SAT Intensive Workshop - Day 18

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1 Today's Events

- Vocabulary quiz and Kahoot.
- Review of Reading section 1 from 2 July.
- Reading section 1 practice exam.
- Math section 3 practice exam.
- Lunch.
- Writing section 2 practice exam.
- Review of Writing section 1 from today.
- Review of Math section 3 from today.
- Logic puzzles.

1.1 Review of Math section 3 from today

The whole point of this section is to build towards proving the *Angle Bisector Theorem* cleanly.

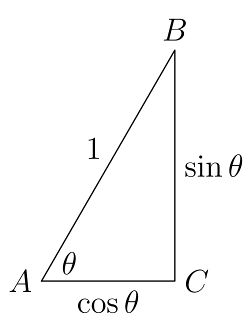
Definition 18.1. The *unit circle* is simply a circle of radius 1 centered at the origin of the xy-plane.

This unit circle is incredibly powerful for dealing with sines and cosines of angles, but it takes a little bit of work to see. First, we start with a crucial identity from trigonometry:

Theorem 18.2. Given any angle θ, then:

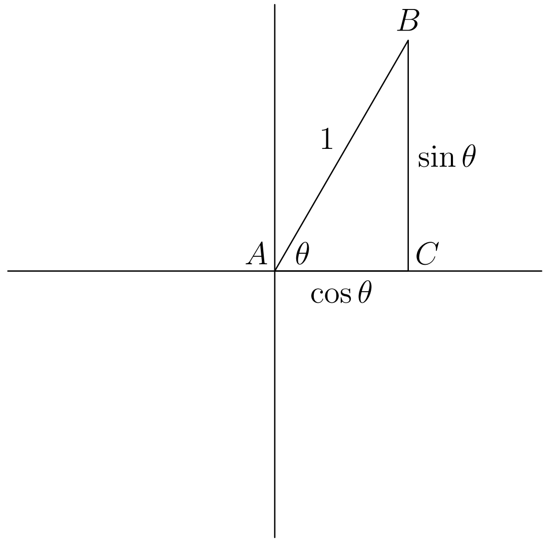
$$\sin^2\theta + \cos^2\theta = 1.$$

Proof. Consider a triangle with hypotenuse 1 such that one of its non-right angles is θ.



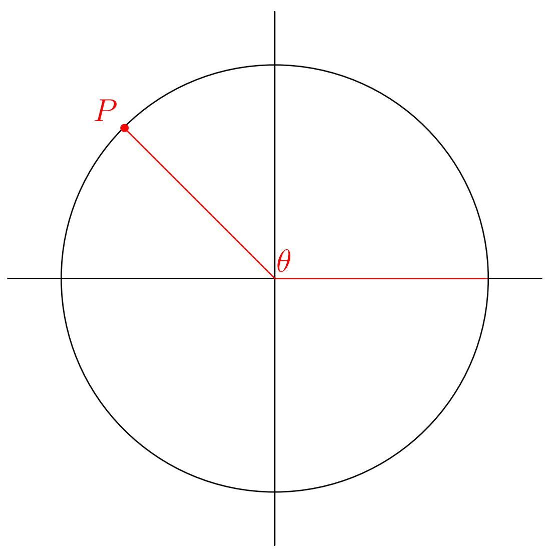
Recall SOHCAHTOA. Applying it to $\angle BAC$, we get that $\sin\theta = \frac{|BC|}{1} = |BC|$, and $\cos\theta = \frac{|AC|}{1} = |AC|$. Applying the Pythagorean Theorem to this triangle, we get the desired $\sin^2\theta + \cos^2\theta = 1$. $\square$

Let's consider the above right triangle on the xy-plane, placing point A at the origin:



Note that the coordinates of point B is $(\cos\theta, \sin\theta)$. This is because the coordinates of A are $(0, 0)$ by default (we set A to be the origin); then the coordinates of point C are $(\cos\theta, 0)$, since we simply go a distance of $\cos\theta$ to the right; then the coordinates of point B are $(\cos\theta, \sin\theta)$, since we go up $\sin\theta$ from the point $(\cos\theta, 0)$.

The key thing to note here is that there's nothing special about the value of θ. This analysis will work for any angle θ formed with the positive x-axis. Here, then, lies the power of the unit circle. Take an arbitrary point on the unit circle – let's call it P. Also, let's draw the line segment from P to the origin, and consider the angle between that line and the positive x-axis.



