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Properties of Solutions of Generalized Sturm–Liouville Discrete Equations

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Abstract

We consider discrete Sturm–Liouville-type equations of the form

$$\Delta(r_n \Delta x_n) = a_n f(x_{\sigma(n)}) + b_n.$$

We present a theory of asymptotic properties of solutions which allows us to control the degree of approximation. Namely, we establish conditions under which for a given sequence y which solves the equation $\Delta(r_n \Delta y_n) = b_n$, the above equation possesses a solution x with the property $x_n = y_n + \mathrm{o}(u_n)$, where u is a given positive, nonincreasing sequence. The obtained results are applied to the study of asymptotically periodic solutions. Moreover, these results also allow us to obtain some nonoscillation criteria for the classical Sturm–Liouville equation.

Keywords Sturm–Liouville difference equation · Asymptotic behavior · Degree of approximation · Harmonic approximation · Geometric approximation · Asymptotically periodic solution

Mathematics Subject Classification 39A22 · 39A10

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1 Introduction

Let $\mathbb{N}$, $\mathbb{R}$ denote the set of positive integers and all real numbers, respectively. In this paper, we assume $a, b : \mathbb{N} \to \mathbb{R}$, $r : \mathbb{N} \to \mathbb{R} \setminus \{0\}$, $f : \mathbb{R} \to \mathbb{R}$, $\sigma : \mathbb{N} \to \mathbb{N}$, $\lim_{n\to\infty} \sigma(n) = \infty$, and consider the second-order discrete equation with quasidifferences of the form

$$\Delta(r_n \Delta x_n) = a_n f(x_{\sigma(n)}) + b_n. \tag{E}$$

By a solution of Eq. (E), we mean a sequence x which satisfies the equality (E) for all large n.

Let us note that a particular case of Eq. (E) is the well-known Sturm–Liouville difference equation

$$\Delta(r_n \Delta x_n) = a_n x_{n+1}, \tag{1}$$

which has many applications in mathematical physics, matrix theory, control theory or discrete variational theory. Equation (1) has been extensively studied by many authors, especially with regard to the oscillation, disconjugacy and boundary value problems, see, for example, [1–3,5,6,8–11,20]. Several results devoted to asymptotic properties of (1) and for slightly more general equations can be found in [2,7,12,17–19,23–26]. In the presented paper, using the Schauder's-type fixed point theorem, we establish conditions under which for a given sequence y, which solves $\Delta(r_n \Delta y_n) = b_n$, Eq. (E) possesses a solution x such that

$$x_n = y_n + \mathrm{o}(u_n), \tag{2}$$

where u is a positive, nonincreasing sequence. If Eq. (2) is satisfied, then x is called a solution with prescribed asymptotic behavior, and y is called an approximative solution of (E). Taking different sequences u, we may control the degree of approximation. In particular, if $u_n = n^s$ for some fixed $s \in (-\infty, 0]$, then we have harmonic approximation. If $u_n = \alpha^n$ for some fixed $\alpha \in (0, 1)$, then we get geometric approximation. We believe that very strong geometric approximation is important from a numerical point of view. It is worth noticing that our results are new even in the case when $u_n = 1$.

We present also an application of the obtained results to the study of asymptotic periodicity of solutions to Eq. (E). The study of the existence of periodic or asymptotically periodic solutions is a very important topic in the qualitative theory of difference equations because of its many applications in mathematical biology, chemistry, physics, economics and other fields, see, e.g., [1,13]. The existence of asymptotically periodic solutions of Eq. (E) was considered, for example, in [4,16,21,22] or [25].

We also apply our results to the study of asymptotic properties of bounded solutions to discrete Sturm–Liouville Eq. (1). Moreover, using the Sturm separation theorem and our results we obtain some nonoscillation criteria for Eq. (1).

The paper is organized as follows. In Sect. 2, we introduce some preliminary lemmas. In Sect. 3, we study the problem of the existence of solutions with prescribed asymptotic behavior. In Sect. 4, we apply our results to the study of asymptotically periodic solutions. Section 5 is devoted to the study of solutions to discrete Sturm–

Liouville Eq. (1). In Sect. 6, the proof of Theorem 1 and some additional remarks are presented.

2 Preliminaries

The space of all sequences $x : \mathbb{N} \to \mathbb{R}$ we denote by $\mathbb{R}^{\mathbb{N}}$. If $x \in \mathbb{R}^{\mathbb{N}}$, then $|x|$ denotes the sequence defined by $|x|(n) = |x_n|$ and

$$\|x\| = \sup\{|x_n| : n \in \mathbb{N}\}.$$

We will use the convention $\sum_{i=k}^{n} d_i = 0$ whenever $n < k$; moreover, we will use the following notation:

$$r^*, \widehat{r} : \mathbb{N} \to \mathbb{R}, \quad r_n^* = \sum_{i=1}^{n-1} \frac{1}{r_i}, \quad \widehat{r}_n = \max\{|r_1^*|, |r_2^*|, \ldots, |r_{n+1}^*|\}.$$

The sequence $\widehat{r}$ we introduce for technical reasons. One of them is the following important property of $\widehat{r}$

$$\left|\sum_{i=n}^{k} \frac{1}{r_i}\right| = \left|\sum_{i=1}^{k} \frac{1}{r_i} - \sum_{i=1}^{n-1} \frac{1}{r_i}\right| \le 2\widehat{r}_k \quad \text{for} \quad k \ge n. \tag{3}$$

Note that if $r > 0$, then $\widehat{r}_n = r_{n+1}^*$ for any n.

