Reading group: Accelerating Variance Reduction for Stochastic Gradient Methods

UNIVERSITY OF OXFORD
DEPARTMENT OF STATISTICS

February 22, 2020

Review of Accelerated Gradient Descent.

Aim: Find an ε-minimizer of an L-smooth, differentiable and convex function $f : \mathcal{X} \to \mathbb{R}$ in $O(\sqrt{\frac{L}{\varepsilon}})$ iterations, for convex $\mathcal{X}$. Equivalently, starting at an arbitrary point x_0, for a minimizer x^* and after T iterations, compute a point x_t such that $f(x_t) - f(x^*) \lesssim \frac{L}{T^2}$.
Reduction: Most of the work is done when going from a 2ε-minimizer to an ε-minimizer. Indeed, assume we can go from $f(x_0) - f(x^*) \leq d$ to $f(x_t) - f(x^*) \leq \frac{d}{2}$ in $t = O(\sqrt{\frac{L}{d}})$. Then we can obtain an ε-minimizer in

$$
\begin{aligned}
T &\lesssim \sqrt{L/d} + \sqrt{L/(d/2)} + \cdots + \sqrt{L/4\varepsilon} + \sqrt{L/2\varepsilon} \\
&< \sum_{i=1}^{\infty} \sqrt{L/2^i\varepsilon} \lesssim \sqrt{L/\varepsilon}.
\end{aligned}
$$

Acceleration can be understood as a compromise between Gradient Descent (builds a primal solution) and Mirror Descent (builds a dual solution).

Review of Gradient Descent and Mirror Descent.

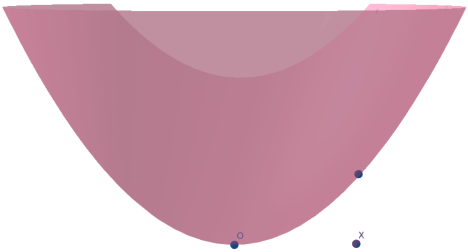

Figure: Parabola $\frac{L}{2}\left\|x - O\right\|^2$.

- Assume gradient at x is w.

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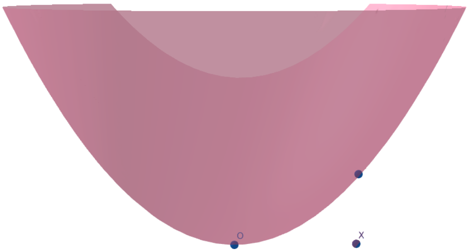

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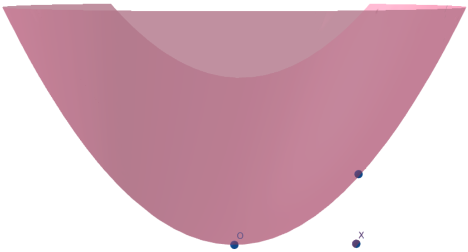

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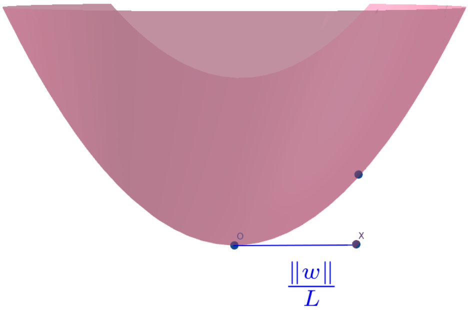

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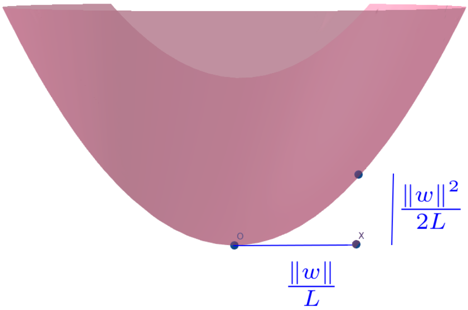

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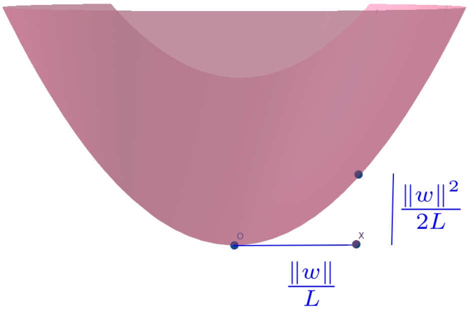

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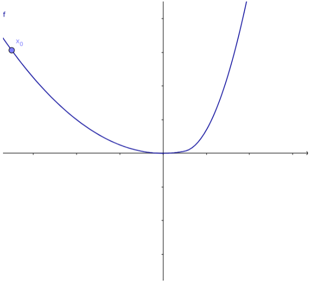
Gradient descent minimizes the upper bound on the function that smoothness yields. For a gradient $\nabla f(x_t)$ it moves to $x_{t+1} = x_t - \frac{1}{L}\nabla f(x_t)$ to decrease the objective $f(x_t) - f(x_{t+1}) \geq \|\nabla f(x_t)\|^2 / 2L$.

Review of Mirror Descent. Intuition.

- Convexity yields linear lower bounds.



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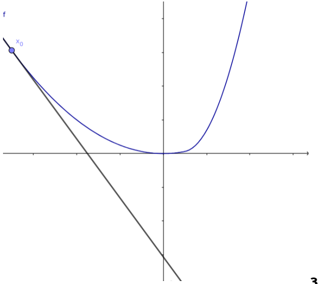

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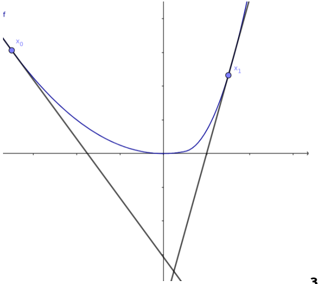

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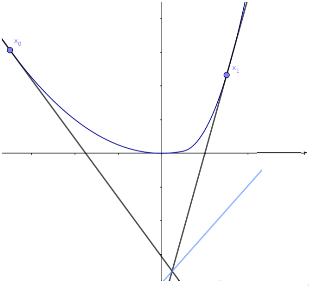

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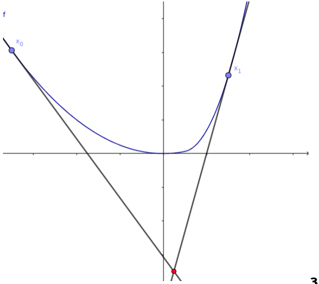

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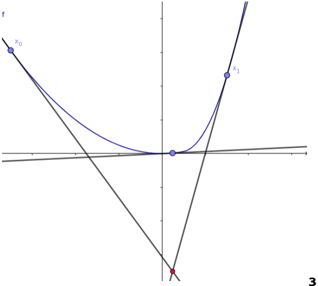

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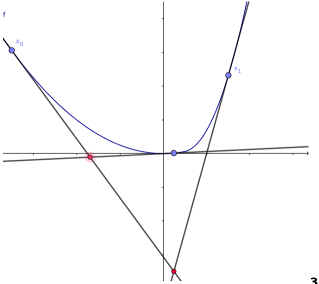

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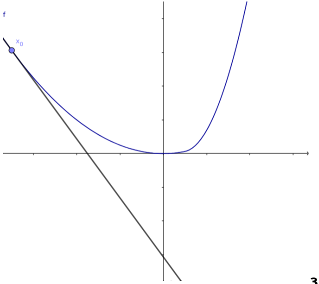
- **This intuition needs refinement:** The **lowest point** of the lower bound **could be undefined**. Also, optimizing the natural lower bound, that is the max of all lower bounds, is expensive in time and memory.



