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ORIGINAL PAPER

On some modifications of *n*-th von Neumann–Jordan constant for Banach spaces

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Abstract

We study, among others, upper, lower, upper modified and lower modified n-th von Neumann–Jordan constant and relationships between them. There are characterized uniformly non-l_n^1 Banach spaces in terms of the upper modified n-th von Neumann–Jordan constant. Moreover, this constant is calculated explicitly for Lebesgue spaces L^p and l^p $(1 \leq p \leq \infty)$. Finally, it is shown that the sequence of n-th upper and modified upper von Neumann–Jordan constants for the space L^p as well as l^p $(2 < p < \infty)$ converges to B_p^2, where B_p is the best type $(2, p)$ constant in the Khinthine inequality for the case $2 \leq p < \infty$.

Keywords von Neumann–Jordan constant · Modified n-th von Neumann–Jordan constant · Uniformly non-$l_n^{(1)}$-Banach space · B-convexity

Mathematics Subject Classification 46E30 · 46E40 · 46B20

1 Introduction

Let $S(X)$ (resp. $B(X)$) be the unit sphere (resp. the unit closed ball) of a real Banach space $(X, \|\cdot\|_X)$. The letters $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{R}$ stand for the sets of integers, positive integers and real numbers, respectively. For any subset $A \subset X$, denote $A^n = \underbrace{A \times \cdots \times A}_{n\text{-times}}$. Let (Ω, Σ, μ) be a measure space with a σ-finite, non-atomic and complete measure μ. Denote by $L^p(\mu)$ $(1 \leq p \leq \infty)$ the Lebesgue space of real Σ-measurable functions

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defined on Ω. The symbol l_m^p $(1 \leq p \leq \infty,\ m \in \mathbb{N} \cup \{\infty\})$ stands for m-dimensional Lebesgue sequence space. Clearly, $l_\infty^p = l^p$.

In 1937 Clarkson [5], on the basis of the famous paper [13] by Jordan and von Neumann, introduced the constant $C_{NJ}(X)$ (called the *von Neumann–Jordan constant or NJ-constant for short*) as the smallest constant $C \geq 1$ such that

$$\frac{1}{C} \leq \frac{\|x+y\|_X^2 + \|x-y\|_X^2}{2\left(\|x\|_X^2 + \|y\|_X^2\right)} \leq C$$

holds for any $x, y \in X$ with $\|x\|_X^2 + \|y\|_X^2 > 0$. An equivalent and more convenient definition of NJ-constant is given in [15] by the formula

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|_X^2 + \|x-y\|_X^2}{2\left(\|x\|_X^2 + \|y\|_X^2\right)} : x \in S(X),\ \ y \in B(X)\right\}.$$

The classical Jordan and von Neumann results [13] state that $1 \leq C_{NJ}(X) \leq 2$ for any Banach space X and $C_{NJ}(X) = 1$ if and only if X is a Hilbert space. Clarkson [5] showed that if $1 \leq p \leq \infty$ and $\dim L^p(\mu) \geq 2$, then $C_{NJ}(L^p(\mu)) = 2^{2/\min\{p,q\}-1}$, where $1/p + 1/q = 1$. Kato and Takahashi [16], observed that $C_{NJ}(X) = C_{NJ}(X^*)$. Moreover, they proved that if the Banach space X is uniformly convex, then $C_{NJ}(X) < 2$ and if $C_{NJ}(X) < 2$, then X admits an equivalent uniformly convex norm. The same authors [16] state that the Banach space X is uniformly non-square if and only if $C_{NJ}(X) < 2$. Some results concerning relationships between von Neumann–Jordan and so called James constant have been obtained among others in [2,15,19,21,23,25].

A similar constant

$$C'_{NJ}(X) = \sup\left\{\frac{\|x+y\|_X^2 + \|x-y\|_X^2}{4} : x, y \in S(X)\right\}$$

was introduced in 2006 by Gao [7] and called the modified von Neumann–Jordan constant. It is clear that $C'_{NJ}(X) \leq C_{NJ}(X)$. It has been shown that $C'_{NJ}(X)$ does not necessarily coincide with $C_{NJ}(X)$ (see [2,8]). These constants have been considered recently also in [22].

The von Neumann Jordan constant has been generalized in many directions (see e.g. [17,24,26]).

To generalize the von Neumann–Jordan constant, denote

$$C^{(n)}(x_1, x_2, \ldots, x_n) = \frac{\sum_{\theta_j=\pm 1}\left\|x_1 + \sum_{j=2}^n \theta_j x_j\right\|_X^2}{2^{n-1}\sum_{j=1}^n \|x_j\|_X^2}$$

for any $x_1, x_2, \ldots, x_n \in X$ such that $\sum_{j=1}^n \|x_j\|_X^2 > 0$.

Definition 1 The smallest, resp., the largest constant $C > 0$ such that

$$C^{(n)}(x_1, x_2, \dots, x_n) \le C, \text{ resp., } C \le C^{(n)}(x_1, x_2, \dots, x_n) \tag{1}$$

for all $x_j \in X$, ($j = 1, 2, \dots, n$ and $n \ge 2$) with $\sum_{j=1}^n \|x_j\|_X^2 > 0$ is called an *upper*, resp., *lower n-th von Neumann–Jordan constant* and denoted by $\overline{C}_{NJ}^{(n)}(X)$, resp., $\underline{C}_{NJ}^{(n)}(X)$. If the infimum, resp., supremum of C satisfying (1) is taken over all $x_j \in S(X)$, ($j = 1, 2, \dots, n$ and $n \ge 2$), then it is called *upper, resp., lower modified n- th von Neumann–Jordan constant* and denoted by $\overline{C}_{mNJ}^{(n)}(X), resp., \underline{C}_{mNJ}^{(n)}(X)$.

It is well known that $\overline{C}_{NJ}^{(2)}(X) = \left[\underline{C}_{NJ}^{(2)}(X)\right]^{-1} = C_{NJ}(X)$ (see [20]). As it is proved below, the equality $\overline{C}_{NJ}^{(n)}(X) = \left[\underline{C}_{NJ}^{(n)}(X)\right]^{-1}$ is not true in general for $n > 2$. Moreover, $\overline{C}_{mNJ}^{(2)}(X) = C'_{NJ}(X)$ and $\overline{C}_{mNJ}^{(n)}(X)$ need not be equal to $\left[\underline{C}_{mNJ}^{(n)}(X)\right]^{-1}$ even for $n = 2$ (see [7]). The n-th von Neumann–Jordan constant introduced and investigated by Kato, Takahashi and Hashimoto in [17] is exactly the upper n-th von Neumann–Jordan constant.

In 1964 James [12] introduced the notion of uniformly non-l_n^1 Banach space. Namely, a Banach space X is called *uniformly non-*l_n^1 if there exists $\delta > 0$ such that for each n elements of the unit ball $B(X)$

$$\min_{\theta_j = \pm 1} \left\| x_1 + \sum\nolimits_{j=2}^n \theta_j x_j \right\|_X \le n\,(1 - \delta)$$

(see [10]). The definition remains the same if we replace the unit ball $B(X)$ by the unit sphere $S(X)$. If X is uniformly non-l_n^1 for $n = 2$, then it is called *uniformly non-square*. It is worth mentioning that uniform non-squareness plays a crucial role in fixed point theory, since any uniformly non-square Banach space has the fixed point property (for more details see [9]). In 1987 Kamińska and Turett [14] proved that the uniform non-l_n^1 for Banach spaces is equivalent to the fact that there exists $\delta > 0$ such that for all $x_1, x_2, \dots, x_n$ in X

$$\min_{\theta_j = \pm 1} \left\| x_1 + \sum\nolimits_{j=2}^n \theta_j x_j \right\|_X \le \left(1 - \frac{\delta n \min_{1 \le i \le n} \|x_i\|_X}{\sum_{j=1}^n \|x_j\|_X} \right) \sum_{j=1}^n \|x_j\|_X .$$

Banach spaces that are uniformly non-l_n^1 for a certain $n \in \mathbb{N}$ have been studied by A. Beck [3]. Such spaces are said to be *B- convex*. Beck [3] proved that a Banach space X is B-convex if and only if a certain strong law of large numbers is valid for random variables with ranges in X. Moreover, B-convexity is a very important property in fixed point theory because every B-convex uniformly monotone Köthe space has the fixed point property (see [1]).

