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Asymptotic behavior of solutions of second-order difference equations of Volterra type

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Abstract: In this paper we investigate the Volterra difference equation of the form

$$\Delta(r_n \Delta x_n) = b_n + \sum_{k=1}^{n} K(n,k) f(x_k).$$

We establish sufficient conditions for the existence of a solution x of the above equation with the property $x_n = y_n + \mathrm{o}(n^s)$, where y is a given solution of the equation $\Delta(r_n \Delta y_n) = b_n$ and s is nonpositive real number. We also obtain sufficient conditions for the existence of asymptotically periodic solutions.

Key words: Volterra difference equation, quasidifference, asymptotic behavior, asymptotically periodic solution, convergent solution

1. Introduction

We consider the nonlinear Volterra sum-difference equations of nonconvolution type

$$\Delta(r_n \Delta x_n) = b_n + \sum_{k=1}^{n} K(n,k) f(x_k), \tag{E}$$

where $n \in \mathbb{N}, r_n, b_n \in \mathbb{R}, r_n \neq 0, f : \mathbb{R} \to \mathbb{R}, K : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$, $\mathbb{N}$ is a set of all positive integers, and $\mathbb{R}$ is the set of all real numbers. By a solution of (E) we mean a sequence of real numbers (x_n) satisfying (E) eventually.

We will use the following notation

$$\mathbb{R}^{\mathbb{N}} = \{x : \mathbb{N} \to \mathbb{R}\}.$$

If $q \in \mathbb{N}$, $Z \subset \mathbb{R}$, $y \in \mathbb{R}^{\mathbb{N}}$, and $\alpha \in (0, \infty)$, then

$$\mathbb{N}_q = \{q, q+1, q+2, \dots\}, \quad \mathrm{B}(Z, \alpha) = \bigcup_{z \in Z} [z - \alpha, z + \alpha], \quad \mathrm{B}(y, q, \alpha) = \mathrm{B}(y(\mathbb{N}_q), \alpha).$$

If x, y in $\mathbb{R}^{\mathbb{N}}$, then

$$xy : \mathbb{N} \to \mathbb{R}, \quad (xy)_n = x_n y_n, \quad \|x\| = \sup\{|x_n| : n \in \mathbb{N}\}.$$

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We also use the convention $\sum_{i=k}^{j} a_j = 0$ whenever $j < k$. If r is a real sequence such that $r_n \neq 0$ for any n, then we define the sequence $r^* \in \mathbb{R}^{\mathbb{N}}$ by

$$r_n^* = \sum_{i=1}^{n-1} \frac{1}{r_i}. \tag{1}$$

Discrete Volterra equations appeared as a discretization of Volterra integral and integro-differential equations. They also often occur during the mathematical modelling of some real life situations because they describe processes whose current state is determined by the whole previous history. The first order discrete Volterra equation are usually represented in the form

$$x(n+1) - x(n) = \sum_{k=1}^{n} A(n,k) f(k, x(k)) + g(n).$$

Asymptotic behavior of solutions of equations of this type has been studied by many authors, i.e. in [1–13, 15, 22, 23]. To the best of our knowledge, there are a few papers dealing with the asymptotic behavior of solutions of higher order discrete Volterra equations, see [14, 17–19].

Note that the difference equation

$$\Delta(r_n \Delta x_n) = a_n f(x_n) + b_n \tag{2}$$

is a special case of (E). Indeed, if $K(n,k) = 0$ for all $k \neq n$, and if we take $a_n = K(n,n)$, then equation (E) takes the form (2). For the results on asymptotic properties of equations of type (2) we refer, for example, to [20, 25–28].

The purpose of this paper is to establish sufficient conditions for the existence of a solution x of equation (E) with the property $x_n = y_n + \mathrm{o}(n^s)$, where y is a given solution of the equation $\Delta(r_n \Delta y_n) = b_n$ and $s \in (-\infty, 0]$. We also obtain sufficient conditions for the existence of asymptotically periodic solutions of equation (E). The presented results generalized some of the results obtained in [19, 20].

2. Main results

In this section, in Theorems 1 and 2, we derive sufficient conditions for the existence of a solution x of equation (E) having the property

$$x_n = y_n + \mathrm{o}(n^s),$$

where y is a given solution of the equation $\Delta(r_n \Delta y_n) = b_n$ and $s \in (-\infty, 0]$. Moreover, in Theorem 3, we give sufficient conditions for the existence of a solution x of equation (E) with the asymptotic property

$$x_n = c r_n^* + d + \mathrm{o}(n^s),$$

where c, d are real constants. We will use the following, easy to prove, remark.

Remark 1 *A sequence y is a solution of the equation $\Delta(r_n \Delta y_n) = b_n$ if and only if there exist real constants c, d such that*

$$y_n = \sum_{j=1}^{n-1} \frac{1}{r_j} \sum_{i=1}^{j-1} b_i + c \sum_{j=1}^{n-1} \frac{1}{r_j} + d$$

for any n.

Moreover, in the proofs of our results we will use three lemmas, which we present below.

Lemma 1 [19, Lemma 1] *Assume that* $u \in \mathbb{R}^{\mathbb{N}}$, $n \in \mathbb{N}$, *and*

$$\sum_{j=1}^{\infty} \frac{1}{|r_j|} \sum_{i=j}^{\infty} |u_i| < \infty.$$

Then

$$\sum_{j=n}^{\infty} \frac{1}{|r_j|} \sum_{i=j}^{\infty} |u_i| \le \sum_{j=n}^{\infty} \sum_{i=1}^{j} \frac{|u_j|}{|r_i|} < \infty.$$

The proof of the following lemma we leave to the reader.

