TECHNISCHE UNIVERSITEIT EINDHOVEN
Department of Mathematics and Computer Science

Exam Calculus B (2WBB0), Monday 6 November 2017, 09:00–12:00
Solutions

Multiple Choice questions

[4] 1. The points (x, y) in the $\mathbb{R}^2$ plane, described by the equation

$$25(x^2 + y^2) - 100(x - y) = -184$$

form a circle with center M and radius r. Give M and r.

(a) $\begin{array}{rcl} r &=& 2/5 \\ M &=& (2,2) \end{array}$ (b) $\begin{array}{rcl} r &=& 4/5 \\ M &=& (-2,2) \end{array}$ (c) $\begin{array}{rcl} r &=& 4/5 \\ M &=& (2,2) \end{array}$ (d) $\begin{array}{rcl} r &=& 4/5 \\ M &=& (2,-2) \end{array}$

Solution. We have

$$\begin{array}{rrl} & 25(x^2 + y^2 - 100(x - y) & = -184 \\ \Rightarrow & 25(x^2 - 4x) + 25(y^2 + 4) & = -184 \\ \Rightarrow & 25(x^2 - 4x + 4) + 25(y^2 + 4y + 4) & = 16 \\ \Rightarrow & 25(x-2)^2 + 25(y+2)^2 & = 16 \\ \Rightarrow & (x-2)^2 + (y+2)^2 & = \frac{16}{25} \end{array} .$$

So, the equation describes a circle with center $M = (2, -2)$ and radius $r = \sqrt{\frac{16}{25}} = \frac{4}{5}$.

The correct answer is **d**.

[4] 2. The value of $\sin(\arctan(\dfrac{1}{9}))$ is equal to:

(a) $\dfrac{1}{\sqrt{82}}$ (b) $\dfrac{1}{81}$ (c) $\dfrac{2}{9}$ (d) $-\dfrac{1}{9}$

Solution. Let $\alpha = \arctan(\frac{1}{9})$. Then

$$\begin{array}{rrl} & \tan(\alpha) & = \frac{1}{9} \\ \Rightarrow & \frac{\sin(\alpha)}{\cos(\alpha)} & = \frac{1}{9} \\ \Rightarrow & \frac{\sin^2(\alpha)}{\cos^2(\alpha)} & = \frac{1}{81} \\ \Rightarrow & 81\sin^2(\alpha) & = \cos^2(\alpha) \\ \Rightarrow & 81\sin^2(\alpha) & = 1 - \sin^2(\alpha) \\ \Rightarrow & 82\sin^2(\alpha) & = 1 \\ \Rightarrow & \sin(\alpha) & = \pm\frac{1}{\sqrt{82}} \end{array}$$

Since α is between 0 and $\frac{pi}{2}$, we find $\sin(\alpha) = \frac{1}{\sqrt{82}}$.

The correct answer is **a**.

[4] 3. Given the three planes in $\mathbb{R}^3$ with equations $U : y = 2$, $V : 3x - \sqrt{3}z = 0$, and $W : x = 1$.
Let ℓ be the intersection line of the planes U and V. Determine the angle between the normal of the plane W and the intersection line ℓ.

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{3}$ (c) $\dfrac{\pi}{4}$ (d) $\dfrac{\pi}{6}$

Solution. A direction vector $\mathbf{v}$ for the intersection line ℓ of U and V is $\begin{pmatrix} \sqrt{3} \\ 0 \\ 3 \end{pmatrix}$. A direction vector $\mathbf{w}$ for the line perpendicular to W is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

For the angle α between ℓ and the normal of W we have $\cos(\alpha) = \frac{\mathbf{v} \bullet \mathbf{w}}{|\mathbf{v}| \cdot |\mathbf{w}|} = \frac{\sqrt{3}}{\sqrt{3+9} \cdot 1} = \frac{1}{2}$.
This implies that $\alpha = \frac{\pi}{3}$.

The correct answer is **b**.