2 Basic Properties

Proposition 1 *Let $n \geq 2$ and X be a Banach space. The lower, upper, modified lower and modified upper n-th von Neumann–Jordan constants have the following properties:*

(a) $1 \leq \overline{C}_{mNJ}^{(n)}(X) \leq \overline{C}_{NJ}^{(n)}(X) \leq n$ *and* $1/n \leq \underline{C}_{NJ}^{(n)}(X) \leq \underline{C}_{mNJ}^{(n)}(X) \leq 1$;
(b) $\overline{C}_{NJ}^{(n)}(X) \leq \overline{C}_{NJ}^{(n+1)}(X)$ *and* $\underline{C}_{NJ}^{(n+1)}(X) \leq \underline{C}_{NJ}^{(n)}(X)$;
(c) $\overline{C}_{mNJ}^{(n)}(X) \leq \frac{n+1}{n}\overline{C}_{mNJ}^{(n+1)}(X)$.

Proof Let $(X, \|\cdot\|_X)$ be a Banach space and $n \geq 2$.

(a) The estimation $\overline{C}_{NJ}^{(n)}(X) \leq n$ is proved in [17]. $\overline{C}_{mNJ}^{(n)}(X) \leq \overline{C}_{NJ}^{(n)}(X)$ by the definition. Putting $x_1 \in S(X)$ and $x_i = x_1$ for $i = 2, 3, \ldots, n$, we have

$$\begin{aligned} C^{(n)}(x_1, x_1, \ldots, x_1) &= \frac{1}{n2^{n-1}} \sum_{\theta_j = \pm 1} \left\| x_1 + \sum\nolimits_{j=2}^{n} \theta_j x_1 \right\|_X^2 \\ &= \frac{1}{n2^{n-1}} \sum\nolimits_{j=0}^{n-1} \binom{n-1}{j} \|(n-2j)x_1\|_X^2 \\ &= \frac{1}{n2^{n-1}} \sum\nolimits_{j=0}^{n-1} \binom{n-1}{j} (n-2j)^2 = 1. \end{aligned}$$

Hence

$$\overline{C}_{mNJ}^{(n)}(X) = \sup\left\{C^{(n)}(x_1, x_2, \ldots, x_n) : x_i \in S(X),\ i = 1, 2, \ldots, n\right\} \geq 1$$

and

$$\underline{C}_{mNJ}^{(n)}(X) = \inf\left\{C^{(n)}(x_1, x_2, \ldots, x_n) : x_i \in S(X),\ i = 1, 2, \ldots, n\right\} \leq 1.$$

Obviously, $\underline{C}_{NJ}^{(n)}(X) \leq \underline{C}_{mNJ}^{(n)}(X)$ by the definition. To prove that $\frac{1}{n} \leq \underline{C}_{NJ}^{(n)}(X)$, we use the mathematical induction principle. For $n = 2$ we have

$$\underline{C}_{NJ}^{(2)}(X) = \frac{1}{\overline{C}_{NJ}^{(2)}(X)} \geq \frac{1}{2}.$$

Suppose that $\underline{C}_{NJ}^{(n-1)}(X) \geq \frac{1}{n-1}$. Notice that

$$\|x+y\|_X^2 + \|x-y\|_X^2 \geq 2\left(\max\left\{\|x\|_X, \|y\|_X\right\}\right)^2 \geq \|x\|_X^2 + \|y\|_X^2 \tag{2}$$

for any $x, y \in X$. Really, since

$$\|x+y\|_X + \|x-y\|_X \geq 2\max\left\{\|x\|_X, \|y\|_X\right\} = 2m_{x,y} \geq \|x+y\|_X - \|x-y\|_X$$

we have

$$\begin{aligned}
\|x+y\|_X^2+\|x-y\|_X^2 &\geq \|x+y\|_X^2+\left(2m_{x,y}-\|x+y\|_X\right)^2\\
&= 2\,\|x+y\|_X^2-4m_{x,y}\,\|x+y\|_X+4\left(m_{x,y}\right)^2\\
&= 2\left[\left(\|x+y\|_X-m_{x,y}\right)^2+\left(m_{x,y}\right)^2\right]\\
&\geq 2\left(\max\left\{\|x\|_X\,,\|y\|_X\right\}\right)^2\geq \|x\|_X^2+\|y\|_X^2\,.
\end{aligned}$$

Hence

$$\begin{aligned}
C^{(n)}(x_1,x_2,\ldots,x_n) &= \frac{\sum_{\theta_j=\pm1}\left\|x_1+\sum_{j=2}^n\theta_jx_j\right\|_X^2}{2^{n-1}\sum_{j=1}^n\left\|x_j\right\|_X^2}\\
&= \frac{\sum_{\theta_j=\pm1}\left\|\sum_{j=1}^n\theta_jx_j\right\|_X^2}{2^n\sum_{j=1}^n\left\|x_j\right\|_X^2}\\
&= \frac{\sum_{\theta_j=\pm1}\left(\left\|\left(\sum_{j\neq i}\theta_jx_j\right)+x_i\right\|_X^2+\left\|\left(\sum_{j\neq i}\theta_jx_j\right)-x_i\right\|_X^2\right)}{2^n\sum_{j=1}^n\left\|x_j\right\|_X^2}\\
&\geq \frac{\sum_{\theta_j=\pm1}\left\|\sum_{j\neq i}\theta_jx_j\right\|_X^2+2^{n-1}\,\|x_i\|_X^2}{2^n\sum_{j=1}^n\left\|x_j\right\|_X^2}\\
&\geq \frac{\frac{2^{n-1}}{n-1}\sum_{j\neq i}\left\|x_j\right\|_X^2+2^{n-1}\,\|x_i\|_X^2}{2^n\sum_{j=1}^n\left\|x_j\right\|_X^2}\\
&= \frac{\frac{1}{n-1}\sum_{j\neq i}\left\|x_j\right\|_X^2+\|x_i\|_X^2}{2\sum_{j=1}^n\left\|x_j\right\|_X^2}\\
&= \frac{\frac{1}{n-1}\sum_{j=1}^n\left\|x_j\right\|_X^2+\frac{n-2}{n-1}\,\|x_i\|_X^2}{2\sum_{j=1}^n\left\|x_j\right\|_X^2}\\
&= \frac{1}{2\,(n-1)}+\frac{(n-2)\,\|x_i\|_X^2}{2\,(n-1)\sum_{j=1}^n\left\|x_j\right\|_X^2}
\end{aligned}$$

for any $i=1,2,\ldots,n$. It follows that

$$\begin{aligned}
C^{(n)}(x_1,x_2,\ldots,x_n) &\geq \frac{1}{n}\sum_{i=1}^n\left(\frac{1}{2\,(n-1)}+\frac{(n-2)\,\|x_i\|_X^2}{2\,(n-1)\sum_{j=1}^n\left\|x_j\right\|_X^2}\right)\\
&= \frac{1}{2\,(n-1)}+\frac{n-2}{2n\,(n-1)}=\frac{1}{n}
\end{aligned}$$

and consequently $\underline{C}_{NJ}^{(n)}(X)\geq\frac{1}{n}$, which finishes the proof of (a).

(b) The inequality $\overline{C}_{NJ}^{(n)}(X) \leq \overline{C}_{NJ}^{(n+1)}(X)$ is proved in [17]. To prove the second inequality it is enough to notice that

$$C^{(n)}(x_1, x_2, \ldots, x_n) = C^{(n+1)}(x_1, x_2, \ldots, x_n, 0)$$

for any elements $x_1, x_2, \ldots, x_n \in X$. Hence

$$\begin{aligned}\underline{C}_{NJ}^{(n+1)}(X) &= \inf\left\{C^{(n+1)}(x_1, x_2, \ldots, x_n, x_{n+1}) : x_1, x_2, \ldots, x_{n+1} \in X\right\} \\ &\leq \inf\left\{C^{(n)}(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in X\right\} = \underline{C}_{NJ}^{(n)}(X).\end{aligned}$$

(c) For any $x_1, x_2, \ldots, x_{n+1} \in S(X)$, by the inequality (2), we have

$$\begin{aligned}C^{(n+1)}(x_1, x_2, \ldots, x_{n+1}) &= \frac{\sum\limits_{\theta_j=\pm 1}\left(\left\|\left(\sum\limits_{j=1}^{n}\theta_j x_j\right) + x_{n+1}\right\|_X^2 + \left\|\left(\sum\limits_{j=1}^{n}\theta_j x_j\right) - x_{n+1}\right\|_X^2\right)}{(n+1)2^{n+1}} \\ &\geq \frac{1}{(n+1)2^n}\sum_{\theta_j=\pm 1}\max\left\{\left\|\sum\nolimits_{j=1}^{n}\theta_j x_j\right\|_X^2, \|x_{n+1}\|_X^2\right\} \\ &\geq \frac{1}{(n+1)2^n}\sum_{\theta_j=\pm 1}\left\|\sum\nolimits_{j=1}^{n}\theta_j x_j\right\|_X^2 = \frac{n}{n+1}C^{(n)}(x_1, x_2, \ldots, x_n),\end{aligned}$$

whence $\overline{C}_{mNJ}^{(n)}(X) \leq \frac{n+1}{n}\overline{C}_{mNJ}^{(n+1)}(X)$.

