Aequat. Math. 93 (2019), 311–343

0001-9054/19/010311-33
published online November 9, 2018
https://doi.org/10.1007/s00010-018-0615-y

Aequationes Mathematicae

CrossMark

Geometric properties of F-normed Orlicz spaces

Yunan Cui, Henryk Hudzik, Radosław Kaczmarek, and Paweł Kolwicz

Dedicated to Professor Karol Baron on the occasion of his 70th Birthday.

Abstract. The paper deals with F-normed functions and sequence spaces. First, some general results on such spaces are presented. But most of the results in this paper concern various monotonicity properties and various Kadec–Klee properties of F-normed Orlicz functions and sequence spaces and their subspaces of elements with order continuous norm, when they are generated by monotone Orlicz functions on $\mathbb{R}_+$ and equipped with the classical Mazur–Orlicz F-norm. Strict monotonicity, lower (and upper) local uniform monotonicity and uniform monotonicity in the classical sense as well as their orthogonal counterparts are considered. It follows from the criteria that are presented for these properties that all the above classical monotonicity properties except for uniform monotonicity differ from their orthogonal counterparts [in contrast to Köthe spaces (see Hudzik et al. in Rocky Mt J Math 30(3):933–950, 2000)]. The Kadec–Klee properties that are considered in this paper correspond to various kinds of convergence: convergence locally in measure and convergence globally in measure for function spaces, uniform convergence and coordinatewise convergence in the case of sequence spaces.

Mathematics Subject Classification. 46E30, 46B20, 46B42.

Keywords. Orlicz spaces, Mazur–Orlicz F-norm, Köthe normed spaces, F-normed Köthe spaces, Symmetric spaces, Symmetric F-normed spaces, Order continuity, Fatou properties, Strict monotonicity, Orthogonal strict monotonicity, Lower local uniform monotonicity, Orthogonal lower local uniform monotonicity, Upper local uniform monotonicity, Orthogonal upper local uniform monotonicity, Uniform monotonicity, Orthogonal uniform monotonicity, Condition Δ_2, Strong condition Δ_2, Kadec–Klee properties H_l, H_g, H_u and H_c.

Yunan Cui gratefully acknowledges the support of NFS of CHINA (11871181).
Paweł Kolwicz was supported by the Ministry of Science and Higher Education of Poland, Grant Number 04/43/DSPB/0094.

Birkhäuser

1. Introduction

The geometry of Banach spaces has found a lot of applications (approximation theory, fixed point theory, ergodic theory, control theory, probability theory, theory of vector analytic functions, theory of generalized inverses) and has been intensively developed during the last decades. Monotonicity properties (strict and uniform monotonicity) play a similar role in the theory of Banach lattices as the respective rotundity properties (rotundity and uniform rotundity) in the theory of Banach spaces. Recall that monotonicity properties are restrictions of appropriate rotundity properties to the set of couples of comparable elements in the positive cone of a Banach lattice (see [17]). Consequently, if we restrict ourselves to Banach spaces being Banach lattices, then in many cases, good rotundity properties can be replaced successfully by respective monotonicity properties. It is known that monotonicity properties (strict and uniform monotonicity) play an analogous role in the best dominated approximation problems in Banach lattices as the respective rotundity properties (strict and uniform rotundity) do in the best approximation problems in Banach spaces (see [7,33]). Moreover, monotonicity properties are applicable in ergodic theory, since they provide a tool for estimating a norm (see [1]).

Uniform monotonicity was introduced and studied by Birkhoff in [3]. Next, the various monotonicity properties and their applications to dominated best approximation and complex rotundities have been intensively investigated by many mathematicians (see [6,7,9,16,20,22,29,30,33,34]). It is worth mentioning that also Kadec–Klee properties are applicable in the best dominated approximation problems in Banach lattices (see [8] and the references therein).

Our paper offers an extension of results on monotonicity and Kadec–Klee properties of Banach function lattices to F-normed function lattices, specifically to F-normed Orlicz spaces (compare with [5,8–12,19,21,22,24,26–28,33,34]).

2. Preliminaries

Denote by $\mathbb{N}, \mathbb{R}$ the sets of natural and real numbers, respectively. Let $\mathbb{R}_+ := [0, \infty)$. Given any real vector space X the functional $X \ni x \mapsto \|x\| \in \mathbb{R}_+$, is called an *$F$-norm* if the following conditions are satisfied:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|-x\| = \|x\|$ for all $x \in X$;
(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$;
(iv) $\|\lambda_n x_n - \lambda x\| \to 0$ whenever $\|x_n - x\| \to 0$ and $\lambda_n \to \lambda$ for any $x \in X$, $(x_n)_{n=1}^{\infty}$ in X, $\lambda \in \mathbb{R}$ and $(\lambda_n)_{n=1}^{\infty}$ in $\mathbb{R}$.

We say that an F-normed space $X = (X, \|\cdot\|)$ is an *F-space* if it is complete with respect to the F-norm topology.

As usual, $S(X)$ (resp. $B(X)$) stands for the unit sphere (resp. the closed unit ball) of a real F-normed space $(X, \|\cdot\|_X)$.

2.1. F-normed Köthe spaces

Let (Ω, Σ, μ) be a complete σ-finite measure space and $L^0 = L^0(\mu)$ be the space of all (equivalence) classes of Σ-measurable real-valued functions defined on Ω.

Definition 2.1. An F-space $(E, \|\cdot\|_E)$ is called an F-normed Köthe space if it is a linear subspace of L^0 satisfying the following conditions.

(i) If $x \in L^0$, $y \in E$ and $|x| \le |y|$ μ-a.e., then $x \in E$ and $\|x\|_E \le \|y\|_E$.
(ii) There exists a strictly positive $x \in E$ (called a weak unit).

Clearly, each F-normed Köthe space is an F-lattice. If, in addition, the measure μ is non-atomic, then we say that E is an F-normed (Köthe) function space. If we consider E over the counting measure space $\left(\mathbb{N}, 2^{\mathbb{N}}, \mu\right)$, then we say that E is an F-normed (Köthe) sequence space.

By E_+ we denote the positive cone of E, that is, $E_+ = \{x \in E : x \ge 0\}$. For a measurable function x set

$$\operatorname{supp} x = \{t \in \Omega : x(t) \neq 0\}.$$

Note for $x \in L^0$ (the equivalence class of measurable functions which are equal $\mu-$a.e.) the set $\operatorname{supp} x$ is determined up to a set of measure zero.

Recall that an F-normed function space E is called *order continuous* ($E \in (OC)$) if for each $x \in E$ and every sequence $0 \le x_n \le |x|$ with $x_n \to 0$ $\mu-$a.e. we have $\|x_n\|_E \to 0$ (see [25,35,46]). The symbol E_a stands for the subspace of all order continuous elements of E.

Recall that E has the Fatou property ($E \in (FP)$) if for any $x \in L^0$ and any $(x_n)_{n=1}^\infty$ in E such that $0 \le x_n \nearrow x$ $\mu-$a.e. in Ω and $\sup_{n \in N} \|x_n\|_E < \infty$, we have $x \in E$ and $\lim_n \|x_n\|_E = \|x\|_E$. Recall that in the definition of a semi-Fatou property ($E \in (s-FP)$) we assume that $x \in E$ instead of $x \in L^0$.

A point $x \in E$ is said to be an H_g-point (resp. H_l-point) in E if for any $(x_n) \subset E$ such that $x_n \to x$ globally (resp. locally) in measure and $\|x_n\|_E \to \|x\|_E$, we have $\|x_n - x\|_E \to 0$. We say that the space E has the Kadec–Klee property globally (resp. locally) in measure if each $x \in E$ is an H_g-point (resp. H_l-point) in E. Note that if we consider the above two Kadec–Klee properties in E being an F-normed Köthe sequence space, then we get the Kadec–Klee properties with respect to uniform (coordinatewise) convergence, respectively. We denote these properties by H_u (H_c), respectively.

Definition 2.2. An F-normed Köthe space $(E, \|.\|_E)$ is said to be strictly monotone ($E \in (SM)$ for short) if for any $x, y \in E$ such that $0 \le y \le x$, we have $\|y\|_E < \|x\|_E$ whenever $y \neq x$ (or equivalently $\|x - y\|_E < \|x\|_E$ whenever $y \neq 0$) (see [3]).

Definition 2.3. An F-normed Köthe space E is said to be orthogonally strictly monotone ($E \in (OSM)$ for short) if for any $x \in E_+ \backslash \{0\}$ and any $A \in \Sigma \cap \operatorname{supp} x$ with $\mu(A) > 0$, we have $\|x\chi_{\Omega\backslash A}\|_E < \|x\|_E$.

Definition 2.4. An F-normed Köthe space $(E, \|.\|_E)$ is said to be lower locally uniformly monotone ($E \in (LLUM)$ for short) if for any $x \in E$ and $(x_n)_{n=1}^{\infty}$ in E such that $0 \leq x_n \leq x$ for all $n \in \mathbb{N}$ and $\|x_n\|_E \to \|x\|_E$ as $n \to \infty$, the condition $\|x - x_n\|_E \to 0$ as $n \to \infty$ holds.

Definition 2.5. An F-normed Köthe space $(E, \|.\|_E)$ is said to be orthogonally lower locally uniformly monotone ($E \in (OLLUM)$ for short) if for any $x \in E_+ \backslash \{0\}$ and any $(A_n)_{n=1}^{\infty}$ in Σ the following implication is satisfied:

$$\left(\|x\chi_{A_n}\|_E \underset{n\to\infty}{\to} \|x\|_E\right) \Rightarrow \left(\|x - x\chi_{A_n}\|_E = \|x\chi_{\Omega\backslash A_n}\|_E \underset{n\to\infty}{\to} 0\right).$$

Definition 2.6. An F-normed Köthe space $(E, \|.\|_E)$ is said to be upper locally uniformly monotone ($E \in (ULUM)$ for short) if for any $x \in E_+$ and $(x_n)_{n=1}^{\infty}$ in E_+ such that $x \leq x_n$ for all $n \in \mathbb{N}$ and $\|x_n\|_E \to \|x\|_E$ as $n \to \infty$, the condition $\|x_n - x\|_E \to 0$ as $n \to \infty$ holds.

Definition 2.7. An F-normed Köthe space $(E, \|.\|_E)$ is said to be orthogonally upper locally uniformly monotone ($E \in (OULUM)$ for short) if for any $x \in E_+ \backslash \{0\}$ such that $\mu(\operatorname{supp} E \backslash \operatorname{supp} x) > 0$ and any $x_n \in E_+$ with $\operatorname{supp} x_n \subset \operatorname{supp} E \backslash \operatorname{supp} x$, if $\|x + x_n\|_E \to \|x\|_E$ then $\|x_n\|_E \to 0$ as $n \to \infty$.

Definition 2.8. An F-normed Köthe space $(E, \|.\|_E)$ is said to be uniformly monotone ($E \in (UM)$ for short) if for any $c > 0$, any $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty} \subset E$ such that $0 \leq y_n \leq x_n$, $\|x_n\|_E \leq c$ for any $n \in \mathbb{N}$ and $\|y_n\|_E \to c$, there holds $\|x_n - y_n\|_E \to 0$ as $n \to \infty$.

Definition 2.9. An F-normed Köthe space $(E, \|.\|_E)$ is said to be orthogonally uniformly monotone ($E \in (OUM)$ for short) if for any $c > 0$, any $(x_n)_{n=1}^{\infty} \subset E_+$ and any $(A_n)_{n=1}^{\infty} \subset \Sigma$ such that $\|x_n\|_E \leq c$ for any $n \in \mathbb{N}$ and $\|x_n\chi_{A_n}\|_E \to c$, the condition $\|x_n - x_n\chi_{A_n}\|_E \to 0$ as $n \to \infty$ holds.

Note that if we consider a normed Köthe space $(E, \|.\|_E)$, then Definition 2.8 gives the well known definition of uniform monotonicity (see [3]). Recall also that if E is a normed Köthe space, then $E \in (SM)$ if and only if $E \in (OSM)$. Similarly, $E \in (UM)$ if and only if $E \in (OUM)$ (see [17]). Such equivalences needn't be true if we consider F-normed Köthe spaces. Namely, in F-normed Orlicz spaces properties SM and OSM differ (see Theorems 2.19 and 4.5). The problem if uniform monotonicity differs from orthogonal uniform monotonicity in F-normed Köthe spaces is still unsolved because we have not yet managed to receive complete necessary and sufficient conditions for these properties in F-normed Orlicz spaces. However, from the results of this paper it also follows that, in the case of F-normed Orlicz spaces, properties $LLUM$ and $ULUM$

differ from their orthogonal counterparts (see Theorem 4.3 and Corollaries 4.6 and 4.8).

Let $L^0 = L^0(I, \Sigma, \mu)$ be the space of all (equivalence classes of) Lebesgue measurable real-valued functions defined on the interval I, where $I = (0, 1)$ or $I = (0, \infty)$. Given any $x \in L^0$ we define its distribution function $d_x : [0, +\infty) \to [0, \mu(I)]$ by

$$d_x(\lambda) = \mu(\{t \in I : |x(t)| > \lambda\})$$

(see [2,32,35]) and the non-increasing rearrangement $x^* : I \to [0, \infty]$ of x as

$$x^*(t) = \inf\{\lambda \geq 0 : d_x(\lambda) \leq t\}$$

(under the convention $\inf \emptyset = \infty$). We agree with the notation $x^*(\infty) := \lim_{t\to\infty} x^*(t)$. We say that two functions $x, y \in L^0$ are equimeasurable if $d_x(\lambda) = d_y(\lambda)$ for all $\lambda \geq 0$. Then we obviously have $x^* = y^*$.

An F-normed Köthe function space $E = (E, \|.\|_E)$ over the Lebesgue measure space (I, Σ, μ) is called a symmetric function F-space if E is rearrangement invariant, which means that if $x \in E$, $y \in L^0$ and $x^* = y^*$, then $y \in E$ and $\|x\|_E = \|y\|_E$.

A symmetric sequence F-space over the counting measure space $\left(\mathbb{N}, 2^{\mathbb{N}}, \mu\right)$ is defined similarly. The only difference is that the rearrangement $x^* = (x^*(i))_{i=1}^\infty$ of $x = (x(i))_{i=1}^\infty$ must be defined in the following way (see [13]):

$$x^*(i) = \inf\{\lambda \geq 0 : d_x(\lambda) < i\}.$$

2.2. *F*-normed Orlicz spaces

Definition 2.10. A function $\Phi : [0, \infty) \to [0, \infty]$ is called a monotone Orlicz function if $\Phi(0) = 0$, $b(\Phi) > 0$, Φ is non-decreasing and continuous on the interval $[0, b(\Phi))$, $\lim_{u\to b(\Phi)^-} \Phi(u) =: \Phi(b(\Phi)) \in (0, \infty]$ and $0 < \lim_{u\to\infty} \Phi(u) \leq \infty$, where

$$b(\Phi) := \sup\{u \geq 0 : \Phi(u) < \infty\}.$$

We also let

$$a(\Phi) := \sup\{u \geq 0 : \Phi(u) = 0\}.$$

Note that a monotone Orlicz function can be neither identically equal to zero nor identically equal to infinity on the interval $(0, \infty)$.