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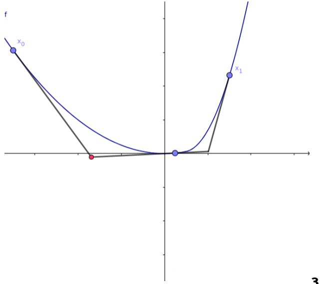

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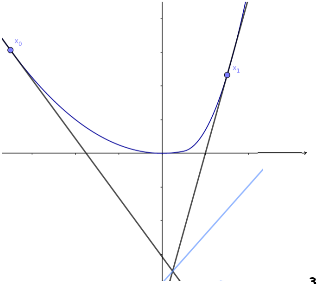
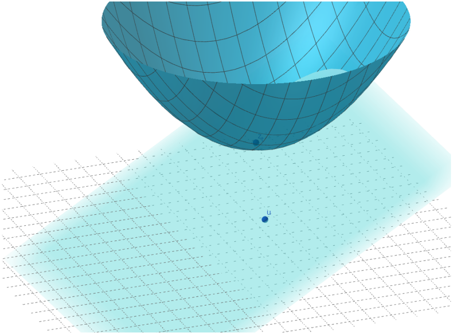
Solutions: Regularize + use average of lower bounds. This is good enough! In equations:
$f(\sum_i x_i/t) - f(x^*)$ (Jensen)
$\leq (\sum_i f(x_i) - f(x^*))/t$ (convexity)
$\leq (\sum_i \langle \nabla f(x_i), x_i - x^* \rangle)/t$.
If $\alpha \|x_{t-1} - x^*\|^2$ is bounded, adding the regularizer $\alpha \|x_{t-1} - x^*\|^2/t$ to the lower bound will not change the rate if we aim for $O(1/t)$ or slower.



Review of Mirror Descent. Mirror Descent Lemma.

Given $z_t, w \in \mathbb{R}^d$ we have a hyperplane $H(\cdot) = \langle w, \cdot - z_t \rangle$. We add the blue parabola $p_b(\cdot) = \frac{1}{2} \|\cdot - z_t\|^2$ as a regularizer. The sum is another parabola p_r **tangent** to H at z_t.



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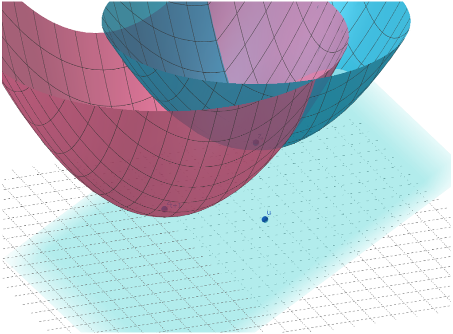

$$p_r = p_b + H.$$

So $\forall u \in \mathbb{R}^d$: $\langle w, z_t - u \rangle = \frac{\|w\|^2}{2} + \frac{1}{2} \|u - z_t\|^2 - \frac{1}{2} \|u - z_{t+1}\|^2$ $(-H = p_b - p_r)$

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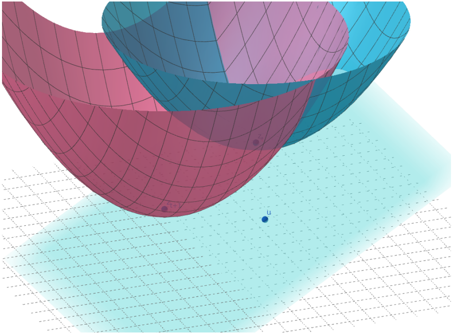

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For $w_t = \alpha \nabla f(z_t)$ we converge for L_1-Lipschitz functions (i.e. $\|\nabla f(\cdot)\| \leq L_1$):

$$f\left(\sum_{i=0}^{T-1} z_i/T\right) - f(x^*) \leq \frac{\sum_i \langle \alpha \nabla f(z_i), z_i - x^* \rangle}{\alpha T} \leq \frac{\alpha^2 L_1^2 T}{2\alpha T} + \frac{1}{2\alpha T}\left\|x^* - z_0\right\|^2$$

$$= \sqrt{L_1^2 \left\|x^* - z_0\right\|^2 / 4T}. \text{(where } \alpha = \sqrt{\left\|x^* - z_0\right\|^2 / T L_1^2}\text{)}$$

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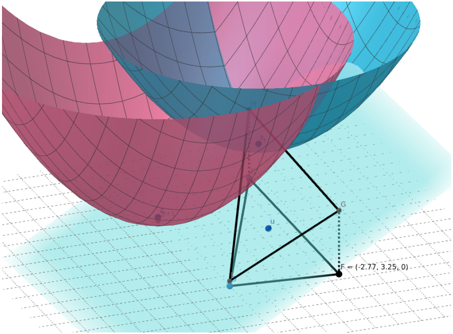

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If we have a constraint $\mathcal{X}$ we need to find the optimum z'_{t+1} of p_r in $\mathcal{X}$.

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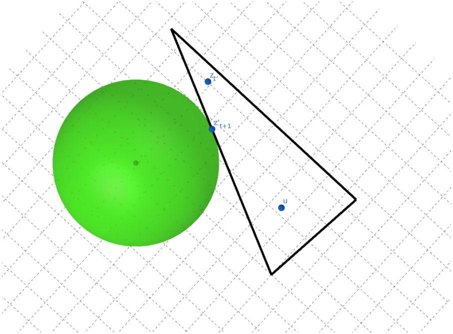

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Projecting z_t onto $\mathcal{X}$ we get z_t'. (Minimum dist. to $z_{t+1} \Rightarrow$ min. value of p_r.)

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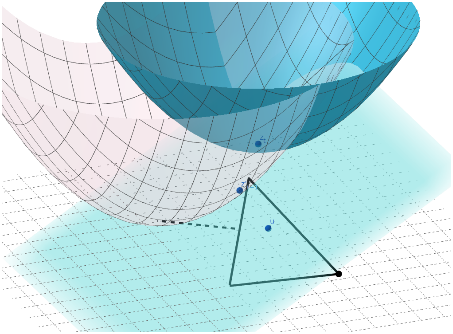

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Here $\forall u \in \mathcal{X}$: $\frac{1}{2}\|u - z'_{t+1}\|^2 \leq p_r(u) - p_r(z'_{t+1})$ (p_r is strongly convex, z'_{t+1} is minimizer. Alternatively, note p_r grows from z'_{t+1} towards $u \in \mathcal{X}$, Hessian is $\succcurlyeq I$ so difference of p_r's is $\geq \frac{1}{2}\text{dist}^2$ ($\frac{1}{2}\text{dist}^2$ if starting with zero directional derivative and hessian in the path in between evaluates 1 at (v, v) with $v = u - z'_{t+1}$. Higher o/w)).

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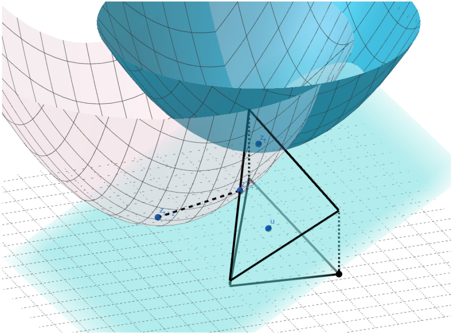

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So $\langle w, z_t - u \rangle \leq -p_r(z'_{t+1}) + \frac{1}{2} \|u - z_t\|^2 - \frac{1}{2} \left\|u - z'_{t+1}\right\|^2$. Term $-p_r(z'_{t+1})$ can be trivially bounded by $-p_r(z_t) \leq \|w\|^2 / 2$ (and convergence follows as before) but sometimes it is better to use its exact value $-p_r(z'_{t+1}) = -p_b(z'_{t+1}) - H(z'_{t+1}) = \langle w, z_t - z'_{t+1} \rangle - \frac{1}{2} \left\|z_t - z'_{t+1}\right\|^2$.