□

Proposition 2 *Let $n \geq 2$ and X be a Banach space.*

(a) *The following conditions are equivalent:*

(i) *X is a Hilbert space;*
(ii) $\overline{C}_{NJ}^{(n)}(X) = 1$;
(iii) $\underline{C}_{NJ}^{(n)}(X) = 1$.

(b) *If X is a Hilbert space, then $\overline{C}_{mNJ}^{(n)}(X) = \underline{C}_{mNJ}^{(n)}(X) = 1$.*

Proof (a) By Theorem 5 (iii) in [17], conditions (i) and (ii) are equivalent. To prove the implication (i) ⇒ (iii) suppose that X is a Hilbert space. By elementary calculations, we get that

$$C^{(n)}(x_1, x_2, \ldots, x_n) = 1 \tag{3}$$

for any elements $x_1, x_2, \ldots, x_n \in X$, whence $\underline{C}_{NJ}^{(n)}(X) = 1$. Conversely, if $\underline{C}_{NJ}^{(n)}(X) = 1$, then, by Proposition 1 (a) and (b), we have

$$1 \geq \frac{1}{C_{NJ}(X)} = \underline{C}_{NJ}^{(2)}(X) \geq \underline{C}_{NJ}^{(n)}(X) = 1.$$

Hence $C_{NJ}(X) = 1$. Consequently, X is a Hilbert space (see [13]).
(b) follows immediately from (3). □

In general, the explicit calculation of various types of n-th von Neumann–Jordan constant is rather a hard problem. Anyway, the next proposition can be helpful to do this.

Proposition 3 *Let $(X, \|\cdot\|_X)$ be a Banach space and $n \geq 2$. Denote $D_1 = [B(X)]^n \setminus \{\mathbf{0}\}$, $D_2 = B(l_n^2(X)) \setminus \{\mathbf{0}\}$, $D_3 = S(l_n^2(X))$, where $\mathbf{0} = (0, 0, \ldots, 0)$. Then*

$$\overline{C}_{NJ}^{(n)}(X) = \sup\left\{C^{(n)}(x_1, x_2, \ldots, x_n) : (x_1, x_2, \ldots, x_n) \in D_j\right\} \tag{4}$$

and

$$\underline{C}_{NJ}^{(n)}(X) = \inf\left\{C^{(n)}(x_1, x_2, \ldots, x_n) : (x_1, x_2, \ldots, x_n) \in D_j\right\} \tag{5}$$

for any $j = 1, 2, 3$.

Proof Since

$$S(l_n^2(X)) \subset B(l_n^2(X)) \setminus \{\mathbf{0}\} \subset [B(X)]^n \setminus \{\mathbf{0}\} \subset X^n \setminus \{\mathbf{0}\},$$

it follows that

$$\sup_{\mathbf{x}\in D_3} C^{(n)}(\mathbf{x}) \leq \sup_{\mathbf{x}\in D_2} C^{(n)}(\mathbf{x}) \leq \sup_{\mathbf{x}\in D_1} C^{(n)}(\mathbf{x}) \leq \overline{C}_{NJ}^{(n)}(X),$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. To show (4), it remains to prove that $\sup_{\mathbf{x}\in D_3} C^{(n)}(\mathbf{x}) \geq \overline{C}_{NJ}^{(n)}(X)$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in X^n \setminus \{\mathbf{0}\}$. Define the sequence $\mathbf{y} = (y_k)_{k=1}^n$ by

$$y_k = \frac{x_k}{\left(\sum_{j=1}^n \|x_j\|_X^2\right)^{\frac{1}{2}}}$$

for $k = 1, 2, \ldots, n$ and $n \geq 2$. Obviously, $\mathbf{y} \in S(l_n^2(X))$. Hence

$$\sup_{\mathbf{x}\in D_3} C^{(n)}(\mathbf{x}) \geq C^{(n)}(\mathbf{y}) = \frac{\sum_{\theta_j=\pm 1} \left\|y_1 + \sum_{j=2}^n \theta_j y_j\right\|_X^2}{2^{n-1}}$$
$$= \frac{\sum_{\theta_j=\pm 1} \left\|x_1 + \sum_{j=2}^n \theta_j x_j\right\|_X^2}{2^{n-1} \sum_{j=1}^n \|x_j\|_X^2} = C^{(n)}(\mathbf{x})$$

for any elements $\mathbf{x} \in X^n \setminus \{\mathbf{0}\}$. Therefore

$$\sup_{\mathbf{x}\in D_3} C^{(n)}(\mathbf{x}) \geq \sup\left\{C^{(n)}(\mathbf{x}) : \mathbf{x} \in X^n \setminus \{\mathbf{0}\}\right\} = \overline{C}_{NJ}^{(n)}(X)$$

which finishes the proof of (4). The equality (5) can be proved similarly. □

Let $n \geq 2$. Define

$$A_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}_{2\times 2}$$

and for each integers $n > 2$

$$A_n = \begin{bmatrix} A_{n-1} & \mathbf{1} \\ A_{n-1} & -\mathbf{1} \end{bmatrix}_{2^{n-1}\times n},$$

where $\mathbf{1}$ denotes the 2^{n-2}-by-1 column vector in which all the elements are equal to 1. The matrix A_n generates a linear operator

$$T_n : l_n^2(X) \to l_{2^{n-1}}^2(X)$$

defined for any $x \in l_n^2(X)$ by the formula

$$T_n(x) = A_n x.$$

A one-to-one correspondence between n-th von Neumann–Jordan constant $\overline{C}_{NJ}^{(n)}(X)$ and the norm of the operator T_n is given by the following result.

Corollary 1 *Let $(X, \|\cdot\|_X)$ be a Banach space and $T_n : l_n^2(X) \to l_{2^{n-1}}^2(X)$ be the linear operator generated by the matrix A_n. Then*

$$\overline{C}_{NJ}^{(n)}(X) = \frac{||T_n||^2}{2^{n-1}}$$

for any integer $n \geq 2$.

Proof Fix $n \geq 2$. Let $x_1, x_2, \ldots, x_n \in X$ and $\sum_{j=1}^n \|x_j\|_X^2 > 0$. Denote $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. By Proposition 3, we have

$$\overline{C}_{NJ}^{(n)}(X) = \sup\left\{\frac{\sum_{\theta_j=\pm 1}\left\|x_1 + \sum_{j=2}^n \theta_j x_j\right\|_X^2}{2^{n-1}\sum_{j=1}^n \|x_j\|_X^2} : \mathbf{x} \in S(l_n^2(X))\right\}$$
$$= \frac{1}{2^{n-1}}\sup\left\{||T_n x||_{l_{2^{n-1}}^2(X)}^2 : \mathbf{x} \in S(l_n^2(X))\right\} = \frac{||T_n||^2}{2^{n-1}}.$$

□

Corollary 2 *Let $(X^*, \|\cdot\|_{X^*})$ be the dual space of the Banach space $(X, \|\cdot\|_X)$. Then*

(a) $\underline{C}_{NJ}^{(n)}(X^*) \geq \frac{1}{\overline{C}_{NJ}^{(n)}(X)}$.

(b) $\underline{C}_{NJ}^{(n)}(X) \geq \frac{1}{\overline{C}_{NJ}^{(n)}(X^*)}$.