[4] 4. The function $P(t)$ measures the level of oxygen in a pond, where $P(t) = 1$ is the normal (un-polluted) level and $0 \leq P(t) \leq 1$. Time t is measured in weeks. At $t = 0$ waste is dumped into the pond and, while the waste material is oxidized, the level of oxygen is given by

$$P(t) = \frac{t^2 - t + 1}{t^2 + 1}, \text{ for } t \geq 0.$$

At which time t is the level of oxygen in the pond the lowest?

(a) 0 weeks (b) $\sqrt{3}$ weeks (c) 1 week (d) 3 weeks

Solution. The critical points for the function P are 0 (border point) and the points where $P'(t) = 0$.

Using the quotient rule we find

$$P'(t) = \frac{(2t-1)(t^2+1) - (t^2 - t + 1)(2t)}{(t^2+1)^2} = \frac{2t^3 + 2t - t^2 - 1 - 2t^3 + 2t^2 - 2t}{(t^2+1)^2} = \frac{t^2 - 1}{(t^2+1)^2}.$$

So, as $t \geq 0$, the only critical point except for 0 is $t = 1$. We see that $P'(t) < 0$ for $0 < t < 1$ and $P'(t) > 0$ for $t > 1$. So, P has a minimum for $t = 1$.

The correct answer is **c**.

[4] 5. The function f given by $f(x) = x^3 + x$ is invertible. If $g(x) = f^{-1}(x)$, determine $g'(2)$.

(a) $\dfrac{1}{13}$ (b) $\dfrac{1}{2}$ (c) $\dfrac{1}{4}$ (d) $-\dfrac{1}{2}$

Solution. We have $f'(x) = 3x^2 + 1$. Moreover, notice that $f(1) = 2$, so $g'(2) = \frac{1}{f'(1)} = \frac{1}{3+1}$.

The correct answer is **c**.

Exam Calculus B (2WBB0), Monday 6 November 2017, 09:00–12:00
Solutions

Open questions

[4] 6. Determine an equation for the tangent line to the curve given by the equation

$$y + x\ln(y) - 2x = 0$$

in the point $\left(\frac{1}{2}, 1\right)$.

Solution. Implicit differentiation yields:

$$y' + \ln(y) + x \cdot \frac{y'}{y} - 2 = 0.$$

So, in the point $(\frac{1}{2}, 1)$, which is a point of the curve, we find $y' + 0 + \frac{1}{2}y' - 2 = 0$, and hence $y' = \frac{4}{3}$.

An equation for the tangent line is then

$$y - 1 = \frac{4}{3}(x - \frac{1}{2}).$$

[4] 7. Determine the Taylor polynomial of degree 3 around $a = 0$ of the function

$$f(x) = \ln(1 + \sin(x))\,.$$

Solution.

We have

$$\ln(1+y) = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 + \mathcal{O}(y^4)$$

and

$$\sin(x) = x - \frac{1}{6}x^3 + \mathcal{O}(x^5).$$

This implies

$$\begin{aligned}
\ln(1+\sin(x)) \quad &= x - \tfrac{1}{6}x^3 + \mathcal{O}(x^5) - \tfrac{1}{2}(x - \tfrac{1}{6}x^3 + \mathcal{O}(x^5))^2 \\
&\quad + \tfrac{1}{3}(x - \tfrac{1}{6}x^3 + \mathcal{O}(x^5))^3 \\
&\quad + \mathcal{O}((x - \tfrac{1}{6}x^3 + \mathcal{O}(x^5))^4) \\
&= x - \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{3}x^3 + \mathcal{O}(x^4) \\
&= x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \mathcal{O}(x^4)
\end{aligned}$$

The Taylor polynomial of f of order 3 is then

$$x - \frac{1}{2}x^2 + \frac{1}{6}x^3.$$

8. Calculate the following limits:

[4] **(a)** $\lim_{x\to 0}\left(\frac{e^x}{e^x-1}-\frac{1}{x}\right)$

[4] **(b)** $\lim_{x\to 0^+}\sin(x)\ln(x)$

Solution.

