Automated Market Makers Designs beyond Constant Functions

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Abstract

Popular automated market makers (AMMs) use constant function markets (CFMs) to clear the demand and supply in the pool of liquidity. A key drawback in the implementation of CFMs is that liquidity providers (LPs) are currently providing liquidity at a loss, on average. In this paper, we propose two new designs for decentralised trading venues, the arithmetic liquidity pool (ALP) and the geometric liquidity pool (GLP). In both pools, LPs choose impact functions that determine how liquidity taking orders impact the marginal exchange rate of the pool, and set the price of liquidity in the form of quotes around the marginal rate. The impact functions and the quotes determine the dynamics of the marginal rate and the price of liquidity. We show that CFMs are a subset of ALP; specifically, given a trading function of a CFM, there are impact functions and quotes in the ALP that replicate the marginal rate dynamics and the execution costs in the CFM. For the ALP and GLP, we propose an optimal liquidity provision strategy where the price of liquidity maximises the LP's expected profit and the strategy depends on the LP's (i) tolerance to inventory risk and (ii) views on the demand for liquidity. Our strategies admit closed-form solutions and are computationally efficient. We show that the price of liquidity in CFMs is suboptimal in the ALP. Also, we give conditions on the impact functions and the liquidity provision strategy to prevent arbitrages from rountrip trades. Finally, we use transaction data from Binance and Uniswap v3 to show that liquidity provision is not a loss-leading activity in the ALP.

Keywords: decentralised finance, automated market making, smart contracts, algorithmic trading, market making, stochastic AMMs.

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1. Introduction

Matching buyers with sellers is the basic function of a financial market. A well-functioning market is underpinned by liquidity providers (LPs) who are willing to take either side of a trade and participate in the price discovery process, in exchange for earning the spread or a fee. In traditional markets, price discovery is typically based on auctions, request-for-quotes, and limit order books (LOBs). Currently, in decentralised finance, most trading happens in constant function markets (CFMs), where market participants interact with the liquidity pool according to a trading function (e.g., the constant product trading function) that determines how the market clears. CFMs are popular exchange venues due to their ability to match trades and set prices with minimal storage and computational requirements; see Angeris et al. (2019), Capponi and Jia (2021).

CFMs use a deterministic trading function and pools of liquidity of two (or more) assets. The number of assets in the pools and the trade function determine how LTs and LPs interact; see Angeris et al. (2019). The liquidity available in the pool and the convexity of the trading function determine the marginal exchange rates and slippage for trades of various sizes executed by LTs in the pool. In most CFMs, LTs pay a fee for each trade they execute and the LPs who are counterparty to the trade share the fee. The revenue from fees is compensation for exposure to adverse selection and for predictable losses (PL) see Cartea et al. (2022b); however, empirical evidence (see Cartea et al., 2022b; Evans, 2021; Loesch et al., 2021) shows that LPs, on average, lose money in CFMs. There are a number of theoretical and experimental works in the literature that explain the losses of LPs in CFMs which are a consequence of their design: (i) LPs do not fully participate in price discovery, which happens mainly with liquidity taking trades, because of their limited strategic behaviour,[1] (ii) the slippage for liquidity taking trades is determined by the convexity of the trading function (i.e., by the available liquidity) which is hardwired and the same for all participants in the pool, and (iii) LPs cannot fully express their individual risk preferences.

In this paper, we propose two new designs for automated market making that do not rely on trading functions, the arithmetic liquidity pool (ALP) and the geometric liquidity pool (GLP). In the ALP and GLP, the LP (i) chooses *impact* functions that determine how liquidity taking orders impact the marginal rate and (ii) sets the price of liquidity in the form of quotes around the marginal rate of the pool. In contrast to CFMs where LPs

[1]The strategic behaviour of LPs in CFMs with concentrated liquidity (CL) is focused on increasing fee revenue. However, CL increases the predictable losses of LPs; see Cartea et al. (2022b).

cannot directly affect the marginal rate, the impact functions and the quotes in the ALP determine the dynamics of the marginal rate and the price of liquidity. In the ALP, the LP's inventory and the marginal rate are additive, and in the GLP, they are multiplicative. To set the quotes, we propose optimal liquidity provision models for ALPs and GLPs. The two models employ a set of parameters that encodes the LP's beliefs over the liquidity taking trading flow and the LP's risk preferences. Our models admit closed-form solutions and are computationally efficient.[2]

In the ALP, LPs can choose any non-negative and deterministic impact functions for which we give conditions that prevent LTs from executing roundtrip arbitrages. Also, we show that CFMs are a special case of the ALP when the LP chooses impact functions and quotes that replicate the marginal rate dynamics and the execution costs implied by the trading function of a CFM. In particular, we show that this equivalence is obtained with a strategy that is sub-optimal for an LP who maximises expected terminal wealth while controlling (or not) exposure to inventory risk.

In both designs, exchange rates form in the pool as a result of LT activity and the strategic behaviour of the LPs. Trading costs of LTs are determined by the LP's quotes, and the dynamics of the marginal rate are determined by the LP's impact functions. The impact functions are key to our models because the LPs can tailor them to account for inventory and adverse selection risks. Thus, the LPs can effectively adjust their quotes based on market conditions, e.g., the toxicity of the trading flow, the intensity of buying and selling liquidity taking orders, and the volatility of the exchange rate. Finally, we use transaction data from Binance to showcase our liquidity provision strategy in the ALP, and transaction data from Uniswap v3 to compare the performance of liquidity provision in the ALP with that of CFMs.

Numerous works study the losses of LPs and strategic liquidity provision. Angeris and Chitra (2020) show that the convexity of the trading function is key in CFMs, Lehar and Parlour (2021) discuss the competition between CFMs and LOBs, Angeris et al. (2021) and Angeris et al. (2022) study the returns of LPs in simple setups, Neuder et al. (2021) and Cartea et al. (2022b) study strategic liquidity provision in CFMs with concentrated liquidity, Li et al. (2023) study strategic liquidity provision in different types of AMMs, Cartea et al. (2023b) derive the predictable losses of LPs in CFMs and in concentrated liquidity AMMs, Milionis et al. (2022) study the arbitrage gains of LTs in CFMs, and Fukasawa et al. (2023)

[2]The code repository is in `https://github.com/leandro-sbetancourt/amm_gym`. The `amm_gym` environment builds on `mbt_gym` in Jerome et al. (2022).

study the hedging of the impermanent losses of LPs. A strand of the literature studies liquidity taking strategies in AMMs; see Cartea et al. (2022a) and Jaimungal et al. (2023).

The literature that studies the design of AMMs mainly explores optimal dynamic fees in CFMs. Evans et al. (2021) study optimal fees in geometric markets, Goyal et al. (2023) study an AMM with a dynamic trading function that incorporates the beliefs of LPs about future asset prices, Sabate-Vidales and Šiška (2022) study variable fees in CPMs, and Cohen et al. (2023) derive an upper bound for fee revenue to make liquidity provision profitable in CFMs. Some works also explore AMM designs that do not rely on a trading function; Bergault et al. (2022a) design an AMM where LPs set quotes around an exogenous oracle price and Lommers et al. (2023) discuss AMM designs where the LP's strategy adjusts dynamically to market information. In contrast to this literature, we design AMMs with computationally efficient rules where prices form endogenously and where LPs can individually express their beliefs and risk preferences. Both of our AMM designs and the corresponding liquidity provision strategies build on the existing literature for liquidity provision in OTC and LOB markets, see Ho and Stoll (1983), Glosten and Milgrom (1985), and Avellaneda and Stoikov (2008). These frameworks have been extended in many directions; see Guéant et al. (2012), Guéant et al. (2013), Cartea et al. (2015), Guéant (2016), and Drissi (2022).

The remainder of this paper proceeds as follows. Section 2 introduces the design of the ALP and the corresponding optimal liquidity provision strategy; Subsection 2.1 shows that the CFM is a particular case of the ALP; Subsection 2.2 introduces the problem of optimal liquidity provision by a representative LP in the ALP and provides closed-form solutions to the value function of the associated stochastic control problem; Subsection 2.3 discusses ALPs with competing LPs and gives conditions on the impact functions and the quotes to prevent roundtrip arbitrages; and finally, Subsection 2.4 studies the sensitivity of the optimal quotes to the LP's strategy parameters, and uses Binance and Uniswap v3 data to study the performance of the LP's optimal liquidity provision strategy. In particular, we compare the profitability of liquidity provision in ALPs with that of liquidity provision in CFMs. Section 3 introduces the design of the GLP and the corresponding optimal liquidity provision strategy. For a particular choice of impact functions, we show that the strategy admits a closed-form solution. Finally, Appendix A collects the proofs.

2. The ALP design

We consider an ALP that provides a venue to trade a pair of assets X and Y. The reference asset is X and we denote by Z the pool's marginal exchange rate of asset Y

in units of X, i.e., the exchange rate for an infinitesimal quantity of asset Y. The marginal exchange rate in the ALP is akin to the midprice in an LOB. The LP initialises the ALP with quantities x_0 and y_0 of both assets and Z_0 is the initial marginal exchange rate.[3] Liquidity takers (LTs) interact with the ALP to buy and sell asset Y over a trading window $[0,T]$, during which the LP does not deposit or withdraw any assets. Next, we specify how the pool quantities and the marginal rates are updated in the ALP.

We work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ that supports the processes we introduce and satisfies the usual conditions. Let $(x_t)_{t\in[0,T]}$ and $(y_t)_{t\in[0,T]}$ be the processes that describe the quantities (which can be negative) of both assets in the ALP, and let $(Z_t)_{t\in[0,T]}$ be the marginal rate process.

In the ALP, the LP chooses the shifts δ^b_t and δ^a_t that determine the rates $Z_{t^-} - \delta^b_t$ and $Z_{t^-} + \delta^a_t$ at which she is willing to buy and sell a constant amount $\zeta > 0$ of asset Y throughout the trading window $[0,T]$. Let $\left(N^b_t\right)_{t\in[0,T]}$ and $(N^a_t)_{t\in[0,T]}$ be the Poisson processes that count the number of sell and buy orders filled by the LP, respectively. As trading evolves, the quantities of assets X and Y in the ALP follow the dynamics

$$\begin{aligned} \mathrm{d}y_t &= \zeta\, \mathrm{d}N^b_t - \zeta\, \mathrm{d}N^a_t\,, \\ \mathrm{d}x_t &= -\,\zeta\, \left(Z_{t^-} - \delta^b_t\right) \mathrm{d}N^b_t + \zeta \left(Z_{t^-} + \delta^a_t\right) \mathrm{d}N^a_t\,. \end{aligned} \tag{1}$$

When the ALP receives a buy or sell order of quantity ζ, the marginal rate Z in the pool is updated, and the LP quotes a bid and an ask around the new marginal rate. The updates in the marginal rate Z depend on the LP's choice of non-negative deterministic impact functions $\eta^a(\,\cdot\,)$ and $\eta^b(\,\cdot\,)$. Specifically, the impact functions determine the pool's marginal rate response to incoming trades as a function of the LP's position as follows

$$\mathrm{d}Z_t = -\,\eta^b(y_{t^-})\,\mathrm{d}N^b_t + \eta^a(y_{t^-})\,\mathrm{d}N^a_t\,. \tag{2}$$

Thus, the position of the LP in the ALP is characterised by a trading horizon T, impact functions $\eta^b(\,\cdot\,)$ and $\eta^a(\,\cdot\,)$, and a (possibly stochastic) strategy δ^b_t and δ^a_t that determines the price of liquidity at the bid and the ask at each time t as a function of the state of the pool.[4]

[3]Below, we discuss the ALP design for competing LPs.

[4]The impact functions and the quoting strategy can be implemented as smart contracts that customise the ALP's actions and encode designated actions. Thus, any impact function and quoting strategy can be implemented in general purpose blockchains.

Below, Subsection 2.1 shows that each CFM is a particular case of the ALP; Subsection 2.2 proposes an optimal closed-form liquidity provision strategy; Subsection 2.3 discusses competition among LPs and arbitrage in the ALP; and Subsection 2.4 showcases the superior performance of liquidity provision in ALPs.

2.1 CFMs are a particular case of ALP

Here, we show that CFMs are a particular case of the ALP for an appropriate choice of the LP's impact functions $\{\eta^b, \eta^a\}$ and an appropriate (suboptimal) provision strategy $\{\delta^b, \delta^a\}$.

Consider a CFM that makes liquidity in the pair of assets X and Y throughout the trading window $[0, T]$. The trading function $f : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ of the CFM is twice differentiable and increasing in both arguments. Let m be the depth of the CFM pool and let $\left(x_t^{\text{CFM}}\right)_{t\in[0,T]}$ and $\left(y_t^{\text{CFM}}\right)_{t\in[0,T]}$ be the processes describing the amount of asset X and of asset Y, respectively, in the CFM pool, where x_0^{CFM} and y_0^{CFM} are known and fixed. Recall that the liquidity taking trading condition $f(x_t, y_t) = m^2$ in the CFM must hold for all $t \in [0, T]$, so one defines an appropriate *level* function $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $x_t = \varphi(y_t)$ for all $t \in [0, T]$.

Next, assume that all trades in the ALP and CFM pools are for the same amount ζ of asset Y and assume the fees in the CFM are zero. By Proposition 1 in Cartea et al. (2022b), the level function φ is convex and the marginal rate in the CFM pool at time t^- is $-\varphi'\left(y_{t^-}^{\text{CFM}}\right)$. The dynamics of the amounts of asset X and asset Y and the marginal rate Z^{CFM} in the CFM pool are given by

$$\begin{aligned}
\mathrm{d}y_t^{\text{CFM}} &= \zeta\, \mathrm{d}N_t^b - \zeta\, \mathrm{d}N_t^a\,, \\
\mathrm{d}x_t^{\text{CFM}} &= -\left(\varphi\left(y_{t^-}^{\text{CFM}} + \zeta\right) - \varphi\left(y_{t^-}^{\text{CFM}}\right)\right)\, \mathrm{d}N_t^b + \left(\varphi\left(y_{t^-}^{\text{CFM}} - \zeta\right) - \varphi\left(y_{t^-}^{\text{CFM}}\right)\right)\, \mathrm{d}N_t^a\,, \\
\mathrm{d}Z_t^{\text{CFM}} &= \left(-\varphi'\left(y_{t^-}^{\text{CFM}} + \zeta\right) + \varphi'\left(y_{t^-}^{\text{CFM}}\right)\right)\, \mathrm{d}N_t^b + \left(-\varphi'\left(y_{t^-}^{\text{CFM}} - \zeta\right) + \varphi'\left(y_{t^-}^{\text{CFM}}\right)\right)\, \mathrm{d}N_t^a\,.
\end{aligned} \tag{3}$$

The next theorem shows that LPs can choose impact functions and quotes in the ALP to replicate the marginal rate dynamics and the execution costs in a CFM.