Any monotone Orlicz function Φ determines a functional $I_\Phi : L^0(\mu) \to [0, \infty]$ defined by the formula $I_\Phi(f) = \int_\Omega \Phi(|f(t)|)d\mu(t)$ and called the modular. The order ideal

$$L^\Phi(\mu) = \{f \in L^0(\mu) : I_\Phi(rf) < \infty \text{ for some } r > 0\}$$

in $L^0(\mu)$ is called an Orlicz space. The space $L^\Phi(\mu)$ is complete with respect to the following lattice F-norm, called the Mazur–Orlicz F-norm (see [38]):

$$\|f\|_\Phi = \inf\{\lambda > 0 : I_\Phi(f/\lambda) \leq \lambda\}, \tag{2.1}$$

so it is an F-normed Köthe space, but not a Banach lattice, in general. If Φ is convex, then $L^\Phi(\mu)$, endowed with the equivalent lattice norm (called the Luxemburg norm) $\|f\|_\Phi = \inf\{\lambda > 0 : I_\Phi(f/\lambda) \le 1\}$, becomes a Banach lattice. Let us define a lattice ideal $E^\Phi(\mu)$ in $L^\Phi(\mu)$ as

$$E^\Phi(\mu) = \{f \in L^0(\mu) : I_\Phi(rf) < \infty \text{ for all } r > 0\}$$

(see [31,35–41]). Note that in the case of a non-atomic complete and $\sigma-$finite measure space (Ω, Σ, μ) and any monotone Orlicz function Φ both spaces: the space of order continuous elements of the space $L^\Phi(\mu)$ denoted by $(L^\Phi(\mu))_a$ and the space $E^\Phi(\mu)$ are non-trivial if and only if $b(\Phi) = \infty$, and they are equal. In the case of the counting measure space, the space $(L^\Phi(\mu))_a = (\ell^\Phi)_a$ is always non-trivial and

$$(\ell^\Phi)_a = \left\{ x \in \ell^0 : \underset{\lambda>0}{\forall} \; \underset{n_\lambda \in \mathbb{N}}{\exists} \sum_{n=n_\lambda}^{\infty} \Phi(\lambda x(n)) < \infty \right\} =: h^\Phi$$

cf. [44,45], see also [16, Theorem 6.5].

Definition 2.11. We say that Φ satisfies the Δ_2-condition for large arguments (we write shortly $\Phi \in \Delta_2(\infty)$) whenever $b(\Phi) = \infty$ and there are constants $K > 0$ and $u_0 > 0$ such that $\Phi(2u) \le K\Phi(u)$ for all $u \ge u_0$. We say that Φ satisfies the Δ_2-condition for small arguments (we write shortly $\Phi \in \Delta_2(0)$) whenever $a(\Phi) = 0$ and there are constants $K > 0$ and $u_0 > 0$ such that $\Phi(2u) \le K\Phi(u)$ for all $u \in (0, u_0]$. We say that Φ satisfies the Δ_2-condition for all arguments (we write shortly $\Phi \in \Delta_2(\mathbb{R}_+)$) if there is a constant $K > 0$ such that $\Phi(2u) \le K\Phi(u)$ for all $u > 0$.

Remark 2.12. If Φ is a monotone Orlicz function such that $L := \lim_{u\to\infty} \Phi(u) < \infty$, then $\Phi \in \Delta_2(\infty)$. Therefore, if $\Phi \in \Delta_2(0)$ and $L < \infty$, then $\Phi \in \Delta_2(\mathbb{R}_+)$.

Proof. Let $L < \infty$ and take arbitrary $u_0 > a(\Phi)$. Then, for any $u \ge u_0$,

$$\Phi(2u) \le L = \frac{L}{\Phi(u_0)} \cdot \Phi(u_0) \le \frac{L}{\Phi(u_0)} \cdot \Phi(u),$$

which means that $\Phi \in \Delta_2(\infty)$. The second statement of the remark is obvious. □

We define $\Phi \in \Delta_2$ (or that Φ satisfies the suitable Δ_2-condition) in the following way:

(i) $\Phi \in \Delta_2(\infty)$ if we consider the Orlicz function space L^Φ over a non-atomic and finite measure space,
(ii) $\Phi \in \Delta_2(\mathbb{R}_+)$ if we consider the Orlicz function space L^Φ over a non-atomic and infinite measure space,
(iii) $\Phi \in \Delta_2(0)$ if we consider the Orlicz sequence space l^Φ.

Remark 2.13. Let $l > 1$. We say that Φ satisfies the Δ_l-condition for all arguments (we write shortly $\Phi \in \Delta_l(\mathbb{R}_+)$) if there is a constant $K > 0$ such that $\Phi(lu) \leq K\Phi(u)$ for all $u > 0$. Then the following statements are equivalent: (i) $\Phi \in \Delta_l(\mathbb{R}_+)$ for each $l > 1$; (ii) $\Phi \in \Delta_l(\mathbb{R}_+)$ for some $l > 1$; (iii) $\Phi \in \Delta_2(\mathbb{R}_+)$. The proof can be done in the same way as for the convex Orlicz function Φ (see [4]). Moreover, we analogously obtain the respective characterization of the condition Δ_2 for small (large) arguments.

In the case of normed Orlicz spaces the characterization of order continuity is well known. We will see that it is still true for F-normed-Orlicz spaces.

Theorem 2.14. *Let $L^\Phi(\mu)$ be the F-normed Orlicz function space. Then $L^\Phi(\mu) \in (OC)$ (equivalently $L^\Phi(\mu) = E^\Phi(\mu)$) if and only if $\Phi \in \Delta_2$. Similarly $l^\Phi \in (OC)$ (equivalently $l^\Phi = h^\Phi$) if and only if $\Phi \in \Delta_2(0)$.*

Proof. The necessity follows from Theorem 2.5 in [14] (the sequence case) and from Theorem 2.6 in [14] (the function case). On the other hand, it is not difficult to show that if $\Phi \in \Delta_2$ then $L^\Phi = E^\Phi$ ($l^\Phi = h^\Phi$), respectively, which can be done in the same way as for normed Orlicz spaces, applying Theorem 6.5 from [16]. $\square$

Definition 2.15. Let Φ be a monotone Orlicz function. We say that Φ satisfies the condition $\Delta_2^{str}(\infty)$ ($\Phi \in \Delta_2^{str}(\infty)$ for short) if $b(\Phi) = \infty$ and for any $\varepsilon > 0$ there exist $\delta > 0$ and $u_0 > 0$ such that

$$\Phi((1+\delta)u) \leq (1+\varepsilon)\Phi(u) \quad (\forall\, u \geq u_0). \tag{2.2}$$

We say that Φ satisfies the condition $\Delta_2^{str}(\mathbb{R}_+)$ ($\Phi \in \Delta_2^{str}(\mathbb{R}_+)$ for short) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\Phi((1+\delta)u) \leq (1+\varepsilon)\Phi(u) \quad (\forall\, u \geq 0). \tag{2.3}$$

We say that Φ satisfies the condition $\Delta_2^{str}(0)$ ($\Phi \in \Delta_2^{str}(0)$ for short) if $a(\Phi) = 0$ and for any $\varepsilon > 0$ there exist $\delta > 0$ and $u_1 > 0$ such that

$$\Phi((1+\delta)u) \leq (1+\varepsilon)\Phi(u) \quad (\forall\, u \in [0, u_1]). \tag{2.4}$$

We say that Φ satisfies the suitable Δ_2^{str}-condition if Φ satisfies the $\Delta_2^{str}(\infty)$-condition in the case of the non-atomic finite measure space, the $\Delta_2^{str}(\mathbb{R}_+)$-condition in the case of the non-atomic infinite (but σ-finite) measure space and the $\Delta_2^{str}(0)$-condition in the case of the counting measure space.

Obviously, if a monotone Orlicz function Φ satisfies the suitable Δ_2^{str}-condition, then it also satisfies the suitable Δ_2-condition. Note that for a convex Orlicz function the conditions Δ_2^{str} and Δ_2 are equivalent (see [4]). However, for non-convex Orlicz functions they needn't be equivalent as the following example shows.

Example. There is a monotone (even strictly increasing) Orlicz function Φ such that $\Phi \in \Delta_2(\infty)$ and $\Phi \notin \Delta_2^{str}(\infty)$. Take $u_n = 2^n$ and $\varepsilon > 0$ arbitrarily small. Let $\Phi(0) = 0, \Phi(1) = 1/2$, $\Phi(2) = 1$ and let Φ be affine on intervals $(0,1)$ and $(1,2)$. Define

$$\Phi\left(u_n\left(1+\frac{1}{2n}\right)\right) = (1+\varepsilon)\,\Phi(u_n) \text{ for each } n = 1,2,\ldots,$$

$$\Phi(u_n) = (1+\varepsilon)\,\Phi\left(u_{n-1}\left(1+\frac{1}{2(n-1)}\right)\right) \text{ for any } n = 2,3,\ldots.$$

Moreover, let Φ be affine on the intervals $\left(u_n, u_n\left(1+\frac{1}{2n}\right)\right)$ and $\left(u_{n-1}\left(1+\frac{1}{2(n-1)}\right), u_n\right)$. Then

$$\frac{\Phi\left((1+1/2n)\,u_n\right)}{\Phi(u_n)} = 1+\varepsilon.$$

Note also that Φ is a strictly increasing Orlicz function. It is easy to see that condition (2.2) is not satisfied. Really, for each $\delta, u_0 > 0$, taking n such that $1+1/2n < 1+\delta$ and $u_n > u_0$, we have $\Phi\left(\left(1+\frac{1}{2n}\right)u_n\right)/\Phi(u_n) = 1+\varepsilon$, where $\varepsilon > 0$ is fixed and $1+\frac{1}{2n} \to 0$ as $n \to \infty$. On the other hand $\Phi(2u) \le K\Phi(u)$ with $K = \max\left\{2, (1+\varepsilon)^2\right\}$.

Below in the whole section, Φ is a monotone Orlicz function, $L^\Phi(\mu)$ is the Orlicz space equipped with the Mazur–Orlicz F-norm $\|.\|_\Phi$. Let us also recall some auxiliary results.

Lemma 2.16. *For $x \in L^\Phi(\mu)\backslash\{0\}$, we have:*

(i) $I_\Phi\left(\frac{x}{\|x\|_\Phi}\right) \le \|x\|_\Phi$,

(ii) $I_\Phi\left(\frac{x}{\|x\|_\Phi}\right) = \|x\|_\Phi$ *whenever* $I_\Phi\left(\lambda\frac{x}{\|x\|_\Phi}\right) < \infty$ *for some* $\lambda > 1$.

(iii) If $I_\Phi\left(\frac{x}{\lambda}\right) = \lambda$ *for* $\lambda > 0$, *then* $\|x\|_\Phi = \lambda$.

Proof. For the proof of statements (i) and (ii) we refer to [16, Lemma 6.1]. The proof of statement (iii). By the definition of the F-norm $\|.\|_\Phi$, we have that $\|x\|_\Phi \le \lambda$. Assuming that $\|x\|_\Phi < \lambda$, by statement (i) of this lemma, we get

$$\lambda = I_\Phi\left(\frac{x}{\lambda}\right) \le I_\Phi\left(\frac{x}{\|x\|_\Phi}\right) \le \|x\|_\Phi < \lambda,$$

a contradiction. □

Corollary 2.17. [16, Corollary 6.2] *The following statements are true.*

(i) $I_\Phi(x) \le \|x\|_\Phi$ *if* $x \in B(L^\Phi(\mu))$:

(ii) *If* $x \in S(L^\Phi(\mu))$ *and* $I_\Phi(\lambda x) < \infty$ *for some* $\lambda > 1$, *then* $I_\Phi(x) = 1$:

(iii) *The equality* $I_\Phi\left(\frac{x}{\|x\|_\Phi}\right) = \|x\|_\Phi$ *holds for any* $x \in L^\Phi(\mu)\backslash\{0\}$ *in the following three situations:*

(a) *μ is non-atomic, finite and* $\Phi \in \Delta_2(\infty)$:
(b) *μ is non-atomic, infinite and* $\Phi \in \Delta_2(\mathbb{R}_+)$:
(c) *μ is the counting measure on* $2^{\mathbb{N}}$, $b(\Phi) = \infty$ *and* $\Phi \in \Delta_2(0)$.

We will need in the sequel the following fact which is not included in the previous corollary (because it was not proved in [16]).

Lemma 2.18. *Assume that either* $b(\Phi) = \infty$ *or* $\Phi(b(\Phi)) = \infty$ *when* $b(\Phi) < \infty$. *Then* $I_\Phi\left(\frac{x}{\|x\|_\Phi}\right) = \|x\|_\Phi$ *for each* $x \in h_\Phi \backslash \{0\}$.

Proof. The case $b(\Phi) = \infty$ is evident. Let $b(\Phi) < \infty$ and $\Phi(b(\Phi)) = \infty$. We have $I_\Phi\left(\frac{x}{\|x\|_\Phi}\right) \leq \|x\|_\Phi$, whence $\frac{x(i)}{\|x\|_\Phi} < b(\Phi)$ for each $i \in \mathbb{N}$. Since $x \in h_\Phi$, $x \in c_0$. Consequently, $\frac{1}{\|x\|_\Phi} \|x\|_\infty < b(\Phi)$, whence there is $\lambda_0 > 1$ such that $\frac{\lambda_0}{\|x\|_\Phi} \|x\|_\infty < b(\Phi)$. Thus $\frac{\lambda_0}{\|x\|_\Phi} x(i) < b(\Phi)$ for each $i \in \mathbb{N}$. Therefore, by the definition of the space h_Φ, we conclude that there is $n_0 = n(\lambda_0)$ such that $\sum_{n=n_0}^{\infty} \Phi(\frac{\lambda_0}{\|x\|_\Phi} x(n)) < \infty$, so $I_\Phi\left(\frac{\lambda_0 x}{\|x\|_\Phi}\right) < \infty$. Thus $I_\Phi\left(\frac{x}{\|x\|_\Phi}\right) = \|x\|_\Phi$, by Lemma 2.16(*ii*). □

Theorem 2.19. *Let* (Ω, Σ, μ) *be a complete, non-atomic* σ*-finite measure space. Then:*

(i) *The Orlicz space* $L^\Phi(\mu)$ *is strictly monotone if and only if* $b(\Phi) = \infty$, Φ *is strictly increasing on* $\mathbb{R}_+$ *and* Φ *satisfies the condition* $\Delta_2(\mathbb{R}_+)$ *if* μ *is infinite and the condition* $\Delta_2(\infty)$ *if* μ *is finite (see* [16, Theorem 6.6]*).*
(ii) *If* $b(\Phi) = \infty$, *the space* $E^\Phi(\mu)$ *equipped with the Mazur–Orlicz* F*-norm* $\|.\|_\Phi$ *is strictly monotone if and only if* Φ *is strictly increasing on* $\mathbb{R}_+$ *(see* [16, Corollary 6.7] *and* [15]*).*

For the sequence spaces ℓ^Φ a similar result has been proved in [16], Theorem 6.8.

3. Results on general F-normed Köthe spaces

Theorem 3.1. *For any* F*-normed Köthe space* E *the following assertions are equivalent:*

(i) E *is orthogonally strictly monotone:*
(ii) *for any* $x, y \in E\backslash\{0\}$ *such that* $\mu(\operatorname{supp} x \cap \operatorname{supp} y) = 0$ *there holds the inequality* $\|x + y\|_E > \max\{\|x\|_E, \|y\|_E\}$.

Proof. It is obvious that (ii) implies (i). Assume now that (i) holds and $x, y \in E\backslash\{0\}$ are such that $\mu(\operatorname{supp} x \cap \operatorname{supp} y) = 0$. Then, $\|x\|_E = \||x|\|_E < \||x| + |y|\|_E = \||x + y|\|_E$ by (i) and, similarly, $\|y\|_E = \||y|\|_E < \||x| + |y|\|_E = \||x + y|\|_E$, so (ii) holds. □

The following result is well known, but we will present its short proof for the sake of completeness.