In the proofs of the main results, we will need some lemmas which are presented below. The straightforward proof of these lemmas we leave to the reader.

Lemma 1 *A sequence y is a solution of the equation $\Delta(r_n \Delta y_n) = b_n$ if and only if there exist real constants c_1, c_2 such that*

$$y_n = \sum_{j=1}^{n-1} \frac{1}{r_j} \sum_{i=1}^{j-1} b_i + c_1 \sum_{j=1}^{n-1} \frac{1}{r_j} + c_2$$

for any n.

Lemma 2 *If $\sum_{k=1}^{\infty} \widehat{r}_k |x_k| < \infty$, then for any $n \in \mathbb{N}$ we have*

$$\Delta\left(r_n \Delta\left(\sum_{k=n}^{\infty} x_k \sum_{i=n}^{k} \frac{1}{r_i}\right)\right) = x_n.$$

Lemma 3 *If $n \in \mathbb{N}$ and*

$$\sum_{k=1}^{\infty} \frac{1}{|r_k|} \sum_{i=k}^{\infty} |x_i| < \infty, \quad \textit{then} \quad \Delta\left(r_n \Delta\left(\sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} x_i\right)\right) = x_n.$$

Lemma 4 *If* $x:\mathbb{N}\to\mathbb{R}$, $u:\mathbb{N}\to(0,\infty)$, $\Delta u\le 0$, *and*

$$\sum_{k=1}^{\infty}\frac{\widehat{r}_k|x_k|}{u_k}<\infty,\quad \textit{then}\quad \sum_{k=n}^{\infty}x_k\sum_{i=n}^{k}\frac{1}{r_i}=\mathrm{o}(u_n).$$

Lemma 5 *If* $x:\mathbb{N}\to\mathbb{R}$, $u:\mathbb{N}\to(0,\infty)$, $\Delta u\le 0$, *and*

$$\sum_{k=1}^{\infty}\frac{1}{|r_k|u_k}\sum_{i=k}^{\infty}|x_i|<\infty,\quad \textit{then}\quad \sum_{k=n}^{\infty}\frac{1}{r_k}\sum_{i=k}^{\infty}x_i=\mathrm{o}(u_n).$$

Lemma 6 *If* $g,w:\mathbb{N}\to[0,\infty)$, $\sum_{j=1}^{\infty}g_j<\infty$, *and* $n\in\mathbb{N}$, *then*

$$\sum_{k=n}^{\infty}w_k\sum_{j=k}^{\infty}g_j=\sum_{k=n}^{\infty}g_k\sum_{j=n}^{k}w_j.$$

3 Solutions with Prescribed Asymptotic Behavior

In this section, in Theorem 1, we present our main result. We establish conditions under which for a given solution y of the equation $\Delta(r_n\Delta y_n)=b_n$ and a given positive, nonincreasing sequence u there exists a solution x of (E) such that $x_n=y_n+\mathrm{o}(u_n)$. Next, we present various consequences of Theorem 1. The proof of Theorem 1 is presented in Sect. 6.

Theorem 1 *Assume* y *is a solution of the equation* $\Delta(r_n\Delta y_n)=b_n$,

$$u:\mathbb{N}\to(0,\infty),\quad \Delta u\le 0, \tag{4}$$

$$q\in\mathbb{N},\quad \alpha\in(0,\infty),\quad U=\bigcup_{n=q}^{\infty}[y_n-\alpha,\,y_n+\alpha], \tag{5}$$

$$(a)\ \sum_{k=1}^{\infty}\frac{\widehat{r}_k|a_k|}{u_k}<\infty\quad or\quad (b)\ \sum_{k=1}^{\infty}\frac{1}{|r_k|u_k}\sum_{i=k}^{\infty}|a_i|<\infty, \tag{6}$$

and f *is continuous and bounded on* U. *Then, there exists a solution* x *of* (E) *such that* $x_n=y_n+\mathrm{o}(u_n)$.

We say that a sequence $y\in\mathbb{R}^{\mathbb{N}}$ is f-regular if there exist an index q and a positive number α such that f is continuous and bounded on the set

$$\bigcup_{n=q}^{\infty}[y_n-\alpha,\,y_n+\alpha].$$

It is clear that if f is continuous on $\mathbb{R}$, then any bounded sequence is f-regular. Hence, the following corollary is an immediate consequence of Theorem 1.

Corollary 1 *Assume* (4), (6), *f is continuous and y is a bounded solution of the equation $\Delta(r_n \Delta y_n) = b_n$. Then, there exists a solution x of* (E) *such that $x_n = y_n + \mathrm{o}(u_n)$.*

In the next corollary, we present a result concerning harmonic approximation.