Theorem 1. *Let $\varphi(\,\cdot\,)$ be the level function of a CFM. Assume the LP in the ALP chooses the impact functions*

$$\eta^a(y) = \varphi'(y) - \varphi'(y - \zeta)\,, \qquad \eta^b(y) = -\varphi'(y) + \varphi'(y + \zeta)\,, \tag{4}$$

and chooses the shifts

$$\delta_t^a = \frac{\varphi(y_{t^-} - \zeta) - \varphi(y_{t^-})}{\zeta} + \varphi'(y_{t^-})\,, \qquad \delta_t^b = \frac{\varphi(y_{t^-} + \zeta) - \varphi(y_{t^-})}{\zeta} - \varphi'(y_{t^-})\,. \tag{5}$$

Then, the marginal rate dynamics, inventory dynamics, and execution costs in the ALP are the same as those in the CFM with level function $\varphi(\,\cdot\,)$.

For a proof, see Appendix A.1.

Thus, for a specific choice of impact functions, the ALP behaves as a CFM if the LP chooses the quoting strategy (5). However, for an LP who maximises some performance criterion, the quotes (5) do not necessarily provide the optimal strategy. Below, we propose a framework to derive an optimal quoting strategy based on the LP's view on the liquidity taking trading flow and her risk preferences.

2.2 The optimal price of liquidity in ALPs

Here, we derive an optimal liquidity provision strategy in ALPs for a fixed pair of impact functions η^b and η^a when the LP assumes that the arrival intensity of liquidity taking orders decays exponentially as a function of the shift from the marginal rate that she quotes at the bid and the ask. Specifically, let $\lambda_t^b\left(\delta_t^b\right)$ and $\lambda_t^a\left(\delta_t^b\right)$ denote the stochastic intensity of order arrivals, i.e., the stochastic intensity of the Poisson processes $\left(N_t^b\right)_{t\in[0,T]}$ and $\left(N_t^a\right)_{t\in[0,T]}$ that count the number of sell and buy orders filled by the LP, respectively, and write

$$\begin{cases} \lambda_t^b\left(\delta_t^b\right) = c^b\, e^{-\kappa\,\delta_t^b}\, \mathbb{1}^b\left(y_{t^-}\right)\,, \\ \lambda_t^a\left(\delta_t^a\right) = c^a\, e^{-\kappa\,\delta_t^a}\, \mathbb{1}^a\left(y_{t^-}\right)\,, \end{cases} \tag{6}$$

where c^a and c^b are two non-negative constants that capture the baseline selling and buying intensity, respectively, and where

$$\mathbb{1}^b(y) = \mathbb{1}_{\{y+\zeta\leq\overline{y}\}} \quad \text{and} \quad \mathbb{1}^a(y) = \mathbb{1}_{\left\{y-\zeta\geq\underline{y}\right\}}\,,$$

indicate that the ALP stops using the LP's liquidity upon reaching her inventory limits $\underline{y}, \overline{y}$.

The positive constant κ determines how sensitive the LT is to the price of liquidity.[5] The ALP is continuously updating the shifts δ_t^b and δ_t^a of the LP's position to optimise a performance criterion that we specify below.

[5]With multiple LPs, everything else being equal, the values of the baseline intensities c^a and c^b would decrease and the value of κ would increase; that is, as competition increases, a given LP's quotes will be filled less frequently.

For $t \in [0,T]$, we define the set $\mathcal{A}_t$ of admissible shifts

$$\mathcal{A}_t = \Big\{ \delta_s = (\delta^b_s, \delta^a_s)_{s\in[t,T]},\ \mathbb{R}^2\text{-valued},\ \mathbb{F}\text{-adapted}, \tag{7}$$
$$\text{square-integrable, and bounded from below by } \underline{\delta} \Big\},$$

where $\underline{\delta} \in \mathbb{R}$ is given and write $\mathcal{A} := \mathcal{A}_0$.

In our liquidity provision model, LTs use the LP's liquidity in the ALP to buy or sell the quantity $\zeta > 0$ of asset Y at any time $t \in [0,T]$ as long as the LP's inventory risk constraint $y_t \in \{\underline{y}, \underline{y}+\zeta, \dots, \overline{y}-\zeta, \overline{y}\} = \mathcal{Y}$ is obeyed, where $\underline{y}$ and $\overline{y}$ are integer multiples of ζ satisfying $\underline{y} < \overline{y}$.[6] As explained above, the quantities of assets X and Y in the pool follow the dynamics in (1) and the pool's marginal rate follows the dynamics in (2). The LP sets the bid and ask quotes $Z - \delta^b$ and $Z + \delta^a$ to optimise her expected final wealth, while controlling her exposure to asset Y by keeping her inventory close to some level $\hat{y} \in \mathcal{Y}$ in units of asset Y.

The performance criterion of the LP implementing the strategy $\delta = \left(\delta^b, \delta^a\right) \in \mathcal{A}$ is a function $w^\delta \colon [0,T] \times \mathbb{R} \times \mathcal{Y} \times \mathbb{R} \to \mathbb{R}$, which is given by

$$w^\delta(t,x,y,z) = \mathbb{E}_{t,x,y,z}\left[x_T + y_T\, Z_T - \alpha\,(y_T - \hat{y})^2 - \phi \int_t^T (y_s - \hat{y})^2\, \mathrm{d}s \right], \tag{8}$$

where $\mathbb{E}_{t,x,y,z}$ is the expectation operator conditional on the values of the controlled (using the strategy δ) processes $(x_s)_{s\in[t,T]}$, $(y_s)_{s\in[t,T]}$, and $(Z_s)_{s\in[t,T]}$ at time t. Here, the constants $\alpha, \phi \in \mathbb{R}^+$ determine the terminal and running penalty (respectively) on inventory deviations from $\hat{y} \in \mathbb{R}$.

The value function $w \colon [0,T] \times \mathbb{R} \times \mathcal{Y} \times \mathbb{R} \to \mathbb{R}$ of the LP is

$$w(t,x,y,z) = \sup_{\delta \in \mathcal{A}_t} w^\delta(t,x,y,z)\,. \tag{9}$$

Proposition 1 shows that the performance criterion (8) is finite and that the associated value function (9) is well defined.

Proposition 1. *There is $C \in \mathbb{R}$ such that for all $(\delta_s)_{s\in[t,T]} \in \mathcal{A}_t$, the performance criterion*

[6] One can impose regularity conditions on the impact functions η^a, η^b to relax our condition that the inventory of the LP lies in a finite set $\mathcal{Y}$ at all times. For example, if $\eta^{a,b}(\cdot)$ are bounded or continuous, then one can consider a compact interval $[\underline{y}, \overline{y}]$.

of the LP satisfies

$$w^{\delta}(t,x,y,z) \le C < \infty\,,$$

so the value function w in (9) is well defined.

For a proof, see Appendix A.2.

Below, Proposition 2 provides a candidate closed-form solution and Theorem 2 is the verification theorem for the optimal liquidity provision problem in the ALP.

Proposition 2 (Candidate closed-form solution: ALP). *Let $\underline{N} = \underline{y}/\zeta$, $\overline{N} = \overline{y}/\zeta$, and $N = \overline{N} - \underline{N} + 1$. Define the matrix $\mathbf{K} \in \mathbb{R}^{N\times N}$ by*

$$\mathbf{K}_{mn} = \begin{cases} c^a\, e^{-1}\, e^{\kappa\,(m-1)\,\eta^a(m\,\zeta)} & \text{if } n = m-1 \text{ and } m > \underline{N}\,, \\ -\kappa\,\phi\,(m\,\zeta - \hat{y})^2\,/\,\zeta & \text{if } n = m\,, \\ c^b\, e^{-1}\, e^{-\kappa\,(m+1)\,\eta^b(m\,\zeta)} & \text{if } n = m+1 \text{ and } m < \overline{N}\,, \end{cases} \tag{10}$$

for $m, n \in \{\underline{N}, \underline{N}+1, \dots, \overline{N}\}$. Let $\mathbf{U} \in C^1([0,T], \mathbb{R}^N)$ be

$$\mathbf{U}(t) = \exp\left(\mathbf{K}\, t\right)\, \mathbf{U}(0)\,, \quad t \in [0,T]\,, \tag{11}$$

where $\exp(\,\cdot\,)$ is matrix exponentiation and

$$\mathbf{U}(0)_m = e^{-\alpha\,\frac{\kappa}{\zeta}\,(\zeta\,m - \hat{y})^2}\,, \quad m \in [\underline{N}, \bar{N}] \cap \mathbb{Z}\,. \tag{12}$$

For $m \in [\underline{N}, \bar{N}] \cap \mathbb{Z}$ let

$$u(t, m\,\zeta) = \mathbf{U}(T-t)_m\,,$$

and define

$$\theta(t,y) = \frac{\zeta}{\kappa}\,\log u(t,y)\,. \tag{13}$$

Then, the function $\omega\colon [0,T]\times\mathbb{R}\times\mathcal{Y}\times\mathbb{R} \to \mathbb{R}$ given by

$$\omega(t,x,y,z) = x + y\,z + \theta(t,y) \tag{14}$$

solves the HJB equation associated with problem (9) which is given by

$$\begin{aligned} 0 = {} & \partial_t \omega - \phi\,(y-\hat{y})^2 \\ & + \sup_{\delta^b} \lambda^b\left(\delta^b\right) \left\{ \omega\left(t, x - \zeta\,\left(z - \delta^b\right), y + \zeta, z - \eta^b(y)\right) - \omega\left(t,x,y,z\right) \right\} \\ & + \sup_{\delta^a} \lambda^a\left(\delta^a\right) \left\{ \omega\left(t, x + \zeta\,\left(z + \delta^a\right), y - \zeta, z + \eta^a(y)\right) - \omega\left(t,x,y,z\right) \right\}, \end{aligned} \tag{15}$$

with the terminal condition

$$\omega(T,x,y,z) = x + y\,z - \alpha\,(y-\hat{y})^2\,.$$

For a proof, see Appendix A.3.

Theorem 2 (Verification: ALP). *Let ω be defined as in Proposition 2. Then the function ω in (14) satisfies that for all $(t,x,y,z) \in [0,T]\times\mathbb{R}\times\mathcal{Y}\times\mathbb{R}$ and $\delta = (\delta_s)_{s\in[t,T]} \in \mathcal{A}_t$,*

$$w^{\delta}(t,x,y,z) \le \omega\,(t,x,y,z)\ . \tag{16}$$

Moreover, equality is obtained in (16) with the admissible optimal Markovian control $(\delta^\star_s)_{s\in[t,T]} = \left(\delta^{b\star}_s, \delta^{a\star}_s\right)_{s\in[t,T]} \in \mathcal{A}_t$ given by the feedback formulae

$$\begin{aligned}
\delta^{b\star}(t,y_{t^-}) &= \frac{1}{\kappa} - \frac{\theta\,(t,y_{t^-}+\zeta) - \theta\,(t,y_{t^-})}{\zeta} - \frac{(y_{t^-}+\zeta)\,\eta^b(y_{t^-})}{\zeta}\,,\\
\delta^{a\star}(t,y_{t^-}) &= \frac{1}{\kappa} - \frac{\theta\,(t,y_{t^-}-\zeta) - \theta\,(t,y_{t^-})}{\zeta} + \frac{(y_{t^-}-\zeta)\,\eta^a(y_{t^-})}{\zeta}\,,
\end{aligned} \tag{17}$$

where θ is in (13). In particular, $\omega = w$ on $[0,T]\times\mathbb{R}\times\mathcal{Y}\times\mathbb{R}$.

For a proof, see Appendix A.4.

The result is valid for any pair of non-negative deterministic functions η^b and η^a. The first two terms in the optimal strategy (17) are similar to those in the classical model of Avellaneda and Stoikov (2008). The term $1/\kappa$ optimises the instantaneous expected profit from a roundtrip trade. Here, the expected profit from a roundtrip trade is $2\,\zeta\,(\delta^b\,\lambda^b(\delta^b) + \delta^a\,\lambda^a(\delta^a))$ which is maximal for $\delta^b = 1/\kappa$ and $\delta^a = 1/\kappa$, where $\lambda^b(\,\cdot\,)$ and $\lambda^a(\,\cdot\,)$ are given by (6). The second term in (17) is an adjustment to the $1/\kappa$ strategy and represents an additional wealth that the LP demands for selling or buying the asset as a function of her current inventory y, the target inventory $\hat{y}$, the remaining time $T-t$, the impact functions η^b and η^a, and the penalty parameters α and ϕ. The last term (impact component) in (17) is an adjustment to the quotes stemming from the impact of LT activity on the marginal pool rate Z. The effect of the impact component of the strategy in (17) is not symmetric; it decreases the bid and increases the ask when $y-\zeta\ge 0$ and it has the opposite effect when $y+\zeta\le 0$. Recall that in our model, the marginal rate is determined by the supply and demand of liquidity in the pool, so changes in the wealth of the LP, which is measured in units of X, stem from variations in the inventory y and from variations in the pool's marginal rate Z.

Finally, the optimal shifts $\{\delta^{b\star}, \delta^{a\star}\}$ can become negative as a result of inventory constraints and impact functions. Without the constraints $\delta^{b\star}, \delta^{a\star} \ge 0$, our strategy admits a closed-form solution so the AMM can rapidly compute the quotes for different levels of the marginal rate and the inventory before providing liquidity.

Below, Subsection 2.3 discusses the ALP for competing LPs, rountrip arbitrage in ALPs, and how ALPs compare with CFMs. Next, the analysis of Subsection 2.4 compares the performance of liquidity provision in a classical constant product market (CPM) with that of our proposed design with a specific choice of the impact functions.

2.3 Discussion

Here, we discuss how competition affects the strategies of the LPs and how the traditional CFM compares with the ALP design we propose.

2.3.1. Multiple liquidity providers

The optimal liquidity provision model of Subsection 2.2 considers a representative LP who provides liquidity in the ALP where the rate Z is the pool's marginal rate of asset Y in terms of asset X and liquidity taking activity determines rate innovations through the LP's impact functions. However, different LPs can set up different liquidity positions in the ALP with different values for the trading horizon T, impact functions, and quotes. In the ALP with different liquidity positions, LPs will compete for order flow and this will affect the price of liquidity that each LP posts on the market, which will also determine which LPs will survive in the long term.

Specifically, as competition in liquidity provision increases, the LPs will have to share the incoming order flow, where the best quotes among all LPs are the first ones to be filled. To quote optimal prices, the LPs will need to adjust the market arrival rates $\lambda^{a,b}(\delta^{a,b})$ they specify in their model, see (6). Recall that the shifts $\delta^{a,b}$ they post from the midprice depend on their choices of impact functions $\{\eta^a, \eta^b\}$ and on the penalty parameters $\{\alpha, \phi\}$. The dynamics of competition will ensure that those LPs who misspecify the arrival rate of liquidity taking orders will be driven out of the market, or will have to reassess model parameters to remain competitive. By a similar argument, LPs who choose impact functions that produce quotes susceptible to arbitrages or that produce quotes that are too expensive to attract enough flow will disappear from the market.

2.3.2. CFM and ALP

In CFMs, the impact of liquidity taking orders on the marginal rate of the pool and the execution costs are determined by the convexity of the trading function; see Cartea et al. (2023a). The convexity of the trading function is proportional to the size of the pool; the more liquid is the pool, the cheaper it is for LTs to trade. Moreover, the predictable losses of LPs scale with the volatility of the exchange rates and with the toxicity (i.e., informed trading) of the liquidity taking trading flow; see Cartea et al. (2023b).

