SAT Intensive Workshop - Day 8

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1 Today's Events

- Vocabulary quizzes and games.
- Finish review of Math section 4 from 17 June.
- Math section 4 practice exam.
- Review of Reading section 1 from 18 June.
- Lunch.
- Review of Writing section 2 from 18 June.
- Reading section 1 practice exam.
- Writing section 2 practice exam.
- Practice essay.

1.1 Review of Math section 4 from 17 June (cont'd)

1.1.1 Completing the Square

Occasionally, you'll want to factor a quadratic as the perfect square of a binomial. Let's say you have a quadratic equation $rx^2 + sx = 0$, and you want to write it in the form $a(x+b)^2 = c$. There are a number of steps through which you must go:

1. Turn the quadratic into a monic polynomial (a polynomial whose leading coefficient is 1). To do so, factor out an r from both terms on the left hand side to get

$$rx^2 + sx = 0 \implies r\left(x^2 + \frac{s}{r}x\right) = 0.$$

   Note that if the quadratic is already monic, it makes our lives much, much easier.

2. Now, we want to turn the expression inside the parentheses into the perfect square of a binomial. So, we add $\frac{s^2}{4r}$ to both sides to get

$$r\left(x^2 + \frac{s}{r}x\right) + \frac{s^2}{4r} = \frac{s^2}{4r}.$$

   We now artificially factor out an r from the $\frac{s^2}{4r}$ term on the left hand side to push it into the parentheses, getting

$$r\left(x^2 + \frac{s}{r}x + \frac{s^2}{4r^2}\right) = \frac{s^2}{4r}.$$

   Finally, we can factor the term inside the parentheses as a perfect square, getting the final answer of

$$r\left(x + \frac{s}{2r}\right)^2 = \frac{s^2}{4r}.$$

This is rather abstract, so let's look at the problem we did in class to try to illuminate us.

Example 8.1. Find the radius of the circle with equation $x^2 + y^2 + 4x - 2y = -1$.

Proof. Recall that the general form of a circle, $(x-h)^2 + (y-k)^2 = r^2$, tells us what the radius of a circle is. So, we want to shoehorn the equation we were given into the format of the general form of a circle, so that we can simply read off the radius. But, in order to do that, we must complete the square in both x and y.

Consider the x terms for a second, and pretend that the y terms don't exist. Our current equation is $x^2 + 4x = -1$. We want to convert $x^2 + 4x$ into something of the form $(x + b)^2$. So, after FOILing and comparing them, we see that $x^2+4x \leftrightarrow x^2+2bx+b^2$. We want these to be equal, so we set $2b = 4 \implies b = 2$. So, we need to add 4 to $x^2 + 4x$ in order to have it factor as the square of a binomial. Doing so, we get:

$x^2 + 4x = -1 \implies x^2 + 4x + 4 = -1 + 4 \implies (x + 2)^2 = 3$. Now, our equation is $(x + 2)^2 + y^2 - 2y = 3$. Doing similarly for y (work it out yourself), we get

$$(x + 2)^2 + (y - 1)^2 = 4 = 2^2.$$

It is clear, then, that the radius of the circle is $\boxed{2}$. $\square$

1.1.2 The Discriminant

Recall from your algebra class that the solutions to a quadratic equation of the form $ax^2 + bx + c = 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The most interesting part of this formula is the discriminant.

Definition 8.2. The *discriminant* of a quadratic $ax^2 + bx + c$ is the expression $b^2 - 4ac$. It is frequently denoted by a capital delta, Δ.

Why is it important, though? Consider the following three scenarios.

1. Suppose a quadratic has $\Delta > 0$. In particular, this means that $\sqrt{\Delta} > 0$. Then, the quadratic equation tells us that the solutions to this quadratic look like $\dfrac{-b \pm \sqrt{\Delta}}{2a}$. So, since all of these numbers are real, we get two distinct real solutions for this quadratic, namely $\dfrac{-b + \sqrt{\Delta}}{2a}$ and $\dfrac{-b - \sqrt{\Delta}}{2a}$.
   $\Delta > 0 \implies$ two distinct real solutions.