Corollary 2 *Assume y is an f-regular solution of the equation $\Delta(r_n \Delta y_n) = b_n$,*

$$s \in (-\infty, 0], \ \ \tau \in [s, \infty), \ \ r_n^{-1} = \mathrm{O}(n^\tau), \ \ \textit{and} \ \sum_{n=1}^{\infty} n^{1+\tau-s}|a_n| < \infty. \tag{7}$$

Then, there exists a solution x of (E) *such that $x_n = y_n + \mathrm{o}(n^s)$.*

Proof Let $u_n = n^s$ for any n. Choose a positive constant M such that $|r_k^{-1}| \leq M k^\tau$ for any k. Using Lemma 6, we get

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{|r_k| u_k} \sum_{i=k}^{\infty} |a_i| &\leq M \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} k^{\tau-s}|a_i| \leq M \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} i^{\tau-s}|a_i| \\ &= M \sum_{k=1}^{\infty} k^{1+\tau-s}|a_k| < \infty.\end{aligned}$$

Now, using Theorem 1, we obtain the result. □

Remark 1 If $\tau \in [s, \infty)$, $\gamma = 2 + \tau - s$, then, by [15, Section 6], any of the following conditions

$$\liminf_{n\to\infty} n \left(\frac{|a_n|}{|a_{n+1}|} - 1 \right) > \gamma, \quad \limsup_{n\to\infty} \frac{\ln |a_n|}{\ln n} < -\gamma, \quad \liminf_{n\to\infty} n \ln \frac{|a_n|}{|a_{n+1}|} > \gamma$$

imply

$$\sum_{n=1}^{\infty} n^{1+\tau-s}|a_n| < \infty.$$

The following result is devoted to geometric approximation.

Corollary 3 *Assume y is an f-regular solution of the equation $\Delta(r_n \Delta y_n) = b_n$,*

$$\tau \in \mathbb{R}, \ \ r_n^{-1} = \mathrm{O}(n^\tau), \ \ \textit{and} \ \limsup_{n\to\infty} \sqrt[n]{|a_n|} < \beta < 1. \tag{8}$$

Then, there exists a solution x of (E) *such that $x_n = y_n + \mathrm{o}(\beta^n)$.*

Proof Let u be a sequence defined by $u_n = \beta^n$. It is easy to see that $\widehat{r}_n = \mathrm{O}(n^{\tau+1})$. Hence,

$$\limsup_{n\to\infty} \sqrt[n]{\frac{\widehat{r}_n |a_n|}{u_n}} < 1.$$

Therefore, the assertion is a consequence of Theorem 1. □

So far, we did not impose any conditions on the sequence b. Now, we assume that b is "small." That allows us to obtain simpler asymptotic behavior of solutions of Eq. (E). Namely, we may replace solutions y of the equation $\Delta(r_n \Delta y_n) = b_n$ by solutions of the equation $\Delta(r_n \Delta y_n) = 0$.

Theorem 2 *Assume* (4),

$$(a') \quad \sum_{k=1}^{\infty} \frac{\widehat{r}_k(|a_k| + |b_k|)}{u_k} < \infty, \quad or \quad (b') \quad \sum_{k=1}^{\infty} \frac{1}{|r_k| u_k} \sum_{i=k}^{\infty} (|a_i| + |b_i|) < \infty. \tag{9}$$

and y is an f-regular solution of the equation $\Delta(r_n \Delta y_n) = 0$. Then, there exists a solution x of (E) such that $x_n = y_n + \mathrm{o}(u_n)$.

Proof Choose an index q and a positive number α such that f is continuous and bounded on

$$U = \bigcup_{n=q}^{\infty} [y_n - \alpha, y_n + \alpha].$$

Choose a number $\alpha' \in (0, \alpha)$, and let $\beta = \alpha - \alpha'$. Define sequences v, y' by

$$v_n = \begin{cases} \sum_{k=n}^{\infty} b_k \sum_{i=n}^{k} \frac{1}{r_i} & \text{in case } (a') \\ \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} b_i & \text{in case } (b') \end{cases}, \qquad y'_n = y_n + v_n.$$

Using Lemma 4 or Lemma 5, we get $v_n = \mathrm{o}(u_n)$. Hence, there exists an index $q' \geq q$ such that $|v_n| \leq \beta$ for any $n \geq q'$. Let

$$U' = \bigcup_{n=q'}^{\infty} [y'_n - \alpha', y'_n + \alpha'].$$

If $t \in U'$, then there exists an index $k \geq q'$ such that $|t - y'_k| \leq \alpha'$. Then,

$$|t - y_k| = |t - y'_k + y'_k - y_k| \leq |t - y'_k| + |y'_k - y_k| \leq \alpha' + |v_k| \leq \alpha' + \beta = \alpha.$$

Hence, $U' \subset U$. Therefore, f is continuous and bounded on U'. Using Lemma 2 or Lemma 3, we get $\Delta(r_n \Delta v_n) = b_n$. Thus,

$$\Delta(r_n \Delta y'_n) = \Delta(r_n \Delta y_n) + \Delta(r_n \Delta v_n) = 0 + b_n = b_n.$$

By Theorem 1, there exists a solution x of (E) such that $x_n = y'_n + \mathrm{o}(u_n)$. Then,

$$x_n = y_n + v_n + \mathrm{o}(u_n) = y_n + \mathrm{o}(u_n).$$

□

The following harmonic approximation case of Theorem 2 generalizes [17, Theorem 1].