In CFMs, an LP who believes that liquidity in the pool is cheap (e.g., in view of the volatility and the trading flow) cannot express her belief through her liquidity provision strategy. In equilibrium, one expects LPs in CFM pools with low profitability to be driven out of the market. Specifically, only illiquid pools (i.e., pools with high convexity) survive when liquidity provision is not profitable. In contrast, the quotes in the ALP are a function of (i) the impact functions, (ii) the beliefs of LPs over the nature of the liquidity taking trading flow, (iii) the individual risk preferences of LPs, and (iv) the inventory held by the LPs. Thus, in equilibrium, one expects the price of liquidity in the ALP to clear the demand of liquidity without driving LPs out of the market.

Finally, within the optimal liquidity provision framework of Subsection 2.2, if the impact functions in the ALP are those in a CFM, then the execution costs in the CFM are sub-optimal for an LP who maximises the performance criterion (8). The next result shows that CFMs are suboptimal.

Proposition 3. *Let $\varphi(\,\cdot\,)$ be the level function of a CFM. Consider an LP with initial wealth (x_0, y_0) who sets a liquidity posititon in the CFM and whose performance criterion is given by*

$$J^{CFM} = \mathbb{E}\left[x_T^{CFM} + y_T^{CFM}\, Z_T^{CFM} - \alpha\,(y_T^{CFM} - \hat{y})^2 - \phi\int_0^T (y_s^{CFM} - \hat{y})^2\,\mathrm{d}s\right],$$

with $J^{CFM} \in \mathbb{R}$. Consider an LP in a ALP with initial wealth (x_0, y_0) and with impact functions $\eta^a(\,\cdot\,)$ and $\eta^b(\,\cdot\,)$ given by (4). Let $\delta_t^{CFM} = \left(\delta_t^{a,CFM}, \delta_t^{b,CFM}\right)$ be given by

$$\delta_t^{a,CFM} = \frac{\varphi(y_{t^-} - \zeta) - \varphi(y_{t^-})}{\zeta} + \varphi'(y_{t^-})\,, \qquad \delta_t^{b,CFM} = \frac{\varphi(y_{t^-} + \zeta) - \varphi(y_{t^-})}{\zeta} - \varphi'(y_{t^-})\,.$$

Consider the performance criterion $J : \mathcal{A}_0 \to \mathbb{R}$

$$J(\delta) = \mathbb{E}\left[x_T + y_T\, Z_T - \alpha\,(y_T - \hat{y})^2 - \phi\int_0^T (y_s - \hat{y})^2\,\mathrm{d}s\right],$$

where $\delta = (\delta^a,\, \delta^b)$ and $\mathcal{A}$ is in (7). Then,

$$J^{CFM} = J\left(\delta^{CFM}\right) \qquad \text{and} \qquad J^{CFM} \leq J\left(\delta^\star\right), \tag{18}$$

where $\delta^\star = (\delta^{a,\star}, \delta^{b,\star})$ is given by (17).

For a proof, see Appendix A.5.

2.3.3. No-arbitrage in the ALP

Consider an ALP with depth $y_0 \in \mathcal{Y}$, marginal rate Z_0, buy/sell quantity ζ, and impact functions $\eta^a(\,\cdot\,)$ and $\eta^b(\,\cdot\,)$. Assume that the ALP trades with only one LT who starts with zero inventory and zero cash. Here, we define arbitrage as any (roundtrip) sequence of trades $\{\epsilon_1, \dots, \epsilon_\mathfrak{m}\}$, where $\epsilon_k = \pm 1$ (buy/sell) for $k \in \{1, \dots, \mathfrak{m}\}$ and $\sum_{k=1}^{\mathfrak{m}} \epsilon_k = 0$, such that the terminal cash of the LT is positive. In AMMs that run on blockchains, LTs use atomic transactions to implement cyclic arbitrage and mitigate the risk of other trades entering the arbitrage sequence; see Wang et al. (2022).[7] Thus, we consider that all the trades ϵ_k of the roundtrip sequence are executed in an orderly manner and simultaneously (with the same timestamp), and we drop the time variable t.

For any liquidity provision strategy $\left(\delta^b, \delta^a\right) = \left(\mathfrak{d}^b(y, Z), \mathfrak{d}^a(y, Z)\right)$, where $\mathfrak{d}^b$ and $\mathfrak{d}^a$ are functions of the pool state (y, Z),[8] there are choices of impact functions $\eta^a(\,\cdot\,)$ and $\eta^b(\,\cdot\,)$ that make arbitrage possible. Assume $\mathfrak{m} = 2$ and the LT does either a roundtrip trade of form (i) buy-then-sell ($\epsilon_1 = 1$ and $\epsilon_1 = -1$) or (ii) sell-then-buy ($\epsilon_1 = -1$ and $\epsilon_1 = 1$). Use the dynamics in (1) and (2) to obtain the P&L of the LT after the roundtrip trade as

$$
\begin{aligned}
&\text{case (i)} \qquad \text{P\&L} = \zeta \left(\eta^a\left(y_0\right) - \mathfrak{d}^a(y_0, Z_0) - \mathfrak{d}^b(y_0 - \zeta, Z + \eta^a(y_0)\right)\,, \\
&\text{case (ii)} \qquad \text{P\&L} = \zeta \left(\eta^b\left(y_0\right) - \mathfrak{d}^b(y_0, Z_0) - \mathfrak{d}^a(y_0 + \zeta, Z_0 - \eta^b(y_0)\right)\,.
\end{aligned}
\tag{19}
$$

If the LP in the ALP pool implements the strategy (17), then

$$
\begin{aligned}
&\text{case (i)} \qquad \text{P\&L} = -\frac{2\,\zeta}{\kappa} + 2\,\zeta\,\eta^a(y_0) + y\left(\eta^b(y_0 - \zeta) - \eta^a(y_0)\right)\,, \\
&\text{case (ii)} \qquad \text{P\&L} = -\frac{2\,\zeta}{\kappa} + 2\,\zeta\,\eta^b(y_0) + y\left(\eta^a(y_0 + \zeta) - \eta^b(y_0)\right)\,.
\end{aligned}
\tag{20}
$$

If the LP chooses impact functions $\eta^a(\,\cdot\,)$ and $\eta^b(\,\cdot\,)$ such that either of the above expressions are positive, then, arbitrageurs can execute these roundtrip trades with risk-free profits and drive the LP out of business. Clearly, the profits in (19) are negative if the bid quote

$$
Z_0 + \eta^a(y_0) - \mathfrak{d}^b(y_0 - \zeta, Z_0 + \eta^a(y_0))
$$

[7]Transactions in the Ethereum blockchain, where the majority of AMMs run, are atomic. A transaction is an ordered set of instructions to different protocols, and it is atomic because each instruction has to be valid for the whole transaction to be executed. In Ethereum, all the instructions of a transaction are executed *simultaneously*.

[8]Below, our results hold for functions $\mathfrak{d}^b$ and $\mathfrak{d}^a$ that depend on exogenous information.

after a buy trade is less than the ask quote

$$Z_0 + \mathfrak{d}^a(y_0, Z_0)$$

before the trade, because it guarantees

$$\eta^a(y_0) \leq \mathfrak{d}^b(y_0 - \zeta, Z_0 + \eta^a(y_0)) + \mathfrak{d}^a(y_0, Z_0)\,,$$

and conversely for a sell trade.

The no-arbitrage condition above (for roundtrip sequences of length $\mathfrak{m} = 2$) does not guarantee that the marginal exchange rate $Z + \eta^a(y) - \eta^b(y - \zeta)$ or $Z - \eta^b(y) + \eta^a(y + \zeta)$ at the end of the arbitrage sequence is Z, thus price manipulation is possible. We restrict the impact functions $\eta^a(\cdot)$ and $\eta^b(\cdot)$ to a set of functions which guarantee that the marginal rate returns to its initial value for any roundtrip sequence of trades. To simplify notation, let $\mathfrak{y}_1 = \underline{y}$, $\mathfrak{y}_2 = \underline{y} + \zeta$, ..., and $\mathfrak{y}_N = \overline{y}$ where $N = \overline{N} - \underline{N} + 1$, $\overline{N} = \overline{y}/\zeta$, and $\underline{N} = \underline{y}/\zeta$. The following proposition provides necessary and sufficient conditions for roundtrip trades not to move the marginal rate, and further helps us provide a general no-arbitrage condition for the ALP.

Proposition 4. *The marginal rate Z takes only the ordered finitely many values $\mathcal{Z} = \{\mathfrak{z}_1, \ldots, \mathfrak{z}_N\}$, with the property that $Z_0 \in \mathcal{Z}$ and for $i \in \{1, \ldots, N-1\}$*

$$\mathfrak{z}_{i+1} - \eta^b(\mathfrak{y}_{N-i}) = \mathfrak{z}_i \qquad \textit{and} \qquad \mathfrak{z}_i + \eta^a(\mathfrak{y}_{N-i} + \zeta) = \mathfrak{z}_{i+1}\,,$$

if and only if $\eta^a(\cdot)$ and $\eta^b(\cdot)$ are such that

$$\eta^b(\mathfrak{y}_i) = \eta^a(\mathfrak{y}_i + \zeta)\,, \tag{21}$$

for $i \in \{1, \ldots, N-1\}$.

Intuitively, the above proposition gives a necessary and sufficient condition for the marginal rate Z to live within a fixed grid with lattice-style dynamics, that is, from $\mathfrak{z}_i$, the marginal rate can only move to $\mathfrak{z}_{i-1}$ or $\mathfrak{z}_{i+1}$ provided it stays in the grid. A simple set of impact functions that satisfy (21) are the constant impact functions, that is, $\eta^a(y) = \eta^b(y) = \eta \in \mathbb{R}^+$. Also, the impact functions in (1) that replicate the marginal rate dynamics in a CFM with level function φ satisfy the condition (21).

Theorem 3. *Let $\eta^a(\cdot)$ and $\eta^b(\cdot)$ satisfy (21) for $i \in \{1, \ldots, N-1\}$. For any liquidity provision strategy of the form $(\delta^b, \delta^a) = (\mathfrak{d}^b(y, Z), \mathfrak{d}^a(y, Z))$, if for all $i \in \{1, \ldots, N-1\}$,*

$$\eta^a(\mathfrak{y}_{i+1}) \leq \mathfrak{d}^a\left(\mathfrak{y}_{i+1}, \mathfrak{z}_{N-i}\right) + \mathfrak{d}^b\left(\mathfrak{y}_{i+1} - \zeta, \mathfrak{z}_{N-i} + \eta^a\left(\mathfrak{y}_{i+1}\right)\right) \tag{22}$$

$$\textit{and} \quad \eta^b(\mathfrak{y}_i) \leq \mathfrak{d}^b\left(\mathfrak{y}_i, \mathfrak{z}_{N-i+1}\right) + \mathfrak{d}^a\left(\mathfrak{y}_i + \zeta, \mathfrak{z}_{N-i+1} - \eta^b\left(\mathfrak{y}_i\right)\right) ,$$

or equivalently

$$\eta^a(\mathfrak{y}_{i+1}) \leq \mathfrak{d}^a\left(\mathfrak{y}_{i+1}, \mathfrak{z}_{N-i}\right) + \mathfrak{d}^b\left(\mathfrak{y}_i, \mathfrak{z}_{N-i+1}\right) \quad \textit{and} \quad \eta^b(\mathfrak{y}_i) \quad \leq \mathfrak{d}^b\left(\mathfrak{y}_i, \mathfrak{z}_{N-i+1}\right) + \mathfrak{d}^a\left(\mathfrak{y}_{i+1}, \mathfrak{z}_{N-i}\right) ,$$

then there is no roundtrip sequence of trades that a liquidity taker can execute to arbitrage the ALP. For the liquidity provision strategy in (17), the condition simplifies to

$$\eta^a(\mathfrak{y}_i) \leq \frac{1}{\kappa}, \quad \textit{and} \qquad \eta^b(\mathfrak{y}_i) \leq \frac{1}{\kappa}, \tag{23}$$

for all $i \in \{1, \ldots, N\}$.

For a proof, see Appendix A.6

The conditions of Theorem 3 are standard in a trading venue that is profitable for LPs. For general liquidity provision strategies, condition (22) implies that in any state of the pool, the bid quote after a buy trade is lower than the ask quote before the trade, and the ask quote after a sell trade is higher than the bid quote before the trade. For the optimal liquidity provision strategy (17), condition (23) reduces to ensuring that the jump in the marginal rate after a trade is less than the baseline half bid-ask spread $1/\kappa$.[9]

In the numerical examples below, we assume that $c^a = c^b = c > 0$ and that the inventory risk constraint is $y \in \{\underline{y}, \ldots, \overline{y}\}$ where $\underline{y} \geq \zeta$. Then, we employ the following impact functions for the liquidity provision strategy in the ALP:

$$\eta^b(y) = \frac{\zeta}{\frac{1}{2}\, y + \zeta}\, L \quad \text{and} \quad \eta^a(y) = \frac{\zeta}{\frac{1}{2}\, y - \zeta}\, L\,, \tag{24}$$

where $L > 0$ is the *impact parameter*.[10] We choose $L < \frac{1}{\kappa}$ and observe that $\eta^a(\,\cdot\,)$ and $\eta^b(\,\cdot\,)$ satisfy (21) for $i \in \{1, \ldots, N-1\}$, so from Theorem 3 the ALP does not admit arbitrage.

2.4 Performance and evaluation using historical data

2.4.1. Numerical implementation

To obtain the optimal quotes in (17) and the value function in (13) for an LP in the ALP, the ALP must compute the solution to the system of ordinary differential equations (ODEs)

[9]Recall that $1/\kappa$ is the component of the spread that optimises the instantaneous expected profit of the LP from a roundtrip trade. The expected profit from a roundtrip trade is $2\,\zeta\,(\delta^b\,\lambda^b(\delta^b) + \delta^a\,\lambda^a(\delta^a))$ and is maximal for $\delta^b = 1/\kappa$ and $\delta^a = 1/\kappa$, where $\lambda^b(\,\cdot\,)$ and $\lambda^a(\,\cdot\,)$ are in (6).

[10]If the inventory is $y = \zeta$, then the LP does not accept trades at the ask.

in (A.5), which involves the exponential of a matrix. In practice, the size of the matrix $\boldsymbol{K}$ is large, and thus matrix exponentiation is not efficient. However, to make the model computationally efficient, we exploit two key features. First, the matrix $\boldsymbol{K}$ is sparse, and second, instead of $\exp(\boldsymbol{K}\,t)$ we require the action of $\exp(\boldsymbol{K}\,t)$ on the vector $\boldsymbol{U}(0)$; see (11). Thus, we use the algorithm in Al-Mohy and Higham (2011) to compute $\exp(\boldsymbol{K}\,t)\,\boldsymbol{U}(0)$. The approach uses the scaling and squaring method and the matrix Padé approximation of the exponential, which significantly improve computational performance in the case of sparse matrices. Moreover, for known impact functions η^b and η^a of the LP's position in the ALP, one can pre-compute (off-chain) the optimal quotes for tuples of the strategy parameters $(T, \zeta, y, \kappa, c, \phi, \alpha)$ and parameters defining the impact functions η^b, η^a. Thus, in practice, the computational burden for the LP position in the ALP results in storage burden.