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So we proved that for all $u \in \mathcal{X} \subseteq \mathbb{R}^d$, $w, z_t \in \mathbb{R}^d$ we have

$$\langle w, z_t - u \rangle \leq \langle w, z_t - z'_{t+1} \rangle - \frac{1}{2} \left\| z_t - z'_{t+1} \right\|^2 + \frac{1}{2} \left\| u - z_t \right\|^2 - \frac{1}{2} \left\| u - z'_{t+1} \right\|^2$$

and in particular

$$\langle w, z_t - u \rangle \leq \frac{\left\| w \right\|^2}{2} + \frac{1}{2} \left\| u - z_t \right\|^2 - \frac{1}{2} \left\| u - z'_{t+1} \right\|^2 .$$

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When w is a multiple of $\nabla f(z_t)$ (what you would normally want) the term $\|w\|^2/2$ can be large sometimes. But recall, if we use gradient descent from z_t, the new point y_{t+1} has a guaranteed progress proportional to $\|w\|^2$. Good GD progress means compensating bad MD performance and bad GD progress would happen only when MD has good performance!! **Problem:** each method will tell us to evaluate a different point . We need to mix them in some way: **Linear coupling**.

Linear Coupling (Unconstrained)

Linear coupling: Run your MD, but compute the next gradient at a convex (linear) combination of the points suggested $x_{t+1} = (1-\tau)y_t + \tau z'_t$, where y_t and z'_t are the gradient and mirror points defined as $y_0 = z'_0 = x_0 \in \mathcal{X}$ arbitrary, $y_t = \operatorname{argmin}_{y \in \mathcal{X}} \{f(x_t) + \langle \nabla f(x_t), y - x_t \rangle + \frac{L}{2} \|y - x_t\|^2\}$ and $z'_t = \operatorname{argmin}_{z \in \mathcal{X}} \{\langle \alpha \nabla f(x_t), z - x_t \rangle + \frac{1}{2} \|z - z'_t\|^2\}$, for α to be chosen later.

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Analysis: $f(\sum_{t=0}^{T-1} x_t/T) - f(x^*) \leq \frac{1}{T}\sum_{t=0}^{T-1}\langle \nabla f(x_{t+1}), x_{t+1} - x^* \pm z'_t \rangle \leq (*)$

Linear Coupling (Unconstrained)

Linear coupling: Run your MD, but compute the next gradient at a convex (linear) combination of the points suggested $x_{t+1} = (1-\tau)y_t + \tau z_t'$, where y_t and z_t' are the gradient and mirror points defined as $y_0 = z_0' = x_0 \in \mathcal{X}$ arbitrary, $y_t = \operatorname{argmin}_{y\in\mathcal{X}}\{f(x_t) + \langle \nabla f(x_t), y - x_t\rangle + \frac{L}{2}\|y - x_t\|^2\}$ and $z_t' = \operatorname{argmin}_{z\in\mathcal{X}}\{\langle \alpha\nabla f(x_t), z - x_t\rangle + \frac{1}{2}\|z - z_t'\|^2\}$, for α to be chosen later.
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And finally, picking $w_t = \alpha \nabla f(x_t)$, and using the unconstrained GD guarantee $\alpha\|\nabla f(x_{t+1})\|^2/2 \leq \alpha L(f(x_{t+1}) - f(y_{t+1})) =: B_t$, we obtain:

Linear Coupling (Unconstrained)

Linear coupling: Run your MD, but compute the next gradient at a convex (linear) combination of the points suggested $x_{t+1} = (1-\tau)y_t + \tau z_t'$, where y_t and z_t' are the gradient and mirror points defined as $y_0 = z_0' = x_0 \in \mathcal{X}$ arbitrary, $y_t = \operatorname{argmin}_{y \in \mathcal{X}}\{f(x_t) + \langle \nabla f(x_t), y - x_t\rangle + \frac{L}{2}\|y - x_t\|^2\}$ and $z_t' = \operatorname{argmin}_{z \in \mathcal{X}}\{\langle \alpha \nabla f(x_t), z - x_t\rangle + \frac{1}{2}\|z - z_t'\|^2\}$, for α to be chosen later.
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Parameter τ is chosen to satisfy $(1-\tau)/\tau = \alpha L$ to balance the constant between the progress of GD and $\|w\|^2$. That is, the coupling implies

$$\langle \nabla f(x_{t+1}), x_{t+1} - z_t'\rangle = \frac{1-\tau}{\tau}\langle \nabla f(x_{t+1}), y_t - x_t\rangle \overset{\text{cvx.}}{\leq} \alpha L(f(y_t) - f(x_{t+1})) =: A_t.$$

And finally, picking $w_t = \alpha \nabla f(x_t)$, and using the unconstrained GD guarantee $\alpha\|\nabla f(x_{t+1})\|^2/2 \leq \alpha L(f(x_{t+1}) - f(y_{t+1})) =: B_t$, we obtain:

$$\langle \alpha \nabla f(x_{t+1}), x_{t+1} - x^* \pm z_t'\rangle/\alpha \leq \frac{\alpha\|\nabla f(x_{t+1})\|^2}{2} + \underbrace{\frac{\|x^* - z_t'\|^2 - \|x^* - z_{t+1}'\|^2}{2\alpha}}_{=:C_t} + A_t$$

$$\leq A_t + B_t + C_t = \alpha L(f(y_t) - f(y_{t+1})) + C_t.$$

Linear Coupling (Unconstrained)

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Parameter τ is chosen to satisfy $(1-\tau)/\tau = \alpha L$ to balance the constant between the progress of GD and $\|w\|^2$. That is, the coupling implies

$$\langle \nabla f(x_{t+1}), x_{t+1} - z'_t\rangle = \tfrac{1-\tau}{\tau}\langle \nabla f(x_{t+1}), y_t - x_t\rangle \overset{\text{cvx.}}{\leq} \alpha L(f(y_t) - f(x_{t+1})) =: A_t.$$

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$$\leq A_t + B_t + C_t = \alpha L(f(y_t) - f(y_{t+1})) + C_t.$$

$(*) \leq \frac{1}{T}\left(\alpha L(f(y_0) - f(y_T)) + \frac{1}{2\alpha}\|x^* - z_0\|^2\right) \leq \frac{\sqrt{dL\|x^*-x_0\|^2}}{T} \leq \frac{d}{2}$, for $T = O((\frac{L}{d})^{\frac{1}{2}})$ and $\alpha = (\frac{\|x^*-x_0\|^2}{2Ld})^{\frac{1}{2}}$. The reduction applies.

Linear Coupling (Constrained)

Mirror lemma. Define v such that $(v - x_{k+1})/\tau = (z_t - z_{t+1})$, for $\tau = \alpha L$.

$$\langle w, z_t - u \rangle \leq \langle w, z_t - z'_{t+1} \rangle - \frac{1}{2} \left\| z_t - z'_{t+1} \right\|^2 + \frac{1}{2} \left\| u - z_t \right\|^2 - \frac{1}{2} \left\| u - z'_{t+1} \right\|^2$$

Finite sum stochastic convex optimization

Problem:

$$\min_{x\in\mathbb{R}^m}\left\{F(x) \overset{\text{def}}{=} f(x)+g(x) \overset{\text{def}}{=} \frac{1}{n}\sum_{i=1}^{n} f_i(x)+g(x)\right\}.$$

For differentiable L-smooth convex $f_i:\mathbb{R}^m\to\mathbb{R}$ and proper, μ-strongly convex ($\mu\geq 0$) $g:\mathbb{R}^m\to\mathbb{R}\cup\{\infty\}$ that can be non-differentiable but it is lower semicontinuous.

MSEB property: For $\rho_B,\rho_F,\rho_M\in(0,1]$:

$$\nabla f\left(x_{k+1}\right)-\mathbb{E}_k\widetilde{\nabla}_{k+1}=\left(1-\rho_B\right)\left(\nabla f\left(x_k\right)-\widetilde{\nabla}_k\right) \quad \leftarrow \text{bias}$$

$$\mathbb{E}\left\|\widetilde{\nabla}_{k+1}-\nabla f\left(x_{k+1}\right)\right\|^2\leq\mathcal{M}_k \quad \leftarrow \text{MSE}$$

$$\mathcal{M}_k\leq\frac{M_1}{n}\sum_{i=1}^{n}\mathbb{E}\left\|\nabla f_i\left(x_{k+1}\right)-\nabla f_i\left(x_k\right)\right\|^2+\mathcal{F}_k+\left(1-\rho_M\right)\mathcal{M}_{k-1}$$

$$\mathcal{F}_k\leq\sum_{\ell=0}^{k}\frac{M_2\left(1-\rho_F\right)^{k-\ell}}{n}\sum_{i=1}^{n}\mathbb{E}\left\|\nabla f_i\left(x_{\ell+1}\right)-\nabla f_i\left(x_\ell\right)\right\|^2$$

Algorithm

Given an estimator of the gradient $\tilde{\nabla}_{k+1}$The algorithm is the following, with proper learning rates γ_k and linear coupling parameters τ_k.