Corollary 4 *Assume (7), y is an f-regular solution of the equation $\Delta(r_n \Delta y_n) = 0$, and*

$$\sum_{n=1}^{\infty} n^{1+\tau-s}|b_n| < \infty.$$

Then, there exists a solution x of (E) such that $x_n = y_n + \mathrm{o}(n^s)$.

Proof Let $u_n = n^s$ for any n. Similarly, as in the proof of Corollary 2 we get

$$\sum_{k=1}^{\infty} \frac{1}{|r_k|u_k} \sum_{i=k}^{\infty}(|a_i| + |b_i|) < \infty.$$

Hence, the assertion is a consequence of Theorem 2. □

Below we get the geometric approximation case of Theorem 2.

Corollary 5 *Assume y is an f-regular solution of the equation $\Delta(r_n \Delta y_n) = 0$,*

$$\tau \in \mathbb{R}, \; r_n^{-1} = \mathrm{O}(n^\tau), \; \textit{and} \; \limsup_{n\to\infty} \sqrt[n]{|a_n| + |b_n|} < \beta < 1.$$

Then, there exists a solution x of (E) such that $x_n = y_n + \mathrm{o}(\beta^n)$.

Proof As in the proof of Corollary 3, we have $\limsup_{n\to\infty} \sqrt[n]{\widehat{r}_n \beta^{-n}(|a_n| + |b_n|)} < 1$. Using Theorem 2, with $u_n = \beta^n$, we get the result. □

If the function f is continuous on $\mathbb{R}$ and the sequence r^* is bounded, we obtain an especially simple case of Theorem 2. The presented result generalizes [16, Theorem 2.1].

Corollary 6 *Assume (4), f is continuous, the sequence r^* is bounded, and*

$$\sum_{k=1}^{\infty} \frac{|a_k| + |b_k|}{u_k} < \infty.$$

Then, for any real constants c, d there exists a solution x of (E) such that

$$x_n = c r_n^* + d + \mathrm{o}(u_n).$$

Proof The boundedness of r^* implies the boundedness of $\widehat{r}$. Hence,

$$\sum_{k=1}^{\infty} \frac{\widehat{r}_k(|a_k| + |b_k|)}{u_k} < \infty.$$

Assume $c, d \in \mathbb{R}$. By Lemma 1, the sequence $y_n = cr^*_n + d$ is a solution of the equation $\Delta(r_n \Delta y_n) = 0$. Moreover, y is bounded. Hence, using Theorem 2, we obtain the result. □

As we mentioned before, Theorem 1 generalizes [17, Theorem 1], [16, Theorem 2.1], and [16, Theorem 2.2]. Below we present an example illustrating Theorem 1. None of these theorems can be applied to the equation given in this example.

Example 1 Assume $s \in (-1, 0]$,

$$r_n = (-1)^n n, \quad a_n = \frac{(-1)^n}{(n+1)^2}\left(\sum_{j=1}^{n-1} \frac{(-1)^j}{j} + \frac{1}{n^2}\right) \quad b_n = (-1)^n \frac{3n+4}{n(n+2)^2},$$

and

$$\sigma(n) = n, \quad u_n = n^s, \quad y_n = \sum_{k=1}^{n-1} \frac{(-1)^k}{k}, \quad f(t) = \begin{cases} \frac{1}{t}, & \text{for } t \neq 0, \\ 1, & \text{for } t = 0 \end{cases} .$$

Then, Eq. (E) takes the form

$$\Delta((-1)^n n \Delta x_n) = a_n f(x_n) + (-1)^n \frac{3n+4}{n(n+2)^2}. \tag{10}$$

By Lemma 1, the sequence y is a solution of $\Delta((-1)^n n \Delta y_n) = 0$. Obviously, y is convergent and f-regular. There exists a positive constant M such that

$$|a_n| \le \frac{M}{(n+1)^2}$$

for any $n \in \mathbb{N}$. Using the formula $\sum_{j=k}^{\infty} \frac{1}{j(j+1)} = \frac{1}{k}$, we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{|r_k| u_k} \sum_{j=k}^{\infty} (|a_j| + |b_j|) &\le \sum_{k=1}^{\infty} \frac{1}{k^{1+s}} \sum_{j=k}^{\infty} \left(\frac{M}{(j+1)^2} + \frac{3j+4}{j(j+2)^2} \right) \\ &\le \sum_{k=1}^{\infty} \frac{1}{k^{1+s}} \sum_{j=k}^{\infty} \frac{M+7}{j(j+1)} = (M+7) \sum_{k=1}^{\infty} \frac{1}{k^{2+s}} < \infty, \end{aligned}$$

Hence, by Corollary 2, there exists a solution x of (10) such that $x_n = y_n + \mathrm{o}(u_n)$. The sequence $x_n = y_n + n^{-2}$ is such a solution. Note that theorem [17, Theorem 1] cannot be applied to Eq. (10) because the condition $r > 0$ is not satisfied. Moreover, since the sequences r^* and b are not periodic, theorems [16, Theorem 2.1] and [16, Theorem 2.2] also cannot be applied to Eq. (10).