Lemma 2 *If* β *is a nonnegative sequence, then* $\sum_{n=1}^{\infty} n\beta_n = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \beta_k$.

Lemma 3 *If* $s \in (-\infty, 0]$ *and* $\sum_{j=1}^{\infty} \frac{1}{j^s |r_j|} \sum_{k=j}^{\infty} \sum_{i=1}^{k} |K(k,i)| < \infty$, *then*

$$\sum_{j=n}^{\infty} \frac{1}{|r_j|} \sum_{i=j}^{\infty} \sum_{k=1}^{i} |K(i,k)| = \mathrm{o}(n^s).$$

Proof For $n \in \mathbb{N}$ let

$$\eta_n = \sum_{j=n}^{\infty} \frac{1}{j^s |r_j|} \sum_{k=j}^{\infty} \sum_{i=1}^{k} |K(k,i)|, \quad \lambda_n = \sum_{j=n}^{\infty} \frac{1}{|r_j|} \sum_{i=j}^{\infty} \sum_{k=1}^{i} |K(i,k)|.$$

By assumption, $\eta_n = \mathrm{o}(1)$. Moreover,

$$\frac{\lambda_n}{n^s} = \frac{1}{n^s} \sum_{j=n}^{\infty} \frac{1}{|r_j|} \sum_{i=j}^{\infty} \sum_{k=1}^{i} |K(i,k)| \le \eta_n = \mathrm{o}(1).$$

Hence, $\lambda_n = \mathrm{o}(n^s)$. □

We will also apply a fixed point lemma of the form.

Lemma 4 [16, Lemma 4.7] *Let* y *be a real sequence and let* ρ *be a positive sequence, which converges to zero. Define a metric* d *on the set*

$$X = \{x \in \mathbb{R}^{\mathbb{N}} : \ |x_n - y_n| \le \rho_n, \ \textit{for any} \ \ n \in \mathbb{N}\}$$

by $d(x,z) = \|x - z\|$. *Then any continuous map* $H : X \to X$ *has a fixed point.*

Theorem 1 *Assume that* $s \in (-\infty, 0]$, $t \in [s, \infty)$, $q \in \mathbb{N}$, $\alpha, L, Q \in (0, \infty)$,

$$\sum_{n=1}^{\infty} n^{1+t-s} \sum_{i=1}^{n} |K(n,i)| < \infty, \quad y \in \mathbb{R}^{\mathbb{N}}, \quad \Delta(r_n \Delta y_n) = b_n, \tag{3}$$

$$|f(u)| \leq L \text{ for } u \in \mathrm{B}(y,q,\alpha), \qquad \frac{1}{|r_n|} \leq Qn^t \text{ for any } n, \tag{4}$$

and f is continuous on $\mathrm{B}(y,q,\alpha)$. Then there exists a solution x of (E) possessing the property

$$x_n = y_n + \mathrm{o}(n^s). \tag{5}$$

Proof For a sequence $x : \mathbb{N} \to \mathbb{R}$ we define a sequence $\hat{x}$ by

$$\hat{x}_n = \sum_{k=1}^{n} K(n,k) f(x_k). \tag{6}$$

Let us denote

$$Y = \{x \in \mathbb{R}^{\mathbb{N}} : |x_n - y_n| \leq \alpha \text{ for any } n \in \mathbb{N}\}.$$

We define a sequence θ by

$$\theta_n = \sum_{k=1}^{n} |K(n,k)|.$$

Then, using Lemma 2, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^s|r_n|} \sum_{j=n}^{\infty} \theta_j \leq Q \sum_{n=1}^{\infty} n^{t-s} \sum_{j=n}^{\infty} \theta_j \leq Q \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} j^{t-s} \theta_j$$

$$= Q \sum_{n=1}^{\infty} n n^{t-s} \theta_n = Q \sum_{n=1}^{\infty} n^{1+t-s} \sum_{k=1}^{n} |K(n,k)|.$$

Hence, by our assumptions, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^s|r_n|} \sum_{j=n}^{\infty} \theta_j < \infty, \qquad \sum_{j=1}^{\infty} \frac{1}{|r_j|} \sum_{i=j}^{\infty} \theta_i < \infty. \tag{7}$$

Now, we define a real sequence ρ by

$$\rho_n = L \sum_{j=n}^{\infty} \frac{1}{|r_j|} \sum_{i=j}^{\infty} \theta_i. \tag{8}$$

By Lemma 3, we have $\rho_n = \mathrm{o}(n^s)$. Hence, there exists an index $p \geq q$ such that

$$\rho_n \leq \alpha \tag{9}$$

for $n \geq p$. Next, we introduce a subset X of $\mathbb{R}^{\mathbb{N}}$

$$X = \{x \in \mathbb{R}^{\mathbb{N}} : |x_n - y_n| \leq \rho \text{ for } n \geq p \text{ and } x_n = y_n \text{ for } n < p\}$$

and we define a mapping H as follows

$$H : Y \to \mathbb{R}^{\mathbb{N}}, \qquad H(x)(n) = \begin{cases} y_n & \text{for } n < p \\ y_n + \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} \hat{x}_i & \text{for } n \geq p. \end{cases}$$

Note that $X \subset Y$. Let $x \in X$. By (4), $|f(x_i)| \le L$ for $i \ge p$. Using (6) and (8) we obtain

$$|H(x)(n) - y_n| = \left| \sum_{j=n}^{\infty} \frac{1}{|r_j|} \sum_{i=j}^{\infty} \hat{x}_i \right| \le \sum_{j=n}^{\infty} \frac{1}{|r_j|} \sum_{i=j}^{\infty} |\hat{x}_i| \le \rho_n$$

for $n \ge p$. Therefore, $HX \subset X$.