Then, the coordinates of point P are $(\cos\theta, \sin\theta)$. So, this entire discussion tells us that:

Theorem 18.3. The coordinates of a point P on the unit circle are $(\cos\theta, \sin\theta)$, where θ is the angle formed by the positive x-axis and the line segment from P to the origin.

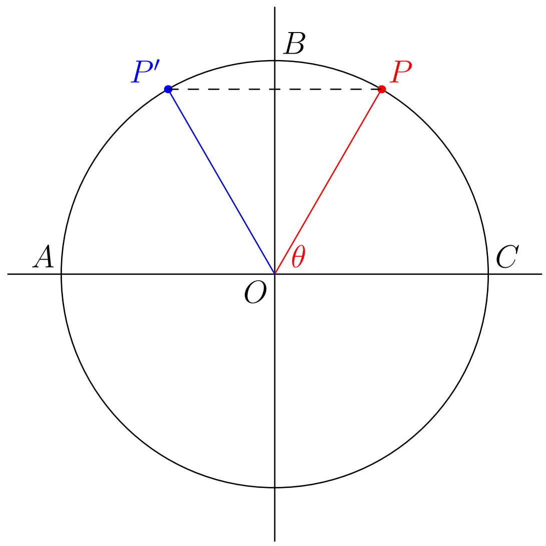
Let's look at how we can use the unit circle to prove some otherwise difficult-to-show identities.

Proposition 18.4. Show that:

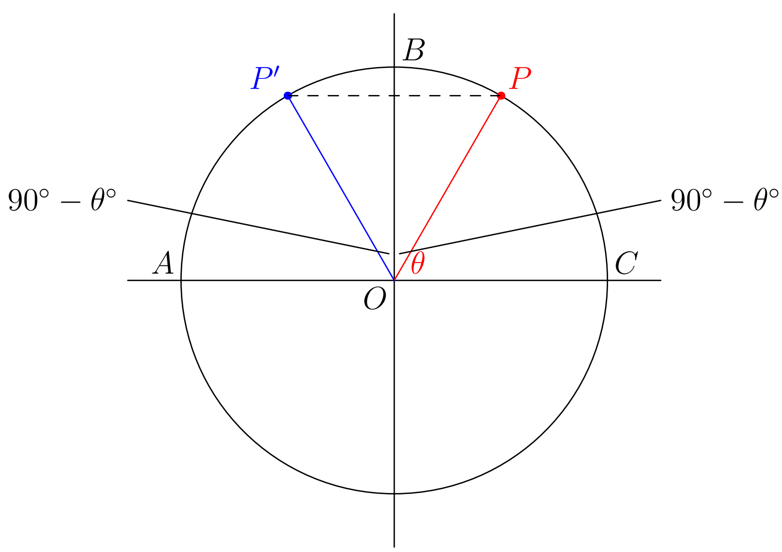
(a) $\sin\theta = \sin(180 - \theta)$

(b) $\cos\theta = -\cos(180 - \theta)$

Proof. There doesn't seem to be an immediately direct way to show these identities algebraically, so let's turn to the unit circle. Let the origin (and the center of the circle) be point O. Consider a point P on the unit circle that forms an angle of θ with the positive x-axis. Also consider its reflection across the x-axis (call that point P'). Furthermore, let the point $(-1, 0)$ be called A, the point $(0, 1)$ be called B, and the point $(1, 0)$ be called C:



By default, $\angle POC = \theta°$. Then, since $90° = \angle BOC = \angle BOP + \angle POC$, we know that $\angle BOP = 90° - \theta°$. But then, since P' is a reflection of P, we know that $\angle P'OB = \angle BOP = 90° - \theta°$.



So, that means that the angle OP' forms with the x-axis is $\theta^\circ + (90^\circ - \theta^\circ) + (90^\circ - \theta^\circ) = 180^\circ - \theta^\circ$.

By Theorem 18.3, we know that the coordinates of point P' are $(\cos(180-\theta), \sin(180-\theta))$, since the angle it forms with the positive x-axis is $180-\theta$. For the same reason, the coordinates of point P are $(\cos\theta, \sin\theta)$.

Since P' is a reflection across the y-axis, its coordinates are $(-\cos\theta, \sin\theta)$ (recall that reflecting the point (a, b) across the y-axis gives the point $(-a, b)$).