Lemma 3.2. *If X, Y are F-normed Köthe spaces and $X \subset Y$ then the inclusion operator $I : X \to Y$ is continuous.*

Proof. Although the proof can be found in [42, Theorem 2.7.2, p. 77] (see also [23, Theorem 1.6, p. 10]), we present the details for the reader's convenience. In view of the closed graph theorem (which is true for F-spaces, see [23, Theorem 1.6, p. 10]) it is enough to show that the inclusion operator I has its graph closed. Assume $\|x_n - x_0\|_X \to 0$ and $\|Ix_n - y_0\|_Y \to 0$. We need to prove that $y_0 = Ix_0$. Since $\|x_n - x_0\|_X \to 0$, $x_n - x_0 \to 0$ μ−a.e. (see [25, Lemma 2, p. 138] , with the same proof for F-normed Köthe spaces, because only the completeness and triangle inequality are important). Similarly, $Ix_n - y_0 \to 0$ μ−a.e. Thus

$$0 \leq |y_0 - Ix_0| \leq |y_0 - Ix_n| + |Ix_n - Ix_0| = |y_0 - Ix_n| + |x_n - x_0| \to 0 \ \ \mu\text{−a.e.},$$

which ends the proof. □

Lemma 3.3. *Let E be a symmetric function F-space over $I = (0, \infty)$. If $x \in E_a$, then $x^* (\infty) = 0$.*

Proof. Recall that for each symmetric function F-space we have $E \subset L^\infty + L^{fin}$, where

$$L^{fin} = \left\{x \in L^0 : m\,(\operatorname{supp} x) < \infty\right\}$$

and

$$L^\infty + L^{fin} = \left\{x \in L^0 : x = y + z \text{ and } y \in L^\infty, z \in L^{fin}\right\}$$

is a symmetric function F-space with the F-norm

$$\|x\|_{L^\infty + L^{fin}} = \inf\left\{\|y\|_\infty + m\,(\operatorname{supp} z) : x = y + z \text{ and } y \in L^\infty, z \in L^{fin}\right\}$$

(see [18, Theorem 1]). Moreover, since E and $L^\infty + L^{fin}$ are complete F-normed function spaces, by Lemma 3.2, the inclusion operator $I : E \to L^\infty + L^{fin}$ is continuous. Assume that $a := x^* (\infty) > 0$. Denoting $A = \{t \in I : |x\,(t)| \geq a/2\}$ we have $m\,(A) = \infty$. We find a sequence (A_n) of pairwise disjoint sets with $A_n \subset A$ and $m\,(A_n) = \infty$ and set $x_n = x\chi_{A_n}$. For each partition $x_n = y_n + z_n$ with $y_n \in L^\infty$ and $z_n \in L^{fin}$ we have $\|y_n\|_\infty \geq a/2$. Thus $\|x_n\|_{L^\infty + L^{fin}} \geq a/2$. If $\|x_n\|_E \to 0$ then we would get $\|Ix_n\|_{L^\infty + L^{fin}} \to 0$, because the operator I is continuous, which is a contradiction. It means that $\|x_n\|_E \not\to 0$ as $n \to \infty$, whence $x \notin E_a$. This finishes the proof. □

Proposition 3.4. *Let E be a symmetric function F-space over $I = (0, \infty)$ with $E \in (FP)$. If $E \in (H_l)$, then $E \in (H_g)$, $E \in (OC)$ and $E \in (OSM)$.*

Proof. Obviously, we need only to show that if $E \in (H_l)$ then $E \in (OC)$ and $E \in (OSM)$. Suppose $E \notin (OC)$. Then there is $x \in E$ and a sequence $0 \leq x_n \leq |x|$ such that $x_n \to 0$ a.e. and $\|x_n\|_E \nrightarrow 0$. Taking $y_n = |x| - x_n$ we get $0 \leq y_n \leq |x|$ and $y_n \to |x|$ a.e., whence, by the Fatou property of E, $\|x\|_E \leq \liminf_{n\to\infty} \|y_n\|_E$ (see [2, Lemma 1.5, p. 5]). On the other hand,

$$\liminf_{n\to\infty} \|y_n\|_E \leq \limsup_{n\to\infty} \|y_n\|_E \leq \|x\|_E ,$$

whence $\|y_n\|_E \to \|x\|_E$. Moreover, since the measure is σ-finite, passing to a subsequence if necessary, we conclude that $y_n \to |x|$ locally in measure. Finally, $\|y_n - |x|\|_E = \|x_n\|_E \nrightarrow 0$. It means that $E \notin (H_l)$.

Assume that $E \notin (OSM)$. Then there is an element $x \in E_+$ and a measurable set $A \in \Sigma \cap \operatorname{supp} x$ such that $m(A) > 0$ and $\|x\|_E = \left\|x\chi_{I\setminus A}\right\|_E$. We may assume that $0 < m(A) < 1$. Let $J = \cup_{n=1}^{\infty}(2n-2, 2n-1]$ and σ be a measure preserving transformation from J onto I (see [43]) and let us define

$$y(t) = (x\chi_A)(\sigma(t)) \text{ and } z(t) = \left(x\chi_{I\setminus A}\right)(\sigma(t)) \text{ for } t \in J.$$

Then $m(\operatorname{supp} y \cap \operatorname{supp} z) = 0$ and $m(\operatorname{supp} y) < 1$. Define

$$y_n(t) = \begin{cases} y^*(t-(2n-1)) & \text{if } \quad t > 2n-1, \\ \quad 0 & \text{if } 0 < t \leq 2n-1, \end{cases}$$

and

$$z_n = z + y_n.$$

Then, by the symmetry of E,

$$\|z_n\|_E = \|x\|_E = \left\|x\chi_{I\setminus A}\right\|_E = \|z\|_E$$

and

$$\|z_n - z\|_E = \|y_n\|_E = \|y\|_E = \|x\chi_A\|_E > 0.$$

Clearly, $z_n \to z$ locally in measure. Thus $E \notin (H_l)$. □

Question Can the implication from Proposition 3.4 be reversed? It is known that for symmetric Banach function spaces the following characterization is true: $E \in (H_l)$ if and only if $E \in (H_g)$, $E \in (OC)$ and $E \in (SM)$ (see [8, Corollary 3.15]). It seems that some methods from [8] may be useful.

Recall that in the counting measure, convergence globally (locally) in measure is just the uniform (coordinatewise) convergence and the property H_g (H_l) is then denoted by H_u (H_c). Similarly to the above Proposition, one can prove the following

Proposition 3.5. *Let E be a symmetric F-normed sequence space with the Fatou property. If $E \in (H_c)$, then $E \in (H_u)$, $E \in (OC)$ and $E \in (OSM)$.*

Lemma 3.6. *Let E be an F-normed function (sequence) space. Then $E \in (H_l)$ ($E \in (H_c)$) if and only if $E_+ \in (H_l)$ ($E_+ \in (H_c)$).*

Proof. Clearly, we need only to prove the sufficiency. First notice that if $E_+ \in (H_l)$ then $E_+ \in (OC)$, see the proof of Proposition 3.4. Then the conclusion follows from Proposition 1 in [19] (the proof for F-lattices is the same as for Banach lattices). □

4. Results on F-normed Orlicz-spaces

In a natural way we are going to consider below the following three cases of the measure space (Ω, Σ, μ):

- μ is non-atomic and finite,
- μ is non-atomic and infinite,
- μ is the counting measure.

Note that in all these cases the Orlicz space $L^\Phi(\mu)$ is symmetric. Thus we restrict ourselves for simplicity to the cases (see [32,35]):

- μ is the Lebesgue measure on the interval $\Omega := (0, \alpha)$, where $\alpha = 1$ or $\alpha = \infty$ (the case of the Orlicz function space $L^\Phi(\mu)$),
- μ is the counting measure on $\left(\mathbb{N}, 2^{\mathbb{N}}, \mu\right)$ (the case of the Orlicz sequence space l^Φ).

We will use the symbol μ in all these cases because it will not cause any misunderstanding. Unless otherwise stated, in the whole section:

- Φ is a monotone Orlicz function on $\mathbb{R}_+$,
- $\Phi(b(\Phi)) := \lim_{u\to b(\Phi)^-} \Phi(u)$,
- $L^\Phi(\mu)$ is an F-normed Orlicz function space equipped with the Mazur–Orlicz F-norm $\|\cdot\|_\Phi$,
- $E^\Phi(\mu)$ is the subspace of order continuous elements in $L^\Phi(\mu)$ with the Mazur–Orlicz F-norm $\|\cdot\|_\Phi$ induced from $L^\Phi(\mu)$,
- we consider the sequence spaces l^Φ and h^Φ (the subspace of order continuous elements in ℓ^Φ) with the Mazur–Orlicz F-norm $\|\cdot\|_\Phi$.

Lemma 4.1. (i) *For any $x \in E^\Phi(\mu)$ the function $f_x(\lambda) = I_\Phi(\lambda x)$ is continuous on the interval $(0,\infty)$ and right-continuous at 0.*

(ii) *Assume that $b(\Phi) = \infty$ or $\Phi(b(\Phi)) = \infty$ whenever $b(\Phi) < \infty$. Then for any $x \in h^\Phi$ the function $f_x(\lambda) = I_\Phi(\lambda x)$ is continuous at each point $\lambda_0 \in (0, b(\Phi))$ satisfying $f_x(\lambda_0) < \infty$.*

Proof. (i) Take any $\lambda_0 \in [0,\infty)$ and any sequence $(\lambda_n)_{n=1}^\infty$ in $[0,\infty)$ such that $\lambda_n \to \lambda_0$ as $n \to \infty$. Then, there exists a constant $\alpha > 0$ such that $0 \le \lambda_n \le \alpha$ for any $n \in \mathbb{N}$. In consequence, $0 \le \Phi(\lambda_n|x(t)|) \le \Phi(\alpha|x(t)|)$ for all $n \in \mathbb{N}$ and $\mu-$a.e. $t \in \Omega$. Since $x \in E^\Phi(\mu)$, we have $\Phi \circ \alpha|x| \in L^1(\Omega)$. By the Lebesgue dominated convergence theorem, we obtain

$$f(\lambda_n) = I_\Phi(\lambda_n x) \to I_\Phi(\lambda_0 x) = f(\lambda_0)$$

as $n \to \infty$, and the proof is finished.

(ii) We consider only the case $b(\Phi) < \infty$ and $\Phi(b(\Phi)) = \infty$, because in the case $b(\Phi) = \infty$ the proof is simpler. Let $x \in h^{\Phi}$, $\lambda_0 \in (0, b(\Phi))$ and a sequence $(\lambda_n)_{n=1}^{\infty} \subset (0, b(\Phi))$ be such that $f_x(\lambda_0) < \infty$ and $\lambda_n \to \lambda_0$ as $n \to \infty$. Since $x \in h^{\Phi}$, $x \in c_0$ (see Lemma 2.18). Thus $\lambda_0 \|x\|_\infty = \lambda_0 x(i_0) < b(\Phi)$ for some $i_0 \in \mathbb{N}$. Take $\alpha > 0$ such that $\alpha x(i_0) \in (\lambda_0 x(i_0), b(\Phi))$. Then, there is $n_0 \in \mathbb{N}$ such that $0 \le \Phi(\lambda_n |x(i)|) \le \Phi(\alpha |x(i)|)$ for all $n \ge n_0$ and $i \in \mathbb{N}$. Applying the assumption that $x \in h^{\Phi}$, we find a number $i = i(\alpha)$ such that $\sum_{i=i(\alpha)}^{\infty} \Phi(\alpha x(i)) < \infty$. Thus $\Phi \circ \alpha |x| \in l^1$ and we can finish the proof in the same way as in case (i). □

Recall that if $b(\Phi) < \infty$ then $E^{\Phi}(\mu) = \{0\}$. Thus the assumption $b(\Phi) = \infty$ in the theorem presented below is natural. Note also that the theorem presented below is an extension of Theorem 2.19(ii).

Theorem 4.2. *Assume that $b(\Phi) = \infty$. Then the following conditions are equivalent.*

(i) *$E^{\Phi}(\mu)$ is lower locally uniformly monotone.*
(ii) *$E^{\Phi}(\mu)$ is strictly monotone.*
(iii) *Φ is strictly increasing on $\mathbb{R}_+$.*

Proof. The implication $(i) \Rightarrow (ii)$ is obvious. In virtue of Theorem 2.19(ii), we have that $(ii) \Leftrightarrow (iii)$. Thus, we only need to prove that (iii) implies (i).

In order to do it, assume that $0 \le x_n \le x \in E_+^{\Phi}(\mu) \backslash \{0\}$ and $\|x_n\|_\Phi \to \|x\|_\Phi$ as $n \to \infty$. We need to prove that $\|x_n - x\|_\Phi \to 0$ as $n \to \infty$. We have that $I_\Phi\left(\frac{x_n}{\|x_n\|_\Phi}\right) = \|x_n\|_\Phi$ for any $n \in \mathbb{N}$ and $I_\Phi\left(\frac{x}{\|x\|_\Phi}\right) = \|x\|_\Phi$ (see Lemma 2.16), whence, by the assumption that $\|x_n\|_\Phi \to \|x\|_\Phi$ as $n \to \infty$, we obtain

$$I_\Phi\left(\frac{x_n}{\|x_n\|_\Phi}\right) \to I_\Phi\left(\frac{x}{\|x\|_\Phi}\right) \tag{4.1}$$

as $n \to \infty$. Furthermore, by Lemma 4.1(i), $I_\Phi\left(\frac{x}{\|x_n\|_\Phi}\right) \to I_\Phi\left(\frac{x}{\|x\|_\Phi}\right)$ as $n \to \infty$, which together with the previous condition gives

$$I_\Phi\left(\frac{x}{\|x_n\|_\Phi}\right) - I_\Phi\left(\frac{x_n}{\|x_n\|_\Phi}\right) \to 0 \tag{4.2}$$

as $n \to \infty$. Since $\Phi \circ \frac{x_n}{\|x_n\|_\Phi} \le \Phi \circ \frac{x}{\|x_n\|_\Phi}$, condition (4.2) implies that

$$\left\|\Phi \circ \frac{x}{\|x_n\|_\Phi} - \Phi \circ \frac{x_n}{\|x_n\|_\Phi}\right\|_{L^1(\Omega)} = I_\Phi\left(\frac{x}{\|x_n\|_\Phi}\right) - I_\Phi\left(\frac{x_n}{\|x_n\|_\Phi}\right) \to 0 \tag{4.3}$$

as $n \to \infty$. Consequently, the sequence $\{\Phi \circ \frac{x}{\|x_n\|_\Phi} - \Phi \circ \frac{x_n}{\|x_n\|_\Phi}\}_{n=1}^{\infty}$ converges to zero in measure. By the σ-finiteness of the measure μ, we obtain that there exists a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ in $\mathbb{N}$ such that the sequence $\{\Phi \circ$

$\frac{x}{\|x_{n_k}\|_\Phi} - \Phi\circ\frac{x_{n_k}}{\|x_{n_k}\|_\Phi}\}_{k=1}^{\infty}$ converges to zero $\mu-$a.e.. Hence, the assumption that Φ is strictly increasing and continuous on $\mathbb{R}_+$ implies that $\{\frac{x}{\|x_{n_k}\|_\Phi} - \frac{x_{n_k}}{\|x_{n_k}\|_\Phi}\}_{k=1}^{\infty}$ converges to zero $\mu-$a.e.. This implies that $\{x - x_{n_k}\}_{k=1}^{\infty}$ converges to zero $\mu-$a.e.. Let us take an arbitrary $\lambda > 0$. Then

$$\Phi \circ \lambda\,(x - x_{n_k}) \to 0 \tag{4.4}$$

as $k \to \infty$ $\mu-$a.e. in Ω. Clearly,

$$\Phi \circ \lambda(x - x_{n_k}) \leq \Phi \circ \lambda x \in L^1(\Omega), \tag{4.5}$$

because $x \in E_+^{\Phi}(\mu)$. Conditions (4.4), (4.5) and the Lebesgue dominated convergence theorem imply that $I_\Phi(\lambda(x - x_{n_k})) \to 0$ as $k \to \infty$ for any $\lambda > 0$. In consequence, $\|x - x_{n_k}\|_\Phi \to 0$ as $k \to \infty$ (see [16, Lemma 6.4]). Finally, by the double extract subsequence theorem, we finish the proof. □

The following theorem is an extension of Theorem 2.19(i). Moreover, the assumption below that $b(\Phi) = \infty$ does not diminish the generality because if $b(\Phi) < \infty$ then none of the conditions (i)–(iv) is satisfied and $E^{\Phi}(\mu) = \{0\}$.

Theorem 4.3. *If $b(\Phi) = \infty$, the following conditions are equivalent.*

(i) *$L^{\Phi}(\mu)$ is lower locally uniformly monotone.*
(ii) *$L^{\Phi}(\mu)$ is upper locally uniformly monotone.*
(iii) *$L^{\Phi}(\mu)$ is strictly monotone.*
(iv) *Φ is strictly increasing and Φ satisfies the suitable Δ_2-condition.*
(v) *$E^{\Phi}(\mu)$ is upper locally uniformly monotone.*

Proof. Applying Theorem 2.19(i) we get $(i) \Rightarrow (iii) \Leftrightarrow (iv)$. Moreover, if $\Phi \in \Delta_2$, then $E^{\Phi}(\mu) = L^{\Phi}(\mu)$. Thus $(iv) \Rightarrow (i)$, by Theorem 4.2. Moreover, $(ii) \Rightarrow (iii) \Rightarrow (iv)$. Consequently, we need only to prove the implication $(iv) \Rightarrow (ii)$ and that (v) is equivalent to any of the conditions $(i) - (iv)$.