Now we show that the mapping H is continuous with respect to the metric defined in Lemma 4. By Lemma 1 and (7), we have

$$\sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{1}{|r_j|} \theta_n < \infty.$$

Let $\varepsilon > 0$. There exists an index m such that

$$m \ge p \quad \text{and} \quad L \sum_{n=m}^{\infty} \sum_{j=1}^{n} \frac{1}{|r_j|} \theta_n < \varepsilon. \tag{10}$$

Moreover, there exists a positive constant γ such that

$$\gamma \sum_{n=1}^{m} \sum_{j=1}^{n} \frac{1}{|r_j|} \theta_n < \varepsilon. \tag{11}$$

Let

$$B = \mathrm{B}(\{y_1, y_2, \ldots, y_m\}, \alpha).$$

Since f is uniformly continuous on B, there exists a positive δ such that if $t_1, t_2 \in B$ and $|t_1 - t_2| < \delta$, then

$$|f(t_1) - f(t_2)| < \gamma. \tag{12}$$

Let us take $z \in X$ such that $\|x - z\| < \delta$. Then, we have

$$\begin{aligned} \|Hx - Hz\| \quad &= \sup_{n \ge p} \left| \sum_{j=n}^{\infty} \tfrac{1}{r_j} \sum_{i=j}^{\infty} (\hat{x}_i - \hat{z}_i) \right| \\ &\le \sup_{n \ge p} \sum_{j=n}^{\infty} \tfrac{1}{|r_j|} \sum_{i=j}^{\infty} |\hat{x}_i - \hat{z}_i| \\ &= \textstyle\sum_{j=p}^{\infty} \tfrac{1}{|r_j|} \sum_{i=j}^{\infty} |\hat{x}_i - \hat{z}_i| \\ &\le \sum_{j=p}^{\infty} \tfrac{1}{|r_j|} \sum_{i=j}^{\infty} \sum_{k=1}^{i} |K(i,k)||f(x_k) - f(z_k)|. \end{aligned}$$

Hence, using Lemma 1, we get

$$\|Hx - Hz\| \le \sum_{j=p}^{\infty} \sum_{i=1}^{j} \frac{1}{|r_i|} \sum_{k=1}^{j} |K(j,k)||f(x_k) - f(z_k)|.$$

Remark that by (4) $|f(x_j)-f(z_j)|\le 2L$ for $j\ge p$, and, by (12),

$$|f(x_j)-f(z_j)|\le\gamma \quad \text{for} \quad j\in\{1,2,\dots,m\}.$$

Hence, using (10) and (11), we obtain

$$\|Hx-Hz\|\le\gamma\sum_{n=1}^{m}\sum_{j=1}^{n}\frac{1}{|r_j|}\theta_n+2L\sum_{n=m}^{\infty}\sum_{j=1}^{n}\frac{1}{|r_j|}\theta_n<3\varepsilon.$$

This proves the continuity of H on X. By Lemma 4 there exists a point $x\in X$ such that $x=Hx$. Then, for $n\ge p$, we have

$$x_n=y_n+\sum_{j=n}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}\hat{x}_i.$$

Hence, for $n\ge p$, we get

$$\begin{aligned}\Delta(r_n\Delta x_n) &= \Delta(r_n\Delta y_n)+\Delta\left(r_n\Delta\left(\sum_{j=n}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}\hat{x}_i\right)\right)\\ &= b_n-\Delta\left(r_n\frac{1}{r_n}\sum_{i=n}^{\infty}\hat{x}_i\right)\\ &= b_n+\hat{x}_n=b_n+\sum_{k=1}^{n}K(n,k)f(x_k).\end{aligned}$$

This means that x is a solution of equation (E). Since $x\in X$ and $\rho_n=\mathrm{o}(n^s)$ we have $x_n-y_n=\mathrm{o}(n^s)$. Hence, $x_n=y_n+\mathrm{o}(n^s)$. □

The following corollary shows, that Theorem 1 extends [19, Theorem 3.1].

Corollary 1 *Assume* $s\in(-\infty,0]$, $t\in[s,\infty)$, $c,d\in\mathbb{R}$, $q\in\mathbb{N}$, $\alpha\in(0,\infty)$,

$$r_n^{-1}=\mathrm{O}(n^t),\qquad \sum_{n=1}^{\infty}n^{1+t-s}\sum_{i=1}^{n}|K(n,i)|<\infty,\qquad \sum_{n=1}^{\infty}n^{1+t-s}|b_n|<\infty,$$

and f *is continuous and bounded on* $\mathrm{B}(cr^*+d,q,\alpha)$. *Then there exists a solution* x *of* (E) *possessing the property*

$$x_n=cr_n^*+d+\mathrm{o}(n^s).$$

Proof Choose a number $\alpha'\in(0,\alpha)$. Define sequences v,y' by

$$v_n=\sum_{j=n}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}b_i,\qquad y_n'=cr_n^*+d+v_n.$$

It is easy to see that $v_n=\mathrm{o}(n^s)$. There exists an index $q'\ge q$ such that $|v_n|\le\alpha-\alpha'$ for any $n\ge q'$. Then

$$\mathrm{B}(y',q',\alpha')\subset\mathrm{B}(cr^*+d,q,\alpha).$$

Hence, f is continuous and bounded on $\mathrm{B}(y', q', \alpha')$. Moreover, by Remark 1, we have $\Delta(r_n\Delta(cr_n^* + d)) = 0$ and $\Delta(r_n\Delta v_n) = b_n$. Therefore, $\Delta(r_n\Delta y_n') = b_n$. By Theorem 1, there exists a solution x of (E) such that $x_n = y_n' + \mathrm{o}(n^s)$. Then

$$x_n = cr_n^* + d + v_n + \mathrm{o}(n^s) = cr_n^* + d + \mathrm{o}(n^s).$$

□

Note that if, in Corollary 1, we assume that $r_n > 0$, then we obtain [19, Theorem 3.1].