4 Asymptotically Periodic Solutions

In this section, we present the application of the previously obtained results to the study of asymptotic periodicity of solutions to Eq. (E). The presented results generalize the main results of [16]. In the first two corollaries, we establish sufficient conditions for the existence of asymptotically periodic solutions. The following result is a consequence of Corollary 6.

Corollary 7 *Assume f is continuous,*

$$u : \mathbb{N} \to (0, \infty), \quad \Delta u \le 0, \quad \omega \in \mathbb{N}, \tag{11}$$

the sequence r^* is ω-periodic, and

$$\sum_{k=1}^{\infty} \frac{|a_k| + |b_k|}{u_k} < \infty. \tag{12}$$

Then, for any real constants c, d there exists an asymptotically ω-periodic solution x of (E) such that $x_n = cr_n^* + d + \mathrm{o}(u_n)$.

Corollary 8 *Assume (11), f is continuous, the sequence r^* is bounded, y is an ω-periodic solution of the equation $\Delta(r_n \Delta y_n) = b_n$, and*

$$\sum_{k=1}^{\infty} \frac{|a_k|}{u_k} < \infty. \tag{13}$$

Then, for any real constant d there exists an asymptotically ω-periodic solution x of (E) such that $x_n = y_n + d + \mathrm{o}(u_n)$.

Proof For any $d \in \mathbb{R}$, the sequence $d + y_n$ is a bounded solution of the equation $\Delta(r_n \Delta y_n) = b_n$. Using Corollary 1, we obtain the result. □

The following example illustrates Corollary 8.

Example 2 Let $r_n = n^2$, $f(t) = t$, $\sigma(n) = n$,

$$a_n = \frac{-1}{(n+1)(n+2)(3 + 2(-1)^n + n^{-1})}, \quad b_n = 4(-1)^n(2n^2 + 2n + 1).$$

Then, Eq. (E) takes the form

$$\Delta(n^2 \Delta x_n) = \frac{-x_n}{(n+1)(n+2)(3 + 2(-1)^n + n^{-1})} + 4(-1)^n(2n^2 + 2n + 1). \tag{14}$$

Let

$$y_n = 3 + 2(-1)^n, \quad u_n = \frac{1}{\sqrt{n}}.$$

Then, y is a 2-periodic solution of the equation $\Delta(n^2\Delta x_n) = b_n$. Obviously, the sequence r^* is bounded and the condition (13) is satisfied. Hence, by Corollary 8, there exists an asymptotically 2-periodic solution x of Eq. (14) such that $x_n = y_n + \mathrm{o}(u_n)$. Indeed, the sequence $x_n = 3 + 2(-1)^n + n^{-1}$ is such a solution.

5 Sturm–Liouville Discrete Equations

Now, we apply our results to discrete Sturm–Liouville equation

$$\Delta(r_n \Delta x_n) = a_n x_{n+1}. \tag{15}$$

First, we present some results concerning asymptotic properties of bounded solutions. Next, we give some nonoscillation criteria.

Theorem 3 *Assume*

$$u : \mathbb{N} \to (0,\infty), \quad \Delta u \le 0, \tag{16}$$

and

$$\sum_{k=1}^{\infty} \frac{\widehat{r}_k |a_k|}{u_k} < \infty \quad or \quad \sum_{k=1}^{\infty} \frac{1}{|r_k| u_k} \sum_{i=k}^{\infty} |a_i| < \infty. \tag{17}$$

Then, for any bounded solution y of the equation

$$\Delta(r_n \Delta y_n) = 0 \tag{18}$$

there exists a solution x of (15) such that $x_n = y_n + \mathrm{o}(u_n)$.

Proof Taking $b_n = 0$, $\sigma(n) = n + 1$ for any $n \in \mathbb{N}$, and $f(t) = t$ for any $t \in \mathbb{R}$ and using Corollary 1, we get the result. □

Note that, by Lemma 1, a sequence y is a solution of the Eq. (18) if and only if there exist real constants c, d such that

$$y_n = c r_n^* + d, \quad \text{where} \quad r_n^* = \sum_{i=1}^{n-1} \frac{1}{r_i}.$$

In particular, any constant sequence is a solution of (18). If the sequence r^* is bounded, then any solution of (18) is bounded. On the other hand, if r^* is unbounded, then the only bounded solutions of (18) are constant.