First, we will prove that $(iv) \Rightarrow (ii)$. Let $0 \leq x \leq x_n \in L^{\Phi}(\mu)$ for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|x_n\|_\Phi = \|x\|_\Phi$, where $x \neq 0$. Since $\Phi \in \Delta_2$, we conclude that $I_\Phi\left(\frac{x_n}{\|x_n\|_\Phi}\right) = \|x_n\|_\Phi$ and $I_\Phi\left(\frac{x}{\|x\|_\Phi}\right) = \|x\|_\Phi$, so $I_\Phi\left(\frac{x_n}{\|x_n\|_\Phi}\right) - I_\Phi\left(\frac{x}{\|x\|_\Phi}\right) \to 0$ as $n \to \infty$. By $\Phi \in \Delta_2$, we have $E^{\Phi}(\mu) = L^{\Phi}(\mu)$. Thus, applying Lemma 4.1(i), we obtain that $I_\Phi\left(\frac{x}{\|x_n\|_\Phi}\right) \to I_\Phi\left(\frac{x}{\|x\|_\Phi}\right) = \|x\|_\Phi$ as $n \to \infty$. Consequently,

$$0 \leq I_\Phi\left(\frac{x_n}{\|x_n\|_\Phi}\right) - I_\Phi\left(\frac{x}{\|x_n\|_\Phi}\right) \to 0$$

as $n \to \infty$. This means that

$$\left\|\Phi \circ \frac{x_n}{\|x_n\|_\Phi} - \Phi \circ \frac{x}{\|x_n\|_\Phi}\right\|_{L^1(\Omega)} \to 0 \tag{4.6}$$

as $n \to \infty$. Therefore, there exist a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers, a sequence of positive numbers $\{\varepsilon_k\}_{k=1}^{\infty} \in (0,1)$ and $y \in L_+^1(\Omega)$ such that $\lim\limits_{k\to\infty} \varepsilon_k = 0$ and

$$0 \le \Phi \circ \frac{x_{n_k}}{\|x_{n_k}\|_\Phi} - \Phi \circ \frac{x}{\|x_{n_k}\|_\Phi} \le \varepsilon_k y \quad (k \in \mathbb{N}) \text{ (see [25])}, \tag{4.7}$$

whence

$$\Phi \circ \frac{x_{n_k}}{\|x_{n_k}\|_\Phi} \le \varepsilon_k y + \Phi \circ \frac{x}{\|x_{n_k}\|_\Phi} \le y + \Phi \circ \frac{x}{\|x\|_\Phi}. \tag{4.8}$$

From condition (4.7), we conclude that

$$\Phi \circ \frac{x_{n_k}}{\|x_{n_k}\|_\Phi} - \Phi \circ \frac{x}{\|x_{n_k}\|_\Phi} \to 0 \;\; \mu - a.e..$$

Since Φ is strictly increasing, $\frac{x_{n_k}}{\|x_{n_k}\|_\Phi} - \frac{x}{\|x_{n_k}\|_\Phi} \to 0$ μ–a.e., whence

$$\Phi \circ \left(\frac{x_{n_k}}{\|x_{n_k}\|_\Phi} - \frac{x}{\|x_{n_k}\|_\Phi} \right) \to 0 \;\; \mu - a.e..$$

Moreover, by (4.8),

$$\Phi \circ \left(\frac{x_{n_k}}{\|x_{n_k}\|_\Phi} - \frac{x}{\|x_{n_k}\|_\Phi} \right) \le \Phi \circ \left(\frac{x_{n_k}}{\|x_{n_k}\|_\Phi} \right) \le y + \Phi \circ \left(\frac{x}{\|x\|_\Phi} \right) \in L^1.$$

Note that $\|x_{n_k}\|_\Phi \le 2\|x\|_\Phi$ for k large enough. By $L^1 \in (OC)$ we conclude that

$$0 \le I_\Phi \left(\frac{x_{n_k} - x}{2\|x\|_\Phi} \right) \le I_\Phi \left(\frac{x_{n_k} - x}{\|x_{n_k}\|_\Phi} \right) \to 0$$

as $k \to \infty$. By the suitable Δ_2-condition, we can hence easily get that $I_\Phi(\lambda(x_{n_k} - x)) \to 0$ as $k \to \infty$ for any $\lambda > 0$, which means that $\|x_{n_k} - x\|_\Phi \to 0$ as $k \to \infty$. By the double extract subsequence theorem, we obtain that $\|x_n - x\|_\Phi \to 0$ as $n \to \infty$.

Now, we will prove that condition (v) is equivalent to any of the conditions (i)–(iv). Clearly, condition (iv) gives the equality $E^\Phi(\mu) = L^\Phi(\mu)$ and condition (ii) implies that $E^\Phi(\mu) \in (ULUM)$. Conversely, if $E^\Phi(\mu) \in (ULUM)$ then $E^\Phi(\mu) \in (SM)$, whence Φ is strictly increasing, by Theorem 2.19(ii). Thus it is enough to show that if $E^\Phi(\mu) \in (ULUM)$ then $\Phi \in \Delta_2$. We divide the proof into two cases.

(a) Let $\Omega = (0,1)$, $b(\Phi) = \infty$ and $\Phi \notin \Delta_2(\infty)$. Then for each $l > 1$, $\Phi \notin \Delta_l(\infty)$ (see Remark 2.13). Consequently, there exists a sequence $\{u_n\}_{n=1}^{\infty}$ of positive numbers such that $u_n \ge n$ and $\Phi\left(\left(1 + \frac{1}{n}\right) u_n\right) > 2^n \Phi(u_n)$ for any $n \in \mathbb{N}$. Let us take an arbitrary set $A \in \Sigma$ such that $\mu(A) > 0$ and $\mu(\Omega \backslash A) > 0$. Next, choose in $\Omega \backslash A$ a sequence $\{B_n\}_{n=1}^{\infty}$ of measurable and pairwise disjoint sets of positive measure (we can even assume that $\mu(B_n) = 2^{-n}\mu(\Omega \backslash A)$ for every $n \in \mathbb{N}$). Take $n_1 \in \mathbb{N}$ such that $\Phi(u_{n_1})\mu(B_1) \ge \frac{1}{2}$ and choose a measurable subset C_1 of B_1 such that $\Phi(u_{n_1})\mu(C_1) = \frac{1}{2}$. Next, we can find $n_2 > n_1$, $n_2 \in \mathbb{N}$, such that $\Phi(u_{n_2})\mu(B_2) \ge \frac{1}{4}$ and let us choose a measurable subset C_2

of B_2 such that $\Phi(u_{n_2})\mu(C_2) = \frac{1}{4}$. Continuing this procedure, we can find, by induction, a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers and a sequence of measurable sets $\{C_k\}_{k=1}^{\infty}$ such that $C_k \subset B_k$ and $\Phi(u_{n_k})\mu(C_k) = 2^{-k}$ for all $k \in \mathbb{N}$. Define

$$y_k = u_{n_k}\chi_{C_k} \quad (\forall\, k \in \mathbb{N}).$$

Then, $y_k \in E^{\Phi}(\mu)$, $I_{\Phi}(y_k) = 2^{-k}$ for any $k \in \mathbb{N}$ and

$$\begin{aligned} I_{\Phi}\left(\left(1+\frac{1}{n_k}\right)y_k\right) &= \Phi\left(\left(1+\frac{1}{n_k}\right)u_{n_k}\right)\mu(C_k) \\ &> 2^{n_k}\Phi(u_{n_k})\mu(C_k) \\ &= 2^{n_k}\cdot 2^{-k} \geq 1, \end{aligned}$$

whence $\left\|\left(1+\frac{1}{n_k}\right)y_k\right\|_{\Phi} \geq 1$. Therefore, by the triangle inequality, $\|2y_k\|_{\Phi} \geq 1$ and $\|y_k\|_{\Phi} \geq \frac{1}{2}$ for any $k \in \mathbb{N}$.

Let $a \in \mathbb{R}_+$ be such that $\Phi(a)\mu(A) = 1$. Then $x := a\chi_A \in E^{\Phi}(\mu)$ and $I_{\Phi}(x) = 1$, whence $\|x\|_{\Phi} = 1$. Let us define

$$x_k = x + y_k$$

for any $k \in \mathbb{N}$. Then $x_k \in E^{\Phi}(\mu)$, $0 \leq x \leq x_k$ and $I_{\Phi}(x_k) = I_{\Phi}(x) + I_{\Phi}(y_k) = 1 + 2^{-k}$ for every $k \in \mathbb{N}$. Hence,

$$1 = \|x\|_{\Phi} \leq \|x_k\|_{\Phi} \leq 1 + 2^{-k}$$

for all $k \in \mathbb{N}$, so $\|x_k\|_{\Phi} \to \|x\|_{\Phi}$, as $k \to \infty$. Simultaneously, $\|x_k - x\|_{\Phi} = \|y_k\|_{\Phi} \geq \frac{1}{2}$ for all $k \in \mathbb{N}$, which means that $E^{\Phi}(\mu)$ is not upper locally uniformly monotone.

(b) Assume that $\Omega = (0,\infty)$, $b(\Phi) = \infty$ and $\Phi \notin \Delta_2(\mathbb{R}_+)$. It means that $\Phi \notin \Delta_2(\infty)$ or $\Phi \notin \Delta_2(0)$. In the first case we follow as in (a). If $\Phi \notin \Delta_2(0)$ then we may also repeat the above proof analogously with suitably changed sequences (u_{n_k}) and (C_k) (see [14]). □

Remark 4.4. In order to prove that the statements (i)-(iv) in Theorem 4.3 are equivalent, we need not assume that $b(\Phi) = \infty$. In Theorem 4.3 we assumed that $b(\Phi) = \infty$ in order to have that $E^{\Phi}(\mu) \neq \{0\}$.

From Theorems 4.2 and 4.3 it follows that properties $(LLUM)$ and $(ULUM)$ are not equivalent in general.

Theorem 4.5. *If $b(\Phi) = \infty$, the following conditions are equivalent.*

(i) *$E^{\Phi}(\mu)$ is orthogonally lower locally uniformly monotone.*
(ii) *$E^{\Phi}(\mu)$ is orthogonally strictly monotone.*
(iii) *$a(\Phi) = 0$.*

Proof. The implication $(i) \Rightarrow (ii)$ is clear. Let us prove the implication $(ii) \Rightarrow (iii)$. Assume that $a(\Phi) > 0$. Let $A, B \in \Sigma$ be such that $0 < \mu(A) < \infty$, $B \subset \Omega\backslash A$ and $0 < \mu(B) < \infty$. Take any $x \in E^{\Phi}_{+}(\mu)\backslash\{0\}$ with $\operatorname{supp} x \subseteq A$ and define $y = x + a(\Phi)\|x\|_\Phi \chi_B$. Then $0 \le x \le y \in E_+^\Phi(\mu)$, so $\|x\|_\Phi \le \|y\|_\Phi$. On the other hand

$$I_\Phi\left(\frac{y}{\|x\|_\Phi}\right) = I_\Phi\left(\frac{x}{\|x\|_\Phi}\right) = \|x\|_\Phi,$$

whence $\|y\|_\Phi \le \|x\|_\Phi$. Therefore, $\|x\|_\Phi = \|y\|_\Phi$. Since $x \neq y$, we proved that $E^\Phi(\mu)$ is not orthogonally strictly monotone if $a(\Phi) > 0$, and our implication is proved.

Let us now prove the implication $(iii) \Rightarrow (i)$. Assume that $x \in E_+^\Phi(\mu)\backslash\{0\}$ and $(A_n)_{n=1}^\infty$ is a sequence in Σ such that $\|x\chi_{A_n}\|_\Phi \to \|x\|_\Phi$ as $n \to \infty$. We need to show that $\|x - x\chi_{A_n}\|_\Phi = \|x\chi_{\Omega\backslash A_n}\|_\Phi \to 0$ as $n \to \infty$. By Lemmas 2.16 and 4.1 (see conditions (4.1)–(4.3) in the proof of Theorem 4.2), we conclude that $\|\Phi \circ x\chi_{\Omega\backslash A_n}\|_{L^1} \to 0$, whence $x\chi_{\Omega\backslash A_n} \to 0$ as $n \to \infty$ in measure (because $a(\Phi) = 0$). In consequence, by the assumption that the measure μ is σ-finite, passing to a subsequence, if necessary, we can assume that $x\chi_{\Omega\backslash A_n} \to 0$ as $n \to \infty$ $\mu-$a.e. in Ω. Let us take an arbitrary $\lambda > 0$. Note that $\Phi \circ \lambda x\chi_{\Omega\backslash A_n} \le \Phi \circ \lambda x \in L^1$, because $x \in E_+^\Phi(\mu)$. Moreover, $\Phi \circ \lambda x\chi_{\Omega\backslash A_n} \to 0$ as $n \to \infty$ μ-a.e. in Ω. Consequently, by the Lebesgue dominated convergence theorem, $\left\|\Phi \circ \lambda x\chi_{\Omega\backslash A_n}\right\|_{L^1} \to 0$. It means that $\left\|x\chi_{\Omega\backslash A_n}\right\|_\Phi \to 0$ as $n \to \infty$, which finishes the proof. □

Corollary 4.6. *The following conditions are equivalent.*

(i) $L^\Phi(\mu)$ *is orthogonally lower locally uniformly monotone.*
(ii) $L^\Phi(\mu)$ *is orthogonally strictly monotone.*
(iii) $a(\Phi) = 0$ *and* Φ *satisfies the suitable* Δ_2*-condition.*

Proof. The implication $(i) \Rightarrow (ii)$ is clear. Let us prove the implication $(ii) \Rightarrow (iii)$. Assuming that condition (ii) holds, the condition $a(\Phi) = 0$ follows from Theorem 4.5. The conditions $b(\Phi) = \infty$ and $\Phi \in \Delta_2$ follow directly from the proof of the necessity part of Theorem 6.6 in [16], so we omit the details here. Finally, if $\Phi \in \Delta_2$, then $E^\Phi(\mu) = L^\Phi(\mu)$. Thus $(iii) \Rightarrow (i)$ by Theorem 4.5. □

Theorem 4.7. *Assume that* $b(\Phi) = \infty$*. Then* $E^\Phi(\mu)$ *is orthogonally upper locally uniformly monotone if and only if* $a(\Phi) = 0$ *and* $\Phi \in \Delta_2$*.*

Proof. Necessity. The condition $a(\Phi) = 0$ follows from Theorem 4.5. In the proof of Theorem 4.3 we have proved in fact that if $\Phi \notin \Delta_2$ then $E^\Phi(\mu) \notin (OULUM)$.