(a)

$$\begin{aligned}\lim_{x\to 0}\left(\frac{e^x}{e^x-1}-\frac{1}{x}\right) &= \lim_{x\to 0}\frac{xe^x-e^x+1}{x(e^x-1)}\\ &= \lim_{x\to 0}\frac{x(1+x+\mathcal{O}(x^2))-(1+x+\frac{x^2}{2}+\mathcal{O}(x^3))+1}{x(x+\mathcal{O}(x^2))}\\ &= \lim_{x\to 0}\frac{\frac{1}{2}x^2+\mathcal{O}(x^3)}{x^2+\mathcal{O}(x^3)}\\ &= \lim_{x\to 0}\frac{\frac{1}{2}+\mathcal{O}(x)}{1+\mathcal{O}(x)}\\ &= \frac{1}{2}\end{aligned}$$

(b) For $x>0$ we have $0\le|\sin(x)\ln(x)|\le|x\ln(x)|$. Now $\lim_{x\to 0^+}x\ln(x)=0$ is a standard limit, so by the Squeeze Theorem we also have $\lim_{x\to 0^+}\sin(x)\ln(x)=0$.

Alternative 1: $\lim_{x\to 0^+}\sin(x)\ln(x)=\lim_{x\to 0^+}\frac{\sin(x)}{x}\cdot x\ln(x)=\lim_{x\to 0^+}\frac{\sin(x)}{x}\cdot\lim_{x\to 0^+}x\ln(x)=1\cdot 0=0$.

Alternative 2: with second l'Hôpital Rule: $\lim_{x\to 0^+}\sin(x)\ln(x)=\lim_{x\to 0^+}\frac{\ln(x)}{1/\sin(x)}=\lim_{x\to 0^+}\frac{1/x}{-\cos(x)/\sin^2(x)}=\lim_{x\to 0^+}-\frac{\sin(x)}{x}\frac{\sin(x)}{\cos(x)}=1\cdot 0.$

9. Calculate the following integrals:

[4] **(a)** $\int_0^\pi 5(5-4\cos(x))^{1/4}\sin(x)\,dx$

[5] **(b)** $\int x^7\sin(2x^4)\,dx$

Solution.

(a) With $u=5-4\cos(x)$ we have $du=4\sin(x)dx$, so

$$\begin{aligned}\int_0^\pi 5(5-4\cos(x))^{1/4}\sin(x)\,dx = &\int_1^9\frac{5}{4}u^{1/4}du\\ &=[u^{5/4}]_1^9\\ &=9^{5/4}-1\\ &=9\sqrt{3}-1\end{aligned}$$

(b) With $u = 2x^4$ we have $du = 8x^3dx$ and we get

$$\begin{aligned}
\int x^7 \sin(2x^4)\,dx &= \int \frac{1}{16} u \sin(u) du \\
&= \tfrac{1}{16} \int u \sin(u) du \\
&= \tfrac{1}{16}[-u\cos(u) + \int \cos(u) du] \\
&= \tfrac{1}{16}[-u\cos(u) + \sin(u)] + C \\
&= \tfrac{1}{16}[-2x^4\cos(2x^4) + \sin(2x^4)] + C
\end{aligned}$$

[5] 10. Determine the solution y of the differential equation

$$\frac{dy}{dx} = -2xy + 3x^2e^{-x^2}$$

with initial value $y(0) = 1$.

Solution.

The equation

$$\frac{dy}{dx} = -2xy + 3x^2e^{-x^2}$$

is a linear differential equation with corresponding homogenoeus equation

$$\frac{dy}{dx} = -2xy.$$

The solution to this homogenous equation are of the form

$$y(x) = Ce^{-x^2}$$

where C is a constant.

Variation of the constant says, try $y = C(x)e^{-x^2}$ for the original equation.

This yields

$$C'(x)e^{-x^2} + C(x)(-2x)e^{-x^2} = (-2x)C(x)e^{-x^2} + 3x^2e^{-x^2}.$$

So $C'(x) = 3x^2$ from which we deduce that $C(x) = x^3 + D$ for some constant D.

This implies that the general solution is

$$y(x) = (x^3 + D)e^{-x^2}.$$

As $y(0) = D = 1$ we find $y(x) = (x^3 + 1)e^{x^2}$.