SAT Intensive Workshop - Day 24

Tomas Matzner

11 July 2019

1 Today's Events

- Review of Math section 3 from 9 July.
- Review of Reading section 1 from 10 July.
- Reading section 1 practice exam.
- Lunch.
- Review of Math section 4 from 10 July.
- Writing section 2 practice exam.
- Practice essay.
- Math section 3 practice exam.

1.1 Review of Math section 3 from 9 July

1.1.1 The Axis of Symmetry for a Parabola

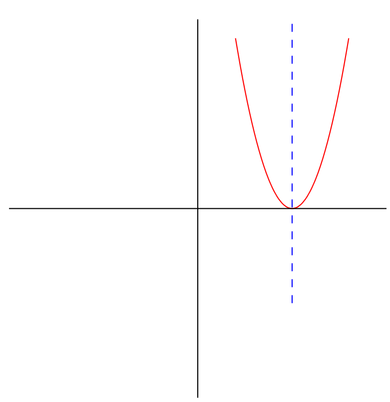
First, let's start off with the general form for a parabola. You should all know this, as well as the general form for a circle.

Definition 24.1. The *standard form* for a parabola is given by the equation

$$y = a(x - h)^2 + k,$$

where a is the scaling factor and (h, k) is the center.

Now, consider a standard parabola – let's say $y = (x-5)^2$, so that its vertex is $(5, 0)$. Note that the parabola has an axis of symmetry, where one side of the parabola is simply the other side reflected across this axis:



In general:

Definition 24.2. The *axis of symmetry* of a parabola $y = a(x-h)^2 + k$ is the line $x = h$.

This means that two points of any parabola that lie on a horizontal line are equidistant from the axis of symmetry. Let's look at an example of this principle in action.

Problem 24.3. The vertex of the parabola given by the equation $y = a(x-2)(x+4)$, where a is a non-zero constant, is (c, d). Find d.

Proof. We will present two solutions to this problem.

i) Completing the square. Let's just force the equation we're given into the standard form for a quadratic. Expanding the parentheses, we get:

$$y = a(x-2)(x+4) = a(x^2 + 2x - 8).$$

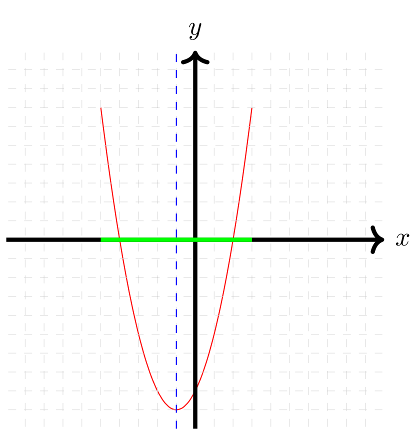
So, completing the square by forcing $x^2 + 2x - 8$ into the closest square of a binomial, which we get by dividing the coefficient on x by 2, we get $x^2 + 2x - 8 = (x+1)^2 - 9$, since $(x+1)^2 = x^2 + 2x + 1$, and $1 - 9 = -8$. Therefore, the equation for the parabola is now $y = a((x+1)^2 - 9)$, and pushing the a through,

$$y = a(x+1)^2 - 9a.$$

It is obvious, then, that the y-coordinate of the vertex is $\boxed{-9a}$.

ii) Using the axis of symmetry. We know that -4 and 2 both lie on the same horizontal line $y = 0$, since they are both roots of the polynomial $a(x-2)(x+4)$. They must be equidistant, then, from the vertical axis of symmetry. The point that is equidistant from -4 and 2 is -1, so we know that the axis of symmetry is the line $x = -1$. Furthermore, this lets us know that the x-coordinate of the vertex of the parabola is -1.

So, plugging -1 into the equation, we get $y = a(-1-2)(-1+4) = a(-3)(3) = \boxed{-9a}$. The below diagram helps to explain exactly what was going on in this solution. The blue dashed line is the axis of symmetry, and the green line is the horizontal line on which -4 and 2 both live.



$\square$

1.1.2 Polynomial Long Division

Let's look at the division algorithm for real numbers first as a roadmap for how to divide polynomials.

Theorem 24.4. The division of any real number p by any real number $s \neq 0$ can be expressed in the following form:

$$\frac{p}{s} = q + \frac{r}{s},$$

where q is the quotient and r is the remainder, so $r < s$. This is often rewritten as

$$p = sq + r.$$

The fact that such an algorithm exists is quite remarkable, but the tools to think about it are well beyond the scope of the SAT. Let's look at some examples to really understand what Theorem 24.4 is saying.

Example 24.5. Dividing real numbers.

i) Consider $65 \div 8$. In fraction form, this is written as $\frac{65}{8}$. We all know that this equals 8 with remainder 1, so

$$65 = 8 \cdot 8 + 1 \text{ or } \frac{65}{8} = 8 + \frac{1}{8}.$$

ii) Consider $39 \div 7$. We get

$$39 = 7 \cdot 5 + 4 \text{ or } \frac{39}{7} = 5 + \frac{4}{7}.$$

Now, let's look at the division algorithm for polynomials.