Proof (a) Define an operator $T_n : l_n^2(X) \to l_{2^{n-1}}^2(X)$ as above. Let T_n^* be the adjoint of operator T_n. Obviously, $T_n^* : l_{2^{n-1}}^2(X^*) \to l_n^2(X^*)$ is generated by the matrix $A_n^* = A_n^T$. Then, by Corollary 1, we get

$$\frac{1}{\overline{C}_{NJ}^{(n)}(X)} = \frac{2^{n-1}}{||T_n||^2} = \frac{2^{n-1}}{||T_n^*||^2} \leq \frac{2^{n-1}\, \|y^*\|_{l_{2^{n-1}}^2(X^*)}^2}{\left\| T_n^* y^* \right\|_{l_n^2(X^*)}^2} \tag{6}$$

for any $y^* = (y_1^*, y_2^*, \ldots, y_{2^{n-1}}^*) \in l_{2^{n-1}}^2(X^*) \setminus \{\mathbf{0}\}$. Let $S_n^* = \frac{1}{2^{n-1}} A_n$. Then, $S_n : l_n^2(X^*) \to l_{2^{n-1}}^2(X^*)$. Since

$$A_n^* \cdot \frac{1}{2^{n-1}} A_n = I_n,$$

where I_n is the identity matrix of size $n \times n$, it follows that

$$T_n^* \left(S_n^* x^* \right) = x^*$$

for any $x^* = (x_1^*, x_2^*, \ldots, x_n^*) \in l_n^2 \left(X^* \right)$. Hence, by (6), we have

$$\begin{aligned} \frac{1}{\overline{C}_{NJ}^{(n)}(X)} &\leq \frac{2^{n-1} \left\| S_n^* x^* \right\|_{l_{2^{n-1}}^2(X^*)}^2}{\left\| T_n^* \left(S_n^* x^* \right) \right\|_{l_n^2(X^*)}^2} = \frac{\|A_n x^*\|_{l_{2^{n-1}}^2(X^*)}^2}{2^{n-1}\, \|x^*\|_{l_n^2(X^*)}^2} \\ &= \frac{\sum_{\theta_j = \pm 1} \left\| x_1^* + \sum_{j=2}^n \theta_j x_j^* \right\|_{X^*}^2}{2^{n-1} \sum_{j=1}^n \left\| x_j^* \right\|_{X^*}^2} = C^{(n)} \left(x_1^*, x_2^*, \ldots, x_n^* \right) \end{aligned}$$

for any $(x_1^*, x_2^*, \ldots, x_n^*) \in (X^*)^n \setminus \{\mathbf{0}\}$. By the definition of the lower n-th von Neumann–Jordan constant, we get the thesis of (a).

(b) Since X can be isometrically embedded into X^{**}, it follows that $\underline{C}_{NJ}^{(n)}(X) \geq \underline{C}_{NJ}^{(n)}(X^{**})$. Hence, by (a), we have

$$\underline{C}_{NJ}^{(n)}(X) \geq \underline{C}_{NJ}^{(n)}(X^{**}) \geq \frac{1}{\overline{C}_{NJ}^{(n)}(X^*)}.$$

□

The n-th von Neumann–Jordan constant for some classical Banach spaces can be calculated effectively.

Proposition 4 *Let $n \geq 2$.*

(a) *If $1 \leq p \leq 2$ and $n \leq m \leq \infty$, then $\overline{C}_{NJ}^{(n)}(l_m^p) = \overline{C}_{mNJ}^{(n)}(l_m^p) = n^{\frac{2}{p}-1}$.*
(b) *If $2^{n-1} \leq m \leq \infty$, then $\overline{C}_{NJ}^{(n)}(l_m^\infty) = \overline{C}_{mNJ}^{(n)}(l_m^\infty) = n$.*
(c) *If $n \leq m \leq \infty$, then $\underline{C}_{NJ}^{(n)}(l_m^\infty) = \underline{C}_{mNJ}^{(n)}(l_m^\infty) = \frac{1}{n}$.*
(d) *Let (Ω, Σ, μ) be a measure space with non-atomic σ-finite and complete measure μ. Then*

$$\overline{C}_{mNJ}^{(n)}(L^1(\mu)) = \overline{C}_{NJ}^{(n)}(L^1(\mu)) = \overline{C}_{NJ}^{(n)}(L^\infty(\mu)) = \overline{C}_{NJ}^{(n)}(L^\infty(\mu)) = n$$

and

$$\underline{C}_{mNJ}^{(n)}(L^\infty(\mu)) = \underline{C}_{NJ}^{(n)}(L^\infty(\mu)) = \frac{1}{n}.$$

Proof (a) Let $n \leq m \leq \infty$. By Theorem 3 (ii) in [17], $\overline{C}_{NJ}^{(n)}(l_m^p) = n^{\frac{2}{p}-1}$. By Proposition 1 (b), $\overline{C}_{mNJ}^{(n)}(l_m^p) \leq n^{\frac{2}{p}-1}$. Taking the canonical basis $(e_i)_{i=1}^m$ ($n \leq m \leq \infty$) in l_m^p, we get

$$\begin{aligned} C^{(n)}(e_1, e_2, \ldots, e_n) &= \frac{\sum_{\theta_j = \pm 1} \left\| e_1 + \sum_{j=2}^n \theta_j e_j \right\|_{l^p}^2}{2^{n-1} \sum_{j=1}^n \left\| e_j \right\|_{l^p}^2} \\ &= \frac{n^{\frac{2}{p}} 2^{n-1}}{n 2^{n-1}} = n^{\frac{2}{p}-1}. \end{aligned}$$

Since $e_i \in S(l_m^p)$ for $i \in \mathbb{N} \cap [1, m]$, it follows that

$$\overline{C}_{mNJ}^{(n)}(l_m^p) \geq C^{(n)}(e_1, e_2, \ldots, e_n) = n^{\frac{2}{p}-1}.$$

Hence $\overline{C}_{mNJ}^{(n)}(l_m^p) = n^{\frac{2}{p}-1}$.

(b) Let $2^{n-1} \leq m \leq \infty$. Then l_m^∞ is not uniformly non-l_n^1. By Theorem 5 (iv) in [17] and Proposition 1 (a), we conclude that $\overline{C}_{NJ}^{(n)}(l_m^\infty) = n$. To prove that $\overline{C}_{mNJ}^{(n)}(l_m^\infty) = n$ take the matrix A_n defined as above. The column j of A_n denote by $\mathbf{a}_j^{(n)}$. Then $\mathbf{a}_j^{(n)} = [1, a_{2j}^{(n)}, \ldots, a_{2^{n-1}j}^{(n)}]$, where $a_{ij}^{(n)} = \pm 1$ for any $1 \leq j \leq n$ and $2 \leq i \leq 2^{n-1}$. Define for $j = 1, 2, \ldots, n$,

$$z_j = \sum_{i=1}^{2^{n-1}} a_{ij}^{(n)} e_i,$$

where $(e_i)_{i=1}^m$ ($n \leq m \leq \infty$) is the canonical basis in l_m^∞. Obviously, $\left\| z_j \right\|_{l_m^\infty} = 1$. Let $(1, \theta_2, \ldots, \theta_n)$ be an arbitrary sequence such that $\theta_j = \pm 1$ for any $2 \leq j \leq n$.

Then there is exactly one row i_0 of A_n such that

$$[1, \theta_2, \ldots, \theta_n] = \left[a_{i_0 1}^{(n)}, a_{i_0 2}^{(n)}, \ldots, a_{i_0 n}^{(n)}\right].$$

Hence

$$1 + \sum_{j=2}^{n} \theta_j a_{i_0 j}^{(n)} = n.$$

Moreover

$$\left| 1 + \sum_{j=2}^{n} \theta_j a_{ij}^{(n)} \right| < n$$

for any $i \neq i_0$. Consequently,

$$\left\| z_1 + \sum_{j=2}^{n} \theta_j z_j \right\|_{l^\infty} = \max_{1 \leq i \leq 2^{n-1}} \left| 1 + \sum_{j=2}^{n} \theta_j a_{ij}^{(n)} \right| = n,$$

whence

$$C^{(n)}(z_1, z_2, \ldots, z_n) = \frac{\sum_{\theta_j = \pm 1} \left\| z_1 + \sum_{j=2}^{n} \theta_j z_j \right\|_{l^\infty}^2}{2^{n-1} \sum_{j=1}^{n} \left\| z_j \right\|_{l^\infty}^2} = \frac{n^2 2^{n-1}}{n 2^{n-1}} = n.$$

Therefore

$$n \leq \overline{C}_{mNJ}^{(n)}(l_m^\infty) \leq \overline{C}_{NJ}^{(n)}(l_m^\infty) = n,$$

which completes the proof of (b).

