Tentamen Calculus B (2WBB1).

Multiple choice questions: 4 points each. **1. b**; **2. b**; **3. b**; **4. c**; **5. c**.

Open Questions

[3] **6.** In $\mathbb{R}^3$ is given the triangle ABC, with $A = (1,1,1)$, $B = (4,3,3)$ and $C = (1,2,5)$. Show that this triangle is isosceles (so two of the three sides have the same length) and determine the cosine of the angle at the apex (so the angle between these two sides).

Answer. $|AB| = \sqrt{9+4+4} = \sqrt{17} = |AC| = \sqrt{0+1+16}$, so these two sides have the same length, and the apex is A.
The cosine of the angle at A equals $(3,2,2) \bullet (0,1,4)/17 = 10/17$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = (x^2+1)\arctan(x) - x$.

[2] **(a)** Determine the derivative $f'(x)$ and show that $f'(x) \geq 0$ for all $x \in \mathbb{R}$ and $f'(x) > 0$ for $x \neq 0$.

[4] **(b)** From **(a)** it follows that f is invertible. Determine the derivative of the inverse f^{-1} of f in the point $1 - \frac{1}{2}\pi$, so $(f^{-1})'(1 - \frac{1}{2}\pi)$.

Answer (a) $f'(x) = 2x\arctan(x)$, x and $\arctan x$ have the same sign, and are 0 only for $x = 0$, so $f'(x) \geq 0$ and $f'(x) > 0$ for $x \neq 0$.

Answer (b) Since $f(-1) = -\frac{1}{2}\pi + 1$ and $f'(-1) = \frac{1}{2}\pi$ the derivative becomes $(f^{-1})'(1 - \frac{1}{2}\pi) = 1/f'(-1) = \frac{2}{\pi}$.

[4] **8.** Determine the Taylor polynomial of degree 3 around $a = \pi$ of $f(x) = e^{\sin(x)}$.

Answer. $f = e^s$, $f' = ce^s$, $f'' = (-s + c^2)e^s$ and $f''' = (-c - 2cs + c(-s + c^2))e^s$. Here $s = \sin x = \sin \pi = 0$ and $c = \cos x = \cos \pi = -1$, so $f(\pi) = 1$, $f'(\pi) = -1$; $f''(\pi) = 1$ and $f''' = 0$. The Taylorpolynomial is $P_3(x) = 1 - (x - \pi) + \frac{1}{2}(x - \pi)^2$.

9. Compute the following integrals (simplify the answer if possible):

[4] **(a)** $\displaystyle\int_{\ln(\pi/6)}^{\ln(\pi/2)} e^x \sin(\tfrac{1}{3}\pi - e^x)\, dx.$

[4] **(b)** $\displaystyle\int \frac{x^4 - 6x^3 + 7x^2}{x^2 - 6x + 8}\, dx.$

Answer (a) We apply the substitution $u = e^x - \dfrac{\pi}{3}$ and $du = e^x dx$, and now $\ln\dfrac{\pi}{6} \le x \le \ln\dfrac{\pi}{2}$ give $-\frac{1}{6}\pi \le u \le \frac{1}{6}\pi$. In terms of u we compute $\displaystyle\int_{-\pi/6}^{\pi/6} -\sin u\, du = 0$.

Answer (b) We start by dividing: $x^4 - 6x^3 + 7x^2 = (x^2-1)(x^2-6x+8) - (6x-8)$, so we compute $I = \displaystyle\int \left(x^2 - 1 - \frac{6x-8}{x^2-6x+8}\right) dx$. Next we solve $\dfrac{6x-8}{x^2-6x+8} = \dfrac{A}{x-2} + \dfrac{B}{x-4}$. This is equivalent to $A(x-4) + B(x-2) = 6x-8$, and we find with $x = 4$ that $B = 8$ and with $x = 2$ that $A = -2$. We finally find

$$I = \frac{1}{3}x^3 - x + 2\ln|x-2| - 8\ln|x-4| (+C) \quad (= \frac{1}{3}x^3 - x + \ln\frac{(x-2)^2}{(x-4)^8} + C).$$

[4] **10.** Determine an equation of the tangent line at the point $(1, 2)$ to the curve, given by the equation

$$2x^2y - e^{xy-2} = 3.$$

Answer. The equation of the tangent line is $(y-2) = m(x-1)$, where $m = y'(1)$. We differentiate with respect to x: $4xy + 2x^2y' - (y + xy')e^{xy-2} = 0$. Next we put $x = 1$, $y = 2$: $8 + 2y' - (2 + y') = 0$, so $y' = -6$ and the equation is $y - 2 = -6(x-1)$, also correct is $6x + y = 8$.

[5] **11.** Determine the solution y of the differential equation

$$\frac{dy}{dx} - \frac{3y}{x-1} = 4,$$

with initial value $y(0) = 1$.

Answer. This is a DE of the form $y' + p(x)y = q(x)$. We start with the solution of the homogeneous equation $y' + p(x)y = 0$, by finding a solution $\mu(x)$ or $P(x)$ of $\mu'(x) = p(x) = -3/(x-1)$. So a solution is $\mu(x) = -3\ln|x-1|$ and now the homogeneous equation has the solution $y = C_1 e^{-\mu} = C(x-1)^3$. Now a particular solution can be found by either using the integrating factor e^{μ} or by variation of constant $y = C(x)(x-1)^3$, and hence $C'(x)(x-1)^3 = 4$, or by guessing that there might be a particular solution of the form $y = c(x-1)$. All methods give a particular solution $y = -2(x-1)$ leading to the answer $y(x) = C(x-1)^3 - 2(x-1)$. Finally $y(0) = 1$ implies $C = 1$, so $y(x) = (x-1)^3 - 2(x-1)$.