Corollary 9 *Assume (16), the sequence r^* is bounded, and*

$$\sum_{k=1}^{\infty} \frac{|a_k|}{u_k} < \infty.$$

Then, for any real constants c, d there exists a solution x of (15) *such that*

$$x_n = cr_n^* + d + \mathrm{o}(u_n).$$

Proof Since the sequence r^* is bounded, the sequence $\widehat{r}_n = \max\{|r_1^*|, |r_2^*|, \dots, |r_{n+1}^*|\}$ is bounded, too. Hence,

$$\sum_{k=1}^{\infty} \frac{\widehat{r}_k |a_k|}{u_k} < \infty.$$

Let $c, d \in \mathbb{R}$. Since the sequence r^* is bounded, the sequence $cr_n^* + d$ is a bounded solution of (18). Using Theorem 3, we obtain the result. □

From Corollary 2, we get the result on harmonic approximation.

Corollary 10 *If $s \in (-\infty, 0]$, $\tau \in [s, \infty)$, $r_n^{-1} = \mathrm{O}(n^\tau)$, and $\sum_{n=1}^{\infty} n^{1+\tau-s}|a_n| < \infty$, then for any bounded solution y of* (18) *there exists a solution x of* (15) *such that $x_n = y_n + \mathrm{o}(n^s)$.*

Analogously, from Corollary 3 we get the geometric approximation.

Corollary 11 *Assume $\tau \in \mathbb{R}$, $r_n^{-1} = \mathrm{O}(n^\tau)$, and $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < \beta < 1$. Then, for any bounded solution y of* (18) *there exists a solution x of* (15) *such that $x_n = y_n + \mathrm{o}(\beta^n)$.*

If $c, d \in \mathbb{R}$ and the sequence r^* is bounded, then the sequence $cr^* + d$ is a bounded solution of (18). Hence, from Corollary 10, we obtain

Corollary 12 *If $s \in (-\infty, 0]$, $\tau \in [s, \infty)$, $r_n^{-1} = \mathrm{O}(n^\tau)$, $\sum_{n=1}^{\infty} n^{1+\tau-s}|a_n| < \infty$, and the sequence r^* is bounded, then for any real constants c, d there exists a solution x of* (15) *such that*

$$x_n = cr_n^* + d + \mathrm{o}(n^s).$$

A sequence $y \in \mathbb{R}^{\mathbb{N}}$ is said to be nonoscillatory if $y_n y_{n+1} > 0$ for any large n. In the other case, the sequence y is said to be oscillatory. We say that Eq. (15) is nonoscillatory if all its nontrivial solutions are nonoscillatory. By the Sturm separation theorem, see, for example, [13, Theorem 6.5], either all solutions of (15) are oscillatory or Eq. (15) is nonoscillatory. In particular, if there exists a convergent solution of (15) with nonzero limit, then Eq. (15) is nonoscillatory. From our results, we can get some sufficient conditions for the existence of convergent solutions. Combining these conditions and the Sturm separation theorem, we obtain some nonoscillation criteria for Eq. (15).

Corollary 13 *Assume $\tau \in [0, \infty)$, $r_n^{-1} = \mathrm{O}(n^\tau)$, and $\sum_{n=1}^{\infty} n^{1+\tau}|a_n| < \infty$. Then, Eq.* (15) *is nonoscillatory.*

Corollary 14 *Assume $\tau \in [0, \infty)$, $r_n^{-1} = \mathrm{O}(n^\tau)$, $\alpha \in (-\infty, -2-\tau)$, and $a_n = \mathrm{O}(n^\alpha)$. Then, Eq.* (15) *is nonoscillatory.*

Corollary 15 *If $\tau \in \mathbb{R}$, $r_n^{-1} = \mathrm{O}(n^\tau)$, and $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then Eq.* (15) *is nonoscillatory.*

6 The proof of Theorem 1 and additional remarks

The proof of Theorem 1 is based on the following Schauder-type fixed point lemma. This lemma is a consequence of the proof of [14, Theorem 1].

Lemma 7 *Let y be a real sequence, and let ρ be a positive sequence which is convergent to zero. Define a metric space (X, d) by*

$$X = \{x \in \mathbb{R}^{\mathbb{N}} : \ |x_n - y_n| \le \rho_n \ \ \textit{for any } n \in \mathbb{N}\}, \quad d(x, z) = \sup_{n \in \mathbb{N}} |x_n - z_n|.$$

Then, for any continuous map $H : X \to X$ there exists a point x in X such that $Hx = x$.

Now, we provide the proof of Theorem 1.