Initialize $x_0 = y_0 = z_0$.
for $k = 0, 1, \ldots, T-1$ **do**
    $x_{k+1} \leftarrow \tau_k z_k + (1-\tau_k) y_k$.
    Compute $\widetilde{\nabla}_{k+1}$, an estimate of $\nabla f(x_{k+1})$.
    $z_{k+1} \leftarrow \operatorname{prox}_{\gamma_k g}\left(z_k - \gamma_k \widetilde{\nabla}_{k+1}\right)$.
    $y_{k+1} \leftarrow \tau_k z_{k+1} + (1-\tau_k)\, y_k$
end for
return y_t.

Finite sum stochastic convex optimization

Common Estimators: (authors explicit rates for these + SARGE)
SVRG:

$$\widetilde{\nabla}_{k+1}^{\mathsf{SVRG}} \stackrel{\text{def}}{=} \frac{1}{|B_k|}\left(\sum_{b_j \in B_k} \nabla f_{b_j}\left(x_{k+1}\right) - \nabla f_{b_j}(\widetilde{x})\right) + \nabla f(\widetilde{x})$$

SAGA:

$$\widetilde{\nabla}_{k+1}^{\mathsf{SAGA}} \stackrel{\text{def}}{=} \frac{1}{|B_k|}\left(\sum_{b_j \in B_k} \nabla f_{b_j}\left(x_{k+1}\right) - \nabla f_{b_j}\left(\varphi_k^{b_j}\right)\right) + \frac{1}{n}\sum_{i=1}^{n} \nabla f_i\left(\varphi_k^i\right)$$

SARAH:

$$\widetilde{\nabla}_{k+1}^{\mathsf{SARAH}} \stackrel{\text{def}}{=} \begin{cases} \frac{1}{|B_k|}\left(\sum_{b_j \in B_k} \nabla f_{b_j}\left(x_{k+1}\right) - \nabla f_{b_j}\left(x_k\right)\right) + \widetilde{\nabla}_k^{\mathsf{SARAH}} & \text{w.p. } 1 - \frac{1}{p}, \\ \nabla f\left(x_{k+1}\right) & \text{w.p. } \frac{1}{p} \end{cases}$$

Bound of Main Result

$$\gamma_k \left(f\left(x_{k+1}\right) - f\left(x^*\right)\right)$$

Convexity.

Bound of Main Result

$$\gamma_k(f(x_{k+1}) - f(x^*))$$

$$\leq \gamma_k\langle \nabla f(x_{k+1}), x_{k+1} - x^*\rangle$$

Convexity.

Bound of Main Result

$$= \gamma_k \langle \nabla f(x_{k+1}), x_{k+1} - x^* \rangle$$

Add and substract z_k. Split.

Bound of Main Result

$$\gamma_k \langle \nabla f(x_{k+1}), x_{k+1} - z_k \rangle + \gamma_k \langle \nabla f(x_{k+1}), z_k - x^* \rangle$$

$$= \gamma_k \langle \nabla f(x_{k+1}), x_{k+1} - x^* \rangle$$

Add and substract z_k. Split.

Bound of Main Result

$$\gamma_k \langle \nabla f\left(x_{k+1}\right), x_{k+1} - z_k \rangle + \gamma_k \langle \nabla f\left(x_{k+1}\right), z_k - x^* \rangle$$

Linear coupling: $x_{k+1} = \tau_k z_k + \left(1 - \tau_k\right) y_k$

Bound of Main Result

$$\gamma_k \langle \nabla f(x_{k+1}), x_{k+1} - z_k \rangle + \gamma_k \langle \nabla f(x_{k+1}), z_k - x^* \rangle$$

$$= (\gamma_k (1-\tau_k)/\tau_k) \langle \nabla f(x_{k+1}), y_k - x_{k+1} \rangle + \gamma_k \langle \nabla f(x_{k+1}), z_k - x^* \rangle$$

Linear coupling: $x_{k+1} = \tau_k z_k + (1-\tau_k) y_k$

Bound of Main Result

$$= (\gamma_k (1 - \tau_k) / \tau_k) \langle \nabla f (x_{k+1}), y_k - x_{k+1} \rangle + \gamma_k \langle \nabla f (x_{k+1}), z_k - x^* \rangle$$

Make $D_f(y_k, x_{k+1})$ appear from the first term. Make $\tilde{\nabla}_{k+1}$ appear in the second term.

Bound of Main Result

$$\gamma_k (1-\tau_k)/\tau_k(f(y_k) - f(x_{k+1})) - \gamma_k (1-\tau_k)/\tau_k D_f(y_k, x_{k+1})$$

$$= (\gamma_k (1-\tau_k)/\tau_k)\langle \nabla f(x_{k+1}), y_k - x_{k+1}\rangle + \gamma_k \langle \nabla f(x_{k+1}), z_k - x^*\rangle$$

$$+\gamma_k \langle \widetilde{\nabla}_{k+1}, z_k - x^*\rangle + \gamma_k \langle \nabla f(x_{k+1}) - \widetilde{\nabla}_{k+1}, z_k - x^*\rangle$$

Make $D_f(y_k, x_{k+1})$ appear from the first term. Make $\tilde{\nabla}_{k+1}$ appear in the second term.

Bound of Main Result

$$\gamma_k\left(1-\tau_k\right)/\tau_k\left(f\left(y_k\right)-f\left(x_{k+1}\right)\right)-\gamma_k\left(1-\tau_k\right)/\tau_k D_f\left(y_k,x_{k+1}\right)$$

$$+\gamma_k\langle\widetilde{\nabla}_{k+1},z_k-x^*\rangle+\gamma_k\langle\nabla f\left(x_{k+1}\right)-\widetilde{\nabla}_{k+1},z_k-x^*\rangle$$

We add and substract z_{k+1} to ease the comparison.

Bound of Main Result

$$\gamma_k \left(1 - \tau_k\right) / \tau_k \left(f\left(y_k\right) - f\left(x_{k+1}\right)\right) - \gamma_k \left(1 - \tau_k\right) / \tau_k D_f \left(y_k, x_{k+1}\right)$$

$$\gamma_k \left(1 - \tau_k\right) / \tau_k \left(f\left(y_k\right) - f\left(x_{k+1}\right)\right) - \gamma_k \left(1 - \tau_k\right) / \tau_k D_f \left(y_k, x_{k+1}\right)$$

$$+\gamma_k \langle \widetilde{\nabla}_{k+1}, z_k - x^* \rangle + \gamma_k \langle \nabla f\left(x_{k+1}\right) - \widetilde{\nabla}_{k+1}, z_k - x^* \rangle$$

$$+\gamma_k \langle \widetilde{\nabla}_{k+1}, z_k - z_{k+1} \rangle + \gamma_k \langle \widetilde{\nabla}_{k+1}, z_{k+1} - x^* \rangle$$

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We add and substract z_{k+1} to ease the comparison.