Corollary 2 *Assume* $s \in (-\infty, 0]$, $t \in [s, \infty)$, $y : \mathbb{N} \to \mathbb{R}$ *is bounded,*

$$r_n^{-1} = \mathrm{O}(n^t), \quad \sum_{n=1}^{\infty} n^{1+t-s} \sum_{i=1}^{n} |K(n,i)| < \infty, \quad \Delta(r_n\Delta y_n) = b_n,$$

and f *is continuous. Then there exists a solution* x *of* (E) *possessing the property*

$$x_n = y_n + \mathrm{o}(n^s).$$

Proof The function f is continuous and bounded on the set $\mathrm{B}(y, 1, 1)$. Hence, the result follows from Theorem 1. □

Corollary 3 *Assume* $s \in (-\infty, 0]$, $t \in [s, \infty)$, $y : \mathbb{N} \to \mathbb{R}$ *is bounded below,*

$$r_n^{-1} = \mathrm{O}(n^t), \quad \sum_{n=1}^{\infty} n^{1+t-s} \sum_{i=1}^{n} |K(n,i)| < \infty, \quad \Delta(r_n\Delta y_n) = b_n,$$

and f *is continuous on* $\mathbb{R}$ *and bounded at infinity. Then there exists a solution* x *of* (E) *possessing the property* $x_n = y_n + \mathrm{o}(n^s)$.

Proof There exists a real constant λ such that

$$\mathrm{B}(y, 1, 1) \subset [\lambda, \infty).$$

The function f is continuous and bounded on $[\lambda, \infty)$. Hence, the result follows from Theorem 1. □

Corollary 4 *Assume* $s \in (-\infty, 0]$, $t \in [s, \infty)$,

$$r_n^{-1} = \mathrm{O}(n^t), \quad \sum_{n=1}^{\infty} n^{1+t-s} \sum_{i=1}^{n} |K(n,i)| < \infty,$$

and f *is continuous and bounded. Then for any solution* y *of the equation* $\Delta(r_n\Delta y_n) = b_n$ *there exists a solution* x *of* (E) *possessing the property*

$$x_n = y_n + \mathrm{o}(n^s).$$

Proof The assertion is an immediate consequence of Theorem 1. □

In the next theorem, we also give sufficient conditions for the existence of a solution x of equation (E) with the property (5), but the assumption for the kernel $K(i, n)$ is weaker than in Theorem 1. However, we must assume the boundedness of the sequence r^*. The method of proof is analogous, but we use another operator H. This theorem generalized Theorem 2.2 from paper [20].

Theorem 2 *Assume* $s \in (-\infty, 0]$, f *is continuous,* r^* *is bounded, and*

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{i=1}^{n} |K(n,i)| < \infty. \tag{13}$$

Then for any bounded solution y *of the equation* $\Delta(r_n \Delta y_n) = b_n$ *there exists a solution* x *of* (E) *possessing the property* $x_n = y_n + \mathrm{o}(n^s)$.

Proof Assume y is a bounded solution of the equation $\Delta(r_n \Delta y_n) = b_n$. For $n \in \mathbb{N}$ and $x : \mathbb{N} \to \mathbb{R}$ let

$$\hat{x}_n = \sum_{k=1}^{n} K(n,k) f(x_k). \tag{14}$$

Let C be a compact subset of $\mathbb{R}$ such that

$$\mathrm{B}(y, 1, 1) \subset C.$$

There exists a constant $L > 0$, such that $|f(t)| \le L$ for any $t \in C$. Since $s \le 0$ we have

$$\sum_{k=1}^{\infty} \sum_{i=1}^{k} |K(k,i)| < \infty. \tag{15}$$

Now, we define a number R and a subset Y of $\mathbb{R}^{\mathbb{N}}$ by $R = \|r^*\|$, $Y = \{x \in \mathbb{R}^{\mathbb{N}} : |x - y| \le 1\}$. Moreover, we define a sequence ρ by

$$\rho_n = 2RL \sum_{k=n}^{\infty} \sum_{i=1}^{k} |K(k,i)|.$$

Since $s \le 0$, we get

$$\frac{\rho_n}{n^s} = 2RL \sum_{k=n}^{\infty} \frac{1}{n^s} \sum_{i=1}^{k} |K(k,i)| \le 2RL \sum_{k=n}^{\infty} \frac{1}{k^s} \sum_{i=1}^{k} |K(k,i)|.$$

By (13) we obtain

$$\sum_{k=n}^{\infty} \frac{1}{k^s} \sum_{i=1}^{k} |K(k,i)| = \mathrm{o}(1),$$

which means that $\rho_n = \mathrm{o}(n^s)$. Therefore, there exists an index $p \in \mathbb{N}$ for which $\rho_n \le 1$ for $n \ge p$. Let

$$X = \{x \in \mathbb{R}^{\mathbb{N}} : |x_n - y_n| \le \rho \text{ for } n \ge p \text{ and } x_n = y_n \text{ for } n < p\}. \tag{16}$$

