EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

Exam Calculus, 2WBB1
Monday 30 January 2017, 18.00-21.00 hours

Answers

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1. c

$$\arcsin(\sin(\tfrac{3}{4}\pi)) = \arcsin(\tfrac{1}{2}\sqrt{2}) = +\tfrac{1}{4}\pi$$

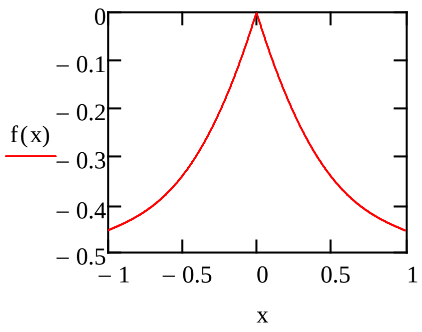
because $-\frac{1}{2}\pi \leq \arcsin(x) \leq +\frac{1}{2}\pi$ by definition

2. c

$$f(x) = \begin{cases} 1+x-\dfrac{\sin(x)\,\cos(x)}{x} & \text{if } x<0 \\ 0 & \text{if } x=0 \\ 1-x-\dfrac{\sin(x)\,\cos(x)}{x} & \text{if } x>0 \end{cases} \quad \text{or} \quad f(x) = \begin{cases} 1+x-\dfrac{\sin(2x)}{2x} & \text{if } x<0 \\ 0 & \text{if } x=0 \\ 1-x-\dfrac{\sin(2x)}{2x} & \text{if } x>0 \end{cases} \quad \text{or}$$

$$f(x) \approx \begin{cases} +x+\dfrac{2}{3}x^2 & \text{if } x<0 \\ 0 & \text{if } x=0 \\ -x+\dfrac{2}{3}x^2 & \text{if } x>0 \end{cases}$$

because $\sin(2x) \approx (2x)-\frac{1}{6}(2x)^3$ and $\dfrac{\sin(2x)}{2x} \approx 1-\frac{2}{3}x^3$



f is continuous in $x = 0$ because $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 0 = f(0)$

f is not differentiable in $x = 0$ because $+1 = \lim_{x\to 0^-} f'(x) \neq \lim_{x\to 0^+} f'(x) = -1$

3. a

swap x and y and solve for y

$$x = \frac{1+e^y}{2+e^y} \iff 2x + x e^y = 1 + e^y \iff (x-1)e^y = 1-2x \iff e^y = \frac{1-2x}{x-1} \iff$$

$$y = \ln\left(\frac{1-2x}{x-1}\right) = \ln\left(-\frac{1-2x}{1-x}\right)$$

4. d

$$\lim_{x\to 0} \frac{\tan(3x)-\sin(3x)}{x^3} = \lim_{x\to 0} \frac{\frac{\sin(3x)}{\cos(3x)} - \sin(3x)}{x^3} = \lim_{x\to 0} \frac{\sin(3x)\left(1-\cos(3x)\right)}{x^3\cos(3x)} =$$

$$\lim_{x\to 0} \frac{\left((3x)-\frac{1}{6}(3x)^3+\ldots\right)\left(\frac{1}{2}(3x)^2-\ldots\right)}{x^3\left(1-\frac{1}{2}(3x)^2+\ldots\right)} = \lim_{x\to 0} \frac{\frac{27}{2}x^3-\ldots}{x^3-\ldots} = \frac{27}{2}$$

Alternative: apply l'Hôpital's rule three times

5. b

Implicit differentiation gives:

$y + x y' + \cos(x+y)(1+y') = 5 - y'$ or $\left(x+\cos(x+y)+1\right) y' = 5 - y - \cos(x+y)$.

Substitution of $x=1$ and $y=-1$ gives: $3y'=5$, from which it follows that $y'(1,-1) = \frac{5}{3}$.

6. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

$\begin{cases} x+y+z=3 \\ x-y+3z=3 \end{cases}$ can be rewritten as $\begin{cases} 2x+4z=6 \\ 2y-2z=0 \\ z=\lambda=\text{free} \end{cases}$ or $\begin{cases} x=3-2\lambda \\ y=\lambda \\ z=\lambda \end{cases}$

Alternative: the line of intersection is perpendicular to the normal vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ of the two planes and can be calculated as $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$, which gives a direction vector $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$. Possible position vectors are $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \ldots$.

7. 7

Multiply with $\dfrac{\sqrt{x^2+13x}+\sqrt{x^2-x}}{\sqrt{x^2+13x}+\sqrt{x^2-x}}$ to obtain $\dfrac{14x}{\sqrt{x^2+13x}+\sqrt{x^2-x}}$ and

divide numerator and denominator by $x=\sqrt{x^2}$ to obtain $\dfrac{14}{\sqrt{1+\dfrac{13}{x}}+\sqrt{1-\dfrac{1}{x}}}$,

which in the limit $x\to\infty$ gives the answer $\dfrac{14}{2}$.

8a $7+\dfrac{1}{14}(x-49)-\dfrac{1}{2744}(x-49)^2$

$h(x)=\sqrt{x}$, $h'(x)=\dfrac{1}{2\sqrt{x}}$ and $h''(x)=\dfrac{-1}{4x\sqrt{x}}$.