Proof For $n \in \mathbb{N}$ and $x \in \mathbb{R}^{\mathbb{N}}$, let

$$F(x)(n) = a_n f(x_{\sigma(n)}). \tag{19}$$

Let

$$Y = \{x \in \mathbb{R}^{\mathbb{N}} : |y - x| \le \alpha\}.$$

Choose a positive constant L, such that

$$|f(t)| \le L \tag{20}$$

for any $t \in U$. For $n \in \mathbb{N}$, let

$$\rho_n = \begin{cases} 2L \sum_{k=n}^{\infty} \widehat{r}_k |a_k| & \text{in case } (a) \\ L \sum_{k=n}^{\infty} \frac{1}{|r_k|} \sum_{i=k}^{\infty} |a_i| & \text{in case } (b). \end{cases}$$

In case (a), it is easy to see that

$$\rho_n = \mathrm{o}(u_n) = \mathrm{o}(1). \tag{21}$$

In case (b), using Lemma 5 we get (21). Therefore, there exists an index p such that

$$\rho_n \le \alpha \quad \text{and} \quad \sigma(n) \ge q$$

for $n \ge p$. Let

$$X = \{x \in \mathbb{R}^{\mathbb{N}} : |x - y| \le \rho \text{ and } x_n = y_n \text{ for } n < p\}.$$

Note that $X \subset Y$. If $x \in Y$ and $n \ge p$, then $|f(x_{\sigma(n)})| \le L$.

Assume condition (a) is satisfied. Define an operator $H : Y \to \mathbb{R}^{\mathbb{N}}$ by

$$H(x)(n) = \begin{cases} y_n & \text{for } n < p \\ y_n + \sum_{k=n}^{\infty} F(x)(k) \sum_{i=n}^{k} \frac{1}{r_i} & \text{for } n \geq p. \end{cases}$$

If $x \in X$, then, by (3), for $n \geq p$ we have

$$|H(x)(n) - y_n| \leq \sum_{k=n}^{\infty} |F(x)(k)| \left| \sum_{i=n}^{k} \frac{1}{r_i} \right| \leq 2L \sum_{k=n}^{\infty} |a_k| \widehat{r}_k = \rho_n.$$

Therefore, $HX \subset X$. Let $x \in X$, and $\varepsilon > 0$. There exist an index $m \geq p$ and a positive constant γ such that

$$4L \sum_{k=m}^{\infty} |a_k| \widehat{r}_k < \varepsilon \quad \text{and} \quad 2\gamma \sum_{k=1}^{m} |a_k| \widehat{r}_k < \varepsilon.$$

Let

$$C = \bigcup_{n=1}^{m} [y_{\sigma(n)} - \alpha, y_{\sigma(n)} + \alpha]. \tag{22}$$

Since C is a compact subset of $\mathbb{R}$, f is uniformly continuous on C. Choose a positive δ such that if $t_1, t_2 \in C$ and $|t_1 - t_2| < \delta$, then

$$|f(t_1) - f(t_2)| < \gamma.$$

Choose $z \in X$ such that $\|x - z\| < \delta$. Then,

$$\begin{aligned} \|Hx - Hz\| &= \sup_{n \geq p} \left| \sum_{k=n}^{\infty} (F(x)(k) - F(z)(k)) \sum_{i=n}^{k} \frac{1}{r_i} \right| \\ &\leq \sum_{k=p}^{\infty} |F(x)(k) - F(z)(k)| \left| \sum_{i=n}^{k} \frac{1}{r_i} \right| \\ &\leq 2 \sum_{k=p}^{\infty} |a_k| |f(x_{\sigma(k)}) - f(z_{\sigma(k)})| \widehat{r}_k \\ &\leq 2\gamma \sum_{k=1}^{m} |a_k| \widehat{r}_k + 4L \sum_{k=m}^{\infty} |a_k| \widehat{r}_k < 2\varepsilon. \end{aligned}$$

Hence, the map $H : X \to X$ is continuous with respect to the metric defined in Lemma 7. By Lemma 7, there exists a point $x \in X$ such that $x = Hx$. Then, for $n \geq p$ we have

$$x_n = y_n + \sum_{k=n}^{\infty} F(x)(k) \sum_{i=n}^{k} \frac{1}{r_i}.$$

Using Lemma 2, we obtain

$$\Delta(r_n \Delta x_n) = \Delta(r_n \Delta y_n) + F(x)(n) = b_n + a_n f(x_{\sigma(n)})$$

for $n \geq p$. Hence, x is a solution of (E). Since $x \in X$ and $\rho_n = \mathrm{o}(u_n)$, we have

$$x_n = y_n + \mathrm{o}(u_n).$$

Now, assume condition (b) is satisfied. Define an operator $G : Y \to \mathbb{R}^{\mathbb{N}}$ by

$$G(x)(n) = \begin{cases} y_n & \text{for } n < p \\ y_n + \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} F(x)(i) & \text{for } n \geq p. \end{cases}$$

If $x \in X$, then for $n \geq p$ we have

$$|G(x)(n) - y_n| \leq \sum_{k=n}^{\infty} \frac{1}{|r_k|} \sum_{i=k}^{\infty} |F(x)(i)| \leq L \sum_{k=n}^{\infty} \frac{1}{|r_k|} \sum_{i=k}^{\infty} |a_i| = \rho_n.$$

Therefore, $GX \subset X$. Let $x \in X$, and $\varepsilon > 0$. There exist an index $m \geq p$ and a positive constant γ such that

$$L \sum_{k=m}^{\infty} \frac{1}{|r_k|} \sum_{i=k}^{\infty} |a_i| < \varepsilon \quad \text{and} \quad \gamma \sum_{k=1}^{m} \frac{1}{|r_k|} \sum_{i=k}^{\infty} |a_i| < \varepsilon.$$