Then $X \subset Y$. For any $x \in Y$ and any $n \in \mathbb{N}$ we have $x_n \in C$. Therefore,

$$|f(x_n)| \le L \tag{17}$$

for any $n \in \mathbb{N}$. Now, we define an operator $H : Y \to \mathbb{R}^{\mathbb{N}}$ by

$$H(x)(n) = \begin{cases} y_n & \text{for } n < p \\ y_n + \sum_{k=n}^{\infty} \hat{x}_k \sum_{i=n}^{k} \frac{1}{r_i} & \text{for } n \ge p. \end{cases} \tag{18}$$

Remark that

$$\left|\sum_{i=n}^{k} \frac{1}{r_i}\right| = |r^*_{k+1} - r^*_n| \le |r^*_{k+1}| + |r^*_n| \le 2R. \tag{19}$$

Let $x \in X$ and $n \ge p$. Then, using (19), we get

$$|H(x)(n) - y_n| \le \sum_{k=n}^{\infty} |\hat{x}_k| \left|\sum_{i=n}^{k} \frac{1}{r_i}\right| \le 2R \sum_{k=n}^{\infty} \left|\sum_{i=1}^{k} K(k,i) f(x_i)\right|.$$

Hence,

$$|H(x)(n) - y_n| \le 2RL \sum_{k=n}^{\infty} \sum_{i=1}^{k} |K(k,i)| = \rho_n$$

and we obtain that $HX \subset X$. Next, we show the continuity of H. Let $x \in X$ and $\varepsilon > 0$. By (15) there is an index $m \ge p$ such that

$$L \sum_{k=m}^{\infty} \sum_{i=1}^{k} |K(k,i)| < \varepsilon. \tag{20}$$

Moreover, there exists a positive constant γ such that

$$\gamma \sum_{k=1}^{m} \sum_{i=1}^{k} |K(k,i)| < \varepsilon. \tag{21}$$

Since f is uniformly continuous on C there exists a positive number δ such that if $t_1, t_2 \in C$ and $|t_1 - t_2| < \delta$, then

$$|f(t_1) - f(t_2)| < \gamma. \tag{22}$$

Let us take $z \in X$ such that $\|x - z\| < \delta$. By (19), we have

$$\|Hx - Hz\| = \sup_{n \ge p} \sum_{k=n}^{\infty} |\hat{x}_k - \hat{z}_k| \left|\sum_{i=n}^{k} \frac{1}{r_i}\right| \le 2R \sum_{k=p}^{\infty} |\hat{x}_k - \hat{z}_k|.$$

Hence, by (14), we get

$$\|Hx - Hz\| \le 2R \sum_{k=p}^{\infty} \sum_{i=1}^{k} |K(k,i)||f(x_i) - f(z_i)|.$$

Using (22) and (17) we get

$$\|Hx - Hz\| \le 2R\gamma \sum_{k=1}^{m} \sum_{i=1}^{k} |K(k,i)| + 4RL \sum_{k=m}^{\infty} \sum_{i=1}^{k} |K(k,i)|.$$

Therefore, by (21) and (20), $\|Hx - Hz\| < 6R\varepsilon$. This means that the map $H : X \to X$ is continuous with respect to the metric defined in Lemma 4. By Lemma 4 there exists a fixed point of H. Denote this fixed point by x. From the definition of H we have

$$x_n = y_n + \sum_{k=n}^{\infty} \hat{x}_k \sum_{i=n}^{k} \frac{1}{r_i} \tag{23}$$

for $n \geq p$. By (23), for $n \geq p$ we get

$$\Delta(r_n \Delta x_n) = b_n + \Delta\left(r_n \Delta\left(\sum_{k=n}^{\infty} \hat{x}_k \sum_{i=n}^{k} \frac{1}{r_i}\right)\right). \tag{24}$$

Moreover,

$$\begin{aligned}
\Delta\left(\sum_{k=n}^{\infty} \hat{x}_k \sum_{i=n}^{k} \tfrac{1}{r_i}\right) &= \sum_{k=n+1}^{\infty} \hat{x}_k \sum_{i=n+1}^{k} \tfrac{1}{r_i} - \sum_{k=n}^{\infty} \hat{x}_k \sum_{i=n}^{k} \tfrac{1}{r_i} \\
&= \sum_{k=n+1}^{\infty} \hat{x}_k \sum_{i=n+1}^{k} \tfrac{1}{r_i} - \sum_{k=n}^{\infty} \hat{x}_k \left(\tfrac{1}{r_n} + \sum_{i=n+1}^{k} \tfrac{1}{r_i}\right) \\
&= \sum_{k=n+1}^{\infty} \hat{x}_k \sum_{i=n+1}^{k} \tfrac{1}{r_i} - \tfrac{1}{r_n} \sum_{k=n}^{\infty} \hat{x}_k - \sum_{k=n}^{\infty} \hat{x}_k \sum_{i=n+1}^{k} \tfrac{1}{r_i}.
\end{aligned}$$