Bound of Main Result

$$\gamma_k \left(1-\tau_k\right)/\tau_k \left(f\left(y_k\right)-f\left(x_{k+1}\right)\right)-\gamma_k \left(1-\tau_k\right)/\tau_k D_f\left(y_k, x_{k+1}\right)$$

$$+\gamma_k \langle \widetilde{\nabla}_{k+1}, z_k - z_{k+1}\rangle + \gamma_k \langle \widetilde{\nabla}_{k+1}, z_{k+1} - x^*\rangle$$

$$+\gamma_k \langle \nabla f\left(x_{k+1}\right) - \widetilde{\nabla}_{k+1}, z_k - x^*\rangle$$

Algorithm: $x_{k+1} - y_{k+1} = \tau_k (z_k - z_{k+1})$

Bound of Main Result

$$\gamma_k\left(1-\tau_k\right)/\tau_k\left(f\left(y_k\right)-f\left(x_{k+1}\right)\right)-\gamma_k\left(1-\tau_k\right)/\tau_k D_f\left(y_k,x_{k+1}\right)$$

$$\gamma_k\left(1-\tau_k\right)/\tau_k\left(f\left(y_k\right)-f\left(x_{k+1}\right)\right)-\gamma_k\left(1-\tau_k\right)/\tau_k D_f\left(y_k,x_{k+1}\right)$$

$$+\gamma_k/\tau_k\langle\widetilde{\nabla}_{k+1},x_{k+1}-y_{k+1}\rangle+\gamma_k\langle\widetilde{\nabla}_{k+1},z_k-x^*\rangle$$

$$+\gamma_k\langle\widetilde{\nabla}_{k+1},z_k-z_{k+1}\rangle+\gamma_k\langle\widetilde{\nabla}_{k+1},z_{k+1}-x^*\rangle$$

$$+\gamma_k\langle\nabla f\left(x_{k+1}\right)-\widetilde{\nabla}_{k+1},z_{k+1}-x^*\rangle$$

$$+\gamma_k\langle\nabla f\left(x_{k+1}\right)-\widetilde{\nabla}_{k+1},z_k-x^*\rangle$$

Algorithm: $x_{k+1}-y_{k+1}=\tau_k(z_k-z_{k+1})$

Bound of Main Result

$$\gamma_k\left(1-\tau_k\right)/\tau_k\left(f\left(y_k\right)-f\left(x_{k+1}\right)\right)-\gamma_k\left(1-\tau_k\right)/\tau_k D_f\left(y_k, x_{k+1}\right)$$

$$+\boxed{\gamma_k/\tau_k\langle\widetilde{\nabla}_{k+1}, x_{k+1}-y_{k+1}\rangle}+\boxed{\gamma_k\langle\widetilde{\nabla}_{k+1}, z_k-x^*\rangle}$$

$$+\gamma_k\langle\nabla f\left(x_{k+1}\right)-\widetilde{\nabla}_{k+1}, z_{k+1}-x^*\rangle$$

We will bound these two terms separately.

Bound of Main Result

$$\boxed{\frac{\gamma_k}{\tau_k}\langle \widetilde{\nabla}_{k+1}, x_{k+1} - y_{k+1} \rangle}$$

Compare to $\nabla f(x_{k+1})$.

Bound of Main Result

$$\boxed{\frac{\gamma_k}{\tau_k}\langle \widetilde{\nabla}_{k+1}, x_{k+1} - y_{k+1} \rangle}$$

$$\frac{\gamma_k}{\tau_k}\langle \nabla f(x_{k+1}), x_{k+1} - y_{k+1} \rangle + \frac{\gamma_k}{\tau_k}\langle \widetilde{\nabla}_{k+1} - \nabla f(x_{k+1}), x_{k+1} - y_{k+1} \rangle$$

Compare to $\nabla f(x_{k+1})$.

Bound of Main Result

$$\frac{\gamma_k}{\tau_k}\langle \nabla f(x_{k+1}), x_{k+1} - y_{k+1}\rangle + \frac{\gamma_k}{\tau_k}\langle \widetilde{\nabla}_{k+1} - \nabla f(x_{k+1}), x_{k+1} - y_{k+1}\rangle$$

We use smoothness.

Bound of Main Result

$$\leq \frac{\gamma_k}{\tau_k}\left(f(x_{k+1}) - f(y_{k+1}) + (L/2)\left\|x_{k+1} - y_{k+1}\right\|^2\right)$$

$$\frac{\gamma_k}{\tau_k}\langle \nabla f(x_{k+1}), x_{k+1} - y_{k+1}\rangle + \frac{\gamma_k}{\tau_k}\langle \widetilde{\nabla}_{k+1} - \nabla f(x_{k+1}), x_{k+1} - y_{k+1}\rangle$$

$$+\langle \widetilde{\nabla}_{k+1} - \nabla f(x_{k+1}), x_{k+1} - y_{k+1}\rangle$$

We use smoothness.

Bound of Main Result

$$\frac{\gamma_k}{\tau_k}\left(f(x_{k+1}) - f(y_{k+1}) + (L/2)\left\|x_{k+1} - y_{k+1}\right\|^2\right)$$

$$+\langle \widetilde{\nabla}_{k+1} - \nabla f(x_{k+1}), x_{k+1} - y_{k+1}\rangle$$

And Young's inequality.

Bound of Main Result

$$\frac{\gamma_k}{\tau_k}\left(f(x_{k+1}) - f(y_{k+1}) + (L/2)\left\|x_{k+1} - y_{k+1}\right\|^2\right)$$

$$\frac{\gamma_k}{\tau_k}\left(f(x_{k+1}) - f(y_{k+1}) + (L/2)\left\|x_{k+1} - y_{k+1}\right\|^2\right)$$

$$+\frac{\gamma_k}{\tau_k}\langle \widetilde{\nabla}_{k+1} - \nabla f(x_{k+1}), x_{k+1} - y_{k+1}\rangle$$

$$\leq \gamma_k^2 \left\|\widetilde{\nabla}_{k+1} - \nabla f(x_{k+1})\right\|^2 + \frac{1}{4\tau_k^2}\left\|x_{k+1} - y_{k+1}\right\|^2$$

And Young's inequality.

Bound of Main Result

$$\frac{\gamma_k}{\tau_k}\left(f(x_{k+1}) - f(y_{k+1}) + (L/2)\left\|x_{k+1} - y_{k+1}\right\|^2\right)$$

$$\gamma_k^2 \left\|\widetilde{\nabla}_{k+1} - \nabla f(x_{k+1})\right\|^2 + \frac{1}{4\tau_k^2}\left\|x_{k+1} - y_{k+1}\right\|^2$$

Group terms and make F appear.

Bound of Main Result

$$\frac{\gamma_k}{\tau_k}(f(x_{k+1}) - f(y_{k+1}) + \frac{\gamma_k}{\tau_k}(f(x_{k+1}) - F(y_{k+1}) + g(y_{k+1}))$$

$$\frac{\gamma_k}{\tau_k}(f(x_{k+1}) - f(y_{k+1}) + (L/2)\left\|x_{k+1} - y_{k+1}\right\|^2)$$

$$\leq \left(\frac{L\gamma_k}{2\tau_k} + \frac{1}{4\tau_k^2}\right)\left\|x_{k+1} - y_{k+1}\right\|^2 + \gamma_k^2\left\|\widetilde{\nabla}_{k+1} - \nabla f(x_{k+1})\right\|^2$$

$$\gamma_k^2\left\|\widetilde{\nabla}_{k+1} - \nabla f(x_{k+1})\right\|^2 + \frac{1}{4\tau_k^2}\left\|x_{k+1} - y_{k+1}\right\|^2$$

Group terms and make F appear.

Bound of Main Result

$$\frac{\gamma_k}{\tau_k}(f(x_{k+1}) - f(y_{k+1}) + \frac{\gamma_k}{\tau_k}(f(x_{k+1}) - F(y_{k+1}) + g(y_{k+1}))$$

$$\left(\frac{L\gamma_k}{2\tau_k} + \frac{1}{4\tau_k^2}\right)\|x_{k+1} - y_{k+1}\|^2 + \gamma_k^2 \left\|\widetilde{\nabla}_{k+1} - \nabla f(x_{k+1})\right\|^2$$

Algorithm: $y_{k+1} = \tau_k z_{k+1} + (1-\tau_k)y_k$. Convexity of g.