Sufficiency. Let $x \in E_+^\Phi\backslash\{0\}$ be such that $\mu(\Omega\backslash \operatorname{supp} x) > 0$. Take a sequence (x_n) in E_+^Φ with $\operatorname{supp} x_n \subset \Omega\backslash \operatorname{supp} x$ and $\|x + x_n\|_E \to \|x\|_E$. We will show that $\|x_n\|_E \to 0$. Similarly as in the proof of Theorem 4.2 (conditions (4.1)–(4.3) in the proof), we get $I_\Phi\left(\frac{x+x_n}{\|x+x_n\|_\Phi}\right) \to I_\Phi\left(\frac{x}{\|x\|_\Phi}\right)$ and $I_\Phi\left(\frac{x}{\|x+x_n\|_\Phi}\right) \to I_\Phi\left(\frac{x}{\|x\|_\Phi}\right)$, whence $I_\Phi\left(\frac{x+x_n}{\|x+x_n\|_\Phi}\right) - I_\Phi\left(\frac{x}{\|x+x_n\|_\Phi}\right) \to 0$. This gives that $I_\Phi\left(\frac{x_n}{\|x+x_n\|_\Phi}\right)\to 0$, because x_n, x have pairwise orthogonal supports. Analogously as in the proof of Theorem 4.3 we find a subsequence (x_{n_k}) of (x_n), a sequence (ε_k) with $0<\varepsilon_k\to 0$ and an element $y\in L^1_+(\Omega)$ such that $\Phi\circ\left(\frac{x_{n_k}}{2\|x\|_\Phi}\right)\le\Phi\circ\left(\frac{x_{n_k}}{\|x+x_{n_k}\|_\Phi}\right)\le\varepsilon_k y$ for k large enough. Consequently, we obtain that $I_\Phi\left(\frac{x_{n_k}}{2\|x\|_\Phi}\right)\to 0$. By $\Phi\in\Delta_2$, we have $\|x_{n_k}\|_\Phi\to 0$. Using the double extract subsequence theorem, we finish the proof. □

Corollary 4.8. *The following conditions are equivalent.*

(i) $L^\Phi(\mu)$ *is orthogonally upper locally uniformly monotone.*
(ii) $L^\Phi(\mu)$ *is orthogonally strictly monotone.*
(iii) $a(\Phi)=0, b(\Phi)=\infty$ *and* $\Phi\in\Delta_2$.

Proof. The implication $(i)\Rightarrow(ii)$ is clear, the implication $(ii)\Rightarrow(iii)$ follows from Corollary 4.6. Finally, if $\Phi\in\Delta_2$ then $E^\Phi(\mu)=L^\Phi(\mu)$ and we can apply the previous theorem. □

Theorem 4.9. *The following statements are equivalent.*

(i) h^Φ *is orthogonally lower locally uniformly monotone.*
(ii) h^Φ *is orthogonally strictly monotone.*
(iii) $a(\Phi)=0$ *and either* $a)$ $b(\Phi)=\infty$ *or* $b)$ $b(\Phi)<\infty$ *and* $\Phi(b(\Phi))=\infty$.

Proof. The implication $(i)\Rightarrow(ii)$ is obvious. Let us prove the implication $(ii)\Rightarrow(iii)$. Suppose $a(\Phi)>0$. Let

$$x=(a(\Phi)+1,a(\Phi),a(\Phi),0,0,\ldots),\quad y=x\chi_{\mathbb{N}\setminus\{2\}}$$

and denote $\Phi(a(\Phi)+1)=c$. Of course $c>0$ and $I_\Phi(x)=I_\Phi(y)=c$, which can be rewritten in the form $I_\Phi\left(\frac{cx}{c}\right)=I_\Phi\left(\frac{cy}{c}\right)=c$. Thus, $\|cx\|_\Phi=\|cy\|_\Phi=c$, whence by $x\ne y$, we obtain that h^Φ is not orthogonally strictly monotone.

We have to show that if h^Φ is orthogonally strictly monotone then $\Phi(b(\Phi)) =\infty$ if $b(\Phi)<\infty$. Assuming that $b(\Phi)<\infty$ and $\Phi(b(\Phi))<\infty$, define

$$x=b(\Phi)e_1,\quad y=b(\Phi)e_1+b(\Phi)e_2$$

and denote $\alpha=\Phi(b(\Phi))$ for short. Then $I_\Phi(y)=2\alpha$, whence $I_\Phi\left(\frac{2\alpha y}{2\alpha}\right)=2\alpha$, so $\|2\alpha y\|_\Phi=2\alpha$. Next, $I_\Phi(x)=\alpha$, so $I_\Phi\left(\frac{2\alpha x}{2\alpha}\right)=\alpha<2\alpha$. Therefore, $\|2\alpha x\|_\Phi\le 2\alpha$. However,

$$I_\Phi\left(\frac{2\alpha x}{\lambda 2\alpha}\right)=I_\Phi\left(\frac{x}{\lambda}\right)=\Phi\left(\frac{b(\Phi)}{\lambda}\right)=\infty>\lambda 2\alpha$$

for any $\lambda\in(0,1)$, whence $\|2\alpha x\|_\Phi\ge 2\lambda\alpha$. By the arbitrariness of $\lambda\in(0,1)$, we get $\|2\alpha x\|_\Phi\ge 2\alpha$, so $\|2\alpha x\|_\Phi=2\alpha$. Therefore, $2\alpha x\le 2\alpha y$, $2\alpha x\ne 2\alpha y$ (in fact $2\alpha x=2\alpha y\chi_{\{1\}}$) and $\|2\alpha x\|_\Phi=\|2\alpha y\|_\Phi$. It means that the space $(h^\Phi,\|.\|_\Phi)$ is not orthogonally strictly monotone, which finishes the proof of the implication $b(\Phi)<\infty\Rightarrow\Phi(b(\Phi))=\infty$.

The implication $(iii) \Rightarrow (i)$. We follow similarly as in the proof of Theorem 4.5, applying additionally Lemmas 2.18 and 4.1(ii). However, we explain how one can conclude in the last part of the proof that $I_\Phi\left(\lambda x\chi_{\mathbb{N}\setminus A_{n_k}}\right) \to 0$ for each $\lambda > 0$. Let us take arbitrary $\lambda, \varepsilon > 0$. Since $x \in h^\Phi$, there is $i_\lambda = i(\lambda)$ such that $\sum_{i=i_\lambda}^{\infty} \Phi(\lambda x(i)) < \infty$. Note that $\Phi \circ \left(\lambda x\chi_{\mathbb{N}\setminus A_{n_k}}\right) \to 0$ coordinatewise. Consequently, there is $k_1 \in \mathbb{N}$ such that $\sum_{i=1}^{i_\lambda} \Phi\left(\lambda x\chi_{\mathbb{N}\setminus A_{n_k}}(i)\right) < \varepsilon/2$ for $k \geq k_1$. Furthermore, for $N_0 = \{i_\lambda + 1, i_\lambda + 2, \ldots\}$, we have $\Phi \circ \left(\lambda x\chi_{(\mathbb{N}\setminus A_{n_k})\cap N_0}\right) \leq \Phi \circ (\lambda x\chi_{N_0}) \in l^1$. Thus, since $l^1 \in (OC)$, there is $k_2 \in \mathbb{N}$ such that $\sum\limits_{i=i_\lambda+1}^{\infty} \Phi\left(\lambda x\chi_{\mathbb{N}\setminus A_{n_k}}(i)\right) < \varepsilon/2$ for $k \geq k_2$, whence

$$I_\Phi\left(\lambda x\chi_{\mathbb{N}\setminus A_{n_k}}\right) = \sum_{i=1}^{i_\lambda} \Phi\left(\lambda x\chi_{\mathbb{N}\setminus A_{n_k}}(i)\right) + \sum_{i=i_\lambda+1}^{\infty} \Phi\left(\lambda x\chi_{\mathbb{N}\setminus A_{n_k}}(i)\right) < \varepsilon$$

for $k \geq \max\{k_1, k_2\}$. □

Corollary 4.10. *The following statements are equivalent.*

(i) l^Φ *is orthogonally lower locally uniformly monotone.*

(ii) l^Φ *is orthogonally strictly monotone.*

(iii) $\Phi \in \Delta_2(0)$ *and either a)* $b(\Phi) = \infty$ *or b)* $b(\Phi) < \infty$ *and* $\Phi(b(\Phi)) = \infty$.

Proof. The implication $(i) \Rightarrow (ii)$ is obvious. Now, we will show the implication $(ii) \Rightarrow (iii)$. The necessity of the alternative a) or b) in condition (iii) for condition (ii) is an immediate consequence of the proof of the necessity of the alternative of condition (ii) from Theorem 4.9. Therefore, based on the previous theorem we need only to prove that $\Phi \in \Delta_2(0)$ is necessary for condition (ii). If $a(\Phi) > 0$, then ℓ^Φ contains an order isometric copy of ℓ^∞ [14, Theorem 2.5 (ii)], so it is not orthogonally strictly monotone. Assuming that $a(\Phi) = 0$ and $\Phi \notin \Delta_2(0)$ and applying the proof of Theorem 6.8 in [16] one can find an element $x \in E_+\setminus\{0\}$ and a set $A \in \Sigma \cap \operatorname{supp} x$ with $\mu(A) > 0$ such that $\|x\chi_{\mathbb{N}\setminus A}\|_E = \|x\|_E$. Thus $(l^\Phi, \|.\|_\Phi)$ is not orthogonally strictly monotone.

Sufficiency. The condition $\Phi \in \Delta_2(0)$ gives that $l^\Phi = h^\Phi$ (see Theorem 2.14). So, it is enough to apply the previous theorem. □

The following two results are essential extensions of Theorem 6.8 from [16], which has been proved under the general assumption that $b(\Phi) = \infty$. We will show below that the result of that theorem remains true under some weaker condition than $b(\Phi) = \infty$ and that this condition is necessary. We will also prove that property SM is equivalent to $LLUM$.

Theorem 4.11. *The following conditions are equivalent.*

(i) h^Φ *is lower locally uniformly monotone.*

(ii) h^Φ *is strictly monotone.*

(iii) Φ *is strictly increasing on* $(0, b(\Phi))$ *and either* $b(\Phi) = \infty$ *or* $b(\Phi) < \infty$ *and* $\Phi(b(\Phi)) = \infty$.

Proof. The implication $(i) \Rightarrow (ii)$ is clear. Let us prove the implication $(ii) \Rightarrow (iii)$. To prove that Φ is strictly increasing we follow in the same way as in the proof of the necessity part of Theorem 6.8 from [16], see also the proof of Theorem 4.9 when $a(\Phi) > 0$. The last statement follows from Theorem 4.9.

Now, we will prove the implication $(iii) \Rightarrow (i)$. The proof can be done analogously as that of the implication $(iii) \Rightarrow (i)$ in Theorem 4.2, by applying Lemma 2.18 and Lemma 4.1(ii). Note also that in the last part of the proof we should apply the techniques from the proof of Theorem 4.9, $(iii) \Rightarrow (i)$ (the problem of the existence of a majorant in l^1). □

Corollary 4.12. *The following conditions are equivalent:*

(i) l^Φ *is lower locally uniformly monotone.*
(ii) l^Φ *is strictly monotone.*
(iii) Φ *is strictly increasing on* $(0, b(\Phi))$, $\Phi \in \Delta_2(0)$ *and either* $b(\Phi) = \infty$ *or* $b(\Phi) < \infty$ *and* $\Phi(b(\Phi)) = \infty$.
(iv) l^Φ *is upper locally uniformly monotone.*
(v) h^Φ *is upper locally uniformly monotone.*

Proof. The fact that (i) implies (ii) and (iv) implies (v) is obvious. Applying the previous theorem and Corollary 4.10, we conclude the implication $(ii) \Rightarrow (iii)$. Finally, if $\Phi \in \Delta_2(0)$ then $h^\Phi = l^\Phi$, so by the previous theorem, $(iii) \Rightarrow (i)$.

We will show that (iii) implies (iv). Let $0 \le x \le x_n \in \ell^\Phi$ for any $n \in \mathbb{N}$ and $\|x_n\|_\Phi \to \|x\|_\Phi$ as $n \to \infty$. Proceeding analogously as in the proof of Theorem 4.3 (the implication $(iv) \Rightarrow (ii)$), and applying Lemma 4.1(ii) and Corollary 2.17, we can finish the proof.

Now we will prove that (v) implies (iii). By Theorem 4.11 we need only to prove that $\Phi \in \Delta_2(0)$ is necessary for (v). Since Φ must be strictly increasing, $a(\Phi) = 0$. Suppose that $\Phi \notin \Delta_2(0)$. Therefore, by Remark 2.13, there exists a sequence $\{u_n\}_{n=1}^\infty$ of positive numbers such that $0 < \Phi(u_n) \le 2^{-(n+1)}$ and $\Phi\left(\left(1 + \frac{1}{n}\right) u_n\right) > 2^{n+1}\Phi(u_n)$ for any $n \in \mathbb{N}$. Let $k_n \in \mathbb{N}$ be the biggest natural number such that $k_n\Phi(u_n) \le 2^{-n}$. Then, by $\Phi(u_n) \le 2^{-(n+1)}$, we have

$$2^{-(n+1)} \le k_n\Phi(u_n) \le 2^{-n} \quad (\forall\, n \in \mathbb{N}).$$

Define

$$y_n = \sum_{i=1}^{k_n} u_n e_{2i+1}.$$

Then $I_\Phi(y_n) \le 2^{-n}$ and $I_\Phi\left(\left(1 + \frac{1}{n}\right) y_n\right) \ge 2^{n+1} I_\Phi(y_n) > 2^{n+1} \cdot 2^{-(n+1)} = 1$, whence $\|\left(1 + \frac{1}{n}\right) y_n\|_\Phi \ge 1$. Therefore, $\|2y_n\|_\Phi \ge 1$ and, by the triangle inequality for the F-norm $\|.\|_\Phi$, we get that $\|y_n\|_\Phi \ge \frac{1}{2}$ for any $n \in \mathbb{N}$.

Denote $\lim_{u\to\infty}\Phi(u)=a$. Let $u_0>0$ be such that $\Phi(u_0)=\frac{1}{2}\min\{1,a\}$. Let k be the biggest natural number such that $k\Phi(u_0)<1$. Then $(k+1)\Phi(u_0)\geq 1$. Consequently, there exists $u_1\in(0,u_0]$ such that $k\Phi(u_0)+\Phi(u_1)=1$. Defining

$$x=\sum_{i=1}^{k}u_0e_{2i}+u_1e_{2(k+1)},$$

we get $I_\Phi(x)=1$, whence $\|x\|_\Phi=1$. Let us define

$$x_n=x+y_n \quad (\forall\, n\in\mathbb{N}).$$

It is obvious that all x_n and x belong to h^Φ. Since $x_n\geq x$, $\|x_n\|_\Phi\geq\|x\|_\Phi$ for any $n\in\mathbb{N}$. On the other hand, since the supports of x and y_n are disjoint, we obtain

$$I_\Phi\left(\frac{x_n}{1+2^{-n}}\right)\leq I_\Phi(x_n)=I_\Phi(x)+I_\Phi(y_n)\leq 1+2^{-n} \quad (\forall\, n\in\mathbb{N}),$$

whence $\|x_n\|_\Phi\leq 1+2^{-n}$ for any $n\in\mathbb{N}$. Consequently, $\|x_n\|_\Phi\to\|x\|_\Phi$ as $n\to\infty$. Simultaneously, $\|x_n-x\|_\Phi=\|y_n\|_\Phi\geq\frac{1}{2}$ for any $n\in\mathbb{N}$. This shows that the space h^Φ is not orthogonally upper locally uniformly monotone, so it is not upper locally uniformly monotone either. □

Theorem 4.13. *The following conditions are equivalent.*

(i) Φ *satisfies the* $\Delta_2(0)$*-condition and either* $b(\Phi)=\infty$ *or* $b(\Phi)<\infty$ *and* $\Phi(b(\Phi))=\infty$.