Define a subset C of $\mathbb{R}$ by (22). Choose a positive δ such that if $t_1, t_2 \in C$ and $|t_1 - t_2| < \delta$, then $|f(t_1) - f(t_2)| < \gamma$. Choose $z \in X$ such that $\|x - z\| < \delta$. Then,

$$\begin{aligned}
\|Gx - Gz\| &= \sup_{n \geq p} \left| \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} (F(x)(i) - F(z)(i)) \right| \\
&\leq \sum_{k=p}^{\infty} \frac{1}{|r_k|} \sum_{i=k}^{\infty} |F(x)(i) - F(z)(i)| = \sum_{k=p}^{\infty} \frac{1}{|r_k|} \sum_{i=k}^{\infty} |a_i| |f(x_{\sigma(k)}) - f(z_{\sigma(k)})| \\
&\leq \gamma \sum_{k=1}^{m} \frac{1}{|r_k|} \sum_{i=k}^{\infty} |a_i| + 2L \sum_{k=m}^{\infty} \frac{1}{|r_k|} \sum_{i=k}^{\infty} |a_i| < 3\varepsilon.
\end{aligned}$$

Hence, the map $H : X \to X$ is continuous. By Lemma 7, there exists a point $x \in X$ such that $x = Hx$. Then, for $n \geq p$ we have

$$x_n = y_n + \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} F(x)(i).$$

By Lemma 3, x is a solution of (E). Since $x \in X$ and $\rho_n = \mathrm{o}(u_n)$, we have

$$x_n = y_n + \mathrm{o}(u_n).$$

□

In our theory, two conditions

$$(a)\quad \sum_{k=1}^{\infty} \frac{\widehat{r}_k |a_k|}{u_k} < \infty, \qquad (b)\quad \sum_{k=1}^{\infty} \frac{1}{|r_k| u_k} \sum_{i=k}^{\infty} |a_i| < \infty$$

play a key role. Below we present a comparison of assumptions (a) and (b).

Remark 2 Assume $a \in \mathbb{R}^{\mathbb{N}}$, $r, u : \mathbb{N} \to (0, \infty)$, $\Delta u \le 0$, and $\sum_{k=1}^{\infty} \frac{\widehat{r}_k |a_k|}{u_k} < \infty$. Then, $\sum_{k=1}^{\infty} |a_k| < \infty$ and using Lemma 6, we obtain

$$\sum_{k=1}^{\infty} \frac{1}{|r_k| u_k} \sum_{j=k}^{\infty} |a_j| = \sum_{k=1}^{\infty} \frac{1}{r_k u_k} \sum_{j=k}^{\infty} |a_j| = \sum_{k=1}^{\infty} |a_k| \sum_{i=1}^{k} \frac{1}{r_i u_i} \le \sum_{k=1}^{\infty} \frac{|a_k|}{u_k} \sum_{i=1}^{k} \frac{1}{r_i} < \infty.$$

Hence, in the case $r > 0$ condition (b) is a consequence of condition (a).

The following two examples show that, in general, conditions (a) and (b) are independent.

Example 3 Assume $\lambda \in (1, \infty)$, $r_n = \lambda^n$, $u_n = \lambda^{-n}$, and $a_n = n^{-3}$. Then,

$$r > 0, \quad \widehat{r}_k = \frac{1 - \lambda^{-k}}{\lambda - 1}, \quad \frac{\widehat{r}_k}{u_k} = \frac{\lambda^k - 1}{\lambda - 1}, \quad \frac{1}{|r_k| u_k} = 1.$$

Hence,

$$\sum_{k=1}^{\infty} \frac{\widehat{r}_k |a_k|}{u_k} = \frac{1}{\lambda - 1} \sum_{k=1}^{\infty} \frac{\lambda^k - 1}{k^3} = \infty,$$

$$\sum_{k=1}^{\infty} \frac{1}{|r_k| u_k} \sum_{i=k}^{\infty} |a_i| = \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} |a_i| = \sum_{k=1}^{\infty} k |a_k| = \sum_{k=1}^{\infty} k^{-2} < \infty.$$

Therefore, in this case, condition (a) is not a consequence of (b).

Example 4 Let $u_k = 1$, $r_k = (-1)^k$, $a_k = \frac{1}{\sqrt{k^3}}$ for any $k \in \mathbb{N}$. Then,

$$\sum_{k=1}^{\infty} \frac{\widehat{r}_k |a_k|}{u_k} \le \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3}} < \infty, \quad \sum_{k=1}^{\infty} \frac{1}{|r_k| u_k} \sum_{j=k}^{\infty} |a_j| = \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \frac{1}{\sqrt{j^3}} = \sum_{k=1}^{\infty} \frac{k}{\sqrt{k^3}} = \infty.$$

Hence, in the case $r_k = (-1)^k$, condition (b) is not a consequence of (a).

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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