(c) Let $n \leq m \leq \infty$ and (e_i) be the canonical basis in l_m^∞. By Proposition 1 (a), we have

$$\begin{aligned} \frac{1}{n} &\leq \underline{C}_{NJ}^{(n)}(l_m^\infty) \leq \underline{C}_{mNJ}^{(n)}(l_m^\infty) \leq C^{(n)}(e_1, e_2, \ldots, e_n) \\ &= \frac{\sum_{\theta_j = \pm 1} \left\| e_1 + \sum_{j=2}^{n} \theta_j e_j \right\|_{l_m^\infty}^2}{2^{n-1} \sum_{j=1}^{n} \left\| e_j \right\|_{l_m^\infty}^2} = \frac{1}{n}. \end{aligned}$$

(d) Since $L^1(\mu)$ contains an isometric copy of l^1, applying Proposition 4 (a) for $p = 1$, we get

$$n = \overline{C}_{mNJ}^{(n)}(l^1) \leq \overline{C}_{mNJ}^{(n)}(L^1(\mu)) \leq \overline{C}_{NJ}^{(n)}(L^1(\mu)) \leq n.$$

Hence $\overline{C}_{mNJ}^{(n)}(L^1(\mu)) = \overline{C}_{NJ}^{(n)}(L^1(\mu)) = n$. Using the same arguments, by Proposition 4 (b) we conclude that $\overline{C}_{NJ}^{(n)}(L^\infty(\mu)) = \overline{C}_{mNJ}^{(n)}(L^\infty(\mu)) = n$. Similarly, by Proposition 4 (c), we obtain

$$\frac{1}{n} \le \underline{C}_{NJ}^{(n)}(L^\infty(\mu)) \le \underline{C}_{mNJ}^{(n)}(L^\infty(\mu)) \le \underline{C}_{mNJ}^{(n)}(l^\infty) = \frac{1}{n},$$

which completes the proof. □

3 Uniformly non-l_n^1 spaces

The next theorem gives some characterizations of the uniform non-l_n^1 property for Banach spaces. Kato, Takahashi and Hashimito proved in [17] that $(X, \|\cdot\|_X)$ is uniformly non-l_n^1 iff $\overline{C}_{NJ}^{(n)}(X) < n$. We will extend their results.

Theorem 1 *Let $(X, \|\cdot\|_X)$ be a Banach space. Then the following conditions are equivalent:*

(a) $\overline{C}_{mNJ}^{(n)}(X) < n$;
(b) $(X, \|\cdot\|_X)$ *is uniformly non-*l_n^1;
(c) *There exists* $\delta \in (0, 1)$ *such that for any element* $(x_1, x_2, \ldots, x_n) \in B\left(l_n^2(X)\right)$, *we have*

$$\min_{\theta_j = \pm 1} \left\| x_1 + \sum\nolimits_{j=2}^n \theta_j x_j \right\|_X \le \sqrt{n}(1 - \delta); \tag{7}$$

(d) *There exists* $\delta \in (0, 1)$ *such that for any element* $(x_1, x_2, \ldots, x_n) \in S\left(l_n^2(X)\right)$, *the inequality* (7) *is satisfied.*

Proof (a) ⇒ (b). Suppose that $\overline{C}_{mNJ}^{(n)}(X) < n$. Then

$$\frac{1}{2^{n-1}} \sum_{\theta_j = \pm 1} \left\| x_1 + \sum\nolimits_{j=2}^n \theta_j x_j \right\|_X^2 \le n\overline{C}_{mNJ}^{(n)}(X)$$

for any $x_1, x_2, \ldots, x_n \in S(X)$. Since on the left hand side we have an arithmetic mean, there is at least one sequence $(1, \overline{\theta}_2, \ldots, \overline{\theta}_n)$ such that

$$\left\| x_1 + \sum\nolimits_{j=2}^n \overline{\theta}_j x_j \right\|_X^2 \le n\overline{C}_{mNJ}^{(n)}(X).$$

Hence

$$\min_{\theta_j=\pm 1}\left\|x_1+\sum\nolimits_{j=2}^{n}\theta_j x_j\right\|_X \leq n\sqrt{\frac{\overline{C}_{mNJ}^{(n)}(X)}{n}} = n\,(1-\delta)\,,$$

where $\delta = \frac{\sqrt{n}-\sqrt{\overline{C}_{mNJ}^{(n)}(X)}}{\sqrt{n}}$. Consequently, $(X, \|\cdot\|_X)$ is uniformly non-l_n^1.

(b) $\Rightarrow$ (c). Assume that $(X, \|\cdot\|_X)$ is uniformly non-l_n^1. Let $(x_1, x_2, \ldots, x_n) \in B\left(l_n^2(X)\right)$. Since $\sum_{j=1}^{n}\left\|x_j\right\|_X^2 \leq 1$, it follows that $\min_{1\leq j\leq n}\left\|x_j\right\|_X \leq \frac{1}{\sqrt{n}}$. Moreover, by the Hölder inequality, we have

$$\sum_{j=1}^{n}\left\|x_j\right\|_X \leq \sqrt{n}\left(\sum_{j=1}^{n}\left\|x_j\right\|_X^2\right)^{1/2} \leq \sqrt{n}. \tag{8}$$

Case 1. Suppose that $\frac{1}{2\sqrt{n}} < \min_{1\leq j\leq n}\left\|x_j\right\|_X \leq \frac{1}{\sqrt{n}}$. By the characterization of uniform non-l_n^1 given in [14] and by the inequality (8), there is $\delta_1 > 0$ such that

$$\begin{aligned}\min_{\theta_j=\pm 1}\left\|x_1+\sum\nolimits_{j=2}^{n}\theta_j x_j\right\|_X &\leq \left(1-\frac{\delta_1 n \min_{1\leq i\leq n}\|x_i\|_X}{\sum_{j=1}^{n}\left\|x_j\right\|_X}\right)\sum_{j=1}^{n}\left\|x_j\right\|_X \\ &\leq \left(1-\frac{\delta_1\sqrt{n}}{2\sum_{j=1}^{n}\left\|x_j\right\|_X}\right)\sum_{j=1}^{n}\left\|x_j\right\|_X \\ &\leq \sqrt{n}\left(1-\frac{\delta_1}{2}\right).\end{aligned}$$

Case 2. Suppose that $0 \leq \min_{1\leq j\leq n}\left\|x_j\right\|_X \leq \frac{1}{2\sqrt{n}}$. Let x_k be the element on which the minimum is taken. Then, by the Hölder inequality, we have

$$\begin{aligned}\|x_1 \pm x_2 \pm \cdots \pm x_n\|_X &\leq \sum_{j=1, j\neq k}^{n}\left\|x_j\right\|_X + \|x_k\|_X \\ &\leq \sqrt{n-1}\sqrt{\sum_{j=1, j\neq k}^{n}\left\|x_j\right\|_X^2} + \|x_k\|_X \\ &\leq \sqrt{n-1}\sqrt{1-\|x_k\|_X^2} + \|x_k\|_X\end{aligned}$$

for any choice of signs. Define

$$f(t) = \sqrt{n-1}\sqrt{1-t^2} + t$$

for any $t \in \left[0, \frac{1}{2\sqrt{n}}\right]$. By elementary calculus, we conclude that f is an increasing function on the interval $\left[0, \frac{1}{2\sqrt{n}}\right]$. Hence, the function $f(t)$ takes its highest value on $\left[0, \frac{1}{2\sqrt{n}}\right]$ at the point $t = \frac{1}{2\sqrt{n}}$. Thus,

$$\begin{aligned}
\|x_1 \pm x_2 \pm \cdots \pm x_n\|_X &\le \sqrt{n-1}\sqrt{1-\left(\frac{1}{2\sqrt{n}}\right)^2} + \frac{1}{2\sqrt{n}} \\
&= \frac{1}{2\sqrt{n}}\left(\sqrt{(4n-1)(n-1)}+1\right) \\
&= \sqrt{n}\left(1 - \frac{(2n-1)-\sqrt{(4n-1)(n-1)}}{2n}\right)
\end{aligned}$$

for any choice of signs. Taking

$$\delta = \min\left\{\frac{\delta_1}{2}, \frac{(2n-1)-\sqrt{(4n-1)(n-1)}}{2n}\right\},$$

we get (c).

(c) $\Rightarrow$ (d). It is obvious.