2.4.2. ALP: choice of impact function

In this section, we assume that $c^a = c^b = c > 0$ and that the inventory risk constraint is $y \in \{\underline{y}, \ldots, \overline{y}\}$ where $\underline{y} \geq \zeta$. We employ the impact functions in (24) for the liquidity provision strategy in the ALP, where L is the *impact parameter* and satisfies $1/\kappa > L > 0$.[11] The optimal quotes in (17) become

$$\delta^{b\star}(t,y) = \frac{1}{\kappa} - \frac{\theta\,(t, y+\zeta) - \theta\,(t,y)}{\zeta} - L\,, \quad \delta^{a\star}(t,y) = \frac{1}{\kappa} - \frac{\theta\,(t, y-\zeta) - \theta\,(t,y)}{\zeta} + L\,,$$

and the matrix $\mathbf{K}$ in (10) is given by

$$\mathbf{K}_{mn} = \begin{cases} c\,e^{-1}\,e^{\kappa\,L} & \text{if } n = m-1 \text{ and } m > \underline{N}\,, \\ -\kappa\,\phi\,(m\,\zeta - \hat{y})^2/\zeta & \text{if } n = m\,, \\ c\,e^{-1}\,e^{-\kappa\,L} & \text{if } n = m+1 \text{ and } m < \overline{N}\,. \end{cases}$$

The impact functions in (24) are inversely proportional to the LP's inventory y, so for small trade sizes, one can write $\eta^b(y) \approx \eta^a(y) \approx \frac{2\zeta}{y}\,L$. Thus, the impact parameter L scales how changes in the inventory of the LP affect the marginal rate Z; small values of L reduce the impact on the rate and large values of L amplify it. This parameter can be used by an LP who assesses the proportion of the order flow that is informed (toxic) as opposed to uninformed (noise); a high value of L leads to a faster rate of adjustment and vice versa.

[11] If the inventory is $y = \zeta$, then the LP does not accept trades at the ask.

Also, for small trade sizes ζ and large inventory y, the bid-ask spread is

$$\delta^{b\star}(t,y) + \delta^{a\star}(t,y) \approx \frac{2}{\kappa}\,.$$

2.4.3. ALP: optimal shifts

Here, we study the properties of our optimal quotes in the ALP as a function of model parameters when the impact functions are those in (24), see Figure 1.

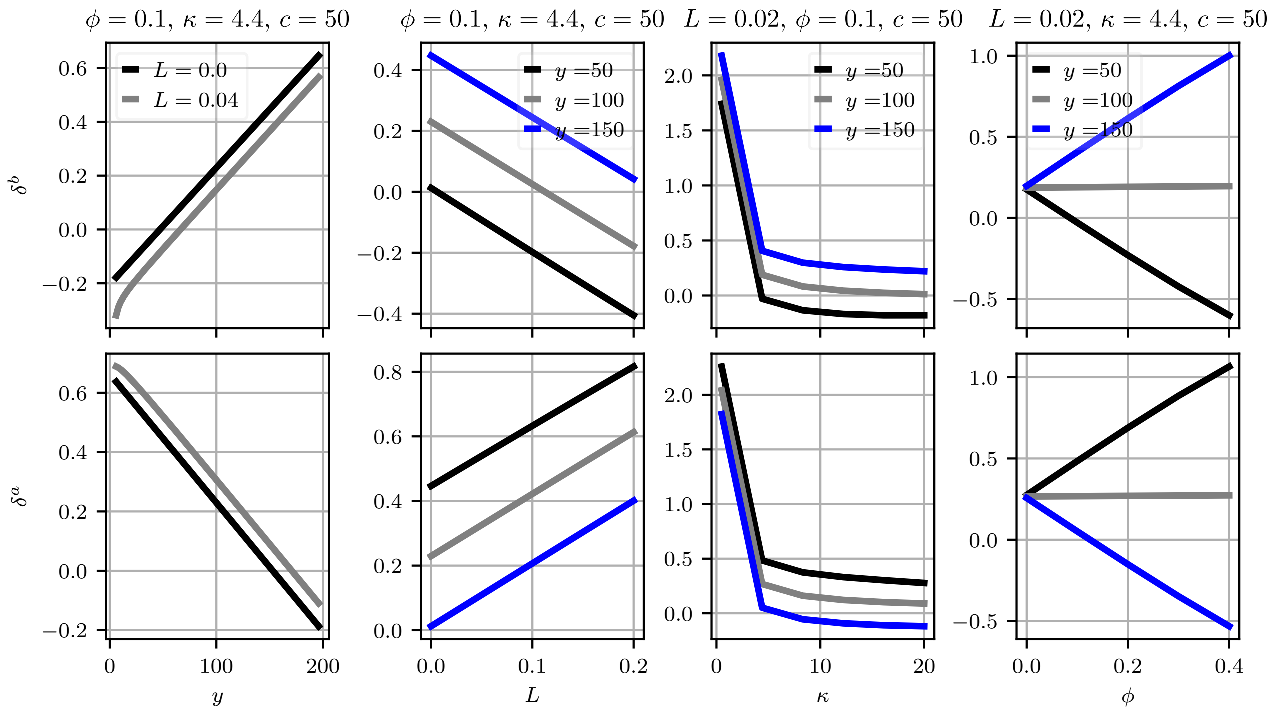

Figure 1: ALP: Optimal shifts as a function of model parameters, where $\hat{y} = 0$ ETH, $[\underline{y}, \overline{y}] = [0, 200]$, and $\alpha = 0\,\text{USDC} \cdot \text{ETH}^{-2}$.

The top left plot of Figure 1 shows how the shifts adjust as a function of the LP's current inventory. In particular, the ask and bid shifts adjust to attract trading flow which induces mean reversion in the inventories to $\hat{y}$. The second plot in the top panel shows the role of the impact parameter L. The third plot shows that high values of κ, i.e., lower expected revenue from roundtrip trades, decrease the value of the shifts δ^a and δ^b. Finally, the last plot shows that, when $y \neq \hat{y}$, the value of the running inventory penalty parameter ϕ skews the shifts to attract trading flow that will push the inventory toward the target inventory $\hat{y}$. The interpretation of the plots in the bottom panel is similar.

2.4.4. LP performance and predictable losses

In this subsection, we use transaction data for the currency pair ETH/USDC from Binance and from the Uniswap v3 pool ETH/USDC 0.05% to illustrate the performance of our optimal liquidity provision strategy in the ALP.[12] Table 1 shows descriptive statistics for both sets of transaction data. Binance is a liquid traditional price-time priority LOB, and Uniswap is a CPM which implements concentrated liquidity; see Adams et al. (2021) and Cartea et al. (2022a).

	ETH/USDC 0.05%		Binance
	LT	LP	
Number of transactions	216,739	42,022	12,341,854
Average transaction size	\$ 109,037	\$ 2,765,499	\$ 1,735
Gross USD volume	$\approx$ \$ 185.57 $\times 10^9$	$\approx$ \$ 116.2 $\times 10^9$	$\approx$ \$ 21.42 $\times 10^9$
Average trading frequency	18.27 seconds	12.3 minutes	2.56 seconds
Median LP holding time	86 minutes		n.a.
Average pool depth	19,788,327 $\sqrt{\text{ETH} \cdot \text{USDC}}$		n.a.

Table 1: LT and LP activity statistics in the Uniswap v3 pool ETH/USDC 0.05% and in Binance between 5 May 2021 (Uniswap inception) and 30 April 2022; see Drissi (2023) for more details.

First, consider an example where only one LP provides liquidity in the ALP for ETH/USDC between 1 August 2021 09:00 and 09:30. The LP's strategy parameters are $\zeta = 1$ ETH, $\kappa = 1\,\text{ETH}^{-1}$, $c = 100$, $L = 0.3$ ETH, $\underline{y} = -500$ ETH, $\overline{y} = 500$ ETH. Moreover, we set $T = 30$ minutes, $\phi = \alpha = 10^{-4}\,\text{USDC} \cdot \text{ETH}^{-2}$, and $y_0 = \hat{y} = 100$.

In CFMs, the execution costs and the rate impact are approximated with the convexity $\varphi''(y)$, where φ is the trading function and y is the amount of the risky asset Y in the pool. In CPMMs, the execution costs for a liquidity taking trade with quantity ζ of asset Y are approximated with $\frac{1}{2}\,\varphi''(y)\,|\zeta| = \frac{1}{\rho}\,Z^{3/2}\,|\zeta|$, and the rate impact is approximated with $\varphi''(y)\,|\Delta y| = \frac{2\,Z^{3/2}}{\rho}\,|\Delta y|$, where ρ is the depth of the pool and Z is the marginal rate. We obtain historical values of ρ and Z from historical LP and LT operations since inception of the pool. Figure 2 compares the execution costs and the marginal rate impact of (i) the liquidity provision strategy in the ALP and (ii) those implied by the available liquidity in the Uniswap pool at $t = 0$ as a function of the trade size ζ. To compute the execution costs

[12] The transaction fee in Uniswap v3 pool ETHU/SDC is 0.05%.

and the marginal rate impact in the ALP, we use the LP's parameters, and to compute those in Uniswap, we use the convexity of the Uniswap pool's trading function on 1 August 2021 09:00.

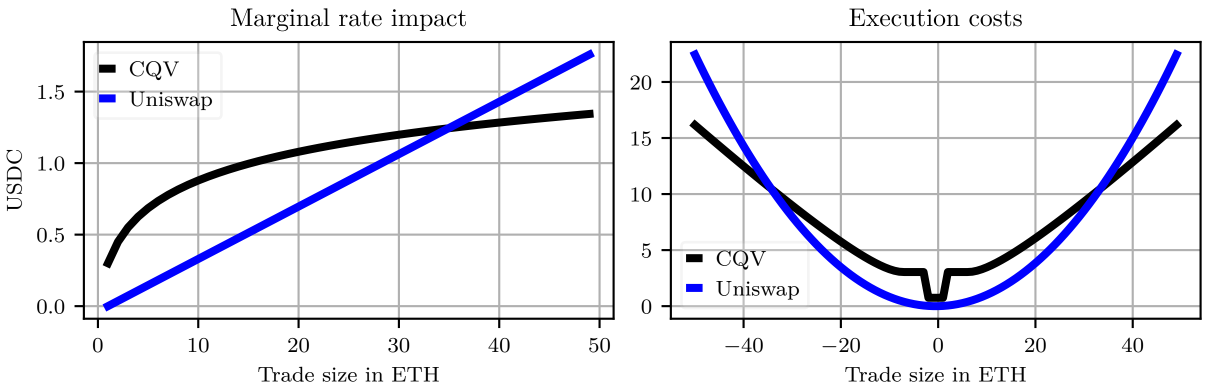

Figure 2: Marginal rate impact and execution costs in the ALP as a function of the size of the trade.

Now, consider a scenario where only arbitrageurs interact with the ALP to align its marginal exchange rate with the exchange rate in Binance, and assume that trading in Binance is frictionless. Figure 3 shows (i) the value of the LP's holdings, (ii) the wealth from holding the LP's inventory outside the ALP, (iii) the earnings of the ALP, and (iv) the total wealth, i.e., the sum of pool value and earnings.

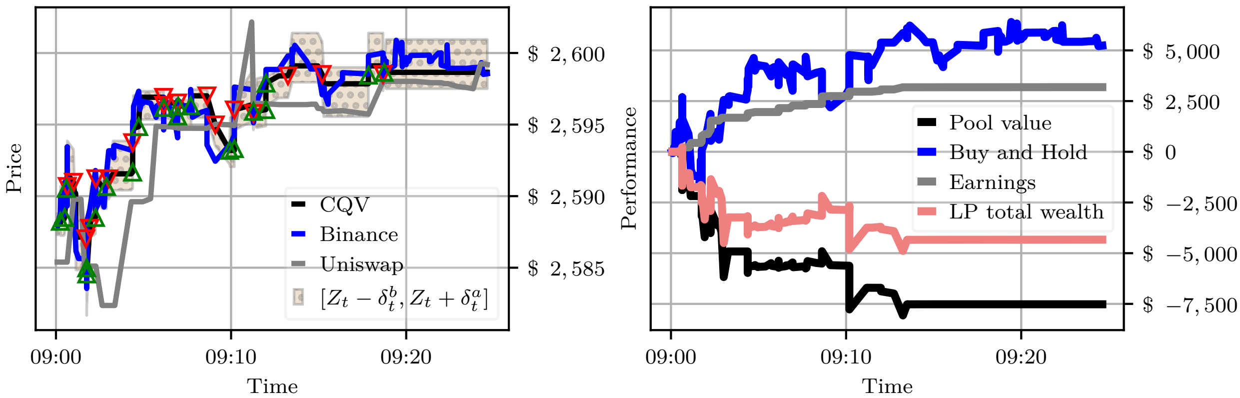

Figure 3: LP wealth when arbitrageurs trade in the ALP and Binance. **Left**: Exchange rates from ALP, Binance, and Uniswap v3. **Right**: *Pool value* is computed as $x_t + y_t\, Z_t$, *Buy and Hold* is computed as the wealth from holding the LP's inventory outside the ALP, i.e., $y_t\, Z_t$, *Earnings* are the revenue from the quotes, and *LP total wealth* is the total LP's wealth.

When the impact function and the execution costs are comparable to those in Uniswap (see Figure 2), the losses of the LP due to adverse selection are significant and are not covered by the revenue from the spread in the ALP. In practice, this scenario is realistic when there is an alternative trading venue where exchange rates form (e.g., Binance) so the LP incurs losses when making liquidity to informed traders (e.g., Uniswap v3). Thus, an LP who expects the trading flow to be mainly driven by informed LTs, as in our example, must increase the price of liquidity in the pool. Everything else being equal, one expects that as the price of liquidity increases, the liquidity taking activity decreases. To increase the price of liquidity, the LP can increase the value of the penalty parameter ϕ to penalise deviations from her target inventory with higher shifts; see Figure 1. Figure 4 shows that an LP can significantly reduce her loss by adjusting the value of the penalty parameter ϕ to 0.1 to increase her provision revenue, which decreases the number of arbitrages and thus reduces the losses from adverse selection.

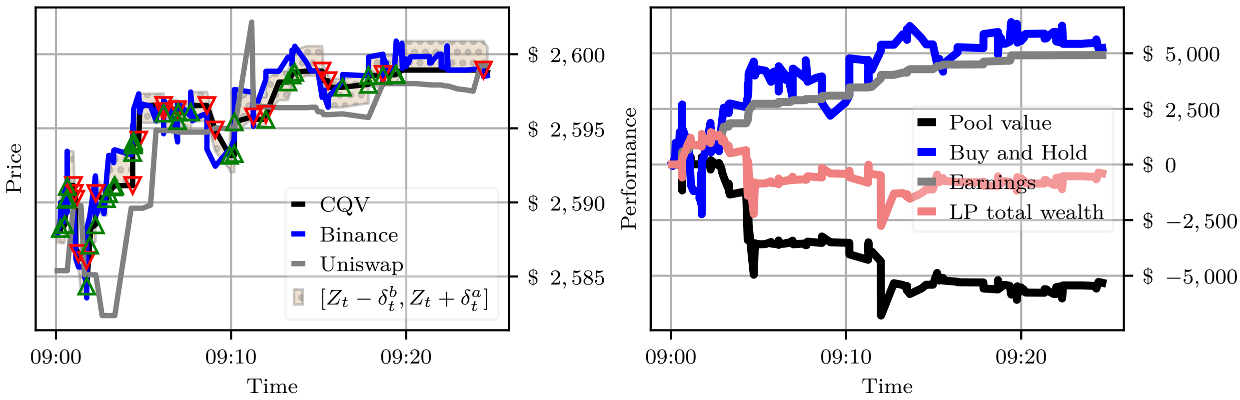

Figure 4: LP wealth when only an arbitrageur interacts in the ALP.

