{du}{u-1} + \frac{1}{\sqrt{a^2-b^2}}\int \frac{du}{u+1}$$
$$= \frac{1}{\sqrt{a^2-b^2}}(-\ln|u-1| + \ln|u+1|)$$
$$= \frac{1}{\sqrt{a^2-b^2}}(-\ln|\frac{b}{\sqrt{a^2-b^2}}t-\frac{a}{\sqrt{a^2-b^2}}-1| + \ln|\frac{b}{\sqrt{a^2-b^2}}t-\frac{a}{\sqrt{a^2-b^2}}+1|)$$
$$= \frac{1}{\sqrt{a^2-b^2}}(-\ln|\frac{b}{\sqrt{a^2-b^2}}\tan(x/2)-\frac{a}{\sqrt{a^2-b^2}}-1| + \ln|\frac{b}{\sqrt{a^2-b^2}}\tan(x/2)-\frac{a}{\sqrt{a^2-b^2}}+1|) + C$$

■

11 v. Resuelva la siguiente integral: $\int \frac{dx}{3+5\operatorname{sen} x}$

Sustitución estereográfica: $= \displaystyle\int \frac{\frac{2}{1+t^2}}{3+5\frac{2t}{1+t^2}}dt = \int \frac{\frac{2}{1+t^2}}{\frac{3+3t^2+10t}{1+t^2}}dt = \int \frac{2}{3t^2+10t+2}dt$

$$\frac{2}{3t^2+10t+3} = \frac{A}{3t+1} + \frac{B}{t+3} = \frac{At+3A+3Bt+B}{3t^2+10t+3}$$

$A + 3B = 0$

$3A + B = 2$

$\rightarrow 8B = -2 \Rightarrow B = -\frac{1}{4} \Rightarrow A = \frac{3}{4}$

$$= \frac{3}{4}\int \frac{dt}{3t+1} - \frac{1}{4}\int \frac{dt}{t+3} = \frac{3}{4} = \int \frac{3}{3(3t+1)}dt - \frac{1}{4}\int \frac{1}{t+3}dt$$

$$= \frac{1}{4}\ln|3t+1| - \frac{1}{4}\ln|t+3| + C = \frac{1}{4}\ln\left|\frac{3t+1}{t+1}\right| + C$$

$$= \frac{1}{4}\ln\left|\frac{3\tan(x/2)+1}{\tan(x/2)+3}\right| + C$$

■