Hence,

$$\Delta\left(\sum_{k=n}^{\infty} \hat{x}_k \sum_{i=n}^{k} \frac{1}{r_i}\right) = -\frac{1}{r_n} \sum_{k=n}^{\infty} \hat{x}_k - \hat{x}_n \sum_{i=n+1}^{n} \frac{1}{r_i} = -\frac{1}{r_n} \sum_{k=n}^{\infty} \hat{x}_k.$$

for $n \geq p$. Therefore,

$$\Delta\left(r_n \Delta\left(\sum_{k=n}^{\infty} \hat{x}_k \sum_{i=n}^{k} \frac{1}{r_i}\right)\right) = -\Delta\left(\sum_{k=n}^{\infty} \hat{x}_k\right) = \hat{x}_n \tag{25}$$

for $n \geq p$. Thus, by (24), (25) and (14), x is a solution of (E). Since $x \in X$ and $\rho_n = \mathrm{o}(n^s)$, we have

$$x_n = y_n + \mathrm{o}(n^s).$$

□

Remark 2 *If the sequence r^* is bounded, then by Remark 1, the existence of a bounded solution of the equation $\Delta(r_n \Delta y_n) = b_n$ is equivalent to the boundedness of the sequence*

$$u_n = \sum_{j=1}^{n-1} \frac{1}{r_j} \sum_{i=1}^{j-1} b_i. \tag{26}$$

Moreover, if the sequence (26) is bounded, then any solution y of the equation $\Delta(r_n \Delta y_n) = b_n$ is bounded.

Below, in our last theorem, we present sufficient conditions under which for any real constants c, d equation (E) possesses solution x with the property $x_n = cr_n^* + d + \mathrm{o}(n^s)$.

Theorem 3 *Assume $s \in (-\infty, 0]$, the function f is continuous, the sequence r^* is bounded,*

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{i=1}^{n} |K(n,i)| < \infty, \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{|b_n|}{n^s} < \infty. \tag{27}$$

Then for any real constants c, d there exists a solution x of equation (E) possessing the property

$$x_n = cr_n^* + d + \mathrm{o}(n^s).$$

Proof Assume $c, d \in \mathbb{R}$. Define a sequence y by $y_n = cr^*_n + d$. It is easy to see that $\Delta(r_n \Delta y_n) = 0$ for any n. For $n \in \mathbb{N}$ and $x : \mathbb{N} \to \mathbb{R}$ let

$$\hat{x}_n = b_n + \sum_{k=1}^{n} K(n,k) f(x_k).$$

Since y is bounded, there is a compact subset C of $\mathbb{R}$ such that

$$\mathrm{B}(y, 1, 1) \subset C.$$

Since f is continuous, there exists a positive constant L, such that $|f(t)| \le L$ for any $t \in C$. We denote $R = \|r^*\|$, $Y = \{x \in \mathbb{R}^{\mathbb{N}} : |x - y| \le 1\}$, and define a sequence ρ by the formula

$$\rho_n = 2R \sum_{k=n}^{\infty} \left(|b_k| + L \sum_{i=1}^{k} |K(k,i)| \right).$$

Similarly as in the proof of Theorem 2, we obtain

$$\sum_{k=n}^{\infty} \sum_{i=1}^{k} |K(k,i)| = \mathrm{o}(n^s).$$

It is easy to see that

$$\sum_{k=n}^{\infty} |b_k| = \mathrm{o}(n^s).$$

Hence, $\rho_n = \mathrm{o}(n^s)$. Therefore, there exists an index $p \in \mathbb{N}$ such that $\rho_n \le 1$ for $n \ge p$. Define a subset X of $\mathbb{R}^{\mathbb{N}}$ by (16), and an operator $H : Y \to \mathbb{R}^{\mathbb{N}}$ by (18). Just like in the proof of Theorem 2, we can show that there exists a point $x \in X$ such that $Hx = x$. Then for $n \ge p$ we have

$$x_n = y_n + \sum_{k=n}^{\infty} \hat{x}_k \sum_{i=n}^{k} \frac{1}{r_i}.$$

Hence,

$$\Delta(r_n \Delta x_n) = \Delta(r_n \Delta y_n) + \Delta \left(\hat{x}_k \sum_{i=n}^{k} \frac{1}{r_i} \right) = 0 + \hat{x}_n$$

$$= b_n + \sum_{k=1}^{n} K(n,k) f(x_k)$$

for $n \ge p$. Therefore, x is a solution of (E). Moreover, since $x \in X$ we get

$$x_n = y_n + \mathrm{o}(n^s) = cr^*_n + d + \mathrm{o}(n^s).$$

□

If, in Theorem 3, we assume that the sequence r^* is ω-periodic, then we get the following criterion for the existence of asymptotically periodic solutions of equation (E).

Corollary 5 *Assume* $s \in (-\infty, 0]$, $\omega \in \mathbb{N}$, f *is continuous,* r^* *is* ω*-periodic,*

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{i=1}^{n} |K(n,i)| < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{|b_n|}{n^s} < \infty.$$

Then equation (E) *has asymptotically* ω*-periodic solution of the form* $x_n = cr_n^* + d + \mathrm{o}(n^s)$ *where* $c, d \in \mathbb{R}$.

Proof This corollary follows directly from Theorem 3. □

As we noted in the Introduction, if $K(n,k) = 0$ for $k \neq n$, denoting $a_n = K(n,n)$ equation (E) takes the form (2). Therefore, Theorem 3 generalizes [20, Theorem 2.1].

3. Examples

In this section we present some examples to illustrate the obtained results.