Next, we perform multiple consecutive in-sample estimation of model parameters and out-of-sample liquidity provision to compare the performance of LPs that use the optimal liquidity provision strategy (17) in the ALP with the historical performance of LPs in Uniswap v3. Specifically, we (i) fix a one hour in-sample window and a 30 minutes out-of-sample window, (ii) estimate in-sample parameters, (iii) compute the LP's out-of-sample performance, and (iv) shift both windows by 30 minutes and repeat the procedure. Throughout the out-of-sample trading window, if the LP's inventory hits the inventory limit $\underline{y}$ or $\overline{y}$, then she liquidates her position, records her accrued PnL, and sets a new liquidity position. We consider two simulation scenarios; scenario I where only arbitrageurs trade in the ALP to align the ALP rate Z with the exchange rate of Binance, and scenario II where half the trading flow is arbitrageurs and the other half is noise traders.

For every run, the LP starts with zero inventory and sets a zero target inventory, i.e.,

$y_0 = \hat{y} = 0$ ETH. The size ζ of trades is the average in-sample traded size in Binance; see Table 1. For the fill ratio exponent, the LP estimates a value $\kappa = 0.05$ in scenario I and $\kappa = 0.1$ in scenario II; recall that the value of κ decreases when LTs react less to the price of liquidity; see Cartea et al. (2015). For the buy/sell pressure parameter c, we use the in-sample average order arrival intensity in Binance. Finally, we fix the rate impact parameter $L = 0.3$ ETH, the inventory limits $\overline{y} = -\underline{y} = 500$ ETH, and the penalty parameters $\phi = 10^{-2}\,\text{USDC} \cdot \text{ETH}^{-2}$ and $\alpha = 10^{-4}\,\text{USDC} \cdot \text{ETH}^{-2}$.

Table 2 shows the average performance of LPs in the ALP for scenarios I and II and the average performance of LPs in the Uniswap v3 pool ETH/USDC 0.05%. LPs in the ALP outperform LPs in Uniswap in both scenarios. Moreover, liquidity provision is more profitable than holding the assets when noise traders interact with the ALP.

	Average	Standard deviation
ALP (scenario I)	−0.006%	0.733%
ALP (scenario II)	0.980%	2.305%
Buy and Hold	0.001%	0.741%
Uniswap v3	−1.485%	7.812%

Table 2: Average and standard deviation of 30-minutes performance of LPs in the ALP for both simulation scenarios, LPs in Uniswap, and buy-and-hold.

3. The GLP design

In this section, we propose the GLP design for AMMs, which is similar to that in Section 2, but here we assume that the marginal rate and the inventory evolve geometrically. The GLP makes liquidity for the pair of assets X and Y and the marginal exchange rate of the asset Y in units of the reference asset X in the GLP is Z. An LP initialises the GLP with quantities (x_0, y_0) of both assets and an initial marginal rate Z_0, and we assume there is no LP activity over a trading window $[0, T]$.

Let $\zeta^{\text{b}} \in (0,1)$ and $\zeta^{\text{a}} \in (0,1)$ be two constants, and let the impact functions at the bid and the ask be $y \mapsto \eta^b(y) \in (0,1)$ and $y \mapsto \eta^a(y) \in (0,\infty)$, respectively.[13] In the GLP, the LP is ready to buy the quantity $\zeta^{\text{b}}\, y_{t^-}$ and to sell the quantity $\zeta^{\text{a}}\, y_{t^-}$ of asset Y at any time

[13]These assumptions are not restrictive because the impact functions in the GLP are relative movements in the marginal rate Z, so a value of 1 means a 100% rate innovation.

$t \in [0, T]$. The quantities of assets X and Y in the pool follow the dynamics

$$
\begin{aligned}
\mathrm{d}y_t &= \zeta^{\mathrm{b}}\, y_{t^-}\, \mathrm{d}N_t^b - \zeta^{\mathrm{a}}\, y_{t^-} \mathrm{d}N_t^a\,,\\
\mathrm{d}x_t &= -\zeta^{\mathrm{b}}\, y_{t^-} Z_{t^-}\, \left(1-\delta_t^b\right)\, \mathrm{d}N_t^b + \zeta^{\mathrm{a}}\, y_{t^-} Z_{t^-} \left(1+\delta_t^a\right) \mathrm{d}N_t^a\,.
\end{aligned}
$$

In contrast to the ALP, the dynamics of the GLP reserves are geometric. The advantage of geometric dynamics is that the values of the strategy parameters of the LP and their interpretation do not depend on the level of the state variables. In particular, in the GLP, the rate impact functions $\{\eta^b, \eta^a\}$ and the shifts $\{\delta^b, \delta^a\}$ denote percentages and do not have units.[14] The values of $\zeta^{\mathrm{b}} \in (0,1)$ and $\zeta^{\mathrm{a}} \in (0,1)$ are the fractions of the inventory in y that the LP offers to buy and sell, respectively; thus, as long as the process $(y_s)_{s\in[t,T]}$ starts with a non-negative initial value, it stays non-negative. Moreover, the shifts δ_t^b and δ_t^a are a fraction of the marginal rate (not the shifts of the bid and ask quotes from the marginal rate). More precisely, the LP's bid and ask quotes are $Z_{t^-}(1-\delta_t^b)$ and $Z_{t^-}(1+\delta_t^a)$, respectively.

The marginal rate in the pool is updated as follows

$$
\mathrm{d}Z_t = Z_{t^-}\, \left(-\eta^b(y_{t^-})\, \mathrm{d}N_t^b + \eta^a(y_{t^-})\, \mathrm{d}N_t^a\right)\,. \tag{25}
$$

From (25), we see that the changes in the marginal rate are proportional to the current rate in the pool. Moreover, the process $(Z_s)_{s\in[t,T]}$ is non-negative as long as $Z_t \geq 0$ because $y \mapsto \eta^b(y) \in (0,1)$.

3.1 The optimal price of liquidity in GLPs

In optimal liquidity provision model in an GLP, the LP is ready to buy the quantity $\zeta^{\mathrm{b}} = \zeta\, y_{t^-}$ and to sell the quantity $\zeta^{\mathrm{a}} = \frac{\zeta}{1+\zeta}\, y_{t^-}$ of asset Y at any time $t \in [0,T]$, for some constant $\zeta \in (0,1)$, as long as her inventory is larger than $\underline{y}$ and smaller than $\underline{y}\,(1+\zeta)^N$ for some fixed $N \in \mathbb{N}$ and some fixed positive constant $\underline{y}$. Thus, her inventory lies within the finite set $\mathcal{Y} = [\underline{y},\, \underline{y}(1+\zeta)^N]$. Thus, the dynamics of the LP's holdings are

$$
\begin{aligned}
\mathrm{d}y_t &= \zeta\, y_{t^-}\, \mathrm{d}N_t^b - \frac{\zeta}{1+\zeta}\, y_{t^-} \mathrm{d}N_t^a\,,\\
\mathrm{d}x_t &= -\zeta\, y_{t^-} Z_{t^-}\, \left(1-\delta_t^b\right)\, \mathrm{d}N_t^b + \frac{\zeta}{1+\zeta}\, y_{t^-} Z_{t^-} \left(1+\delta_t^a\right) \mathrm{d}N_t^a\,.
\end{aligned}
$$

[14] A similar statement holds for the penalty parameters $\{\alpha, \phi\}$ that we define below in Subsection 3.1 for the optimal liquidity provision problem in GLP.

Similar to the ALP, the LP in the GLP assumes that the arrival intensity decays exponentially as a function of the shifts δ^a and δ^b. However, the order size at the ask is smaller than that at the bid by an overall factor equal to $(1+\zeta)^{-1}$, thus the LP assumes that the exponential decay of the liquidity trading flow at the ask is slower by the same fraction, and she writes

$$\begin{cases} \lambda_t^b\left(\delta_t^b\right) = c^b\, e^{-\kappa\,\delta_t^b}\,\mathbb{1}^b\left(y_{t^-}\right)\,, \\ \lambda_t\left(\delta_t^a\right) = c^a\, e^{-\frac{\kappa}{1+\zeta}\delta_t^a}\,\mathbb{1}^a\left(y_{t^-}\right)\,, \end{cases} \tag{26}$$

for some positive constant κ.

The LP is continuously updating the shifts δ_t^b and δ_t^a until a fixed horizon $T>0$. The performance criterion of the LP using the strategy $\delta = \left(\delta^b, \delta^a\right) \in \mathcal{A}$, where the admissible set is in (7), is a function $w^\delta\colon [0,T]\times\mathbb{R}\times\mathcal{Y}\times\mathbb{R}^+\to\mathbb{R}$, which is given by

$$w^\delta(t,x,y,z) = \mathbb{E}_{t,x,y,z}\left[x_T + y_T\,Z_T - \alpha\,Z_T\,(y_T-\hat{y})^2 - \phi\int_t^T Z_s\,(y_s-\hat{y})^2\,\mathrm{d}s\right]. \tag{27}$$

Note that in contrast to the performance criterion (8) in the ALP, the aversion to inventory deviations from $\hat{y}$ in (27) is proportional to the marginal pool rate.

The value function $w\colon [0,T]\times\mathbb{R}\times\mathcal{Y}\times\mathbb{R}^+\to\mathbb{R}$ of the LP in the GLP is given by

$$w(t,x,y,z) = \sup_{\delta\in\mathcal{A}_t} w^\delta(t,x,y,z)\,. \tag{28}$$

Proposition 5 shows that the performance criterion (27) is finite and that the associated value function (28) is well defined.

Proposition 5. *There is $C\in\mathbb{R}$ such that for all $(\delta_s)_{s\in[t,T]}\in\mathcal{A}_t$, the performance criterion of the LP satisfies*

$$w^\delta(t,x,y,z)\le C<\infty\,,$$

so the value function w in (28) is well defined.

For a proof, see Appendix A.7.

Proposition 6 provides a candidate closed-form solution when the impact functions are

$$\eta^b(y) = \frac{\zeta}{1+\zeta}\in(0,1)\,,\quad \eta^a(y)=\zeta\in(0,1)\,, \tag{29}$$

and Theorem 4 is the verification theorem for the optimal liquidity provision problem in an GLP. Note that for small values of ζ, the LP's impact function at the bid is approximately ζ. Thus, the LP impacts the marginal rate of the GLP by the same fraction as her inventory.

Proposition 6 (Candidate closed-form solution: GLP). *Let $\eta^b(\,\cdot\,)$ and $\eta^a(\,\cdot\,)$ be given by (29). Define the matrix $\mathbf{K}\in\mathbb{R}^{N\times N}$ by*

$$\mathbf{K}_{mn}=\begin{cases} c^a e^{-1+\frac{\kappa}{1+\zeta}} & \text{if } m=n-1\,,\\ -\frac{\phi\,\kappa}{\zeta\,\underline{y}(1+\Delta)^m}\left(\underline{y}\,(1+\Delta)^m-\hat{y}\right)^2 & \text{if } m=n\,,\\ c^b e^{-1-\kappa} & \text{if } m=n+1\,,\end{cases} \tag{30}$$

for $m,n\in\{1,\dots,N\}$. Let $\mathbf{U}\in C^1([0,T],\mathbb{R}^N)$ be

$$\mathbf{U}(t)=\exp\left(\mathbf{K}\,t\right)\,\mathbf{U}(0)\,,\quad t\in[0,T]\,, \tag{31}$$

where $\exp(\,\cdot\,)$ is matrix exponentiation and

$$\mathbf{U}(0)_m=e^{-\frac{\alpha\,\kappa}{\zeta\,(1+\zeta)^m\,\underline{y}}}\,,\quad m\in\{1,\dots,N\}\,. \tag{32}$$

For $m\in\{1,\dots,N\}$, let $u(t,(1+\zeta)^m\underline{y})=\mathbf{U}(T-t)_m$ and define

$$\theta(t,y,z)=\frac{\zeta\,y\,z}{\kappa}\log u(t,y)\,. \tag{33}$$

Then, the function $\omega\colon[0,T]\times\mathbb{R}\times\mathcal{Y}\times\mathbb{R}\to\mathbb{R}$ given by

$$\omega(t,x,y,z)=x+y\,z+\theta(t,y,z) \tag{34}$$

solves the HJB equation associated with problem (28) which is given by

$$\begin{aligned}0=\;&\partial_t\omega-\phi\,z\,(y-\hat{y})^2\\ &+\sup_{\delta^b}\lambda\left(\delta^b\right)\left(\omega\left(t,x-\zeta\,y\,z\,\left(1-\delta^b\right),y+\zeta\,y,z-z\,\eta^b(y)\right)-\omega\left(t,x,y,z\right)\right)\\ &+\sup_{\delta^a}\lambda\left(\delta^a\right)\left(\omega\left(t,x+\frac{\zeta}{1+\zeta}\,y\,z\,\left(1+\delta^a\right),y-\frac{\zeta}{1+\zeta}\,y,z+z\,\eta^a(y)\right)-\omega\left(t,x,y,z\right)\right)\end{aligned} \tag{35}$$

on $[0,T)\times\mathbb{R}\times\mathcal{Y}\times\mathbb{R}^+$ with the terminal condition $\omega(T,x,y,z)=x+y\,z-\alpha\,z\,(y-\hat{y})^2$.

For a proof, see Appendix A.8.

Theorem 4 (Verification: GLP). *Let $\eta^b(\,\cdot\,)$ and $\eta^a(\,\cdot\,)$ be given by (29). Let ω be defined as in Proposition 6. Then the function ω in (34) satisfies that for all $(t,x,y,z)\in[0,T]\times\mathbb{R}\times\mathcal{Y}\times\mathbb{R}$ and $\delta=(\delta_s)_{s\in[t,T]}\in\mathcal{A}_t$,*

$$w^\delta(t,x,y,z)\le\omega\left(t,x,y,z\right)\,. \tag{36}$$

Moreover, equality is obtained in (36) with the admissible optimal Markovian control $(\delta^\star_s)_{s\in[t,T]}=$

$\left(\delta_s^{b\star}, \delta_s^{a\star}\right)_{s\in[t,T]} \in \mathcal{A}_t$ *given by the feedback formulae*

$$\begin{aligned}
\delta^{b\star}(t,y) &= 1 + \frac{1}{\kappa} - \frac{\log u\,(t, y + y\,\zeta) - \log u\,(t,y)}{\kappa}\,, \\
\delta^{a\star}(t,y) &= -1 + \frac{1+\zeta}{\kappa} - (1+\zeta)\,\frac{\log u(t, y - \frac{\zeta}{1+\zeta}\,y) - \log u(t,y)}{\kappa}\,,
\end{aligned} \tag{37}$$

where u is in (33). In particular, $\omega = w$ on $[0,T] \times \mathbb{R} \times \mathcal{Y} \times \mathbb{R}$.

