Answers final exam Calculus B (2WBB0), 29-01-2018

1 (c): Draw a right-angled triangle with angle α for which $\sin(\alpha) = \frac{2}{5}$, so opposite side 2 and hypotenuse 5. Then the near side is $\sqrt{21}$, and $\cos(\alpha) = \frac{1}{5}\sqrt{21}$.

2 (b): The solution to the differential equation is $y(t) = 10\, e^{t/10}$.
Solving $10\, e^{t/10} = 4000$ gives (use the hint) $\frac{t}{10} \approx 6$.

3 (b) An easy way is $(x + \mathcal{O}(x^3) - 1 + \frac{1}{2}x^2 + \mathcal{O}(x^4))^2 = 1 + x^2 - 2x - x^2 + \mathcal{O}(x^3) = 1 - 2x + \mathcal{O}(x^3)$.
One may also compute the result using the definition of a Taylor series.

4 (d) Implicit differentiation yields $3x^2 + y^3 + 2xyy' + 3y^2y' = 0$
so $3x^2 + y^3 + (2xy + 3y^2)y' = 0$.

5 (c) The normal vectors are $v = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

The cosine of these vectors is $\dfrac{v \cdot w}{\|v\|\, \|w\|} = \frac{7}{\sqrt{14}\,\sqrt{14}} = \frac{1}{2}$.

6 The inner product of $x - \alpha y = \begin{bmatrix} 1 - 2\alpha \\ 2 - 3\alpha \\ 3 - 4\alpha \end{bmatrix}$ and y should be 0,
so $2(1 - 2\alpha) + 3(2 - 3\alpha) + 4(3 - 4\alpha) = 0$, so $\alpha = \frac{20}{29}$.

7 (a) $y = \frac{3}{\ln(x) - 1}$ so $y\,(\ln(x) - 1) = 3$ so $\ln(x) = \frac{y+3}{y}$ and $x = e^{\frac{y+3}{y}} = f^{-1}(y)$.

(b) The range of f^{-1} is the domain of f, so (e, ∞). The domain of f^{-1} is the range of f. Since $\lim\limits_{x \to e^+} f(x) = \infty$ and $\lim\limits_{x \to \infty} f(x) = 0$, this is $(0, \infty)$.
Note that the domain of f^{-1} is not $\mathbb{R}\backslash\{0\}$; this would correspond to domain of $(0, e) \cup (e, \infty)$ of f.

8 (a) This is a limit of type 0 times "oscillating factor", so we use the squeeze theorem: $0 = \lim\limits_{x \to 0} -x \leq \lim\limits_{x \to 0} x \sin\left(\dfrac{1}{x^2}\right) \leq \lim\limits_{x \to 0} x = 0$, so the answer is 0.

(b) This is a limit of type 1^∞, so we use $e^{\ln}$: $\lim\limits_{n \to \infty} e^{n \ln(1 + \frac{1}{\sqrt{n}})}$.
Consider the exponent: $\lim\limits_{n \to \infty} n \ln(1 + \frac{1}{\sqrt{n}})$.

This is a limit of type $\infty \cdot 0$, so we rewrite it to type 0/0: $\lim\limits_{n \to \infty} \dfrac{\ln(1 + \frac{1}{\sqrt{n}})}{1/n}$.

We can now use l'Hôpital: $\lim\limits_{n \to \infty} \dfrac{1}{1 + \frac{1}{\sqrt{n}}} \cdot -\dfrac{1}{2n^{3/2}} \Big/ -\dfrac{1}{n^2} = \lim\limits_{n \to \infty} \dfrac{\sqrt{n}}{2(1 + \frac{1}{\sqrt{n}})} = \infty$.
So the answer is $e^\infty = \infty$.
As an alternative way, instead of using l'Hôpital, it is more elegant to substitute $x = \frac{1}{\sqrt{n}}$, then the limit becomes $\lim\limits_{x \to 0^+} \dfrac{\ln(1 + x)}{x^2}$.

Now use l'Hôpital: $\lim\limits_{x \to 0^+} \dfrac{1}{1 + x} \Big/ 2x = \infty$.

9 **(a)** With substitution $u = x^2 + 1$, $du = 2xdx$, and $x^2 = u - 1$ we get
$\int x \cdot (x^2)^2 \sqrt{x^2+1} = \frac{1}{2} \int (u-1)^2 \sqrt{u}\, du = \frac{1}{2} \int (u^{5/2} - 2u^{3/2} - u^{1/2})\, du$
$= \frac{1}{2}(\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} - \frac{2}{3}u^{3/2}) = \sqrt{x^2+1} \cdot (\frac{1}{7}(x^2+1)^3 - \frac{2}{5}(x^2+1)^2 - \frac{1}{3}(x^2+1)) + C.$

(b) Twice partial integration:

$$\begin{aligned}\int e^{2x} \sin(x)\, dx &= -e^{2x}\cos(x) + \int 2e^{2x}\cos(x)\, dx \\ &= -e^{2x}\cos(x) + 2e^{2x}\sin(x) - \int 4e^{2x}\sin(x)\, dx\end{aligned}$$

so $5\int e^{2x}\sin(x)\, dx = -e^{2x}\cos(x) + 2e^{2x}\sin(x)$ and therefore
$\int e^{2x}\sin(x)\, dx = -\frac{1}{5}e^{2x}\cos(x) + \frac{2}{5}e^{2x}\sin(x) + C.$

10 The homogeneous part $\frac{dy}{dx} = -2\frac{y}{x}$ gives $\int \frac{dy}{y} = -2\int \frac{dx}{x}$ so $\ln|y| = -2\ln|x| + C$ so $y(x) = C\frac{1}{x^2}$.

Solving the inhomogeneous differential equation by, e.g., variation of constants yields $C'(x)\frac{1}{x^2} + C(x) \cdot -\frac{2}{x^3} + 2\frac{C(x)}{x^2 \cdot x} = 2\frac{e^{x^2}}{x}$.
Therefore, $C'(x) = 2xe^{x^2}$, and $C'(x) = \int 2xe^{x^2}\, dx + D = e^{x^2} + D$ (by substitution $u = x^2$).
So the general solution is $y(x) = \frac{1}{x^2}(e^{x^2} + D)$.
Using the initial condition we get $D = 2 - e$.