Example 1 Let $s = 0$, $t = 1$, $r_n = \frac{1}{n+2}$, $K(n,k) = \frac{1}{n^4 k^2}$, $f(x) = \frac{1}{x}$ for $x \neq 0$ and $b_n = \frac{3}{(n+3)^{\underline{4}}} - \frac{1}{(n+1)n^3}$. Then equation (E) takes the form

$$\Delta\left(\frac{1}{n+2}\Delta x_n\right) = \frac{3}{(n+3)^{\underline{4}}} - \frac{1}{(n+1)n^3} + \sum_{k=1}^{n} \frac{1}{n^4 k^2} \frac{1}{x_k} \tag{28}$$

for $x \neq 0$ and we have $r_n^{-1} = \mathrm{O}(n^t)$. Let $c = 0$, $d = 1$. Then $cr^* + d = 1$. Let $q = 1$, $\alpha = \frac{1}{2}$, then

$$B(cr^* + d, q, \alpha) = \bigcup_{n=1}^{\infty} \left[1 - \frac{1}{2}, 1 + \frac{1}{2}\right] = \left[\frac{1}{2}, \frac{3}{2}\right].$$

Hence, f is bounded and continuous on $B(cr^* + d, q, \alpha)$. It is easy to check that

$$\sum_{n=1}^{\infty} n^{1+t-s}|b(n)| = \sum_{n=1}^{\infty} n^2 \left|\frac{3}{(n+3)^{\underline{4}}} - \frac{1}{(n+1)n^4}\right| < \infty$$

and

$$\sum_{n=1}^{\infty} n^{1+t-s} \sum_{i=1}^{n} |K(n,i)| = \sum_{n=1}^{\infty} n^2 \sum_{k=1}^{n} \frac{1}{n^4 k^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{k^2} < \infty.$$

Hence, by Corollary 1, there exists a solution x of (28) such that $x_n = cr^* + d + \mathrm{o}(1)$. Indeed the sequence $x_n = 1 + \frac{1}{n}$ is a solutions of (28) with such property.

Example 2 Let f be a continuous function, $s \in (-1, 0]$. For $n \in \mathbb{N}$ let

$$r_n = n^2, \quad b_n = -2n^2 \cos\frac{n\pi}{2} - (2n+1)\left(\sin\frac{n\pi}{2} - \cos\frac{n\pi}{2}\right), \quad K(n,k) = \frac{k}{n^4}.$$

Then equation (E) takes the form

$$\Delta\left(n^2 \Delta x_n\right) = -2n^2 \cos\frac{n\pi}{2} - (2n+1)\left(\sin\frac{n\pi}{2} - \cos\frac{n\pi}{2}\right) + \sum_{i=1}^{n} \frac{k}{n^5} f(x_k). \tag{29}$$

The sequence $r^*_n = \sum_{k=1}^{n-1} \frac{1}{k^2}$ is bounded, the sequence $y_n = \sin \frac{n\pi}{2}$ is a 4-periodic solution of the equation

$$\Delta(n^2 \Delta y_n) = -2n^2 \cos \frac{n\pi}{2} - (2n+1)\left(\sin \frac{n\pi}{2} - \cos \frac{n\pi}{2}\right).$$

Moreover, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{i=1}^{n} |K(n,i)| = \sum_{n=1}^{\infty} \frac{1}{n^{s+4}} \sum_{i=1}^{n} k = \sum_{n=1}^{\infty} \frac{n+1}{2n^{s+3}} < \infty.$$

Thus, all assumptions of Theorem 2 are satisfied. Hence, equation (29) has asymptotically 4-periodic solution of the form

$$x_n = \sin \frac{n\pi}{2} + \mathrm{o}(n^s).$$

In the next two examples we show that Theorem 1 is independent of Theorem 2.

Example 3 Let $r_n = \frac{1}{n}$, $K(n,k) = \frac{2k}{n^6}$, f is a continuous function. Then equation (E) takes the form

$$\Delta\left(\frac{1}{n}\Delta x_n\right) = b_n + \sum_{k=1}^{n} \frac{2k}{n^6} f(x_k). \tag{30}$$

Let b be a sequence of real numbers such that $\sum_{n=1}^{\infty} n \sum_{i=1}^{n-1} b_i < \infty$. Then, by Remark 1, there exists a bounded solution of equation

$$\Delta\left(\frac{1}{n}\Delta y_n\right) = b_n.$$

Let us take $s \in (-1, 0]$, $t = 1$. Then we have $r_n^{-1} = \mathrm{O}(n^t)$ and

$$\sum_{n=1}^{\infty} n^{1+t-s} \sum_{i=1}^{n} |K(n,i)| = \sum_{n=1}^{\infty} n^{2-s} \sum_{k=1}^{n} \frac{2k}{n^6} = \sum_{n=1}^{\infty} \frac{n+1}{n^{3+s}} < \infty.$$

By Theorem 1, there exists a solution x of (30) such that $x_n = y_n + \mathrm{o}(n^s)$.
Let us remark, that the sequence $r^*_n = \sum_{k=1}^{n-1} k$ is unbounded. Hence, Theorem 2 can not be applied to (30).

Example 4 Let $r_n = n^2$, $K(n,k) = \frac{2k}{n^4}$, f is a continuous function, b is a sequence of real numbers such that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{i=1}^{n-1} b_i < \infty.$$

Then equation (E) takes the form

$$\Delta\left(n^2 \Delta x_n\right) = b_n + \sum_{k=1}^{n} \frac{2k}{n^4} f(x_k). \tag{31}$$

Let $s \in (-1, 0]$. Then we have

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{i=1}^{n} |K(n,i)| = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{k=1}^{n} \frac{2k}{n^4} = \sum_{n=1}^{\infty} \frac{n+1}{n^{3+s}} < \infty.$$

Moreover, by Remark 1, all solutions of the equation $\Delta(n^2\Delta y_n) = b_n$ are bounded. Hence, by Theorem 2, for any solution y of the equation $\Delta(n^2\Delta y_n) = b_n$ there exists a solution x of (31) such that $x_n = y_n + \mathrm{o}(n^s)$. Note, that for $t \geq s$ we have

$$\sum_{n=1}^{\infty} n^{1+t-s} \sum_{i=1}^{n} |K(n,i)| \geq \sum_{n=1}^{\infty} n \sum_{k=1}^{n} \frac{2k}{n^4} = \sum_{n=1}^{\infty} \frac{n+1}{n^2} = \infty.$$

Hence, Theorem 1 cannot be applied to (31).