2. Suppose now that a quadratic has $\Delta = 0$, so that $\sqrt{\Delta} = 0$. Thus, the quadratic formula tells us that the solutions to the quadratic are $\dfrac{-b \pm 0}{2a} = -\dfrac{b}{2a}$. We get exactly one real solution for the quadratic.
   $\Delta = 0 \implies$ exactly one real solution.

3. Finally, suppose that a quadratic has $\Delta < 0$, so that $\sqrt{\Delta}$ is complex. Then, for the same reasons as the first case, this quadratic will have two distinct complex solutions.

   $\Delta < 0 \implies$ two distinct complex solutions.

Let's look at an example of where the discriminant is useful.

Example 8.3. Suppose $3 = rx^2 + s$. Find an ordered pair (r, s) such that that equation has two distinct real solutions.

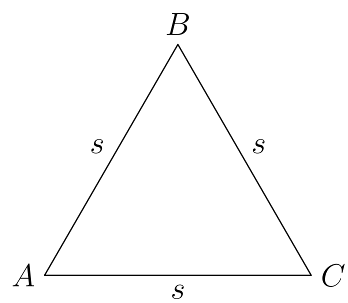
Proof. We have a quadratic equation, $rx^2 + (s - 3) = 0$, and we want to know when it has two real roots. The discriminant is perfect for this – we just want to solve $\Delta > 0$. Doing so, we get

$$\Delta = 0^2 - 4r(s-3) > 0 \implies -4r(s-3) > 0 \implies r(s-3) < 0.$$

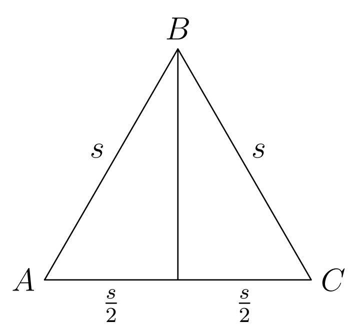
Remember that we have to flip the inequality in the last step since we are dividing by a negative number, -4. So, we just want the product of r and $s - 3$ to be negative – let's take $s - 3$ to be positive and r to be negative. An ordered pair that works here is $\boxed{(r, s) = (-1, 4)}$. $\square$

1.1.3 Deriving the Equation for the Area of an Equilateral Triangle

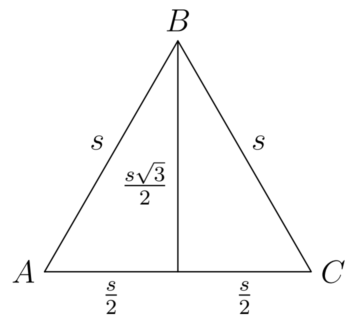
Recall that an equilateral triangle is a triangle whose sides are all equal. As a consequence, all of its angles are also equal. Consider the following equilateral triangle with side length s:



We want to find the area of this triangle. We already have an easy formula to find it, $\frac{1}{2}$(base)(height). So, we try to find the height of this triangle. We drop an altitude from B down to $\overline{AC}$:



Using the Pythagorean Theorem, or noticing that we have a 30-60-90 right triangle, we get that the altitude is $\sqrt{s^2 - \frac{s^2}{4}} = \sqrt{\frac{3s^2}{4}} = \frac{s\sqrt{3}}{2}$ (Pythagorean Theorem).