(ii) ℓ^Φ *is orthogonally upper locally uniformly monotone.*

(iii) h^Φ *is orthogonally upper locally uniformly monotone.*

Proof. The implication $(ii)\Rightarrow(iii)$ is obvious. The implication $(iii)\Rightarrow(i)$ follows from Theorem 4.9 and from the proof of Corollary 4.12, where it was proved that if $\Phi\notin\Delta_2(0)$, then h^Φ is not orthogonally upper locally uniformly monotone. So, we need only to prove that (i) implies (ii). Assume that condition (i) holds, $x\in\ell^\Phi_+\backslash\{0\}$ and $(x_n)_{n=1}^\infty$ in ℓ^Φ_+ is such that $\operatorname{supp}x\cap\operatorname{supp}x_n=\emptyset$ for any $n\in\mathbb{N}$ and $\|x+x_n\|_\Phi\to\|x\|_\Phi$ as $n\to\infty$. We can assume that $\|x+x_n\|_\Phi\leq 2\|x\|_\Phi$ for any $n\in\mathbb{N}$. By Lemmas 2.16 and 2.18 we know that

$$I_\Phi\left(\frac{x}{\|x\|_\Phi}\right)=\|x\|_\Phi \qquad\text{and}\qquad I_\Phi\left(\frac{x+x_n}{\|x+x_n\|_\Phi}\right)=\|x+x_n\|_\Phi \quad (n\in\mathbb{N}).$$

In the same way as in the proof of Theorem 4.7, we obtain that $I_\Phi\left(\frac{x_{n_k}}{2\|x\|_\Phi}\right)\to 0$ as $k\to\infty$ for some subsequence (x_{n_k}) of (x_n). Therefore, for k large enough, we have $I_\Phi\left(\frac{x_{n_k}}{2\|x\|_\Phi}\right)\leq\Phi(u_0)$, where u_0 is the constant from the definition of condition $\Delta_2(0)$. Hence, $0\leq\frac{x_{n_k}(i)}{2\|x\|_\Phi}\leq u_0$ for all $i\in\mathbb{N}$ and for $k\in\mathbb{N}$ large enough. By the $\Delta_2(0)-$condition, we get $I_\Phi\left(\frac{2x_{n_k}}{2\|x\|_\Phi}\right)\leq KI_\Phi\left(\frac{x_{n_k}}{2\|x\|_\Phi}\right)$ for $k\in\mathbb{N}$ large enough. By induction, we obtain that for any natural l, $I_\Phi\left(\frac{2^lx_{n_k}}{2\|x\|_\Phi}\right)\leq$

$K^l I_\Phi\left(\frac{x_{n_k}}{2\|x\|_\Phi}\right)$ for all $k\in\mathbb{N}$ large enough. This yields that $I_\Phi(\lambda x_{n_k})\to 0$ as $k\to\infty$ for any $\lambda>0$, that is, $\|x_{n_k}\|_\Phi\to 0$ as $k\to\infty$, and by the double extract subsequence theorem, $\|x_n\|_\Phi\to 0$ as $n\to\infty$, that is, ℓ^Φ is OULUM. □

Lemma 4.14. *Assume that Φ is strictly increasing on $\mathbb{R}_+$. Then $\Phi\in\Delta_2^{str}(\infty)$ if and only if for any $u_1>0$ and each $\varepsilon>0$ there exists $\sigma=\sigma(\varepsilon,u_1)>0$ such that*

$$\Phi((1+\sigma)u)\le(1+\varepsilon)\Phi(u)\quad(\forall\ u\ge u_1). \tag{4.9}$$

Proof. We only need to prove that the $\Delta_2^{str}(\infty)$-condition implies condition (4.9). Let us take arbitrary $u_1,\varepsilon>0$. Take $u_0=u_0(\varepsilon)$ and $\delta=\delta(\varepsilon)$ from Definition 2.15. If $u_1\ge u_0$, then we have nothing to prove, so assume that $u_1<u_0$. We will explain condition $\Delta_2^{str}(\infty)$ in terms of the inverse function Φ^{-1}. Let $u\ge u_0$ and $v=\Phi(u)$. Then $\Phi^{-1}(v)=u\ge u_0$ and $v\ge\Phi(u_0)$. Applying Definition 2.15, we get

$$\Phi((1+\delta)\Phi^{-1}(v))\le(1+\varepsilon)v\quad(\forall\ v\ge\Phi(u_0)). \tag{4.10}$$

Acting both sides in (4.10) with Φ^{-1}, we get the equivalent condition

$$(1+\delta)\Phi^{-1}(v)\le\Phi^{-1}((1+\varepsilon)v)\quad(\forall\ v\ge\Phi(u_0)). \tag{4.11}$$

It is obvious that by the continuity of the function

$$f(v)=\frac{\Phi^{-1}((1+\varepsilon)v)}{\Phi^{-1}(v)}$$

on the interval $[\Phi(u_1),\Phi(u_0)]$, there exists $\bar v\in[\Phi(u_1),\Phi(u_0)]$ such that $\inf_{v\in[\Phi(u_1),\Phi(u_0)]}f(v)=f(\bar v)$. Consequently,

$$\frac{\Phi^{-1}((1+\varepsilon)v)}{\Phi^{-1}(v)}\ge f(\bar v)\qquad(\forall\ v\in[\Phi(u_1),\Phi(u_0)]),$$

that is,

$$f(\bar v)\Phi^{-1}(v)\le\Phi^{-1}((1+\varepsilon)v)\quad(\forall\ v\in[\Phi(u_1),\Phi(u_0)]),$$

which together with (4.11) gives

$$\min\{1+\delta,f(\bar v)\}\Phi^{-1}(v)\le\Phi^{-1}((1+\varepsilon)v)\quad(\forall\ v\in[\Phi(u_1),\infty)\,).$$

Obviously, there exists $\sigma>0$ such that $\min\{1+\delta,f(\bar v)\}=1+\sigma$, so

$$(1+\sigma)\Phi^{-1}(v)\le\Phi^{-1}((1+\varepsilon)v)\quad(\forall\ v\ge\Phi(u_1)\,),$$

which is equivalent to

$$\Phi((1+\sigma)u)\le(1+\varepsilon)\Phi(u)\quad(\forall\ u\ge u_1),$$

which ends the proof. □

Although the proof of the fact that the condition $\Delta_2^{str}(0)$ can be extended to the interval $[0, u_2]$ (with arbitrary $u_2 > u_0$ such that $(1+\varepsilon)u_2 < b(\Phi)$, u_0 from the definition of the $\Delta_2^{str}(0)$-condition) is similar to the proof of Lemma 4.14, we will present a proof because we must use some simultaneous restrictions on ε and u_2, namely, $0 < (1+\varepsilon)u_2 < b(\Phi)$.

Lemma 4.15. *Assume that Φ is an Orlicz function such that $\Phi(b(\Phi)) = \infty$ whenever $b(\Phi) < \infty$, Φ is strictly increasing on the interval $[0, b(\Phi))$ and $\Phi \in \Delta_2^{str}(0)$. Then for any $u_2 \in (0, b(\Phi))$ and $\varepsilon > 0$ such that $(1+\varepsilon)u_2 < b(\Phi)$ there exists $\sigma = \sigma(\varepsilon, u_2) \in (0, \varepsilon)$ such that*

$$\Phi((1+\sigma)u) \le (1+\varepsilon)\Phi(u) \quad (\forall\, u \in [0, u_2]). \tag{4.12}$$

Proof. Let u_2, ε satisfy the assumptions of the lemma. Applying Definition 2.15 we find suitable $\delta = \delta(\varepsilon)$ and $u_1 = u_1(\varepsilon)$. If $u_2 \le u_1$, then we have nothing to prove. Let $u_2 > u_1$. By Definition 2.15 we get

$$(1+\delta)\Phi^{-1}(v) \le \Phi^{-1}((1+\varepsilon)v) \quad (\forall\, v \in [0, \Phi(u_1)]) \tag{4.13}$$

(see the proof of Lemma 4.14). The function

$$f(v) = \frac{\Phi^{-1}((1+\varepsilon)v)}{\Phi^{-1}(v)}$$

has values strictly greater than 1 and it is continuous on the interval $[\Phi(u_1), \Phi(u_2)]$. Therefore, there exists $\widetilde{v} \in [\Phi(u_1), \Phi(u_2)]$ such that $\inf_{v \in [\Phi(u_1), \Phi(u_2)]} f(v) = f(\widetilde{v})$. Consequently,

$$f(\widetilde{v})\Phi^{-1}(v) \le \Phi^{-1}((1+\varepsilon)v) \quad (\forall\, v \in [\Phi(u_1), \Phi(u_2)]),$$

which together with (4.13) yields

$$\min\{1+\delta, f(\widetilde{v})\}\Phi^{-1}(v) \le \Phi^{-1}((1+\varepsilon)v) \quad (\forall\, v \in [0, \Phi(u_2)]).$$

If $\sigma > 0$ is such that $\min\{1+\delta, f(\widetilde{v})\} = 1+\sigma$, we have

$$(1+\sigma)\Phi^{-1}(v) \le \Phi^{-1}((1+\varepsilon)v) \quad (\forall\, v \in [0, \Phi(u_2)]),$$

which is equivalent to

$$\Phi((1+\sigma)u) \le (1+\varepsilon)\Phi(u) \quad (\forall\, u \in [0, u_2])$$

and the proof is finished. □

Lemma 4.16. *Assume that an Orlicz function Φ is strictly increasing and satisfies the suitable Δ_2^{str}-condition. In the sequence case we assume additionally that $\Phi(b(\Phi)) = \infty$ whenever $b(\Phi) < \infty$ and Φ is strictly increasing on the interval $[0, b(\Phi))$. Then, for any sequence $(x_n)_{n=1}^{\infty}$ in $L^{\Phi}(\mu)$ (or, in ℓ^{Φ} resp.) such that $\|x_n\|_{\Phi} \le c$ for any $n \in \mathbb{N}$ and $\|x_n\|_{\Phi} \to c$ as $n \to \infty$, where c is a fixed positive number, we have $I_{\Phi}\left(\frac{x_n}{c}\right) \to c$ as $n \to \infty$.*

Proof. Let us start with the proof in the case when $\Omega = (0,1)$.

Let $c > 0$. We have, by the assumptions of the Lemma, that

$$c \geq \|x_n\|_\Phi = I_\Phi\left(\frac{x_n}{\|x_n\|_\Phi}\right) \geq I_\Phi\left(\frac{x_n}{c}\right) \quad (\forall\, n \in \mathbb{N}). \tag{4.14}$$

Let $\varepsilon > 0$ and take $u_1 = u_1(\varepsilon) > 0$ such that $\Phi(2u_1)\mu(\Omega) < \varepsilon$. Define

$$A_n = \left\{t \in \Omega : \frac{x_n(t)}{c} \geq u_1\right\}.$$

By Lemma 4.14, using the condition $\Phi \in \Delta_2^{str}(\infty)$, we know that there exists $\delta \in (0,1)$ such that

$$\Phi((1+\delta)u) \leq (1+\varepsilon)\Phi(u) \qquad (\forall\, u \geq u_1). \tag{4.15}$$

Since $\frac{c}{\|x_n\|_\Phi} \to 1$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $\frac{c}{\|x_n\|_\Phi} \leq 1+\delta$ for all $n \geq n_0$. Therefore, by $\Phi(2u_1)\mu(\Omega) < \varepsilon$ and conditions (4.14) and (4.15), we get

$$\begin{aligned}
\|x_n\|_\Phi &= I_\Phi\left(\frac{x_n}{\|x_n\|_\Phi}\right) = I_\Phi\left(\frac{c}{\|x_n\|_\Phi}\cdot\frac{x_n}{c}\right) \\
&\leq I_\Phi\left((1+\delta)\frac{x_n}{c}\right) = I_\Phi\left((1+\delta)\frac{x_n}{c}\chi_{A_n}\right) + I_\Phi\left((1+\delta)\frac{x_n}{c}\chi_{\Omega\setminus A_n}\right) \\
&\leq (1+\varepsilon)I_\Phi\left(\frac{x_n}{c}\right) + \Phi((1+\delta)\,u_1)\mu(\Omega) \leq (1+\varepsilon)I_\Phi\left(\frac{x_n}{c}\right) + \varepsilon \\
&\leq I_\Phi\left(\frac{x_n}{c}\right) + c\varepsilon + \varepsilon.
\end{aligned}$$

By the arbitrariness of $\varepsilon > 0$, this implies

$$\liminf_{n\to\infty} I_\Phi\left(\frac{x_n}{c}\right) \geq \liminf_{n\to\infty}\|x_n\|_\Phi = \lim_{n\to\infty}\|x_n\|_\Phi = c.$$

On the other hand, if follows from inequalities (4.14) that $\limsup_{n\to\infty} I_\Phi\left(\frac{x_n}{c}\right) \leq c$, so consequently, $\lim_{n\to\infty} I_\Phi\left(\frac{x_n}{c}\right) = c$.

In the case of $\Omega = (0,\infty)$, the proof of the Lemma is almost the same, but easier because we have then inequality (4.15) for any $u \geq 0$. Therefore, it is omitted.

Now, we will present a proof for the sequence case. Since Φ satisfies the $\Delta_2^{str}(0)$-condition, it also satisfies the $\Delta_2(0)$-condition, whence $I_\Phi\left(\frac{x_n}{\|x_n\|_\Phi}\right) = \|x_n\|_\Phi \leq c$ (see Lemma 2.18 and Theorem 2.14). Therefore, $I_\Phi\left(\frac{x_n}{c}\right) \leq c$ for any $n \in \mathbb{N}$. Taking $u_c > 0$ such that $\Phi(u_c) = c$, we obtain that $\frac{x_n(i)}{c} \leq u_c < b(\Phi)$ for any $i, n \in \mathbb{N}$. Therefore, taking $\varepsilon > 0$ such that $(1+\varepsilon)u_c < b(\Phi)$, by Lemma 4.15, there exists $\sigma \in (0,\varepsilon)$ such that

$$\Phi((1+\sigma)u) \leq (1+\varepsilon)\Phi(u) \quad (\forall\, u \in [0,u_c]). \tag{4.16}$$

Since $\frac{c}{\|x_n\|_\Phi} \le 1+\sigma$ for n large enough, we obtain

$$\begin{aligned} \|x_n\|_\Phi &= I_\Phi\left(\frac{x_n}{\|x_n\|_\Phi}\right) = I_\Phi\left(\frac{c}{\|x_n\|_\Phi}\cdot\frac{x_n}{c}\right) \\ &\le I_\Phi\left((1+\sigma)\frac{x_n}{c}\right) \le (1+\varepsilon)I_\Phi\left(\frac{x_n}{c}\right) \le I_\Phi\left(\frac{x_n}{c}\right)+c\varepsilon \end{aligned} \tag{4.17}$$

for n large enough. By the fact that $\varepsilon > 0$ satisfying the inequality $(1+\varepsilon)u_c < b(\Phi)$ can be arbitrarily small, inequalities (4.17) imply that

$$\liminf_{n\to\infty} I_\Phi\left(\frac{x_n}{\|x_n\|_\Phi}\right) \ge \lim_{n\to\infty}\|x_n\|_\Phi.$$

Since $\limsup_{n\to\infty} I_\Phi\left(\frac{x_n}{c}\right) \le c$ [see inequalities (4.14)], we obtain $\lim_{n\to\infty} I_\Phi\left(\frac{x_n}{c}\right) = c$. □

In the further part of the paper we will need the following

Lemma 4.17. *Assume that $\Phi \in \Delta_2$ (suitable for the measure space) and $a(\Phi) = 0$. Then $\|x_n\|_\Phi \to 0$ if and only if $I_\Phi(x_n) \to 0$.*

In [16, Lemma 6.4] it was proved that $\|x_n\|_\Phi \to 0$ if and only if $I_\Phi(\lambda x_n) \to 0$ as $n\to\infty$ for any $\lambda > 0$. We can easily prove that if $\Phi \in \Delta_2$ and $a(\Phi) = 0$, then $I_\Phi(x_n) \to 0$ as $n\to\infty$ implies that $I_\Phi(\lambda x_n) \to 0$ for any $\lambda > 0$ as $n\to\infty$.