(d) $\Rightarrow$ (a). Let $(x_1, x_2, \ldots, x_n) \in S\left(l_n^2(X)\right)$. By the assumption (d) there exists $\delta \in (0, 1)$ such that

$$\|x_1 \pm x_2 \pm \cdots \pm x_n\|_X \le \sqrt{n}(1-\delta)$$

for some choice of signs. Moreover, by (8), $\|x_1 \pm x_2 \pm \cdots \pm x_n\|_X \le \sqrt{n}$ for any choice of signs. Hence, we have

$$\begin{aligned}
\frac{\sum_{\theta_j=\pm1}\left\|x_1 + \sum_{j=2}^{n}\theta_j x_j\right\|_X^2}{2^{n-1}\sum_{j=1}^{n}\|x_j\|_X^2} &\le \frac{n(1-\delta)^2 + n\left(2^{n-1}-1\right)}{2^{n-1}} \\
&= n - \frac{\delta n(2-\delta)}{2^{n-1}}.
\end{aligned}$$

By the definition of the upper n-th von Neumann–Jordan constant $\overline{C}_{NJ}^{(n)}(X)$ and Proposition 3, we conclude

$$\overline{C}_{mNJ}^{(n)}(X) \le \overline{C}_{NJ}^{(n)}(X) \le n - \frac{\delta n(2-\delta)}{2^{n-1}} < n,$$

which finishes the proof. □

By Theorem 1 and the definition of B-convexity, we get immediately

Corollary 3 *A Banach space $(X, \|\cdot\|_X)$ is B-convex if and only if there is $n \ge 2$ $(n \in N)$ such that $\overline{C}_{mNJ}^{(n)}(X) < n$.*

Notice that $\overline{C}_{mNJ}^{(n)}(X)$ is not equal to $\overline{C}_{NJ}^{(n)}(X)$ in general (for $n = 2$ see [18]).

Corollary 4 *$\overline{C}_{mNJ}^{(n)}(X) = n$ if and only if $\overline{C}_{NJ}^{(n)}(X) = n$.*

Proof Since $(X, \|\cdot\|_X)$ is uniformly non-l_n^1 iff $\overline{C}_{NJ}^{(n)}(X) < n$, it follows, by Theorem 1, that $\overline{C}_{NJ}^{(n)}(X) < n$ iff $\overline{C}_{mNJ}^{(n)}(X) < n$. Hence, by Proposition 1 (a), we get the thesis. $\square$

Remark 1 Let us notice that the above corollary can be reformulated equivalently as follows

$$\overline{C}_{mNJ}^{(n)}(X) < n \quad \text{if and only if} \quad \overline{C}_{NJ}^{(n)}(X) < n.$$

4 Upper and lower *n*-th von Neumann–Jordan constant for L^p-spaces

Now we will calculate the upper n-th von Neumann–Jordan constant for Lebesgue spaces $L^p(\mu)$ and l_m^p $(1 < p < \infty, 1 \le m \le \infty)$. To prove the next lemma, we will apply the following results given by Figiel, Iwaniec and Pełczyński in [6]. Namely, for arbitrary scalars $c_1, c_2, \dots, c_n$ and $2 < p < \infty$ we have

$$\int_0^1 \left|\sum_{j=1}^n c_j r_j(t)\right|^p dt \le n^{-1} \int_0^1 \left|\sum_{j=1}^n r_j(t)\right|^p dt \sum_{j=1}^n \left|c_j\right|^p, \tag{9}$$

where $r_1, r_2, \dots, r_n$ $(n = 1, 2, \dots)$ are Rademacher functions, that is $r_n(t) = \operatorname{sign}(\sin 2^n \pi t)$.

Let $\lfloor\cdot\rfloor : \mathbb{R} \to \mathbb{Z}$ be the floor function, i.e. $\lfloor x\rfloor = \max\{k \in \mathbb{Z} : k \le x\}$ for any $x \in \mathbb{R}$.

Lemma 1 *Let $2 < p < \infty$ and $X = L^p(\mu)$ or $X = l^p$. Then*

$$\sum_{\theta_j=\pm 1} \left\|x_1 + \sum_{j=2}^n \theta_j x_j\right\|_X^p \le n^{-1} \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{k} (n-2k)^p \sum_{j=1}^n \|x_j\|_X^p$$

for any $x_1, x_2, \dots, x_n \in X$ and any integer $n \ge 1$.

Proof Fix an integer $n \ge 1$. Notice that

$$\int_0^1 \left|\sum_{j=1}^n c_j r_j(t)\right|^p dt = 2^{1-n} \sum_{\theta_j=\pm 1} \left|c_1 + \sum_{j=2}^n \theta_j c_j\right|^p$$

for any scalars $c_1, c_2, \dots, c_n$. On the other hand, it can be proved elementarily that

$$\int_0^1 \left|\sum_{j=1}^n r_j(t)\right|^p dt = 2^{1-n} \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{k} (n-2k)^p.$$

Suppose that $x_k = \left(t_i^{(k)}\right)_{i=1}^{\infty} \in l^p$ for $k = 1, 2, \ldots, n$. By the inequality (9), we get

$$\sum_{\theta_j=\pm 1} \left| t_i^{(1)} + \sum_{j=2}^{n} \theta_j t_i^{(j)} \right|^p \leq n^{-1} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} (n-2k)^p \sum_{j=1}^{n} \left| t_i^{(j)} \right|^p$$

for any $i \in \mathbb{N}$. Summing by sides from $i = 1$ to ∞ and reversing the order of summation, we obtain the thesis. Similarly, for $X = L^p(\mu)$ take $x_1, x_2, \ldots, x_n \in L^p(\mu)$. Then, by the inequality (9), we get

$$\sum_{\theta_j=\pm 1} \left| x_1(t) + \sum_{j=2}^{n} \theta_j x_j(t) \right|^p \leq n^{-1} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} (n-2k)^p \sum_{j=1}^{n} \left| x_j(t) \right|^p$$

for almost every $t \in \Omega$. Integrating by sides this inequality and reversing the order of summation and integration, we obtain the desired inequality. □

Theorem 2 *Let* $1 \leq p < \infty$ *and* $X = L^p(\mu)$ *or* $X = l_m^p$ $(1 \leq m \leq \infty)$. *Then*

$$\overline{C}_{mNJ}^{(n)}(X) = \begin{cases} n^{\frac{2}{p}-1} & \text{if } 1 \leq p \leq 2 \text{ and } m \geq n, \\ n^{-1} \left(2^{1-n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} (n-2k)^p \right)^{\frac{2}{p}} & \text{if } 2 < p < \infty \text{ and } m \geq 2^{n-1}. \end{cases}$$

Proof Case 1. Let $1 \leq p \leq 2$. By Theorem 3 from [17] and Proposition 1 (a), for all $n \geq 2$, we have

$$\overline{C}_{mNJ}^{(n)}(X) \leq \overline{C}_{NJ}^{(n)}(X) = n^{\frac{2}{p}-1}.$$

whenever $X = L^p(\mu)$ or $X = l_m^p$, $m \in \mathbb{N} \cup \{\infty\}$. The opposite inequality follows immediately from Proposition 4 (a) whenever $X = l_m^p$ with $m \geq n$.

Now consider $X = L^p(\mu)$. Let $A \subset \Omega$ be a set of positive finite measure. Divide the set A into n pairwise disjoint subsets $A_1, A_2, \ldots, A_n$ such that $\bigcup_{i=1}^{n} A_i = A$ and $\mu(A_i) = \frac{1}{n}\mu(A)$. Define $z_i = \mu(A_i)^{-1/p}\chi_{A_i}$ for $i = 1, 2, \ldots, n$. Then

$$\|z_i\|_{L^p} = \left(\int_{A_i} \left(\mu(A_i)^{-1/p} \right)^p d\mu \right)^{\frac{1}{p}} = 1$$

for any $i \in \{1, 2, \ldots, n\}$ and

$$\begin{aligned}\overline{C}_{mNJ}^{(n)}(L^p(\mu)) &\geq C^{(n)}(z_1, z_2, \ldots, z_n)\\ &= \frac{1}{n2^{n-1}} \sum_{\theta_j=\pm1} \left\| z_1 + \sum_{j=2}^{n} \theta_j z_j \right\|_{L^p}^2\\ &= \frac{1}{n2^{n-1}} \sum_{\theta_j=\pm1} \left\| \mu(A_1)^{-1/p} \chi_{A_1} + \sum_{j=2}^{n} \theta_j \mu(A_j)^{-1/p} \chi_{A_j} \right\|_{L^p}^2\\ &= \frac{1}{n2^{n-1}} 2^{n-1} \left\| \left(\frac{1}{n}\mu(A)\right)^{-1/p} \chi_A \right\|_{L^p}^2\\ &= \frac{1}{n} \left[\mu(A) \left(\frac{1}{n}\mu(A)\right)^{-1} \right]^{\frac{2}{p}} = n^{\frac{2}{p}-1}.\end{aligned}$$

Hence $\overline{C}_{mNJ}^{(n)}(L^p(\mu)) = \overline{C}_{NJ}^{(n)}(L^p(\mu)) = n^{\frac{2}{p}-1}$, whenever $1 \leq p \leq 2$.