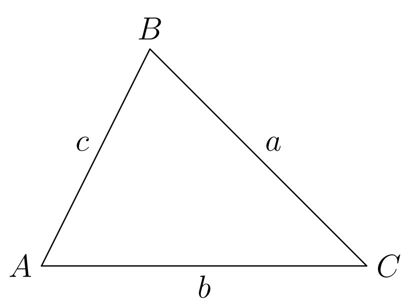
So we have two expressions for the coordinates of point P' – $(\cos(180-\theta), \sin(180-\theta))$ and $(-\cos\theta, \sin\theta)$. Since both these coordinate pairs represent the same point, the coordinates must be equal. Thus, we set the x-coordinates equal to each other and the y-coordinates equal to each other to get the desired:

$$\cos(180-\theta) = -\cos\theta, \ \ \sin(180-\theta) = \sin\theta.$$

□

Armed with this knowledge of trigonometry, we turn to plane geometry. Recall the following theorem from geometry, which will be stated but not proven:

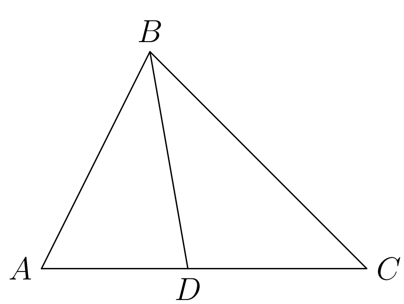
Theorem 18.5 [Law of Sines]. Given an arbitrary triangle ABC:



Then, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

We finally have the tools to prove the Angle Bisector Theorem:

Theorem 18.6 [Angle Bisector Theorem]. Given a triangle ABC and a point D such that BD bisects angle B:



Then,

$$\frac{AB}{AD} = \frac{CB}{CD}.$$

Proof. Applying the Law of Sines (Theorem 18.5) to $\triangle BDC$, we get $\dfrac{BD}{\sin\angle BCD} = \dfrac{CD}{\sin\angle DBC} = \dfrac{CB}{\sin\angle BDC}$. In particular,

$$\frac{CD}{\sin\angle DBC} = \frac{CB}{\sin\angle BDC}.$$

Multiplying both sides by $\sin\angle BDC$ and dividing both sides by CD gives:

$$\frac{CB}{CD} = \frac{\sin\angle BDC}{\sin\angle DBC}. \tag{1}$$

Doing similarly for $\triangle ADB$ gives $\dfrac{AB}{\sin\angle ADB} = \dfrac{BD}{\sin\angle ADB} = \dfrac{AD}{\sin\angle ABD}$, and in particular,

$$\frac{AB}{\sin\angle ADB} = \frac{AD}{\sin\angle ABD}.$$

Multiplying both sides by $\sin\angle ADB$ and dividing both sides by AD gives:

$$\frac{AB}{AD} = \frac{\sin\angle ADB}{\sin\angle ABD}. \tag{2}$$

Now, recall that $\angle ABD = \angle DBC$ by the assumption that BD is an angle bisector. Thus,

$$\sin\angle ABD = \sin\angle DBC = p,$$

for some real value p. Continuing, note that $\angle ADB = 180 - \angle BDC$, since they are supplementary angles. Applying Proposition 18.4(a), we know that

$$\sin\angle ADB = \sin(180 - \angle BDC) = \sin\angle BDC = s,$$

for some real value s. Substituting these values into equations (1) and (2), we get that

$$\frac{CB}{CD} = \frac{s}{p}, \ \frac{AB}{AD} = \frac{s}{p}.$$

Since $\dfrac{CB}{CD}$ and $\dfrac{AB}{AD}$ are equal to the same thing, we conclude that:

$$\frac{AB}{AD} = \frac{CB}{CD}.$$

□

Don't be intimidated by these proofs. They are long because I explained every step explicitly in minute detail. These proofs are actually saying very simple things – make sure you understand them.

2 Homework

Know the following words, as well as their definitions, their parts of speech, and how to use them in a sentence:

- interminable (adj) - never-ending.
- misrepresentation (n) - a deliberate deception of a subject or person.
- peremptory (adj) - in a commanding manner.
- quaint (adj) - picturesque; cute.

- stevedore (n) - someone who works at a dock.
- virtuoso (n) - an accomplished musician.
- augment (v) - to increase or make bigger.
- complacent (adj) - self-satisfied; smug.
- droll (adj) - dryly amusing.
- galleon (n) - ancient type of sailing ship.
- intermittent (adj) - sporadic or irregular.
- mitigate (v) - to lessen or make less sever.
- quandary (n) - dilemma or puzzle.
- stifle (v) - to suppress.
- virulent (adj) - dangerous or harmful.