Theorem 4.18. *Assume that Φ is a strictly increasing Orlicz function satisfying the suitable Δ_2^{str}-condition. In the sequence case, assume additionally that $\Phi(b(\Phi)) = \infty$ whenever $b(\Phi) < \infty$ and that Φ is strictly increasing on the interval $[0, b(\Phi))$. Then the Orlicz function (resp. sequence) space $L^\Phi(\mu)$ (resp. l^Φ) is orthogonally uniformly monotone.*

Proof. Let $c > 0$ and assume that $(A_n)_{n=1}^\infty \subset \Sigma$, $(x_n)_{n=1}^\infty$ from $L^\Phi(\mu)$ are such that $0 \le x_n \in L^\Phi(\mu)$ and $\|x_n\|_\Phi \le c$ for any $n \in \mathbb{N}$ and $\lim_{n\to\infty}\|x_n\chi_{A_n}\|_\Phi = c$. Then, by Corollary 2.17 and Lemma 4.16, we get that $\lim_{n\to\infty} I_\Phi\left(\frac{x_n}{c}\right) = \lim_{n\to\infty} I_\Phi\left(\frac{x_n\chi_{A_n}}{c}\right) = c$. Consequently,

$$\lim_{n\to\infty} I_\Phi\left(\frac{x_n\chi_{\Omega\setminus A_n}}{c}\right) = \lim_{n\to\infty}\left[I_\Phi\left(\frac{x_n}{c}\right) - I_\Phi\left(\frac{x_n\chi_{A_n}}{c}\right)\right] = 0.$$

Recall that $\Phi \in \Delta_2^{str}$ implies that $\Phi \in \Delta_2$. By $a(\Phi) = 0$, $\Phi \in \Delta_2$ and Lemma 4.17, we obtain that $\|x_n - x_n\chi_{A_n}\|_\Phi = \|x_n\chi_{\Omega\setminus A_n}\|_\Phi \to 0$ as $n\to\infty$, which finishes the proof. In the sequence case the proof is similar. □

Corollary 4.19. *If $a(\Phi) = 0$, then the Orlicz function space $L^\Phi(\mu)$ is orthogonally uniformly monotone in the following cases.*

(i) *Φ is a convex function and Φ satisfies the suitable Δ_2-condition.*
(i) *$\Phi(u) = \Psi(u^s)$ for any $u \in \mathbb{R}_+$, where $0 < s \le 1$, Ψ is a convex Orlicz function satisfying the suitable Δ_2-condition.*
(iii) *Φ is a concave and strictly increasing function on $\mathbb{R}_+$.*

Proof. (i) It is well known (see [4, Theorem 1.13]) that if Φ is a convex Orlicz function satisfying the suitable Δ_2-condition, then Φ satisfies the suitable Δ_2^{str}-condition.

(ii) The suitable Δ_2-condition for Ψ means the suitable Δ_2-condition for Φ. Since Ψ is convex, we have the suitable Δ_2^{str}-condition for Ψ, so we also have the suitable Δ_2^{str}-condition for Φ.

(iii) If Φ is concave, then by $\Phi(0)=0$, we have for any fixed $\varepsilon>0$ and any $u\in\mathbb{R}_+$ that $\Phi((1+\varepsilon)u)\le(1+\varepsilon)\Phi(u)$, which means that Φ satisfies the Δ_2^{str}-condition on the whole $\mathbb{R}_+$.

Therefore, in any case among these three cases the thesis follows directly from Theorem 4.18. □

Theorem 4.20. *Assume that Φ is a convex Orlicz function and $E^\Phi(\mu)$ and $L^\Phi(\mu)$ are equipped with the Mazur–Orlicz F-norm. Then the following conditions are equivalent.*

(i) $L^\Phi(\mu)\in(UM)$.
(ii) $b(\Phi)=\infty$ *and* $E^\Phi(\mu)\in(UM)$.
(iii) $a(\Phi)=0$ *and* $\Phi\in\Delta_2$.
(iv) $L^\Phi(\mu)\in(OUM)$.
(v) $L^\Phi(\mu)\in(OLUM)$.
(vi) $L^\Phi(\mu)\in(OSM)$.

Proof. It is obvious that (i) implies (ii) (see Theorem 4.3). The fact that (ii) implies (iii) follows from the implication $(v)\Rightarrow(iv)$ in Theorem 4.3. We will prove that (iii) implies (i). It is known that $\Phi\in\Delta_2$ implies that $\Phi\in\Delta_2^{str}$ when Φ is convex (see [4, Theorem 1.13]). Let us take any $c\in(0,\infty)$ and any two sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$ in $L^\Phi(\mu)$ such that $0\le x_n\le y_n$ and $\|y_n\|_\Phi\le c$ for any $n\in\mathbb{N}$ as well as $\lim_{n\to\infty}\|x_n\|_\Phi=\lim_{n\to\infty}\|y_n\|_\Phi=c$. Then, by $\Phi\in\Delta_2^{str}$ and by Lemma 4.16, we have

$$\lim_{n\to\infty} I_\Phi\left(\frac{x_n}{c}\right)=\lim_{n\to\infty} I_\Phi\left(\frac{y_n}{c}\right)=c.$$

By the convexity of Φ and by $\Phi(0)=0$, the function Φ is superadditive on $\mathbb{R}_+$, whence the modular I_Φ is superadditive on $(L^0(\mu))_+$, so also on $(L^\Phi(\mu))_+$. Therefore,

$$0\le I_\Phi\left(\frac{y_n-x_n}{c}\right)\le I_\Phi\left(\frac{y_n}{c}\right)-I_\Phi\left(\frac{x_n}{c}\right)\to 0$$

as $n\to\infty$, whence $I_\Phi\left(\frac{y_n-x_n}{c}\right)\to 0$ as $n\to\infty$. By the assumption that $a(\Phi)=0$ and $\Phi\in\Delta_2$, we obtain that $\|y_n-x_n\|_\Phi\to 0$. Clearly, $(i)\Rightarrow(iv)\Rightarrow(v)\Rightarrow(vi)$. Finally, $(vi)\Rightarrow(iii)$, by Corollary 4.8, which finishes the proof. □

Theorem 4.21. *Let Φ be a convex Orlicz function. Then the Orlicz sequence space l^Φ equipped with the Mazur–Orlicz F-norm is uniformly monotone if and only if $\Phi\in\Delta_2(0)$ and $\Phi(b(\Phi))=\infty$ whenever $b(\Phi)<\infty$.*

Proof. The necessity follows from Corollary 4.12. The sufficiency can be done in a similar way as the proof of the implication $(iii) \Rightarrow (i)$ from the previous theorem. $\square$

Now, we are going to study the Kadec–Klee properties in Orlicz spaces equipped with the Mazur–Orlicz F-norm. Recall that these properties have already been investigated in some Banach and quasi-Banach lattices by many authors (see for instance, [5,8,11,12,19,21,26–28]).

It is known that properties H_l and the Kadec–Klee property with respect to $\mu-$a.e. convergence ($H_{\mu-a.e.}$ for short) are the same properties for any $\sigma-$finite measure. Moreover, properties H_l and H_g are the same in the case of a non-atomic and finite measure. Consequently, we assume below that $\Omega = (0,\infty)$, because the property H_g will be characterized later.

Theorem 4.22. *Assume that* $\Omega = (0,\infty)$. *The Orlicz function space* $L^\Phi(\mu)$ *has the Kadec–Klee property with respect to local convergence in measure (i.e.* $L^\Phi(\mu) \in (H_l)$*) if and only if* $\Phi \in \Delta_2(\mathbb{R}_+)$.

Proof. Sufficiency. Take any $x \in L^\Phi(\mu)\backslash\{0\}$ and any sequence $\{x_n\}_{n=1}^\infty$ in $L^\Phi(\mu)$ such that $\|x_n\|_\Phi \to \|x\|_\Phi$ and $x_n \to x$ locally in measure μ. By the assumption that $\Phi \in \Delta_2(\mathbb{R}_+)$, we have that $I_\Phi\left(\frac{x}{\|x\|_\Phi}\right) = \|x\|_\Phi$ and $I_\Phi\left(\frac{x_n}{\|x_n\|_\Phi}\right) = \|x_n\|_\Phi$ for any $n \in \mathbb{N}$, which means that $\|x\|_\Phi = \left\|\Phi \circ \frac{x}{\|x\|_\Phi}\right\|_{L^1}$ and $\|x_n\|_\Phi = \left\|\Phi \circ \frac{x_n}{\|x_n\|_\Phi}\right\|_{L^1}$. Since $\|x_n\|_\Phi \to \|x\|_\Phi$, we also have that $\frac{x_n}{\|x_n\|_\Phi} \to \frac{x}{\|x\|_\Phi}$ locally in measure μ. Since μ is σ-finite, we get that there exists a strictly increasing sequence $\{n_k\}_{k=1}^\infty$ of natural numbers such that $\frac{x_{n_k}}{\|x_{n_k}\|_\Phi} \to \frac{x}{\|x\|_\Phi}$ μ-a.e.. By the continuity of Φ, we have that $\Phi \circ \frac{x_{n_k}}{\|x_{n_k}\|_\Phi} \to \Phi \circ \frac{x}{\|x\|_\Phi}$ μ-a.e.. Since the space $L^1(\mu)$ has the Kadec–Klee property with respect to the μ-a.e. convergence, we obtain that

$$\left\|\Phi \circ \frac{x_{n_k}}{\|x_{n_k}\|_\Phi} - \Phi \circ \frac{x}{\|x\|_\Phi}\right\|_{L^1} \to 0.$$

In consequence, the sequence $\left\{|\Phi \circ \frac{x_{n_k}}{\|x_{n_k}\|_\Phi} - \Phi \circ \frac{x}{\|x\|_\Phi}|\right\}_{k=1}^\infty$ has a majorant $y \in L^1_+(\mu)$ (see [25]), whence we get

$$\Phi \circ \frac{x_{n_k}}{\|x_{n_k}\|_\Phi} - \Phi \circ \frac{x}{\|x\|_\Phi} \le |\Phi \circ \frac{x_{n_k}}{\|x_{n_k}\|_\Phi} - \Phi \circ \frac{x}{\|x\|_\Phi}| \le y.$$

Applying the assumption that $\Phi \in \Delta_2(\mathbb{R}_+)$, we conclude that

$$\begin{aligned}\Phi \circ \left(\frac{x_{n_k}}{\|x_{n_k}\|_\Phi} - \frac{x}{\|x\|_\Phi}\right) &= \Phi \circ \left(\frac{1}{2}\frac{2x_{n_k}}{\|x_{n_k}\|_\Phi} - \frac{1}{2}\frac{2x}{\|x\|_\Phi}\right) \\ &\le \Phi \circ \max\left(\frac{2|x_{n_k}|}{\|x_{n_k}\|_\Phi}, \frac{2|x|}{\|x\|_\Phi}\right) \\ &\le K\left\{\Phi \circ \max\left(\frac{|x_{n_k}|}{\|x_{n_k}\|_\Phi}, \frac{|x|}{\|x\|_\Phi}\right)\right\} \\ &\le K\left\{\Phi \circ \frac{|x_{n_k}|}{\|x_{n_k}\|_\Phi} + \Phi \circ \frac{|x|}{\|x\|_\Phi}\right\} \\ &\le K\left\{\Phi \circ \frac{|x|}{\|x\|_\Phi} + y\right\} + K\Phi \circ \frac{|x|}{\|x\|_\Phi} \in L^1_+(\mu).\end{aligned}$$

Therefore, the sequence $\left\{\Phi \circ \left(\frac{x_{n_k}}{\|x_{n_k}\|_\Phi} - \frac{x}{\|x\|_\Phi}\right)\right\}_{k=1}^\infty$ has a majorant in $L^1(\mu)$ and it is convergent to zero μ-a.e.. By the Lebesgue dominated convergence theorem, $I_\Phi\left(\frac{x_{n_k}}{\|x_{n_k}\|_\Phi} - \frac{x}{\|x\|_\Phi}\right) \to 0$ as $k \to \infty$. Next, by $\Phi \in \Delta_2(\mathbb{R}_+)$, we get that $I_\Phi\left(\lambda\left(\frac{x_{n_k}}{\|x_{n_k}\|_\Phi} - \frac{x}{\|x\|_\Phi}\right)\right) \to 0$ for any $\lambda > 0$, that is, $\left\|\frac{x_{n_k}}{\|x_{n_k}\|_\Phi} - \frac{x}{\|x\|_\Phi}\right\|_\Phi \to 0$ as $k \to \infty$. Since $\|x_{n_k}\|_\Phi \to \|x\|_\Phi$, this yields that $\|x_{n_k} - x\|_\Phi \to 0$ as $k \to \infty$.

The necessity follows from Proposition 3.4 and Corollary 4.6. □

Theorem 4.23. *The Orlicz function space $L^\Phi(\mu)$ has the property H_g if and only if $\Phi \in \Delta_2$.*

Proof. Necessity. Suppose $\Phi \notin \Delta_2$.

Let $\Omega = (0, 1)$. If $\Phi \notin \Delta_2(\infty)$, then taking the sequence $(x_n)_{n=1}^\infty$ from the proof of Theorem 2.6 in [14], we have that $m(\operatorname{supp} x_n) \to 0$. Set $u_n = x - x_n$. Then $u_n \to x$ globally in measure and $\|x\|_\Phi = \|u_n\|_\Phi = 1 = \|x - u_n\|_\Phi$ for any $n \in \mathbb{N}$. Thus $L^\Phi(\mu) \notin (H_g)$.

Let $\Omega = (0, \infty)$. If $\Phi \notin \Delta_2(\infty)$, we follow as above. If $\Phi \notin \Delta_2(0)$, then we can find the respective sequence $(x_n)_{n=1}^\infty$ such that $x_n \to 0$ uniformly (see the proof of Theorems 2.5 and 2.6 in [14]). The rest of the proof is the same.

Sufficiency. Let $\Omega = (0, 1)$. Take $x, x_n \in L^\Phi(\mu)$, $n = 1, 2, \ldots$ such that $\|x_n\|_\Phi \to \|x\|_\Phi$ and $x_n \to x$ globally in measure. Clearly, $x_n \to x$ locally in measure, so, applying the proof of sufficiency of Theorem 4.22, we conclude that

$$I_\Phi\left(\frac{x_{n_k}}{\|x_{n_k}\|_\Phi} - \frac{x}{\|x\|_\Phi}\right) \to 0 \text{ as } k \to \infty \tag{4.18}$$

for some subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$. Based on the double extract subsequence theorem we may denote this subsequence $(x_{n_k})_{k=1}^\infty$ by $(x_n)_{n=1}^\infty$. Recall that $\|z_n - z\|_\Phi \to 0$ if and only if $I_\Phi(\lambda(z_n - z)) \to 0$ for each $\lambda > 0$ (see

Lemma 6.4 in [16]). Let $\lambda > 1$ and $\varepsilon > 0$. We will prove that

$$I_\Phi\left(\lambda\left(\frac{x_n}{\|x_n\|_\Phi} - \frac{x}{\|x\|_\Phi}\right)\right) < \varepsilon \text{ for sufficiently large } n.$$

Note that from the assumption that $\Phi \in \Delta_2(\infty)$ it follows that for each $u_1 > a(\Phi)$ and every $l > 1$ there is a constant $K = K(u_1, l) > 1$ such that $\Phi(lu) \le K\Phi(u)$ for all $u \ge u_1$ (in the proof we apply only the continuity of Φ, see also [4]). Take $K = K(u_1, \lambda)$, where $u_1 > a(\Phi)$ satisfies the inequality $\Phi(u_1) < \varepsilon/3$. Denote

$$\begin{aligned} A_n &= \left\{t \in \Omega : \lambda\left|\frac{x_n(t)}{\|x_n\|_\Phi} - \frac{x(t)}{\|x\|_\Phi}\right| > u_1\right\}, \\ A_n^1 &= \left\{t \in A_n : \left|\frac{x_n(t)}{\|x_n\|_\Phi} - \frac{x(t)}{\|x\|_\Phi}\right| > u_1\right\}, \\ A_n^2 &= \left\{t \in A_n : \left|\frac{x_n(t)}{\|x_n\|_\Phi} - \frac{x(t)}{\|x\|_\Phi}\right| \le u_1\right\}. \end{aligned}$$

We have

$$\begin{aligned} I_\Phi\left(\lambda\left(\frac{x_n}{\|x_n\|_\Phi} - \frac{x}{\|x\|_\Phi}\right)\right) &= I_\Phi\left(\lambda\left(\frac{x_n}{\|x_n\|_\Phi} - \frac{x}{\|x\|_\Phi}\right)\chi_{A_n^1}\right) \\ &\quad + I_\Phi\left(\lambda\left(\frac{x_n}{\|x_n\|_\Phi} - \frac{x}{\|x\|_\Phi}\right)\chi_{A_n^2}\right) + I_\Phi\left(\lambda\left(\frac{x_n}{\|x_n\|_\Phi} - \frac{x}{\|x\|_\Phi}\right)\chi_{\Omega/A_n}\right) \\ &\le K \cdot I_\Phi\left(\left(\frac{x_n}{\|x_n\|_\Phi} - \frac{x}{\|x\|_\Phi}\right)\chi_{A_n^1}\right) + \Phi(\lambda u_1)\, m\left(A_n^2\right) + \varepsilon/3. \end{aligned}$$

Since $x_n \to x$ globally in measure, $\frac{x_n}{\|x_n\|_\Phi} \to \frac{x}{\|x\|_\Phi}$ globally in measure, by Lemma 3.6 from [26], which works also with the same proof for symmetric F-spaces. In consequence, $m\left(A_n^2\right) \le m(A_n) \to 0$ and, by (4.18), we get

$$I_\Phi\left(\lambda\left(\frac{x_n}{\|x_n\|_\Phi} - \frac{x}{\|x\|_\Phi}\right)\right) \le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

for sufficiently large n.