For a proof, see Appendix A.9.

4. Conclusion

In this work, we showed that stochastic optimal control can be used to build AMMs that are viable alternative trading venues. We provided two models and solved the associated control problems explicitly. Finally, we carried out simulations to validate the theoretical results.

There are a number of ways this work can be extended. First, one can consider impact functions that are driven by predictive signals. Second, in practice one can incorporate the significant delays of blockchains into our framework to improve trading performance of LPs; see Cartea and Sánchez-Betancourt (2023). Finally, our models can be extended to multiple related pools or multiple currency pairs within the same pool using multivariate models, see Bergault et al. (2022b).

Appendix A. Proofs

Appendix A.1 Proof of Theorem 1

Let $\left(x_t^{\text{CFM}}\right)_{t\in[0,T]}$, $\left(y_t^{\text{CFM}}\right)_{t\in[0,T]}$, and $\left(Z_t^{\text{CFM}}\right)_{t\in[0,T]}$ be the processes describing the amount of asset X, amount of asset Y, and marginal rate respectively, in a CFM pool.

From the trading condition of CFMs we know that

$$\begin{aligned} & f\left(x_{t^-}^{\text{CFM}} - \zeta^x, y_{t^-}^{\text{CFM}} + \zeta\right) = f\left(x_{t^-}^{\text{CFM}}, y_{t^-}^{\text{CFM}}\right) = m^2\,, \\ \text{and} \quad & f\left(x_{t^-}^{\text{CFM}} + \zeta^x, y_{t^-}^{\text{CFM}} - \zeta\right) = f\left(x_{t^-}^{\text{CFM}}, y_{t^-}^{\text{CFM}}\right) = m^2\,, \end{aligned}$$

for a sell order of size ζ and a buy order of size ζ of asset Y, respectively, where m is the fixed depth of the CFM pool, and $\zeta^x > 0$ is in terms of asset X.

Thus, for a sell order of size ζ, $x_{t^-}^{\text{CFM}} - \zeta^x = \varphi\left(y_{t^-}^{\text{CFM}} + \zeta\right)$ and $x_{t^-}^{\text{CFM}} = \varphi\left(y_{t^-}^{\text{CFM}}\right)$ hold, so the (bid) execution price is

$$\frac{\zeta^x}{\zeta} = \frac{\varphi\left(y_{t^-}^{\text{CFM}}\right) - \varphi\left(y_{t^-}^{\text{CFM}} + \zeta\right)}{\zeta}\,.$$

Thus, the execution cost in the CFM pool is

$$\frac{\varphi\left(y_{t^-}^{\text{CFM}} + \zeta\right) - \varphi\left(y_{t^-}^{\text{CFM}}\right)}{\zeta} - \varphi'\left(y_{t^-}^{\text{CFM}}\right)\,.$$

Similarly, the execution cost at the ask is

$$\frac{\varphi\left(y_{t^-}^{\text{CFM}}\right) - \varphi\left(y_{t^-}^{\text{CFM}} + \zeta\right)}{\zeta} + \varphi'\left(y_{t^-}^{\text{CFM}}\right)\,,$$

so the execution costs in the CFM and the ALPs are the same when (5) holds.

Next, following a sell trade of size ζ in the CFM pool, the innovation in the marginal rate is $-\varphi'(y_{t^-}^{\text{CFM}} + \zeta) + \varphi'\left(y_{t^-}^{\text{CFM}}\right)$. Similarly, following a buy trade of size ζ in the CFM pool, the innovation in the marginal rate is $-\varphi'(y_{t^-}^{\text{CFM}} - \zeta) + \varphi'\left(y_{t^-}^{\text{CFM}}\right)$. Thus, the marginal rate dynamics in the CFM and the ALPs are the same when (4) holds and $Z_0^{\text{CFM}} = Z_0$. Similarly, compare (1) and (3) for the choice of $\delta^{a,b}$ above, to see that if $y_0^{\text{CFM}} = y_0$ and $x_0^{\text{CFM}} = x_0$ then $y_t^{\text{CFM}} = y_t$ and $x_t^{\text{CFM}} = x_t$ for all $t \in [0,T]$. $\square$

Appendix A.2 Proof of Proposition 1

To prove that (8) is finite we observe that for $\delta \in \mathcal{A}_t$ it follows that $|y_s| \le y^*$ for $s \in [t, T]$ where $y^* = \max\{|\underline{y}|, |\bar{y}|\}$. Next, define $\eta^* = \max\{\eta^b(\underline{y}), \dots, \eta^b(\bar{y}), \eta^a(\underline{y}), \dots, \eta^a(\bar{y})\}$, and $c^* = \max\{c^a, c^b\}$, then, a simple calculation yields the bound

$$\mathbb{E}[|Z_t|] \le |Z_0| + 2\,T\,\eta^*\,c^*\,,$$

for $t \in [0, T]$, which we use to show that

$$\begin{aligned}
&\left|\mathbb{E}_{t,x,y,z}\left[x_T + y_T\,Z_T - \alpha\,(y_T - \hat{y})^2 - \phi\int_t^T (y_s - \hat{y})^2\,\mathrm{d}s\right]\right| \\
&\quad \le |x_0| + 2\,\left(T\,\zeta\,\left(|Z_0| + 2\,T\,\eta^*\,c^*\right)\,c^*\right) + \zeta\,c^*\,\mathbb{E}\left[\int_t^T |\delta^a_s| + |\delta^b_s|\mathrm{d}s\right] \\
&\qquad + y^*\,\left(|Z_0| + 2\,T\,\eta^*\,c^*\right) + \alpha\,\left(y^* + |\hat{y}|\right)^2 + \phi\,T\,\left(y^* + |\hat{y}|\right)^2 < \infty
\end{aligned}$$

because $\delta \in \mathcal{A}$. Next, use the inequality

$$\delta^{a,b}\,e^{-\kappa\,\delta^{a,b}} \le \frac{1}{\kappa}\,e^{-1}\,,$$

to show that $w^\delta(t, x, y, z)$ is bounded from above:

$$\begin{aligned}
&\left|\mathbb{E}_{t,x,y,z}\left[x_T + y_T\,Z_T - \alpha\,(y_T - \hat{y})^2 - \phi\int_t^T (y_s - \hat{y})^2\,\mathrm{d}s\right]\right| \\
&\quad \le |x_0| + 2\,\left(T\,\zeta\,\left(|Z_0| + 2\,T\,\eta^*\,c^*\right)\,c^* + T\,\zeta\,c^*\,\frac{1}{\kappa}e^{-1}\right) \\
&\qquad + y^*\,\left(|Z_0| + 2\,T\,\eta^*\,c^*\right) + \alpha\,\left(y^* + \hat{y}\right)^2 + \phi\,T\,\left(y^* + \hat{y}\right)^2 < \infty.
\end{aligned}$$

Thus, there is $C \in \mathbb{R}$ such that for all $\delta \in \mathcal{A}_t$, $w^\delta(t, x, y, z) \le C < \infty$ which implies that the supremum over $\mathcal{A}_t$ is finite. Thus, the value function $w\colon [0, T] \times \mathbb{R} \times \mathcal{Y} \times \mathbb{R} \to \mathbb{R}$

$$w(t, x, y, z) = \sup_{\delta \in \mathcal{A}_t} w^\delta(t, x, y, z)$$

is well defined. □

Appendix A.3 Proof of Proposition 2

The function $\omega(t,x,y,z) = x + y\,z + \theta(t,y)$ solves the HJB in (15) if $\theta(t,y)$ solves

$$
\begin{aligned}
0 &= \partial_t \theta - \phi (y-\hat{y})^2 \\
&\quad + \sup_{\delta^b} \lambda^b\left(\delta^b\right) \left\{ \theta\left(t, y+\zeta\right) - \theta\left(t,y\right) + \zeta\, \delta^b - y\, \eta^b(y) - \zeta\, \eta^b(y) \right\} \mathbb{1}^b(y) \\
&\quad + \sup_{\delta^a} \lambda^a\left(\delta^a\right) \left\{ \theta\left(t, y-\zeta\right) - \theta\left(t,y\right) + \zeta \delta^a + y\, \eta^a(y) - \zeta\, \eta^a(y) \right\} \mathbb{1}^a(y)
\end{aligned}
\tag{A.1}
$$

with the terminal condition $\theta(T,y) = -\alpha(y-\hat{y})^2$. Let

$$
\begin{aligned}
g^b(t,y) &= \theta\left(t, y+\zeta\right) - \theta\left(t,y\right) - \left(y+\zeta\right) \eta^b(y)\,, \\
g^a(t,y) &= \theta\left(t, y-\zeta\right) - \theta\left(t,y\right) + \left(y-\zeta\right) \eta^a(y)\,,
\end{aligned}
\tag{A.2}
$$

then, the supremum for the buy side in the HJB equation (A.1) satisfies

$$
\begin{aligned}
&\sup_{\delta^b} \lambda^b\left(\delta^b\right) \Big(\theta\left(t, y+\zeta\right) - \theta\left(t,y\right) + \zeta\, \delta^b - y\, \eta^b(y) - \zeta\, \eta^b(y) \Big) \\
&\qquad = \sup_{\delta^b} \lambda^b\left(\delta^b\right) \Big(\zeta\, \delta^b + g^b(t,y) \Big)\,,
\end{aligned}
$$

where g^b is given by (A.2). The expression for maximising δ^a is similar. In both cases we maximise concave functions where the first order conditions are satisfied by

$$
\delta^{b\star} = \frac{1}{\kappa} - \frac{g^b(t,y)}{\zeta}\,, \qquad \delta^{a\star} = \frac{1}{\kappa} - \frac{g^a(t,y)}{\zeta}\,.
$$

Next, use the expressions for the feedback controls to evaluate the supremum in (A.1), and obtain

$$
\sup_{\delta^b} \lambda^b\left(\delta^b\right) \Big(\zeta\, \delta^b + g^b(t,y) \Big) = c^b\, \frac{\zeta}{\kappa}\, \exp\left(-1 + \frac{\kappa\, g^b(t,y)}{\zeta} \right)\,.
$$

A similar calculation for the ask side yields

$$
\sup_{\delta^a} \lambda^a\left(\delta^a\right) \Big(\zeta\, \delta^a + g^a(t,y) \Big) = c^a\, \frac{\zeta}{\kappa}\, \exp\left(-1 + \frac{\kappa\, g^a(t,y)}{\zeta} \right)\,.
$$

The above implies that $\omega(t,x,y,z) = x + y\,z + \theta(t,y)$ solves the HJB in (15) if $\theta(t,y)$ solves the PDE

$$
0 = \partial_t \theta - \phi\left(y-\hat{y}\right)^2 \tag{A.3}
$$

$$+ c^b \frac{\zeta}{\kappa} \exp\left(-1 + \frac{\kappa\, g^b(t,y)}{\zeta}\right) \mathbb{1}^b(y) + c^a \frac{\zeta}{\kappa} \exp\left(-1 + \frac{\kappa\, g^a(t,y)}{\zeta}\right) \mathbb{1}^a(y)\,,$$

with terminal condition $\theta(T,y) = -\alpha(y-\hat{y})^2$. Recall that $\kappa > 0$, and $\theta(t,y)$ is given by (13). Then $g^b(t,y)$ and $g^a(t,y)$ become

$$\begin{aligned} g^b(t,y) &= \frac{\zeta}{\kappa} \log\left(\frac{u(t,y+\zeta)}{u(t,y)}\right) - (y+\zeta)\,\eta^b(y)\,, \\ g^a(t,y) &= \frac{\zeta}{\kappa} \log\left(\frac{u(t,y-\zeta)}{u(t,y)}\right) + (y-\zeta)\,\eta^a(y)\,. \end{aligned}$$

Thus, θ solves (A.3) with the terminal condition $\theta(T,y) = -\alpha(y-\hat{y})^2$ if u satisfies

$$\begin{aligned} 0 = \frac{\zeta}{u(t,y)\,\kappa}\,\partial_t u(t,y) - \phi\,(y-\hat{y})^2 + c^b \frac{\zeta}{\kappa}\, e^{-1}\, \frac{u(t,y+\zeta)}{u(t,y)} \exp\left(-\frac{\kappa}{\zeta}(y+\zeta)\,\eta^b(y)\right) \mathbb{1}^b(y) \\ + c^a \frac{\zeta}{\kappa}\, e^{-1}\, \frac{u(t,y-\zeta)}{u(t,y)} \exp\left(\frac{\kappa}{\zeta}(y-\zeta)\,\eta^a(y)\right) \mathbb{1}^a(y)\,, \end{aligned} \tag{A.4}$$

with the terminal condition $u(T,y) = \exp(-\frac{\kappa\,\alpha}{\zeta}\,(y-\hat{y})^2)$. This ODE system is equivalent to

$$\begin{aligned} 0 = \partial_t u(t,y) - \frac{\kappa\, u(t,y)}{\zeta}\,\phi\,(y-\hat{y})^2 + c^b\, e^{-1}\, u(t,y+\zeta) \exp\left(-\frac{\kappa}{\zeta}\,(y+\zeta)\,\eta^b(y)\right) \mathbb{1}^b(y) \\ + c^a\, e^{-1}\, u(t,y-\zeta) \exp\left(\frac{\kappa}{\zeta}\,(y-\zeta)\,\eta^a(y)\right) \mathbb{1}^a(y)\,. \end{aligned}$$

Recall that $\underline{N} = \underline{y}/\zeta$ and $\overline{N} = \overline{y}/\zeta$. For $i \in (\underline{N}, \overline{N}) \cap \mathbb{Z}$ and $y = i\,\zeta$, we have that

$$\begin{aligned} 0 = \partial_t u(t,y) - \frac{\kappa\, u(t,y)}{\zeta}\,\phi\,(y-\hat{y})^2 + c^b\, e^{-1}\, u(t,y+\zeta)\, e^{-\frac{\kappa}{\zeta}(y+\zeta)\,\eta^b(y)} \\ + c^a\, e^{-1}\, u(t,y-\zeta)\, e^{\frac{\kappa}{\zeta}(y-\zeta)\eta^a(y)}\,, \end{aligned}$$

and when $i = \underline{N}$ or $i = \overline{N}$ we have that

$$\begin{aligned} 0 &= \partial_t u(t,\underline{N}\zeta) - \frac{\kappa\, u(t,\underline{N}\zeta)}{\zeta}\,\phi\,(\underline{N}\zeta - \hat{y})^2 + c^b\, e^{-1}\, u(t,(\underline{N}+1)\zeta)\, e^{-\frac{\kappa}{\zeta}(\underline{N}+1)\zeta\,\eta^b((\underline{N}+1)\zeta)}\,, \\ 0 &= \partial_t u(t,\overline{N}\zeta) - \frac{\kappa\, u(t,\overline{N}\zeta)}{\zeta}\,\phi\,(\overline{N}\,\zeta - \hat{y})^2 + c^a\, e^{-1}\, u\left(t,\left(\overline{N}-1\right)\zeta\right)\, e^{\frac{\kappa}{\zeta}(\overline{N}-1)\zeta\,\eta^a((\overline{N}-1)\zeta)}\,. \end{aligned}$$

Recall that $N = \overline{N} - \underline{N} + 1$, and that the vector-valued function $\mathbf{U} : [0,T] \to \mathbb{R}^N$ in (31) is given by

$$\mathbf{U}(t) = \begin{pmatrix} u(T-t, \underline{N}\,\zeta) \\ u(T-t, (\underline{N}+1)\,\zeta) \\ \dots \\ u(T-t, \overline{N}\,\zeta) \end{pmatrix},$$

and thus, u solves (A.4) with terminal condition $u(T,y) = e^{-\frac{\kappa\,\alpha}{\zeta}\,(y-\hat{y})^2}$ if $\mathbf{U}$ solves

$$0 = \partial_t \mathbf{U} - \mathbf{K}\,\mathbf{U}\,, \quad t \in [0,T]\,, \quad \mathbf{U}(0) \text{ given in (12)}\,,$$

where the matrix $\mathbf{K} \in \mathbb{R}^{N\times N}$ is given by (10). It is well-known that the unique solution to the previous equation is given by

$$\mathbf{U}(t) = \exp\left(\mathbf{K}\,t\right)\,\mathbf{U}(0)\,, \tag{A.5}$$

where $\exp(\,\cdot\,)$ is matrix exponentiation and this concludes the proof. $\square$

Appendix A.4 Proof of Theorem 2

First, we show that the system is well defined and that the controls we propose are admissible. The optimal shifts $\delta_t^{a\star}, \delta_t^{b\star}$ in (17) are measurable functions of time and y_{t^-}. Note that θ is $\mathcal{C}^1$ in time and y_{t^-} can only take finitely many values in $\mathcal{Y}$. Thus, (i) the function ω is also $\mathcal{C}^1$ in time, (ii) the optimal shifts $\delta_t^{a\star}, \delta_t^{b\star}$ are bounded, square integrable, and thus admissible, and (iii) the stochastic intensity functions $\lambda_t^b = c^b\,e^{-\kappa\,\delta_t^{b\star}}$, $\lambda_t^a = c^a\,e^{-\kappa\,\delta_t^{a\star}}$ are $\mathbb{F}$-adapted and bounded. This implies that the jump processes $N^{a,b}$ with stochastic intensities $\lambda^{a,b}$ are well defined.