Case 2. Let $2 < p < \infty$ and $X = L^p(\mu)$ or $X = l_m^p$ $(2^{n-1} \leq m \leq \infty)$. By the Hölder-Rogers inequality for $p > 2$ and by Lemma 1, we have

$$\begin{aligned}\sum_{\theta_j=\pm1} \left\| x_1 + \sum_{j=2}^{n} \theta_j x_j \right\|_X^2 &\leq 2^{2(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)} \left(\sum_{\theta_j=\pm1} \left\| x_1 + \sum_{j=2}^{n} \theta_j x_j \right\|_X^p \right)^{\frac{2}{p}}\\ &\leq 2^{(n-1)\left(\frac{p-2}{p}\right)} \left(n^{-1} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} (n-2k)^p \sum_{j=1}^{n} \|x_j\|_X^p \right)^{\frac{2}{p}} \qquad (10)\end{aligned}$$

for any $x_1, x_2, \ldots, x_n \in X$. Assume that $x_1, x_2, \ldots, x_n \in S(X)$. Then, by inequality (10), we have

$$\begin{aligned}C^{(n)}(x_1, x_2, \ldots, x_n) =& \frac{\sum_{\theta_j=\pm1} \left\| x_1 + \sum_{j=2}^{n} \theta_j x_j \right\|_X^2}{n2^{n-1}}\\ \leq& \frac{1}{n2^{n-1}} 2^{(n-1)\left(\frac{p-2}{p}\right)} \left(n^{-1} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} (n-2k)^p \sum_{j=1}^{n} \|x_j\|_X^p \right)^{\frac{2}{p}}\\ =& \frac{1}{n} 2^{(n-1)\left(\frac{p-2}{p}-1\right)} \left(\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} (n-2k)^p \right)^{\frac{2}{p}}\\ =& n^{-1} \left(2^{1-n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} (n-2k)^p \right)^{\frac{2}{p}},\end{aligned}$$

whence

$$\overline{C}_{mNJ}^{(n)}(X) \leq n^{-1} \left(2^{1-n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} (n-2k)^p \right)^{\frac{2}{p}}. \tag{11}$$

Let the matrix A_n be defined as in the proof of Proposition 3 (b). Denote by y_i column i of the matrix A_n $(i = 1, 2, \ldots, n)$. For any $i \in \{1, 2, \ldots, n\}$ define

$$z_i = \frac{1}{\left(2^{n-1}\right)^{1/p}} y_i^T,$$

where y_i^T denotes the transpose of the column y_i. Then

$$\|z_i\|_{l^p_{2^{n-1}}} = \left(\sum_{m=1}^{2^{n-1}} \left(\frac{1}{\left(2^{n-1}\right)^{1/p}} \right)^p \right)^{1/p} = 1$$

for any $i \in \{1, 2, \ldots, n\}$. Hence $z_1, z_2, \ldots, z_n \in S\left(l^p_{2^{n-1}}\right)$. For any element $x = (t_1, t_2, \ldots, t_{2^{n-1}}) \in l^p_{2^{n-1}}$ denote by x^* its non-increasing rearrangement, i.e. a non-increasing sequence obtained from $\{|t_i|\}_{i=1}^{2^{n-1}}$ by a suitable permutation of the integers. Notice that for all sequences $(1, \theta_2, \ldots, \theta_n)$ such that $\theta_j = \pm 1$, $(j = 2, 3, \ldots, n)$ the non-increasing rearrangements $\left(z_1 + \sum_{j=2}^n \theta_j z_j\right)^*$ coincide. Denoting by $\left(v_1, v_2, \ldots, v_{2^{n-1}}\right)$ the non-increasing sequence such that

$$\left(z_1 + \sum\nolimits_{j=2}^n \theta_j z_j\right)^* = \left(v_1, v_2, \ldots, v_{2^{n-1}}\right)$$

for any sequences $(1, \theta_2, \ldots, \theta_n)$. Hence

$$\begin{aligned} C^{(n)}(x_1, x_2, \ldots, x_n) &= \frac{\sum_{\theta_j = \pm 1} \left\| z_1 + \sum_{j=2}^n \theta_j z_j \right\|^2_{l^p_{2^{n-1}}}}{n 2^{n-1}} \\ &= \frac{1}{n} \left\| \left(v_1, v_2, \ldots, v_{2^{n-1}}\right) \right\|^2_{l^p_{2^{n-1}}}. \end{aligned}$$

Notice that $v_1 = \frac{n}{\left(2^{n-1}\right)^{1/p}}$, $v_l = \frac{n-2k}{\left(2^{n-1}\right)^{1/p}}$ for $\binom{n}{k}$ subsequent integers l, $(k = 1, 2, \ldots, \lfloor n/2 \rfloor)$. Consequently,

$$
\begin{aligned}
\overline{C}^{(n)}_{mNJ}\left(l^p_{2^{n-1}}\right) &\geq C^{(n)}(x_1, x_2, \ldots, x_n) \\
&= \frac{1}{n}\left(\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\left(\frac{n-2k}{\left(2^{n-1}\right)^{1/p}}\right)^p\right)^{2/p} \\
&= n^{-1}\left(2^{1-n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}(n-2k)^p\right)^{\frac{2}{p}}.
\end{aligned}
$$

Since $l^p_{2^{n-1}}$ can be embedded isometrically in any l^p_m with $m \geq 2^{n-1}$, by inequality (11) applied for $X = l^p_m$, it follows that

$$
\overline{C}^{(n)}_{mNJ}\left(l^p_{2^{n-1}}\right) = \overline{C}^{(n)}_{mNJ}\left(l^p_m\right) = n^{-1}\left(2^{1-n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}(n-2k)^p\right)^{\frac{2}{p}}
$$

whenever $2 < p \leq \infty$ and $m \geq 2^{n-1}$.

Since $L^p(\mu)$ contains an isometric copy of $l^p_{2^{n-1}}$, by inequality (11), we obtain the thesis for $X = L^p(\mu)$, which completes the proof. □

Haagerup [11] proved that the best type $(2, p)$ constant in the Khinthine inequality for $2 \leq p < \infty$ is $B_p = \sqrt{2}\left(\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right)\right)^{\frac{1}{p}}$. Kato, Takahashi and Hashimoto proved in [17] that $\overline{C}^{(n)}_{NJ}(X) \leq \min\left\{n^{\frac{2}{q}-1}, B_p^2\right\}$. Combining this result with Theorem 2, we get two hand side estimation of upper von Neumann–Jordan constant for Lebesgue spaces with $p \in (2, \infty)$.

Corollary 5 *Let $2 < p < \infty$, q be conjugate to p and $X = L^p(\mu)$ or $X = l^p_m$. If $m \geq 2^{n-1}$, then*

$$
n^{-1}\left(2^{1-n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}(n-2k)^p\right)^{\frac{2}{p}} \leq \overline{C}^{(n)}_{NJ}(X) \leq \min\left\{n^{\frac{2}{q}-1}, B_p^2\right\}.
$$

Proof The left hand side inequality follows immediately from Theorem 2. Namely,

$$
n^{-1}\left(2^{1-n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}(n-2k)^p\right)^{\frac{2}{p}} = \overline{C}^{(n)}_{mNJ}(X) \leq \overline{C}^{(n)}_{NJ}(X),
$$

whenever $2 < p < \infty$, $X = L^p(\mu)$ or $X = l^p_m$ and $m \geq 2^{n-1}$. The right hand side inequality was proved in [17]. □

Theorem 3 *Let $2 < p < \infty$. If $X = L^p(\mu)$ or $X = l^p$, then*

$$\lim_{n\to\infty} \overline{C}^{(n)}_{mNJ}(X) = \lim_{n\to\infty} \overline{C}^{(n)}_{NJ}(X) = B_p^2.$$