If $\Omega = (0, \infty)$, the proof is analogous and simpler, because the condition $\Phi \in \Delta_2(\infty)$ is replaced by $\Phi \in \Delta_2(\mathbb{R}_+)$. □

Theorem 4.24. *The Orlicz sequence space $l^\Phi \in (H_c)$ if and only if $\Phi \in \Delta_2(0)$ and $\Phi(b(\Phi)) = \infty$ when $b(\Phi) < \infty$.*

Proof. The necessity follows from Proposition 3.5 and Corollary 4.10.

Sufficiency. By Lemma 3.6 we may take $x, x_n \in \left(l^\Phi\right)_+ \setminus \{0\}$, $n = 1, 2, \ldots,$ such that $\|x_n\|_\Phi \to \|x\|_\Phi$ and $x_n \to x$ coordinatewise. Without loss of generality, we can also assume that $\|x\|_\Phi > 0$. Since $\Phi \in \Delta_2(0)$, $h_\Phi = l_\Phi$. By Lemma 2.18, $I_\Phi\left(\frac{z}{\|z\|_\Phi}\right) = \|z\|_\Phi$ for each $z \in l_\Phi$. Then $I_\Phi\left(\frac{x_n}{\|x_n\|_\Phi}\right) \to I_\Phi\left(\frac{x}{\|x\|_\Phi}\right)$ and

$x_n \to x$ coordinatewise. Moreover, $\frac{|x_n|}{\|x_n\|_\Phi} \to \frac{|x|}{\|x\|_\Phi}$ coordinatewise and consequently $\Phi\left(\frac{|x_n|}{\|x_n\|_\Phi}\right) \to \Phi\left(\frac{|x|}{\|x\|_\Phi}\right)$ coordinatewise, because Φ is continuous. Since $l^1 \in (H_c)$, it follows that

$$\left\|\Phi\left(\frac{|x_n|}{\|x_n\|_\Phi}\right) - \Phi\left(\frac{|x|}{\|x\|_\Phi}\right)\right\|_{\ell_1} \to 0.$$

Therefore, there is $z \in l^1$ satisfying

$$\Phi\left(\frac{|x_n|}{\|x_n\|_\Phi}\right) - \Phi\left(\frac{|x|}{\|x\|_\Phi}\right) \leq \left|\Phi\left(\frac{|x_n|}{\|x_n\|_\Phi}\right) - \Phi\left(\frac{|x|}{\|x\|_\Phi}\right)\right| \leq z.$$

Consequently,

$$\begin{aligned}\Phi\left(\left|\frac{x_n}{\|x_n\|_\Phi} - \frac{x}{\|x\|_\Phi}\right|\right) &\leq \Phi\left(\max\left\{\frac{|x_n|}{\|x_n\|_\Phi}, \frac{|x|}{\|x\|_\Phi}\right\}\right)\\ &\leq \left[\Phi\left(\frac{|x_n|}{\|x_n\|_\Phi}\right) + \Phi\left(\frac{|x|}{\|x\|_\Phi}\right)\right]\\ &\leq \left[z + \Phi\left(\frac{|x|}{\|x\|_\Phi}\right) + \Phi\left(\frac{|x|}{\|x\|_\Phi}\right)\right] \in l^1.\end{aligned}$$

Since $\Phi\left(\frac{|x_n|}{\|x_n\|_\Phi} - \frac{|x|}{\|x\|_\Phi}\right) \to 0$ coordinatewise and $l^1 \in (OC)$, it follows that $I_\Phi\left(\frac{x_n}{\|x_n\|_\Phi} - \frac{x}{\|x\|_\Phi}\right) \to 0$. Applying the assumption that $\Phi \in \Delta_2(0)$ and Lemma 4.17, we conclude that $I_\Phi\left(\lambda\left(\frac{x_n}{\|x_n\|_\Phi} - \frac{x}{\|x\|_\Phi}\right)\right) \to 0$ for any $\lambda > 0$, that is, $\left\|\frac{x_n}{\|x_n\|_\Phi} - \frac{x}{\|x\|_\Phi}\right\|_\Phi \to 0$ (see Lemma 6.4 from [16]). Combining this together with the assumption $\|x_n\|_\Phi \to \|x\|_\Phi > 0$, we get $\|x_n - x\|_\Phi \to 0$ as desired. $\square$

Theorem 4.25. *Suppose Φ is an Orlicz function such that $\Phi(b(\Phi)) = \infty$ when $b(\Phi) < \infty$. The Orlicz sequence space l^Φ has the property H_u if and only if $a(\Phi) > 0$ or $\Phi \in \Delta_2(0)$.*

Proof. Necessity. Suppose $a(\Phi) = 0$ and $\Phi \notin \Delta_2(0)$. Applying the proof of Theorem 2.5 in [14], we find a sequence $(x_n)_{n=1}^\infty$ in $S\left(l^\Phi\right)_+ := S\left(l^\Phi\right) \cap l^\Phi_+$ with pairwise disjoint supports such that $x_n \to 0$ uniformly and $x := \sup x_n \in S\left(l^\Phi\right)$. Taking $u_n = x - x_n$ we have that $\|u_n\|_\Phi = \|x\|_\Phi = 1 = \|x - u_n\|_\Phi$ and $u_n \to x$ uniformly. It means that $l^\Phi \notin (H_u)$.

Sufficiency. Take $x, x_n \in l^\Phi$, $n = 1, 2, \ldots$ such that $\|x_n\|_\Phi \to \|x\|_\Phi$ and $x_n \to x$ uniformly. If $a(\Phi) > 0$ then for each $\lambda > 0$, we have that $I_\Phi(\lambda(x_n - x)) = 0$ for sufficiently large n, which gives $\|x_n - x\|_\Phi \to 0$. Suppose $a(\Phi) = 0$ and $\Phi \in \Delta_2(0)$. Since $\Phi(b(\Phi)) = \infty$ when $b(\Phi) < \infty$, by Theorem 4.24, we conclude that $l^\Phi \in (H_c)$, which is even stronger than we need to have. $\square$

Question Does the previous result remains true without the assumption $\Phi(b(\Phi)) = \infty$ when $b(\Phi) < \infty$?

References

[1] Akcoglu, M.A., Sucheston, L.: On uniform monotonicity of norms and ergodic theorems in function spaces. Rend. Circ. Mat. Palermo II Ser. Suppl. **8**, 325–335 (1985)
[2] Bennett, C., Sharpley, R.: Interpolation of Operators, Pure and Applied Mathematics Series, vol. 129. Academic Press Inc., New York (1988)
[3] Birkhoff, G.: Lattice Theory. Providence, RI (1967)
[4] Chen, S.: Geometry of Orlicz spaces. Diss. Math. **356**, 1–204 (1996)
[5] Chilin, V.I., Dodds, P.G., Sedaev, A.A., Sukochev, F.A.: Characterisations of Kadec–Klee properties in symmetric spaces of measurable functions. Trans. Am. Math. Soc. **348**(12), 4895–4918 (1996)
[6] Ciesielski, M., Kamińska, A., Kolwicz, P., Płuciennik, R.: Monotonicity and rotundity properties of Lorentz spaces $\Gamma_{p,w}$. Nonlinear Anal. **75**, 2713–2723 (2012)
[7] Ciesielski, M., Kolwicz, P., Panfil, A.: Local monotonicity structure of symmetric spaces with applications. J. Math. Anal. Appl. **409**, 649–662 (2014)
[8] Ciesielski, M., Kolwicz, P., Płuciennik, R.: Local approach to Kadec–Klee properties in symmetric function spaces. J. Math. Anal. Appl. **426**, 700–726 (2015)
[9] Cui, Y., Hudzik, H., Szymaszkiewicz, L., Wang, T.: Criteria for monotonicity properties of Musielak–Orlicz spaces equipped with the Amemiya norm. J. Math. Anal. Appl. **303**(2), 376–390 (2005)
[10] Cui, Y., Hudzik, H., Wisła, M.: Monotonicity properties and dominated best approximation problems in Orlicz spaces equipped with the p-Amemiya norm. J. Math. Anal. Appl. **432**, 1095–1105 (2015)
[11] Dominguez, T., Hudzik, H., López, G., Mastyło, M., Sims, B.: Complete characterizations of Kadec–Klee properties in Orlicz spaces. Houst. J. Math. **29**(4), 1027–1044 (2003)
[12] Foralewski, P., Hudzik, H.: Some basic properties of generalized Calderón-Lozanovskiĭ spaces. In: Fourth International Conference on Function Spaces (Zielona Góra, 1995). Collect. Math., vol. 48, no. 4–6, pp. 523–538 (1997)
[13] Foralewski, P., Hudzik, H., Szymaszkiewicz, L.: On some geometric and topological properties of generalized Orlicz–Lorentz sequence spaces. Math. Nachr. **281**(2), 181–198 (2008)
[14] Hudzik, H., Kaczmarek, R., Wang, Y., Wójtowicz, M.: Problems of existence of order copies of l^{∞} and $L_p(\upsilon)$ in some non-Banach Köthe spaces (submitted)
[15] Hudzik, H., Kaczmarek, R., Wójtowicz, M.: Corrigendum to: "Some monotonicity properties in certain s-normed $(0 < s < 1)$ and F-normed lattices" (Vol. 17 (10) (2016), 1985–2011). J. Nonlinear Convex Anal. 18(12), 2275 (2017)
[16] Hudzik, H., Kaczmarek, R., Wójtowicz, M.: Some monotonicity properties in certain s-normed $(0 < s < 1)$ and F-normed lattices. J. Nonlinear Convex Anal. **17**(10), 1985–2011 (2016)
[17] Hudzik, H., Kamińska, A., Mastyło, M.: Monotonicity and rotundity properties in Banach lattices. Rocky Mt. J. Math. **30**(3), 933–950 (2000)

[18] Hudzik, H., Kurc, W.: Monotonicity properties of Musielak-Orlicz spaces and dominated best approximation in Banach lattices. J. Approx. Theory **95**(3), 353–368 (1998)
[19] Hudzik, H., Maligranda, L.: An interpolation theorem in symmetric function F-spaces. Proc. Am. Math. Soc. **110**(1), 89–96 (1990)
[20] Hudzik, H., Mastyło, M.: Strongly extreme points in Köthe–Bochner spaces. Rocky Mt. J. Math. **23**(3), 899–909 (1993)
[21] Hudzik, H., Narloch, A.: Relationships between monotonicity and complex rotundity properties with some consequences. Math. Scand. **96**, 289–306 (2005)
[22] Hudzik, H., Pallaschke, D.: On some convexity properties of Orlicz sequence spaces equipped with the Luxemburg norm. Math. Nachr. **186**, 167–185 (1997)
[23] Kalton, N.J., Peck, N.T., Roberts, J.W.: An F-Space Sampler, London Mathematical Society Lecture Note Series, vol. 89. Cambridge University Press, Cambridge (1984)
[24] Kamińska, A.: Indices, convexity and concavity in Musielak–Orlicz spaces. Funct. Approx. **26**, 67–84 (1998)
[25] Kantorovich, L.V., Akilov, G.P.: Functional Analysis (English translation), 2nd edn. Pergamon Press, Oxford (1982)
[26] Kolwicz, P.: Kadec–Klee properties of Calderón–Lozanovskiĭ function spaces, J. Funct. Spaces Appl. (2012), article ID 314068. https://doi.org/10.1155/2012/314068
[27] Kolwicz, P.: Kadec–Klee properties of Calderón–Lozanovskiĭ sequence spaces. Collect. Math. **63**, 45–58 (2012)
[28] Kolwicz, P.: Kadec–Klee properties of some quasi-Banach function spaces. Positivity. https://doi.org/10.1007/s11117-018-0555-8
[29] Kolwicz, P.: Local structure of symmetrizations $E^{(*)}$ with applications. J. Math. Anal. Appl. **440**, 810–822 (2016)
[30] Kolwicz, P., Płuciennik, R.: Points of upper local uniform monotonicity in Calderón–Lozanowskiĭ spaces. J. Convex Anal. **17**(1), 111–130 (2010)
[31] Krasnoselskiĭ, M.A., Rutickiĭ, Y.B.: Convex Functions and Orlicz Spaces, Groningen, Nordhoff, 1961 (English translation); Original Russian edition: Gos. Izd. Fiz. Mat. Lit, Moskva (1958)
[32] Krein, S.G., Petunin, Y.I., Semenov, E.M.: Interpolation of Linear Operators. Nauka, Moscow (1978). **(in Russian)**
[33] Kurc, W.: Strictly and uniformly monotone Musielak–Orlicz spaces and applications to best approximation. J. Approx. Theory **69**(2), 173–187 (1992)
[34] Kurc, W.: Strictly and uniformly monotone sequential Musielak–Orlicz spaces. Collect. Math. **50**(1), 1–17 (1999)
[35] Lindenstrauss, J., Tzafriri, L.: Classical Banach Spaces II, Function Spaces. Springer, Berlin (1979)
[36] Luxemburg, W.A.J.: Banach Function Spaces. PhD Dissertation, Delft (1955)
[37] Maligranda, L.: Orlicz Spaces and Interpolation, Seminars in Mathematics 5, Universidade Estadual de Campinas, Campinas, SP, Brazil (1989)
[38] Mazur, S., Orlicz, W.: On some classes of linear spaces. Stud. Math. **17**, 97–119 (1958); reprinted in: W. Orlicz, Collected Papers, PWN, Warszawa, 981–1003 (1988)
[39] Musielak, J.: Orlicz Spaces and Modular Spaces. Lecture Notes in Mathematics, vol. 1034. Springer, Berlin (1983)
[40] Musielak, J., Orlicz, W.: On modular spaces. Stud. Math. **18**, 49–65 (1959); reprinted in: W. Orlicz, Collected Papers, PWN, Warszawa, 1052–1068 (1988)
[41] Rao, M.M., Ren, Z.D.: Theory of Orlicz Spaces, Monographs and Textbooks in Pure and Applied Mathematics, vol. 146. Marcel Dekker Inc, New York (1991)
[42] Rolewicz, S.: Metric Linear Spaces, 2nd edn, Mathematics and its Applications, vol. 20, D. Reidel Publishing Co., Dordrecht; PWN - Polish Scientic Publisher, Warsaw (1985)
[43] Royden, H.L.: Real Analysis, 3rd edn. Macmillan Publishing Company, New York (1988)
[44] Wnuk, W.: On the order-topological properties of the quotient space L/L_A. Stud. Math. **79**, 139–149 (1984)

[45] Wnuk, W.: Representations of Orlicz lattices. Diss. Math. (Rozprawy Mat.) **235**, 1–62 (1984)
[46] Wnuk, W.: Banach Lattices with Order Continuous Norms. Polish Scientific Publishers PWN, Warsaw (1999)

Yunan Cui
Department of Mathematics
Harbin University of Science and Technology
Harbin 150080
China
e-mail: yunan_cui@aliyun.com

Henryk Hudzik
Faculty of Economics and Information Technology
The State University of Applied Sciences in Płock
Nowe Trzepowo 55
09-402 Płock
Poland

and

Faculty of Mathematics and Computer Science
Adam Mickiewicz University in Poznań
Umultowska 87
61-614 Poznań
Poland
e-mail: hudzik@amu.edu.pl

Radosław Kaczmarek
Faculty of Mathematics and Computer Science
Adam Mickiewicz University in Poznań
Umultowska 87
61-614 Poznań
Poland
e-mail: radekk@amu.edu.pl

Paweł Kolwicz
Institute of Mathematics, Faculty of Electrical Engineering
Poznań University of Technology
Piotrowo 3A
60-965 Poznań
Poland
e-mail: pawel.kolwicz@put.poznan.pl

Received: May 22, 2018