Let $t \in [0,T)$ and let $\left(\delta_s^b\right)_{s\in[t,T]}$ and $\left(\delta_s^a\right)_{s\in[t,T]}$ be two processes in $\mathcal{A}_t$. Use Itô's lemma to write

$$\begin{aligned}
\omega(T, x_T, y_T, Z_T) &= \omega\left(t,x,y,z\right) + \int_t^T \partial_s \omega\left(s, x_{s^-}, y_{s^-}, Z_{s^-}\right) \mathrm{d}s \\
&+ \int_t^T \left(\omega\left(s, x_{s^-} - \zeta\,\left(Z_{s^-} - \delta_s^b\right), y_{s^-} + \zeta, Z_{s^-} - \eta^b(y_{s^-})\right) - \omega\left(s, x_{s^-}, y_{s^-}, Z_{s^-}\right)\right) \mathrm{d}N_s^b \\
&+ \int_t^T \left(\omega\left(s, x_{s^-} + \zeta\,\left(Z_{s^-} + \delta_s^a\right), y_{s^-} - \zeta, Z_{s^-} + \eta^a(y_{s^-})\right) - \omega\left(s, x_{s^-}, y_{s^-}, Z_{s^-}\right)\right) \mathrm{d}N_s^a\,,
\end{aligned} \tag{A.6}$$

where the processes $(x_s)_{s\in[t,T]}$, $(y_s)_{s\in[t,T]}$, and $(Z_s)_{s\in[t,T]}$ start at time t with initial values

x, y, and z and we employ the controls $\left(\delta^b_s\right)_{s\in[t,T]}$ and $(\delta^a_s)_{s\in[t,T]}$.[15] Given that $\delta^{a\star}$ and $\delta^{b\star}$ attain the equality in (15), then, for any time $s \in [t,T]$ the following holds

$$
\begin{aligned}
&\phi\left(y_{s^-}-\hat{y}\right)^2 \geq\ \partial_s\omega\left(s, x_{s^-}, y_{s^-}, Z_{s^-}\right)\\
&\quad+\lambda^b\left(\delta^b_s\right)\left\{\omega\left(s, x_{s^-}-\zeta\left(Z_{s^-}-\delta^b_s\right), y_{s^-}+\zeta, Z_{s^-}-\eta^b(y_{s^-})\right)-\omega\left(t, x_{s^-}, y_{s^-}, Z_{s^-}\right)\right\}\\
&\quad+\lambda^a\left(\delta^a_s\right)\left\{\omega\left(s, x_{s^-}+\zeta\left(Z_{s^-}+\delta^a_s\right), y_{s^-}-\zeta, Z_{s^-}+\eta^a(y_{s^-})\right)-\omega\left(s, x_{s^-}, y_{s^-}, Z_{s^-}\right)\right\}.
\end{aligned}
$$

Next, take expectations in (A.6) and rearrange terms to obtain

$$
\begin{aligned}
\mathbb{E}\left[\omega(T, x_T, y_T, Z_T)\right] &= \mathbb{E}\left[x_T + y_T\, Z_T - \alpha\left(y_T-\hat{y}\right)^2\right]\\
&\leq \omega\left(t,x,y,z\right)+\mathbb{E}\left[\phi\int_t^T\left(y_{s^-}-\hat{y}\right)^2\mathrm{d}s\right].
\end{aligned}
$$

Thus, we have that $w^\delta(t,x,y,z) \leq \omega(t,x,y,z)$. Given that $\delta^{a,b}$ is arbitrary we have that $w(t,x,y,z)\leq\omega(t,x,y,z)$.

Furthermore, by Proposition 2, equality in (A.6) is obtained with $\delta^\star = \left(\delta^{a\star},\delta^{b\star}\right)$ which is an admissible control. Thus, $\omega(t,x,y,z) = w^{\delta^\star}(t,x,y,z) \leq w(t,x,y,z)$ which completes the proof. $\square$

Appendix A.5 Proof of Proposition 3

Use Theorem 1 to write the performance criterion (A.7) of the LP in the CFM pool as

$$
\begin{aligned}
J^{\text{CFM}} &= \mathbb{E}\left[x^{\text{CFM}}_T + y_T\, Z^{\text{CFM}}_T - \alpha\left(y^{\text{CFM}}_T-\hat{y}\right)^2 - \phi\int_0^T\left(y^{\text{CFM}}_s-\hat{y}\right)^2\mathrm{d}s\right] \qquad \text{(A.7)}\\
&= \mathbb{E}\left[x_T + y_T\, Z_T - \alpha\left(y_T-\hat{y}\right)^2 - \phi\int_0^T\left(y_s-\hat{y}\right)^2\mathrm{d}s\right]\\
&= J\left(\delta^{CFM}\right).
\end{aligned}
$$

Finally, (18) follows from Theorem 2. $\square$

[15] For ease of notation, we drop the superscript in the controlled process notation $\left(x^{t,x,y,z,\delta}_s\right)_{s\in[t,T]}$, $\left(y^{t,y,\delta}_s\right)_{s\in[t,T]}$ and $\left(Z^{t,y,z,\delta}_s\right)_{s\in[t,T]}$.

Appendix A.6 Proof of Theorem 3

Let $y_0 \in \mathcal{Y}$ be the initial inventory of the LP and without loss of generality assume that $\underline{y} < y_0 = \mathfrak{y}_j < \bar{y}$ for $j \in \{2, \ldots, N-1\}$.[16] A roundtrip sequence of trades contains an even number of individual trades, that is, the total number of trades used by the LT is $\mathfrak{m} = 2\,m$ for some $m \in \mathbb{N}$. We proceed by induction over $m \in \mathbb{N}$; our proof is similar to the proof of Lemma 3 in Bouzianis et al. (2023). For $m = 1$ the LT either executes a (i) buy-then-sell or (ii) sell-then-buy strategy. The P&L of the LT is given by (19) for general liquidity provision strategies, which is negative if (22) is satisfied. The P&L of the LT is given by (20) for the strategy (17). Recall that $\eta^a(\,\cdot\,)$ and $\eta^b(\,\cdot\,)$ satisfy (21) so we have that (20) becomes

$$\begin{aligned} &\text{case (i)} \qquad \text{P\&L} = -\,\frac{2\,\zeta}{\kappa} + 2\,\zeta\,\eta^a(y_0)\,, \\ &\text{case (ii)} \qquad \text{P\&L} = -\,\frac{2\,\zeta}{\kappa} + 2\,\zeta\,\eta^b(y_0)\,. \end{aligned} \tag{A.8}$$

It follows that (23) implies that the P&L in both cases of (A.8) is negative.

Our goal is to show that if there is no roundtrip arbitrage in a sequence of $\mathfrak{m} = 2\,(m-1)$ trades, then there is no roundtrip arbitrage in a sequence of $\mathfrak{m} = 2\,m$ trades. Let $\{\epsilon_1, \ldots, \epsilon_{\mathfrak{m}}\}$ be a roundtrip sequence of trades, i.e., $\epsilon_k \in \{1, -1\}$ and $\sum_{k=1}^{\mathfrak{m}} \epsilon_k = 0$, such that $\mathfrak{m} = 2\,m$.

Define the inventory of the LT as $(Q_k)_{k\in\{0,1,\ldots,n\}}$ by setting $Q_0 = 0$ and $Q_k = Q_{k-1} + \epsilon_k$ for $k = 1, \ldots, n$. Consider the case $\epsilon_1 = 1$ (the case $\epsilon_1 = -1$ is analogous) and observe that if an element of the series $(Q_k)_{k\in\{1,\ldots,n\}}$ is negative, then $\exists\, m \in \{2, \ldots, n-1\}$ such that $Q_m = 0$ and $Q_{m+1} = -1$. Define the series $(R_k)_{k\in\{0,\ldots,n-m\}}$ by $R_0 = 0$ and $R_k = R_{k-1} + \epsilon_{k+m}$ for $k \in \{1, \ldots, n-m\}$. Then the inductive hypothesis that both sequences of roundtrip trades are not profitable holds because $m < n$ and $n - m < n$. Thus, it suffices to restrict our attention to the case where all elements of the series $(Q_k)_{k\in\{1,\ldots,n\}}$ are non-negative. It follows that $Q_{\mathfrak{m}-1} = 1$ and the sequence of trades $\{\epsilon_2, \ldots, \epsilon_{\mathfrak{m}-1}\}$ form a roundtrip sequence of $\mathfrak{m} - 2$ trades. By the inductive hypothesis it follows that this sequence of $\mathfrak{m} - 2$ trades is not profitable. Last, observe that the P&L associated with $\epsilon_1 = +1$ and $\epsilon_{\mathfrak{m}} = -1$ is given by

$$\text{P\&L} = \zeta\left(\eta^a\left(\mathfrak{y}_j\right) - \delta^a_{j,-} - \delta^b_{j,+}\right)\,,$$

for general liquidity provision strategies, which is negative as a consequence of (22). For the

[16] If y_0 is at either of the boundaries we just not consider an initial buy (resp. an initial sell) from the LT when y_0 is $\underline{y}$ (resp. $\bar{y}$).

strategy (17), the P&L associated with $\epsilon_1 = +1$ and $\epsilon_{\mathfrak{m}} = -1$ is given by

$$\text{P\&L} = -\frac{2\,\zeta}{\kappa} + 2\,\zeta\,\eta^a(y_0)\,,$$

which is non-positive as a consequence of (23). Thus, the LT is not profitable with a sequence of $\mathfrak{m}$ steps.

By induction, it follows that there is not a roundtrip sequence of trades of any length that would be profitable for the LT. □

Appendix A.7 Proof of Proposition 5

To prove that (27) is finite, observe that for $\delta \in \mathcal{A}_t$, we have $|y_s| \leq y^*$ for $s \in [t, T]$ where $y^* = \underline{y}\,(1+\zeta)^N$. Next, note that y lies within a compact set and define $\eta^* = \max\left\{\sup_{y\in\mathcal{Y}}\{\eta^b(y)\},\,\sup_{y\in\mathcal{Y}}\{\eta^a(y)\}\right\}$. Let $c^* = \max\{c^a, c^b\}$. The solution to the stochastic differential equation in (25) is

$$Z_t = Z_0\, e^{\int_0^t\left(-\eta^b(y_{t-}) - \frac{1}{2}\eta^b(y_{t-})^2\right)\mathrm{d}N_t^b + \int_0^t\left(\eta^a(y_{t-}) - \frac{1}{2}\eta^a(y_{t-})^2\right)\mathrm{d}N_t^a}\,,$$

so

$$\mathbb{E}\left[|Z_t|\right] \leq Z_0\,\mathbb{E}\left[e^{\eta^\star\left(N_t^b + N_t^a\right)}\right]\,,$$

where $\eta^\star = \eta^* + \frac{1}{2}\,\eta^{*2} < \infty$. Next, use the conditioning method in Snyder and Miller (2012) and $\delta \geq \underline{\delta}$ to write

$$\begin{aligned}
\mathbb{E}\left[e^{\eta^\star N_t^b}\right] &= \sum_{k=0}^{\infty} e^{\eta^\star k}\,\mathbb{P}\left[N_t^b = k\right] \\
&= \sum_{k=0}^{\infty} \frac{e^{\eta^\star k}}{k!}\,\mathbb{E}\left[\left(\int_0^t c^b\,e^{-\kappa\,\delta_s}\,\mathrm{d}s\right)^k e^{-\int_0^t c^b\,e^{-\kappa\,\delta_s}\mathrm{d}s}\right] \\
&\leq \sum_{k=0}^{\infty} \frac{e^{\eta^\star k}}{k!}\left(c^b\,e^{-\kappa\,\underline{\delta}}\,t\right)^k \\
&= e^{c^b\,e^{\eta^\star - \kappa\,\underline{\delta}}\,t} \\
&\leq e^{c^*\,e^{\eta^\star - \kappa\,\underline{\delta}}\,t}\,,
\end{aligned}$$

and similarly, $\mathbb{E}\left[e^{\eta^\star N_t^a}\right] \leq e^{c^*\,e^{\eta^\star - \kappa\,\underline{\delta}}\,t}$, so

$$\mathbb{E}\left[|Z_t|\right] \leq Z_0\,e^{2\,c^*\,e^{\eta^\star - \kappa\,\underline{\delta}}\,T} = Z^*\,.$$

Next, recall that

$$\mathrm{d}x_t = -\zeta\, y_{t^-} Z_{t^-} \left(1-\delta_t^b\right) \mathrm{d}N_t^b + \zeta\, y_{t^-} Z_{t^-} \left(1+\delta_t^a\right) \mathrm{d}N_t^a \,,$$

so

$$\begin{aligned}
|\mathbb{E}\,[x_t]| &\leq |x_0| + \zeta \left|\mathbb{E}\left[\int_0^t y_{s^-}\, Z_{s^-} \left(1-\delta_s^b\right)\, c^*\, e^{-\kappa\,\delta_s^b}\, \mathrm{d}s\right]\right| + \zeta \left|\mathbb{E}\left[\int_0^t y_{s^-}\, Z_{s^-}\, (1+\delta_s^a)\, c^*\, e^{-\kappa\,\delta_s^a}\, \mathrm{d}s\right]\right| \\
&\leq |x_0| + \zeta \left|\mathbb{E}\left[\int_0^t y_{s^-}\, Z_{s^-}\, c^*\, e^{-\kappa\,\delta_s^b}\, \mathrm{d}s\right]\right| + \zeta \left|\mathbb{E}\left[\int_0^t c^*\, y_{s^-}\, Z_{s^-} \delta_s^b\, e^{-\kappa\,\delta_s^b}\, \mathrm{d}s\right]\right| \\
&\quad + \zeta \left|\mathbb{E}\left[\int_0^t y_{s^-}\, Z_{s^-}\, c^*\, e^{-\kappa\,\delta_s^a}\, \mathrm{d}s\right]\right| + \zeta \left|\mathbb{E}\left[\int_0^t y_{s^-}\, Z_{s^-} c^*\, \delta_s^a\, e^{-\kappa\,\delta_s^a}\, \mathrm{d}s\right]\right| \\
&\leq |x_0| + 2\,\zeta\, y^*\, c^*\, e^{-\kappa\,\underline{\delta}} \left|\mathbb{E}\left[\int_0^t Z_{s^-}\, \mathrm{d}s\right]\right| + 2\,\zeta\, \frac{c^*\, y^*\, e^{-1}}{\kappa} \left|\mathbb{E}\left[\int_0^t Z_{s^-}\, \mathrm{d}s\right]\right| \\
&\leq |x_0| + 2\,\zeta\, y^*\, c^* \left(e^{-\kappa\,\underline{\delta}} + \frac{e^{-1}}{\kappa}\right) Z^*\, T\,,
\end{aligned}$$

where we use $\delta^{a,b} \geq \underline{\delta}$ and $\delta^{a,b}\, e^{-\kappa\,\delta^{a,b}} \leq \frac{1}{\kappa}\, e^{-1}$.