Proof Assume that $2 < p < \infty$. By Theorem 2, for every $n \in \mathbb{N}$, we get

$$\begin{aligned}
\overline{C}^{(2n)}_{mNJ}(X) &= (2n)^{-1}\left(2^{1-2n}\sum_{k=0}^{n}\binom{2n}{k}(2n-2k)^p\right)^{2/p}\\
&= \frac{1}{2n}\left(\frac{2^{p+1}}{2^{2n}}\sum_{k=0}^{n}\binom{2n}{k}(n-k)^p\right)^{2/p}\\
&= \frac{2^{1+2/p}}{n}\left(\frac{1}{2^{2n}}\sum_{j=0}^{n}\binom{2n}{n-j}j^p\right)^{2/p},
\end{aligned}$$

where $j = n - k$. On the other hand, by elementary asymptotic method (see [4]), we have

$$\sum_{|j|<x\sqrt{n/2}} \frac{1}{2^{2n}}\binom{2n}{n-j} = (1+o(1)) \sum_{|j|<x\sqrt{n/2}} \frac{1}{\sqrt{\pi n}} e^{-j^2/n}$$

for any $x > 0$. Hence, letting $t_j = j\sqrt{2/n}$, we obtain

$$\sum_{|j|<x\sqrt{n/2}} \frac{1}{2^{2n}}\binom{2n}{n-j} = (1+o(1)) \sum_{|t_j|<x} \frac{1}{\sqrt{2\pi}} e^{-t_j^2/2}\Delta t_j$$

for any $x > 0$. Passing to the limit gives the classic de Moivre–Laplace theorem (see [4]). In our case, we get

$$\sum_{j=0}^{n} \frac{1}{2^{2n}}\binom{2n}{n-j} j^p = (1+o(1))\left(\frac{n}{2}\right)^{p/2} \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-t^2/2}t^p dt. \tag{12}$$

Moreover, by the definition of the gamma function, it follows that

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-t^2/2}t^p dt &= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-u}(2u)^{(p-1)/2}du\\
&= \frac{2^{p/2}}{2\sqrt{\pi}}\int_0^\infty e^{-u}u^{(p-1)/2}du\\
&= \frac{2^{p/2}}{2\sqrt{\pi}}\int_0^\infty e^{-u}u^{\left(\frac{p+1}{2}\right)-1}du\\
&= \frac{2^{p/2}}{2\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right).
\end{aligned}$$

Thus, by the equality (12), we obtain

$$
\begin{aligned}
\overline{C}_{mNJ}^{(2n)}(X) =& \frac{2^{1+2/p}}{n}\left(\frac{1}{2^{2n}}\sum_{j=0}^{n}\binom{2n}{n-j}j^p\right)^{2/p}\\
=& \frac{2^{1+2/p}}{n}\left((1+o(1))\left(\frac{n}{2}\right)^{p/2}\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-t^2/2}t^p dt\right)^{2/p}\\
=& \frac{n2^{2/p}}{n}(1+o(1))^{2/p}\left(\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-t^2/2}t^p dt\right)^{2/p}\\
=& 2^{2/p}(1+o(1))^{2/p}\left(\frac{2^{p/2}}{2\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right)\right)^{2/p}\\
=& 2(1+o(1))^{2/p}\left(\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right)\right)^{2/p}\\
=& (1+o(1))^{2/p}\left[\sqrt{2}\left(\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right)\right)^{1/p}\right]^2\\
=& (1+o(1))^{2/p}B_p^2
\end{aligned}
$$

for any $n \in \mathbb{N}$. Hence

$$\lim_{n\to\infty}\overline{C}_{mNJ}^{(2n)}(X) = B_p^2.$$

By Proposition 1 (a), (c) and Corollary 5, we have

$$\overline{C}_{mNJ}^{(2n)}(X) \le \frac{2n+1}{2n}\overline{C}_{mNJ}^{(2n+1)}(X) \le \overline{C}_{NJ}^{(2n+1)}(X) \le B_p^2.$$

By the squeeze theorem, it follows that

$$\lim_{n\to\infty}\overline{C}_{mNJ}^{(2n)}(X) = \lim_{n\to\infty}\overline{C}_{mNJ}^{(2n+1)}(X) = \lim_{n\to\infty}\overline{C}_{NJ}^{(2n+1)}(X) = B_p^2,$$

whence $\lim_{n\to\infty}\overline{C}_{mNJ}^{(n)}(X) = B_p^2$. Furthermore, $\lim_{n\to\infty}\overline{C}_{NJ}^{(n)}(X) = B_p^2$ because the sequence $\left(\overline{C}_{mNJ}^{(n)}(X)\right)$ is increasing. $\square$

The proof of Theorem 3 presented here is a modification of idea given by Cecil Rousseau from The University of Memphis in private communication.

Corollary 6 *Let $2 \le p \le \infty$ and $X = L^p(\mu)$ or $X = l_m^p$. If $m \ge n$, then $\underline{C}_{NJ}^{(n)}(X) = n^{\frac{2}{p}-1}$.*

Proof Let $2 \leq p \leq \infty$ and q denote the conjugate number of p. By Corollary 2 and Theorem 2, we have

$$\underline{C}_{NJ}^{(n)}(L^p(\mu)) \geq \frac{1}{\overline{C}_{NJ}^{(n)}((L^p(\mu))^*)} = \frac{1}{\overline{C}_{NJ}^{(n)}(L^q(\mu))} = n^{1-\frac{2}{q}} = n^{\frac{2}{p}-1}.$$

On the other hand, taking the canonical basis $\{e_i\}_{i=1}^n$ in l_n^p we have

$$\begin{aligned} C^{(n)}(e_1, e_2, \ldots, e_n) &= \frac{\sum_{\theta_j=\pm 1} \left\| e_1 + \sum_{j=2}^n \theta_j e_j \right\|_{l_n^p}^2}{2^{n-1} \sum_{j=1}^n \left\| e_j \right\|_{l_n^p}^2} \\ &= \frac{2^{n-1} n^{\frac{2}{p}}}{2^{n-1} n} = n^{\frac{2}{p}-1}. \end{aligned}$$

Hence, by the definition of lower n-th von Neumann–Jordan constant, we get

$$\underline{C}_{NJ}^{(n)}(L^p(\mu)) \leq \underline{C}_{NJ}^{(n)}(l_m^p) \leq \underline{C}_{NJ}^{(n)}(l_n^p) \leq n^{\frac{2}{p}-1}$$

whenever $m \geq n$. Combining the both inequalities, we get the thesis. $\square$

Remark 2 It is known that $C_{NJ}(X) = C_{NJ}(X^*)$ (see [16]) and in general $\overline{C}_{NJ}^{(n)}(X) \neq \overline{C}_{NJ}^{(n)}(X^*)$ for $n \geq 3$ (see [17]). Theorem 2 shows that $\overline{C}_{mNJ}^{(2)}(X) = \overline{C}_{mNJ}^{(2)}(X^*)$ whenever $X = L^p(\mu)$ or $X = l_m^p$ with $m \geq 2^{n-1}$. Really, fix $1 < p < 2$ and consider $X = L^p(\mu)$ or $X = l_m^p$ with $m \geq 2^{n-1}$. Let q be conjugate to p. Then $q > 2$ and $X^* = L^q(\mu)$ or $X^* = l_m^q$ with $m \geq 2^{n-1}$, respectively. Applying Theorem 2 for $n = 2$, we have

$$\overline{C}_{mNJ}^{(2)}(X) = 2^{\frac{2}{p}-1}.$$

Since $q = \frac{p}{p-1} > 2$, it follows from Theorem 2 that

$$\begin{aligned} \overline{C}_{mNJ}^{(2)}(X^*) &= \frac{1}{2} \left(\frac{1}{2} \sum_{k=0}^{1} \binom{2}{k} (2-2k)^{\frac{p}{p-1}} \right)^{\frac{2(p-1)}{p}} \\ &= \frac{1}{2} \left(2^{\frac{p}{p-1}-1} \right)^{\frac{2(p-1)}{p}} = 2^{\frac{2}{p}-1}, \end{aligned}$$

whence $\overline{C}_{mNJ}^{(2)}(X) = \overline{C}_{mNJ}^{(2)}(X^*)$.

The equality $\overline{C}_{mNJ}^{(n)}(X) = \overline{C}_{mNJ}^{(n)}(X^*)$ does not hold in general for $n \geq 3$. By Remark 9 (ii) in [17] and Theorem 2, we have

$$\overline{C}_{mNJ}^{(n)}(X^*) \le \overline{C}_{NJ}^{(n)}(X^*) < \overline{C}_{NJ}^{(n)}(X) = n^{\frac{2}{p}-1} = \overline{C}_{mNJ}^{(n)}(X),$$

whence $\overline{C}_{mNJ}^{(n)}(X) \neq \overline{C}_{mNJ}^{(n)}(X^*)$ for $n \ge 3$.

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