Bound of Main Result

$$\frac{\gamma_k}{\tau_k}(f(x_{k+1}) - f(y_{k+1}) + \frac{\gamma_k}{\tau_k}(f(x_{k+1}) - F(y_{k+1}) + g(y_{k+1}))$$

$$\leq \frac{\gamma_k}{\tau_k}(f(x_{k+1}) - F(y_{k+1})) + \gamma_k g\left(z_{k+1}\right) + \frac{\gamma_k\left(1-\tau_k\right)}{\tau_k} g\left(y_k\right)$$

$$\left(\frac{L\gamma_k}{2\tau_k} + \frac{1}{4\tau_k^2}\right)\|x_{k+1} - y_{k+1}\|^2 + \gamma_k^2\left\|\widetilde{\nabla}_{k+1} - \nabla f(x_{k+1})\right\|^2$$

$$+\frac{\gamma_k}{\tau_k}(f(x_{k+1}) - f(y_{k+1})$$

$$\left(\frac{L\gamma_k}{2\tau_k} + \frac{1}{4\tau_k^2}\right)\|x_{k+1} - y_{k+1}\|^2 + \gamma_k^2\left\|\widetilde{\nabla}_{k+1} - \nabla f(x_{k+1})\right\|^2$$

Algorithm: $y_{k+1} = \tau_k z_{k+1} + (1-\tau_k) y_k$. Convexity of g.

Bound of Main Result

$$\frac{\gamma_k}{\tau_k}(f(x_{k+1}) - F(y_{k+1})) + \gamma_k g\left(z_{k+1}\right) + \frac{\gamma_k\left(1-\tau_k\right)}{\tau_k} g\left(y_k\right)$$

$$+\frac{\gamma_k}{\tau_k}(f(x_{k+1}) - f(y_{k+1}))$$

$$\left(\frac{L\gamma_k}{2\tau_k} + \frac{1}{4\tau_k^2}\right)\left\|x_{k+1} - y_{k+1}\right\|^2 + \gamma_k^2\left\|\widetilde{\nabla}_{k+1} - \nabla f(x_{k+1})\right\|^2$$

Now we bound the other term.

Bound of Main Result

$$\boxed{\gamma_k \langle \widetilde{\nabla}_{k+1}, z_k - x^* \rangle}$$

Proximal Lemma.

Bound of Main Result

$$\boxed{\gamma_k \langle \widetilde{\nabla}_{k+1}, z_k - x^* \rangle}$$

$$\leq \frac{1}{2} \left\| z_k - x^* \right\|^2 - \frac{1 + \mu\gamma_k}{2} \left\| z_{k+1} - x^* \right\|^2 - \frac{1}{2} \left\| z_{k+1} - z_k \right\|^2$$

$$-\gamma_k g\left(z_{k+1}\right) + \gamma_k g\left(x^*\right)$$

Proximal Lemma.

Bound of Main Result

$$\frac{1}{2}\left\|z_k - x^*\right\|^2 - \frac{1+\mu\gamma_k}{2}\left\|z_{k+1} - x^*\right\|^2 - \frac{1}{2}\left\|z_{k+1} - z_k\right\|^2$$

$$-\gamma_k g\left(z_{k+1}\right) + \gamma_k g\left(x^*\right)$$

Algorithm: $x_{k+1} - y_{k+1} = \tau_k(z_{k+1} - z_k)$.

Bound of Main Result

$$\leq \frac{1}{2}\left\|z_k - x^*\right\|^2 - \frac{1+\mu\gamma_k}{2}\left\|z_{k+1} - x^*\right\|^2 - \frac{1}{2\tau_k^2}\left\|x_{k+1} - y_{k+1}\right\|^2$$

$$\frac{1}{2}\left\|z_k - x^*\right\|^2 - \frac{1+\mu\gamma_k}{2}\left\|z_{k+1} - x^*\right\|^2 - \frac{1}{2}\left\|z_{k+1} - z_k\right\|^2$$

$$-\gamma_k g\left(z_{k+1}\right) + \gamma_k g\left(x^*\right)$$

$$-\gamma_k g\left(z_{k+1}\right) + \gamma_k g\left(x^*\right)$$

Bound of Main Result

$$\frac{1}{2}\left\|z_k - x^*\right\|^2 - \frac{1+\mu\gamma_k}{2}\left\|z_{k+1} - x^*\right\|^2 - \frac{1}{2\tau_k^2}\left\|x_{k+1} - y_{k+1}\right\|^2$$

$$-\gamma_k g\left(z_{k+1}\right) + \gamma_k g\left(x^*\right)$$

Going back to the original inequality and adding all up we have:

Bound of Main Result

$$
\begin{aligned}
F(y_k) - F(x^*) &\overset{H_k}{\leq} \frac{1}{\tau_k} F(y_k) - \frac{1}{\tau_k} F(y_{k+1}) \\
&+ \frac{1}{\tau_k}\left(\frac{L}{2} - \frac{1}{4\tau_k\gamma_k}\right)\|x_{k+1} - y_{k+1}\|^2 + \frac{1}{2\gamma_k}\|z_k - x^*\|^2 \\
&- \frac{1+\mu\gamma_k}{2\gamma_k}\|z_{k+1} - x^*\|^2 + \left\langle \nabla f(x_{k+1}) - \widetilde{\nabla}_{k+1}, z_k - x^* \right\rangle \\
&- \frac{(1-\tau_k)}{\tau_k} D_f(y_k, x_{k+1}) + \gamma_k \left\| \widetilde{\nabla}_{k+1} - \nabla f(x_{k+1}) \right\|^2
\end{aligned}
$$

We will add up $\sum_{k=0}^{T-1} \gamma_k H_k$.

Bound on MSE

Remember the bound on the MSE

$$\mathbb{E}\left\|\widetilde{\nabla}_{k+1} - \nabla f\left(x_{k+1}\right)\right\|^2 \leq \mathcal{M}_k$$

$$\mathcal{M}_k \leq \frac{M_1}{n} \sum_{i=1}^{n} \mathbb{E}\left\|\nabla f_i\left(x_{k+1}\right) - \nabla f_i\left(x_k\right)\right\|^2 + \mathcal{F}_k + \left(1 - \rho_M\right) \mathcal{M}_{k-1}$$

$$\mathcal{F}_k \leq \sum_{\ell=0}^{k} \frac{M_2\left(1 - \rho_F\right)^{k-\ell}}{n} \sum_{i=1}^{n} \mathbb{E}\left\|\nabla f_i\left(x_{\ell+1}\right) - \nabla f_i\left(x_\ell\right)\right\|^2$$

Bound on MSE

Let $\rho = \min\{\rho_M, \rho_B, \rho_F\}$ and let $\{s_k\}_{k=1}^T \geq_0$ be any sequence s.t. $s_k^2(1-\rho) \leq s_{k-1}^2(1-\frac{\rho}{2})$. Let $\Theta_2 = \frac{M_1\rho_F + 2M_2}{\rho_M \rho_F}$.

Bound on MSE

Let $\rho = \min\{\rho_M, \rho_B, \rho_F\}$ and let $\{s_k\}_{k=1}^{T} \geq 0$ be any sequence s.t. $s_k^2(1-\rho) \leq s_{k-1}^2(1-\frac{\rho}{2})$. Let $\Theta_2 = \frac{M_1\rho_F + 2M_2}{\rho_M \rho_F}$.

$$\sum_{k=0}^{T-1} s_k^2 \mathcal{M}_k \leq \sum_{k=0}^{T-1} s_k^2 \mathbb{E}\left\|\nabla f\left(x_{k+1}\right) - \widetilde{\nabla}_{k+1}\right\|^2$$

Use the definition of MSE.

Bound on MSE

Let $\rho = \min\{\rho_M, \rho_B, \rho_F\}$ and let $\{s_k\}_{k=1}^{T} \geq_0$ be any sequence s.t. $s_k^2(1-\rho) \leq s_{k-1}^2(1-\frac{\rho}{2})$. Let $\Theta_2 = \frac{M_1\rho_F + 2M_2}{\rho_M \rho_F}$.

$$\sum_{k=0}^{T-1} s_k^2 \mathcal{M}_k \leq \sum_{k=0}^{T-1} s_k^2 \mathbb{E}\left\|\nabla f\left(x_{k+1}\right) - \widetilde{\nabla}_{k+1}\right\|^2$$

$$\leq \sum_{k=0}^{T-1} \frac{M_1 s_k^2}{n} \sum_{i=1}^{n} \mathbb{E}\left\|\nabla f_i\left(x_{k+1}\right) - \nabla f_i\left(x_k\right)\right\|^2 + s_k^2 \mathcal{F}_k + s_k^2\left(1-\rho_M\right)\mathcal{M}_{k-1}$$

Use the definition of MSE.