The previous bounds imply that

$$\begin{aligned}
&\left|\mathbb{E}_{t,x,y,z}\left[x_T + y_T\, Z_T - \alpha\, Z_T\, (y_T - \hat{y})^2 - \phi \int_t^T Z_s\, (y_s - \hat{y})^2\, \mathrm{d}s\right]\right| \\
&\quad \leq |x_0| + 2\,\zeta\, y^*\, c^* \left(e^{-\kappa\,\underline{\delta}} + \frac{e^{-1}}{\kappa}\right) Z^*\, T + y^*\, Z^* \\
&\qquad + \alpha\, \left(y^* + |\hat{y}|\right)^2 Z^* + \phi\, \left(y^* + |\hat{y}|\right)^2 Z^*\, T < \infty\,.
\end{aligned}$$

Thus, there is $C \in \mathbb{R}$ such that for all $\delta \in \mathcal{A}_t$, $w^\delta(t,x,y,z) \leq C < \infty$ which implies that the supremum over $\mathcal{A}_t$ is finite. Thus, the value function $w\colon [0,T] \times \mathbb{R} \times \mathcal{Y} \times \mathbb{R} \to \mathbb{R}$ in (28) is well defined. □

Appendix A.8 Proof of Proposition 6

The function $\omega(t,x,y,z) = x + y\,z + \theta(t,y,z)$ solves the HJB in (35) if $\theta(t,y,z)$ solves

$$
\begin{aligned}
0 = {} & \partial_t \theta - \phi\, z\, (y-\hat{y})^2 \\
& + \sup_{\delta^b} c^b \exp\left(-\kappa\,\delta^b\right) \Bigg(\zeta\, y\, z\, \delta^b - y\, z\, \eta^b(y)\,(1+\Delta) \\
& \qquad\qquad + \theta\left(t, y+\zeta y, z - z\,\eta^b(y)\right) - \theta(t,y,z) \Bigg) \mathbb{1}^b(y) \\
& + \sup_{\delta^a} c^a \exp\left(-\frac{\kappa}{1+\zeta}\,\delta^a\right) \left(\frac{\zeta}{1+\zeta}\, y\, z\, \delta^a + yz\,\eta^a(y) \left(1 - \frac{\zeta}{1+\zeta}\right) \right. \\
& \qquad\qquad \left. + \theta\left(t, y - \frac{\zeta}{1+\zeta}\, y, z + z\,\eta^a(y)\right) - \theta(t,y,z) \right) \mathbb{1}^a(y)
\end{aligned}
\tag{A.9}
$$

with terminal condition $\theta(T,y,z) = -\alpha\, z\,(y-\hat{y})^2$.

The supremum in (A.9) are achieved with feedback controls of the form

$$
\delta^{b,*}(t,y,z) = \frac{1}{\kappa} + \eta^b(y)\,\frac{1+\zeta}{\zeta} - \frac{\theta\left(t, y+\zeta\, y, z - z\,\eta^b(y)\right) - \theta(t,y,z)}{\zeta\, y\, z}\,, \tag{A.10}
$$

$$
\delta^{a,*}(t,y,z) = \frac{1+\zeta}{\kappa} - \frac{1}{\zeta}\,\eta^a(y) - \frac{\theta\left(t, y - \frac{\zeta}{1+\zeta}\, y, z + z\,\eta^a(y)\right) - \theta(t,y,z)}{\frac{\zeta}{1+\zeta}\, y\, z}\,.
$$

Substitute (A.10) in (A.9) to obtain

$$
\begin{aligned}
0 = {} & \partial_t \theta - \phi\, z\,(y-\hat{y})^2 \\
& + \frac{\zeta\, y\, z\, c^b}{\kappa}\, e^{-1 - \frac{1+\zeta}{\zeta}\,\kappa\,\eta^b(y) + \kappa\,\frac{\theta\left(t, y+\zeta\, y, z - z\,\eta^b(y)\right) - \theta(t,y,z)}{\zeta\, y\, z}}\, \mathbb{1}^b(y) \\
& + \frac{\zeta\, y\, z\, c^a}{\kappa}\, e^{-1 + \kappa\,\frac{\eta^a(y)}{\zeta\,(1+\zeta)} + \kappa\,\frac{\theta\left(t, y - \frac{\zeta}{1+\zeta}\, y, z + z\,\eta^a(y)\right) - \theta(t,y,z)}{\zeta\, y\, z}}\, \mathbb{1}^a(y)\,.
\end{aligned}
$$

By the definition of $u(t,y)$ in (33), the function $\theta(t,y,z)$ solves (A.9) with terminal condition $\theta(T,y,z) = -\alpha\, z\,(y-\hat{y})^2$ if $u(t,y)$ solves

$$
\begin{aligned}
0 = {} & \frac{\partial_t u(t,y)}{u(t,y)} - \frac{\kappa\,\phi}{\zeta\, y}\,(y-\hat{y})^2 \\
& + c^b\, e^{-1 - \kappa\,\frac{1+\zeta}{\zeta}\,\eta^b(y)}\, e^{(1+\zeta)\left(1-\eta^b(y)\right)\log u(t, y+\zeta\, y) - \log u(t,y)}\, \mathbb{1}^b(y) \\
& + c^a\, e^{-1 + \kappa\,\frac{1}{\zeta\,(1+\zeta)}\,\eta^a(y)}\, e^{\frac{1+\eta^a(y)}{1+\zeta}\,\log u\left(t, y - \frac{\zeta}{1+\zeta}\, y\right) - \log u(t,y)}\, \mathbb{1}^a(y)\,,
\end{aligned}
\tag{A.11}
$$

with terminal condition $u(T,y)=e^{-\frac{\alpha\,\kappa}{\zeta\,y}(y-\hat{y})^2}$.

Next, use

$$\eta^b(y)=1-\frac{1}{1+\zeta}=\frac{\zeta}{1+\zeta}\,,\quad \eta^a(y)=\zeta$$

to simplify (A.11) into

$$\begin{aligned}0=\partial_t u(t,y)-\frac{\kappa\,\phi}{\zeta\,y}\,(y-\hat{y})^2\,u(t,y)+c^b e^{\,-1-\kappa}u\,(t,y+\zeta\,y)\,\mathbb{1}^b(y)\\ +c^a\,e^{-1+\frac{\kappa}{1+\zeta}}u\left(t,y-\frac{\zeta}{1+\zeta}\,y\right)\mathbb{1}^a(y)\end{aligned}\tag{A.12}$$

on $[0,T)\times\mathcal{Y}$ with the terminal condition $u(T,y)=e^{-\frac{\alpha\,\kappa}{\zeta\,y}\,(y-\hat{y})^2}$. Recall that $\mathbf{U}:[0,T]\to\mathbb{R}^N$ is given by

$$\mathbf{U}(t)_m=u(T-t,(1+\zeta)^m\,\underline{y})\ ,\ \ m\in\{1,\dots,N\}\,.$$

Thus, u solves (A.12) with terminal condition $u(T,y)=e^{-\frac{\alpha\,\kappa}{\zeta\,y}\,(y-\hat{y})^2}$ if $\mathbf{U}$ solves

$$0=\partial_t\mathbf{U}-\mathbf{K}\,\mathbf{U}\,,\ \ t\in[0,T]\,,\ \ \mathbf{U}(0)\ \text{ given in (32)}\,,$$

where the matrix $\mathbf{K}\in\mathbb{R}^{N\times N}$ is given by (30). It is well-known that the unique solution to the previous equation is given by

$$\mathbf{U}(t)=\exp\left(\mathbf{K}\,t\right)\,\mathbf{U}(0)\,,$$

and recall that $\exp(\,\cdot\,)$ is matrix exponentiation and this concludes the proof. □

Appendix A.9 Proof of Theorem 4

First, we show that the system is well defined and that the controls we propose are admissible. The optimal shifts $\delta_t^{a\star},\delta_t^{b\star}$ in (37) are measurable functions of time and y_{t^-}. Note that θ is $\mathcal{C}^1$ in time and y_{t^-} can only take finitely many values in $\mathcal{Y}$. Thus, (i) the function ω is also $\mathcal{C}^1$ in time, (ii) the optimal shifts $\delta_t^{a\star},\delta_t^{b\star}$ are bounded, square integrable, and thus admissible, and (iii) the stochastic intensity functions $\lambda_t^b=c^b\,e^{-\kappa\,\delta_t^{b\star}}$, $\lambda_t^a=c^a\,e^{-\frac{\kappa}{1+\zeta}\,\delta_t^a}$ are $\mathbb{F}$-adapted and bounded. This implies that the jump processes $N^{a,b}$ with stochastic intensities $\lambda^{a,b}$ in (26) are well defined.

Let $t\in[0,T)$ and let $\left(\delta_s^b\right)_{s\in[t,T]}$ and $\left(\delta_s^a\right)_{s\in[t,T]}$ be two processes in $\mathcal{A}_t$. Use Itô's lemma to write

$$
\begin{aligned}
\omega(T, x_T, y_T, Z_T) &= \omega\left(t, x, y, z\right) + \int_t^T \partial_s \omega\left(s, x_{s^-}, y_{s^-}, Z_{s^-}\right) \mathrm{d}s \\
&+ \int_t^T \Big(\omega\left(s, x_{s^-} - \zeta\, y_{s^-}\, Z_{s^-}\, \left(1-\delta^b_s\right), y_{s^-} + \zeta\, y_{s^-}, Z_{s^-} - Z_{s^-}\, \eta^b(y_{s^-})\right) \\
&\qquad\qquad - \omega\left(s, x_{s^-}, y_{s^-}, Z_{s^-}\right) \Big) \mathrm{d}N^b_s \\
&+ \int_t^T \left(\omega\left(s, x_{s^-} + \frac{\zeta}{1+\zeta}\, y_{s^-}\, Z_{s^-}\, \left(1+\delta^a_s\right), y_{s^-} - \frac{\zeta}{1+\zeta}\, y_{s^-}, Z_{s^-} + Z_{s^-}\, \eta^a(y_{s^-})\right) \right. \\
&\qquad\qquad \left. - \omega\left(s, x_{s^-}, y_{s^-}, Z_{s^-}\right) \right) \mathrm{d}N^a_s\,,
\end{aligned} \tag{A.13}
$$

where the processes $(x_s)_{s\in[t,T]}$, $(y_s)_{s\in[t,T]}$, and $(Z_s)_{s\in[t,T]}$ start at time t with initial values x, y, and z and we employ the controls $\left(\delta^b_s\right)_{s\in[t,T]}$ and $\left(\delta^a_s\right)_{s\in[t,T]}$.[17] Given that $\delta^{a\star}$ and $\delta^{b\star}$ attain the equality of (35), then, for any time $s \in [t,T]$ the following holds

$$
\begin{aligned}
\phi\, Z_{s^-}\left(y_{s^-} - \hat{y}\right)^2 \geq\ & \partial_s \omega\left(s, x_{s^-}, y_{s^-}, Z_{s^-}\right) \\
&+ \lambda^b\left(\delta^b_s\right) \Big\{ \omega\left(s, x_{s^-} - \zeta\, y_{s^-}\, Z_{s^-}\, \left(1-\delta^b_s\right), y_{s^-} + \zeta\, y_{s^-}, Z_{s^-} - Z_{s^-}\, \eta^b(y_{s^-})\right) \\
&\qquad - \omega\left(t, x_{s^-}, y_{s^-}, Z_{s^-}\right) \Big\} \\
&+ \lambda^a\left(\delta^a_s\right) \left\{ \omega\left(s, x_{s^-} + \frac{\zeta}{1+\zeta}\, y_{s^-}\, Z_{s^-}\, \left(1+\delta^a_s\right), y_{s^-} - \frac{\zeta}{1+\zeta}\, y_{s^-}, Z_{s^-} + Z_{s^-}\, \eta^a(y_{s^-})\right) \right. \\
&\qquad \left. - \omega\left(s, x_{s^-}, y_{s^-}, Z_{s^-}\right) \right\}.
\end{aligned}
$$

Next, take expectations in (A.13) and rearrange terms to obtain

$$
\begin{aligned}
\mathbb{E}\left[\omega(T, x_T, y_T, Z_T)\right] &= \mathbb{E}\left[x_T + y_T\, Z_T - \alpha\left(y_T - \hat{y}\right)^2\right] \\
&\leq \omega\left(t, x, y, z\right) + \mathbb{E}\left[\phi \int_t^T Z_{s^-}\, \left(y_{s^-} - \hat{y}\right)^2 \mathrm{d}s\right].
\end{aligned}
$$

Thus, we have that $w^\delta(t,x,y,z) \leq \omega(t,x,y,z)$. Given that $\delta^{a,b}$ is arbitrary we have that $w(t,x,y,z) \leq \omega(t,x,y,z)$.

Furthermore, by Proposition 6, equality in (A.13) is obtained with $\delta^\star = \left(\delta^{a\star}, \delta^{b\star}\right)$ which is an admissible control. Thus, $\omega(t,x,y,z) = w^{\delta^\star}(t,x,y,z) \leq w(t,x,y,z)$ which completes

[17] For ease of notation, we drop the superscript in the controlled process notation $\left(x^{t,x,y,z,\delta}_s\right)_{s\in[t,T]}$, $\left(y^{t,y,\delta}_s\right)_{s\in[t,T]}$ and $\left(Z^{t,y,z,\delta}_s\right)_{s\in[t,T]}$.

the proof. ∎

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