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References

[1] Baker CTH, Song Y. Periodic solutions of non-linear discrete Volterra equations with finite memory. Journal of Computational and Applied Mathematics 2010; 234: 2683-2698.

[2] Berezansky L, Migda M, Schmeidel E. Some stability conditions for scalar Volterra difference equations. Opuscula Mathematica 2016; 36 (4): 459-470.

[3] Bohner M, Sultana N. Subexponential solutions of linear Volterra difference equations. Nonautonomous Dynamical Systems 2015; 2: 63-76.

[4] Crisci MR, Kolmanovskii VB, Russo E, Vecchio A. Boundedness of discrete Volterra equations. Journal of Mathematical Analysis and Applications 1997; 211: 106-130.

[5] Diblík J, Růžičková M, Schmeidel E. Asymptotically periodic solutions of Volterra difference equations. Tatra Mountains Mathematical Publications 2009; 43: 43-61.

[6] Diblík J, Růžičková M, Schmeidel E, Zbaszyniak M. Weighted asymptotically periodic solutions of linear Volterra difference equations. Abstract and Applied Analysis 2011; Art. ID 370982, 14 pp.

[7] Diblík J, Schmeidel E. On the existence of solutions of linear Volterra difference equations asymptotically equivalent to a given sequence. Applied Mathematics and Computation 2012; 218 (18): 9310-9320.

[8] Elaydi S, Messina E, Vecchio A. On the asymptotic stability of linear Volterra difference equations of convolution type. Journal of Difference Equations and Applications 2007; 13 (12): 1079-1084.

[9] Elaydi S. Stability and asymptoticity of Volterra difference equations. A progress report. Journal of Computational and Applied Mathematics 2009; 228 (2): 504-513.

[10] Győri I, Horvath L. Asymptotic representation of the solutions of linear Volterra difference equations. Advances in Difference Equations 2008; ID 932831: 22 pp.

[11] Győri I, Reynolds DW. On asymptotically periodic solutions of linear discrete Volterra equations. Fasciculi Mathematici 2010; 44: 53-67.

[12] I. Győri, Awwad E. On the boundedness of the solutions in nonlinear discrete Volterra difference equations. Advances in Difference Equations 2012; 2: 1-20.

[13] Kolmanovskii V, Shaikhet L. Some conditions for boundedness of solutions of difference Volterra equations. Applied Mathematics Letters 2003; 16: 857-862.

[14] Medina R. Asymptotic behavior of Volterra difference equations. Computers & Mathematics with Applications 2001; 41 (5-6): 679-687.

[15] Messina E, Vecchio A. Boundedness and asymptotic stability for the solution of homogeneous Volterra discrete equations. Discrete Dynamics in Nature and Society. 2018; Article ID 6935069, 8 pages.

[16] Migda J. Approximative solutions of difference equations. Electronic Journal of Qualitative Theory of Differential Equations 2014; 13: 1-26.

[17] Migda J, Migda M. Asymptotic behavior of discrete Volterra equations. Opuscula Mathematica 2016; 36 (2): 265-278.

[18] Migda J, Migda M. Qualitative approximation of solutions to discrete Volterra equations. Electronic Journal of Qualitative Theory of Differential Equations 2018; 3: 1-27.

[19] Migda J, Migda M, Nockowska-Rosiak M. Asymptotic properties of solutions to second-order difference equations of Volterra type. Discrete Dynamics in Nature and Society. 2018; Article ID 2368694, 10 pages.

[20] Migda J, Migda M, Zbaszyniak Z. Asymptotically periodic solutions of second order difference equations. Applied Mathematics and Computation 2019: 350: 181-189.

[21] Migda M. Asymptotic behaviour of solutions of nonlinear difference equations. Fasciculi Mathematici 2001; 31: 57-63.

[22] Migda M, Migda J. Bounded solutions of nonlinear discrete Volterra equations. Mathematica Slovaca 2016; 66 (5): 1169-1178.

[23] Migda M, Morchało J. Asymptotic properties of solutions of difference equations with several delays and Volterra summation equations. Applied Mathematics and Computation 2013; 220: 365-373.

[24] Migda M, Ružičková M, Schmeidel E. Boundedness and stability of discrete Volterra equations. Advances in Difference Equations 2015; 2015: 47.

[25] Řehák P. Asymptotic formulae for solutions of linear second order difference equations. Journal of Difference Equations and Applications 2016; 22 (1): 107-139.

[26] Schmeidel E, Zbaszyniak Z. An application of Darbo's fixed point theorem in the investigation of periodicity of solutions of difference equations. Applied Mathematics and Computation 2012; 64: 2185-2191.

[27] Szafrański Z, Szmanda B. Oscillation of solutions of some nonlinear difference equations. Publicacions Matemàtiques 1996; 40 (1): 127-133.

[28] Thandapani E, Manuel MMS, Graef JR, Spikes PW. Monotone properties of certain classes of solutions of second-order difference equations. Computers & Mathematics with Applications 1998; 36: 291-297.