Bound on MSE

Let $\rho = \min\{\rho_M, \rho_B, \rho_F\}$ and let $\{s_k\}_{k=1}^{T} \geq_0$ be any sequence s.t. $s_k^2(1-\rho) \leq s_{k-1}^2(1-\frac{\rho}{2})$. Let $\Theta_2 = \frac{M_1\rho_F + 2M_2}{\rho_M \rho_F}$.

$$\sum_{k=0}^{T-1} \frac{M_1 s_k^2}{n} \sum_{i=1}^{n} \mathbb{E}\left\|\nabla f_i\left(x_{k+1}\right) - \nabla f_i\left(x_k\right)\right\|^2 + s_k^2 \mathcal{F}_k + s_k^2\left(1-\rho_M\right)\mathcal{M}_{k-1}$$

Substitute the value of $\mathcal{F}_k$. Use $\sum_{k=0}^{j}(1-\rho_F)^k \leq \frac{1}{\rho_F}$.

Bound on MSE

Let $\rho = \min\{\rho_M, \rho_B, \rho_F\}$ and let $\{s_k\}_{k=1}^T \geq_0$ be any sequence s.t. $s_k^2(1-\rho) \leq s_{k-1}^2(1-\frac{\rho}{2})$. Let $\Theta_2 = \frac{M_1\rho_F + 2M_2}{\rho_M \rho_F}$.

$$\leq \sum_{k=0}^{T-1} \frac{\left(M_1\rho_F + 2M_2\right) s_k^2}{n\rho_F} \sum_{i=1}^{n} \mathbb{E}\left\|\nabla f_i\left(x_{k+1}\right) - \nabla f_i\left(x_k\right)\right\|^2 + s_k^2\left(1-\rho_M\right)\mathcal{M}_{k-1}$$

$$\sum_{k=0}^{T-1} \frac{M_1 s_k^2}{n} \sum_{i=1}^{n} \mathbb{E}\left\|\nabla f_i\left(x_{k+1}\right) - \nabla f_i\left(x_k\right)\right\|^2 + s_k^2\mathcal{F}_k + s_k^2\left(1-\rho_M\right)\mathcal{M}_{k-1}$$

Substitute the value of $\mathcal{F}_k$. Use $\sum_{k=0}^{j}(1-\rho_F)^k \leq \frac{1}{\rho_F}$.

Bound on MSE

Let $\rho = \min\{\rho_M, \rho_B, \rho_F\}$ and let $\{s_k\}_{k=1}^T \geq_0$ be any sequence s.t. $s_k^2(1-\rho) \leq s_{k-1}^2(1-\frac{\rho}{2})$. Let $\Theta_2 = \frac{M_1\rho_F+2M_2}{\rho_M\rho_F}$.

$$\sum_{k=0}^{T-1} \frac{(M_1\rho_F + 2M_2)\, s_k^2}{n\rho_F} \sum_{i=1}^{n} \mathbb{E}\left\|\nabla f_i\left(x_{k+1}\right) - \nabla f_i\left(x_k\right)\right\|^2 + s_k^2\left(1-\rho_M\right)\mathcal{M}_{k-1}$$

Use definition of Θ_2. Recur the inequality over $\mathcal{M}_{k-1}, \mathcal{M}_{k-2}, \ldots$

Bound on MSE

Let $\rho = \min\{\rho_M, \rho_B, \rho_F\}$ and let $\{s_k\}_{k=1}^{T} \geq_0$ be any sequence s.t. $s_k^2(1-\rho) \leq s_{k-1}^2(1-\frac{\rho}{2})$. Let $\Theta_2 = \frac{M_1\rho_F + 2M_2}{\rho_M \rho_F}$.

$$\sum_{k=0}^{T-1} \frac{(M_1\rho_F + 2M_2)\, s_k^2}{n\rho_F} \sum_{i=1}^{n} \mathbb{E}\left\|\nabla f_i(x_{k+1}) - \nabla f_i(x_k)\right\|^2 + s_k^2(1-\rho_M)\,\mathcal{M}_{k-1}$$

$$\leq \sum_{k=0}^{T-1} \sum_{\ell=1}^{k} \frac{\Theta_2 s_k^2 (1-\rho_M)^{k-\ell} \rho_M}{n} \sum_{i=1}^{n} \mathbb{E}\left\|\nabla f_i(x_{\ell+1}) - \nabla f_i(x_\ell)\right\|^2$$

Use definition of Θ_2. Recur the inequality over $\mathcal{M}_{k-1}, \mathcal{M}_{k-2}, \ldots$

Bound on MSE

Let $\rho = \min\{\rho_M, \rho_B, \rho_F\}$ and let $\{s_k\}_{k=1}^{T} \geq_0$ be any sequence s.t. $s_k^2(1-\rho) \leq s_{k-1}^2(1-\frac{\rho}{2})$. Let $\Theta_2 = \frac{M_1\rho_F + 2M_2}{\rho_M \rho_F}$.

$$\sum_{k=0}^{T-1} \sum_{\ell=1}^{k} \frac{\Theta_2 s_k^2 \left(1-\rho_M\right)^{k-\ell} \rho_M}{n} \sum_{i=1}^{n} \mathbb{E} \left\| \nabla f_i\left(x_{\ell+1}\right) - \nabla f_i\left(x_\ell\right) \right\|^2$$

Use property of $\{s_k\}$.

Bound on MSE

Let $\rho = \min\{\rho_M, \rho_B, \rho_F\}$ and let $\{s_k\}_{k=1}^T \geq_0$ be any sequence s.t. $s_k^2(1-\rho) \leq s_{k-1}^2(1-\frac{\rho}{2})$. Let $\Theta_2 = \frac{M_1\rho_F + 2M_2}{\rho_M \rho_F}$.

$$\leq \sum_{k=0}^{T-1} \sum_{\ell=1}^{k} \frac{\Theta_2 s_\ell^2 \left(1 - \frac{\rho_M}{2}\right)^{k-\ell} \rho_M}{n} \sum_{i=1}^{n} \mathbb{E} \left\| \nabla f_i \left(x_{\ell+1}\right) - \nabla f_i \left(x_\ell\right) \right\|^2$$

$$\sum_{k=0}^{T-1} \sum_{\ell=1}^{k} \frac{\Theta_2 s_k^2 \left(1 - \rho_M\right)^{k-\ell} \rho_M}{n} \sum_{i=1}^{n} \mathbb{E} \left\| \nabla f_i \left(x_{\ell+1}\right) - \nabla f_i \left(x_\ell\right) \right\|^2$$

Use property of $\{s_k\}$.

Bound on MSE

Let $\rho = \min\{\rho_M, \rho_B, \rho_F\}$ and let $\{s_k\}_{k=1}^T \geq 0$ be any sequence s.t. $s_k^2(1-\rho) \leq s_{k-1}^2(1-\frac{\rho}{2})$. Let $\Theta_2 = \frac{M_1\rho_F + 2M_2}{\rho_M \rho_F}$.

$$\sum_{k=0}^{T-1} \sum_{\ell=1}^{k} \frac{\Theta_2 s_\ell^2 \left(1 - \frac{\rho_M}{2}\right)^{k-\ell} \rho_M}{n} \sum_{i=1}^{n} \mathbb{E} \left\| \nabla f_i \left(x_{\ell+1}\right) - \nabla f_i \left(x_\ell\right) \right\|^2$$

Use inequality $\sum_{k=1}^T \sum_{\ell=1}^k (1-\delta)^{k-\ell} \sigma_\ell \leq \frac{1}{\delta} \sum_{k=1}^T \sigma_k$, if $\delta \in (0,1]$.