Theorem 24.6. The division of any real polynomial p by any real polynomial $s \neq 0$ can be expressed in the following form:

$$\frac{p(x)}{s(x)} = q(x) + \frac{r(x)}{s(x)},$$

where $q(x)$ is the quotient and $r(x)$ is the remainder, so $\deg r(x) < \deg s(x)$. This is often rewritten as

$$p(x) = s(x)q(x) + r(x).$$

Example 24.7. Polynomial long division.

i) Consider the division of $x^3 + 4x^2 + 5x + 6$ by $x - 4$.

$$\begin{array}{r|rrrr}
\multicolumn{2}{r}{x^2} & +8x & +37 & \\
\cline{2-5}
x-4 & x^3+4x^2 & +5x & +6 & \\
 & -x^3+4x^2 & & & \\
\cline{2-2}
 & 8x^2 & +5x & & \\
 & -8x^2 & +32x & & \\
\cline{2-3}
 & & 37x & +6 & \\
 & & -37x & +148 & \\
\cline{3-4}
 & & & 154 &
\end{array}$$

We can write this as $\frac{x^3 + 4x^2 + 5x + 6}{x - 4} = (x^2 + 8x + 37) + \left(\frac{154}{x - 4}\right)$, or $x^3 + 4x^2 + 5x + 6 = (x^2 + 8x + 37)(x - 4) + 154$.

ii) Consider the division of $4x^2+8x+3$ by $2x+3$.

$$
\begin{array}{r|l}
 & \quad\;\; 2x+1 \\
\hline
2x+3 & 4x^2+8x+3 \\
 & -4x^2-6x \\
\hline
 & \qquad\;\; 2x+3 \\
 & \qquad -2x-3 \\
\hline
 & \qquad\qquad 0
\end{array}
$$

We can write this as $\dfrac{4x^2+8x+3}{2x+3} = (2x+1)+\left(\dfrac{0}{2x+3}\right) \implies \dfrac{4x^2+8x+3}{2x+3} = 2x+1$, or $4x^2+8x+3 = (2x+1)(2x+3)$.

Problem 24.8. The equation

$$\frac{24x^2+25x-47}{ax-2} = -8x-3-\frac{53}{ax-2}$$

is true for all values of $x \neq \dfrac{2}{a}$, where a is a constant. Find a.

Proof. Let's divide $24x^2+25x-47$ by $ax-2$. However, we don't have to do the full polynomial long division. Note that ax fits $\dfrac{24}{a}x$ times into $24x^2$, since $ax \cdot \dfrac{24}{a}x = 24x^2$, so we know that the coefficient of the x term of the quotient will be $\dfrac{24}{a}$. But then, note that the quotient is $-8x-3$, while the remainder is 53.

So, we know that the coefficient in front of x in the quotient is -8. Therefore, we can set $\dfrac{24}{a} = -8 \implies a = \boxed{-3}$. □

1.1.3 Other Problems

Problem 24.9. Assuming $x > 0$, find a solution to the equation

$$x^3(x^2-5) = -4x.$$

Proof. Expanding both sides and moving the $-4x$ to the left-hand-side, we get

$$x^5-5x^3+4x = 0.$$

Factoring and dividing through by x (which we can do since we assume $x > 0$ – we don't have to worry about dividing by 0), we get

$$x^4-5x^2+4 = 0.$$

Make the substitution $y = x^2$, which transforms our quartic into a manageable quadratic, $y^2-5y+4 = 0$. This has solutions $y = 4$ and $y = 1$, which means that $x^2 = 4$ and $x^2 = 1$. So, two possible solutions to this equation are $x = \boxed{1, 2}$. □

1.2 Review of Reading section 1 from 10 July

1.2.1 New words

• delusion (n) - a belief that is held despite being contradicted by reason or generally accepted facts.

- avowed (adj) - something that has been asserted or said publicly.
- unanimity (n) - consensus or agreement by all those involved.
- arduous (adj) - difficult; strenuous.
- irrefragable (adj) - indisputable.
- calamity (n) - a disaster; an event causing great harm or damage.

1.3 Review of Math section 4 from 10 July

Problem 24.10. The sum of three numbers is 855. One of the numbers, x, is 50% more than the sum of the other two numbers. What is the value of x?

Proof. Let the three numbers be x, y, z. Then, the information that they give us is that $x + y + z = 855$ and $x = 1.5(y + z)$. Dividing both sides of the second equation by 1.5 gives that $\frac{x}{1.5} = y + z$, and substituting for $y + z$ into the first equation,

$$x + \frac{x}{1.5} = 855.$$

Solving this gives $x = \frac{855}{1 + \frac{1}{1.5}} \implies x = \boxed{513}$. □

Problem 24.11. Given that $0 < a, b < 90$, $\sin(a^\circ) = \cos(b^\circ)$, and that $a = 4k - 22$ and $b = 6k - 13$, find k.

Proof. Recall the following identity, that $\sin(\theta) = \cos(90 - \theta)$. Keeping that in mind, that means that $\sin a = \cos(90 - a)$. But then, $\sin a = \cos b$ as given, so $\cos(90 - a) = \cos b$. Since $90 - a$ and b are both less than 90, we conclude that $90 - a = b$. Substituting the expressions for a and b in terms of k, we get:

$$90 - (4k - 22) = 6k - 13.$$

Solving for k gives $k = \boxed{12.5}$. □

2 Homework

You should know all of the words in the New words section for today, as well as yesterday's, in addition to their definitions and how to use them in a sentence. Like I said, the point of the sentence is so that you understand the nuance of each word and how to use it correctly.

Furthermore, please don't forget to bring your SAT book every day. If you don't want to carry it back and forth each time, you are more than welcome to leave it in the classroom. If you own a calculator of any sort, it would be a good idea to also bring that every day.