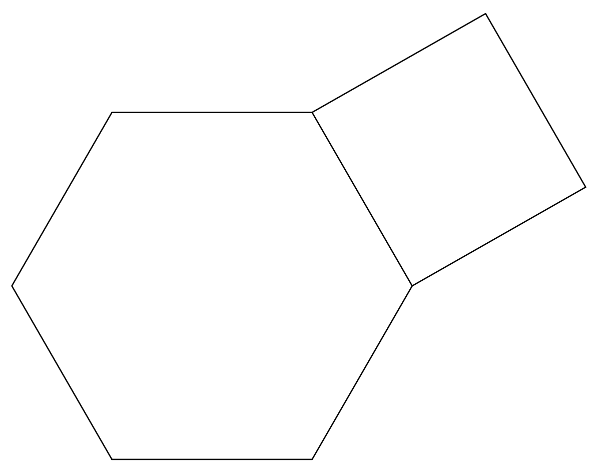


Now, we have both the base and the height, so we can find the area:

$$\frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(s)\left(\frac{s\sqrt{3}}{2}\right) = \boxed{\frac{s^2\sqrt{3}}{4}}.$$

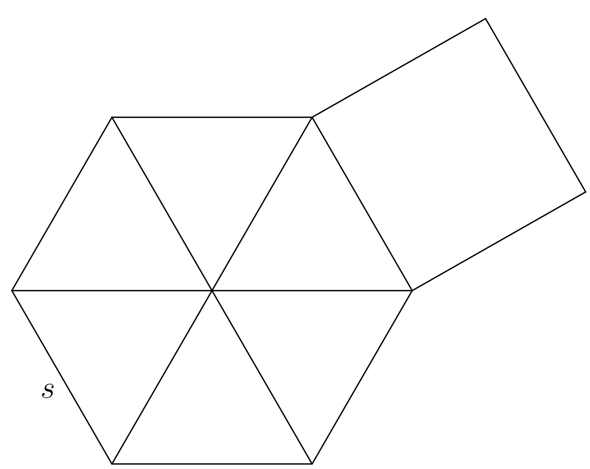
Armed with this knowledge, we can tackle the hexagon problem from class.

Example 8.4. Given the following diagram consisting of a square and a regular hexagon:



If the area of the hexagon is $384\sqrt{3}$, then find the area of the square.

Proof. Note that a regular hexagon is actually made up of 6 equilateral triangles. Let the side length of the hexagon be s.



So, we know that the area of each of the individual equilateral triangles is $\frac{s^2\sqrt{3}}{4}$, so the area of the hexagon is $6 \cdot \frac{s^2\sqrt{3}}{4} = \frac{3s^2\sqrt{3}}{2}$. Setting this equal to $384\sqrt{3}$, we get:

$\frac{3s^2\sqrt{3}}{2} = 384\sqrt{3} = \frac{3}{2}s^2 = 384 \implies s^2 = 256$. Note that the area of the square is given by s^2, so the area of the square is just $\boxed{256}$. $\square$

1.2 Review of Reading section 1 from 18 June

1.2.1 New words

- prodigal (adj) - spending money or resources recklessly.
- profuse (adj) - extremely plentiful and abundant.
- ingenuous (adj) - innocent and unsuspecting.
- imputation (n) - a claim that someone has done something undesirable.
- shrewd (adj) - clever; calculated; smart.
- mercenary (adj) - concerned with making money, often at the expense of ethical behavior.
- base (adj) - immoral.
- sordid (adj) - involving not noble actions or motives, often arousing moral contempt.
- execrate (v) - to feel great hate towards.
- inculcate (v) - to teach by persistent instruction.
- gaudy (adj) - brightly colored.
- indelible (adj) - about an object or mark that cannot be removed.
- iota (n) - a really small amount.
- felicity (n) - happiness.
- annihilation (n) - complete destruction.
- faculty (n) - an inherent mental or physical power.
- adage (n) - a proverb expressing a general truth.
- hinder (v) - to prevent from happening; to get in the way of something happening.
- perturbation (n) - a small change in something.
- reconcile (v) - to make amends, to make up.
- relinquish (v) - to voluntarily give up.
- submit (v) - to yield to the will of another. Also to subject to a particular process.
- cultivate (v) - to acquire or develop, especially of a quality, sentiment, or skill.
- furnish (v) - to supply with information.

2 Homework

Be sure to know the above words, their definitions, and how to use them in a *well-thought out and proper* sentence. Also, it might be advisable to review old words, as there will be a Kahoot on old words tomorrow.