Bound on MSE

Let $\rho = \min\{\rho_M, \rho_B, \rho_F\}$ and let $\{s_k\}_{k=1}^{T} \geq_0$ be any sequence s.t. $s_k^2(1-\rho) \leq s_{k-1}^2(1-\frac{\rho}{2})$. Let $\Theta_2 = \frac{M_1\rho_F + 2M_2}{\rho_M \rho_F}$.

$$\sum_{k=0}^{T-1} \sum_{\ell=1}^{k} \frac{\Theta_2 s_\ell^2 \left(1 - \frac{\rho_M}{2}\right)^{k-\ell} \rho_M}{n} \sum_{i=1}^{n} \mathbb{E} \left\| \nabla f_i\left(x_{\ell+1}\right) - \nabla f_i\left(x_\ell\right) \right\|^2$$

$$\leq \sum_{k=0}^{T-1} \frac{2\Theta_2 s_k^2}{n} \sum_{i=1}^{n} \mathbb{E} \left\| \nabla f_i\left(x_{k+1}\right) - \nabla f_i\left(x_k\right) \right\|^2$$

Use inequality $\sum_{k=1}^{T} \sum_{\ell=1}^{k} (1-\delta)^{k-\ell} \sigma_\ell \leq \frac{1}{\delta} \sum_{k=1}^{T} \sigma_k$, if $\delta \in (0,1]$.

Bound on MSE

Let $\rho = \min\{\rho_M, \rho_B, \rho_F\}$ and let $\{s_k\}_{k=1}^{T} \geq_0$ be any sequence s.t. $s_k^2(1-\rho) \leq s_{k-1}^2(1-\frac{\rho}{2})$. Let $\Theta_2 = \frac{M_1\rho_F + 2M_2}{\rho_M\rho_F}$.

$$\sum_{k=0}^{T-1} \frac{2\Theta_2 s_k^2}{n} \sum_{i=1}^{n} \mathbb{E}\left\|\nabla f_i\left(x_{k+1}\right) - \nabla f_i\left(x_k\right)\right\|^2$$

To make $D_f(y_k, x_k + 1)$ and $\left\|x_k - y_k\right\|^2$ appear, use $\left\|a-c\right\|^2 \leq \left\|a-b\right\|^2 + \left\|b-c\right\|^2$ (triangular $+$ quadratic-arithmetic inequality).

Bound on MSE

Let $\rho = \min\{\rho_M, \rho_B, \rho_F\}$ and let $\{s_k\}_{k=1}^{T} \geq 0$ be any sequence s.t. $s_k^2(1-\rho) \leq s_{k-1}^2(1-\frac{\rho}{2})$. Let $\Theta_2 = \frac{M_1\rho_F + 2M_2}{\rho_M \rho_F}$.

$$\leq \sum_{k=0}^{T-1} \frac{4\Theta_2 s_k^2}{n} \sum_{i=1}^{n} \mathbb{E} \left\| \nabla f_i\left(x_{k+1}\right) - \nabla f_i\left(y_k\right) \right\|^2$$

$$\sum_{k=0}^{T-1} \frac{2\Theta_2 s_k^2}{n} \sum_{i=1}^{n} \mathbb{E} \left\| \nabla f_i\left(x_{k+1}\right) - \nabla f_i\left(x_k\right) \right\|^2$$

$$+ \frac{4\Theta_2 s_k^2}{n} \sum_{i=1}^{n} \mathbb{E} \left\| \nabla f_i\left(y_k\right) - \nabla f_i\left(x_k\right) \right\|^2$$

To make $D_f(y_k, x_k + 1)$ and $\|x_k - y_k\|^2$ appear, use $\|a - c\|^2 \leq \|a - b\|^2 + \|b - c\|^2$ (triangular $+$ quadratic-arithmetic inequality).

Bound on MSE

Let $\rho = \min\{\rho_M, \rho_B, \rho_F\}$ and let $\{s_k\}_{k=1}^{T} \geq_0$ be any sequence s.t. $s_k^2(1-\rho) \leq s_{k-1}^2(1-\frac{\rho}{2})$. Let $\Theta_2 = \frac{M_1\rho_F + 2M_2}{\rho_M \rho_F}$.

$$\sum_{k=0}^{T-1} \frac{4\Theta_2 s_k^2}{n} \sum_{i=1}^{n} \mathbb{E}\left\|\nabla f_i\left(x_{k+1}\right) - \nabla f_i\left(y_k\right)\right\|^2$$

$$+\frac{4\Theta_2 s_k^2}{n} \sum_{i=1}^{n} \mathbb{E}\left\|\nabla f_i\left(y_k\right) - \nabla f_i\left(x_k\right)\right\|^2$$

Apply Lipschitz continuity of ∇f_i and $\|\nabla f_i(x) - \nabla f_i(y)\|^2 \leq 2LD_f(y,x)$.

Bound on MSE

Let $\rho = \min\{\rho_M, \rho_B, \rho_F\}$ and let $\{s_k\}_{k=1}^{T} \geq_0$ be any sequence s.t. $s_k^2(1-\rho) \leq s_{k-1}^2(1-\frac{\rho}{2})$. Let $\Theta_2 = \frac{M_1\rho_F + 2M_2}{\rho_M \rho_F}$.

$$\sum_{k=0}^{T-1} \frac{4\Theta_2 s_k^2}{n} \sum_{i=1}^{n} \mathbb{E} \left\| \nabla f_i\left(x_{k+1}\right) - \nabla f_i\left(y_k\right) \right\|^2$$

$$\leq 8 \sum_{k=0}^{T-1} \Theta_2 L s_k^2 \mathbb{E} D_f\left(y_k, x_{k+1}\right)$$

$$+ \frac{4\Theta_2 s_k^2}{n} \sum_{i=1}^{n} \mathbb{E} \left\| \nabla f_i\left(y_k\right) - \nabla f_i\left(x_k\right) \right\|^2$$

$$\sum_{k=0}^{T-1} + 4\Theta_2 L^2 s_k^2 \mathbb{E} \left\| x_k - y_k \right\|^2$$

Apply $\|\nabla f_i(x) - \nabla f_i(y)\|^2 \leq 2LD_f(y,x)$ and Lipschitz continuity of ∇f_i.

Bound on MSE

Let $\rho = \min\{\rho_M, \rho_B, \rho_F\}$ and let $\{s_k\}_{k=1}^{T} \geq 0$ be any sequence s.t. $s_k^2(1-\rho) \leq s_{k-1}^2(1-\frac{\rho}{2})$. Let $\Theta_2 = \frac{M_1\rho_F + 2M_2}{\rho_M \rho_F}$.

$$8\sum_{k=0}^{T-1} \Theta_2 L s_k^2 \mathbb{E} D_f\left(y_k, x_{k+1}\right)$$

$$\sum_{k=0}^{T-1} +4\Theta_2 L^2 s_k^2 \mathbb{E}\left\|x_k - y_k\right\|^2$$

Bias is bounded similarly.

Finite sum stochastic convex optimization

Parameters of usual estimators and rates under the accelerated meta theorem of this paper, up to constants (and complexity of a near optimal algorithm (Katyusha)):

	Full	aSVRG	aSAGA	aSARAH	aSARGE	Katyusha
M_1	0	$O(n/b^2)$	$O(n/b^2)$	$O(1)$	$O(1/n)$	-
M_2	0	0	0	0	$O(1/n^2)$	-
ρ_M	1	$O(b/n)$	$O(b/n)$	$O(1/n)$	$O(b/n)$	-
ρ_B	1	1	1	$O(1/n)$	$O(b/n)$	-
ρ_F	1	1	1	1	$O(b/n)$	-
cvx	$\frac{n}{\sqrt{\varepsilon}}$	$\frac{n}{\sqrt{\varepsilon}}$	$\frac{n}{\sqrt{\varepsilon}}$	$\frac{n^2}{\sqrt{\varepsilon}}$	$\frac{n^2}{\sqrt{\varepsilon}}$	$n \log 1/\varepsilon + \sqrt{n/\varepsilon}$
$\frac{\text{st. cvx}}{\log 1/\varepsilon}$	$n\sqrt{\kappa}$	$n^{2/3}\sqrt{\kappa}$	$n^{2/3}\sqrt{\kappa}+n$	$n^2\sqrt{\kappa}$	$n^2\sqrt{\kappa}$	$n+\sqrt{n\kappa}$