The required Taylor polynomial is:

$$P_2(x) \ = \ \sqrt{49} \ + \ \frac{1}{2\sqrt{49}}(x-49) \ - \ \frac{1}{2}\frac{1}{4\cdot 49\sqrt{49}}(x-49)^2$$

8b. $7\dfrac{1}{14} \ > \ \sqrt{50}$

$$\sqrt{50} \ \approx \ P_1(50) \ = \ 7+\frac{1}{14}(50-49) \ = \ 7+\frac{1}{14}$$

This linear approximation is larger than $\sqrt{50}$ because $h''(49)<0$; $P_1(50)>P_2(50)\approx\sqrt{50}$.

Alternative: $\left(7+\dfrac{1}{14}\right)^2 \ = \ 49+2\cdot 7\cdot\dfrac{1}{14}+(\dfrac{1}{14})^2 \ = \ 50+(\dfrac{1}{14})^2 \ > \ 50$

9. $\dfrac{1}{6}\pi-\dfrac{1}{8}\sqrt{3}$

Substitute $\begin{cases} x=\sin(u) \\ \mathrm{d}x=\cos(u)\,\mathrm{d}u \end{cases}$ to obtain

$$\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}\sqrt{1-\sin^2(u)}\ \cos(u)\ \mathrm{d}u \ = \ \int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}\cos^2(u)\ \mathrm{d}u \ = \ \frac{1}{2}\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}\big(1+\cos(2u)\big)\ \mathrm{d}u \ = \ \frac{1}{2}\left[u+\frac{1}{2}\sin(2u)\right]_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}$$

Alternative: This is the area of 1/6 of a circle of radius 1 minus the area of a triangle with base 1/2 and height $1/2\sqrt{3}$.

10. $10\dfrac{2}{3}\ln(2) \ + \ \dfrac{1}{2}$

Integrate by parts ($\int u\,\mathrm{d}v=uv-\int v\,\mathrm{d}u$ with $u=\ln(1+x)$ and $v=\dfrac{1}{3}x^3-x$). This gives:

$$\int_0^3 \ln(1+x)\ d(\frac{1}{3}x^3 - x) = \left[\ln(1+x)\ (\frac{1}{3}x^3 - x) \right]_0^3 - \int_0^3 (\frac{1}{3}x^3 - x)\ d\big(\ln(1+x)\big) = 6\ \ln(4) - \int_0^3 \frac{\frac{1}{3}x^3 - x}{1+x} dx =$$

$12\ \ln(2) - \frac{1}{3}\int_0^3 \frac{x^3 - 3x}{x+1} dx$. The remaining integral can be solved after long division:

$$\int_0^3 \frac{x^3 - 3x}{x+1} dx = \int_0^3 \left(x^2 - x - 2 + \frac{2}{x+1} \right) dx = \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 2\ln(x+1) \right]_0^3 = 9 - \frac{9}{2} - 6 + 2\ln(4) = -\frac{3}{2} + 4\ \ln(2)$$

Hence the answer is: $12\ \ln(2) - \frac{1}{3}\left(-\frac{3}{2} + 4\ \ln(2) \right) = (12 - \frac{4}{3})\ \ln(2) + \frac{1}{2}$

Alternative: Substitute $\begin{cases} x = u - 1 \\ dx = du \end{cases}$ to obtain

$$\int_1^4 (u^2 - 2u)\ln(u)\ du = \left[\ln(u)\ (\frac{1}{3}u^3 - u^2) \right]_1^4 - \int_1^4 (\frac{1}{3}u^3 - u^2)\ d\big(\ln(u)\big) = \left(\frac{64}{3} - 16 \right) \ln(4) - \int_1^4 \frac{\frac{1}{3}u^3 - u^2}{u} du =$$

$$\left(\frac{16}{3} \right) \ln(4) - \int_1^4 \frac{\frac{1}{3}u^3 - u^2}{u} du = \left(\frac{32}{3} \right) \ln(2) - \int_1^4 \left(\tfrac{1}{3}u^2 - u \right) du = \left(\frac{32}{3} \right) \ln(2) - \left[\frac{1}{9}u^3 - \frac{1}{2}u^2 \right]_1^4 = \left(\frac{32}{3} \right) \ln(2) - \left(\frac{63}{9} - \frac{15}{2} \right)$$

11. $y = \tan\left(\frac{\pi}{4} - \frac{1}{x} \right)$

Separation of variables gives $\frac{dy}{1+y^2} = \frac{dx}{x^2}$ so that $\int \frac{1}{1+y^2} dy = \int \frac{1}{x^2} dx$ from which it follows that $\arctan(y) = -\frac{1}{x} + C$ and $y = \tan\left(-\frac{1}{x} + C \right)$.

The condition $\lim_{x \to \infty} y(x) = 1$ requires that $C = \frac{\pi}{4}$.

Marks

4 points for multiple-choice questions and 5 points for open questions.

Total of 50 points.