Verjetnostne metode v računalništvu - zapiski s predavanj prof. Marca

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Kazalo

Poglavje 1

Introduction

1.1 Probability

(Ω, F, P_r):

- $\emptyset \in F$,
- $A \in F \implies A^c \in F$,
- $A_1, A_2 \dots \in F \implies \cup_{i=1}^{\infty} A_i \in F$.

$P_r(A) \geq 0$,
$P_r\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P_r(A_i)$ if A_i disjoint,
$P_r\left(\cup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P_r(A_i)$,
$\Omega = \{\omega_1, \omega_2 \dots\}$ - countable case.
$\begin{pmatrix} \omega_1 & \omega_2 & \dots \\ p_1 & p_2 & \dots \end{pmatrix}$

Primer.

```
Alg():
  while True:
    B = sample as random from {0,1}  # 1 with probability p
    if B = 1:
```

```
return
```

$\Omega = \{1, 01, 001, 0001\dots\}$
$\begin{pmatrix} 1 & 01 & 001 & 0001 & \dots \\ p & (1-p)p & (1-p)^2p & (1-p)^3p & \dots \end{pmatrix}$.

1.2 Random variables

$X : \Omega \to \mathbb{Z}$.
$E[X] = \sum_{c\in\mathbb{Z}} c \cdot P_r(X = c)$ expected value of X.
Properties:

- $E[f(X)] = \sum_{c\in\mathbb{Z}} f(c) \cdot P_r(X = c)$,
- $E[aX + bY] = aE[X] + bE[Y]$,
- $E[X \cdot Y] = E[X] \cdot E[Y]$ if X, Y independent,
- $P_r(X \geq a) \leq \frac{E[X]}{a}$ $\forall a > 0$ $X \geq 0$ Markov inequality.

Primer. (Continuing from before).
$X =$ number of trials before return.
$X : \Omega \to \mathbb{Z}$.
$X : 1 \to 1, 01 \to 2, 003 \to 3\dots$
$\begin{pmatrix} 1 & 2 & 3 & 4 & \dots \\ p & (1-p)p & (1-p)^2p & (1-p)^3p & \dots \end{pmatrix}$ - geometric distribution.

Trditev 1.2.1. $E[X] = \frac{1}{p}$.

Dokaz 1.2.2. $X = \sum_{i=1}^{\infty} X_i$.
$X_i = \begin{cases} 1 \text{ if trial } i \text{ is executed} \\ 0 \text{ else} \end{cases}$

$$E[X] = E[\sum_{i=1}^{\infty} X_i] = \sum_{i=1}^{\infty} E[X_i] =$$
$$= \sum_{i=1}^{\infty} (1-p)^{i-1} = \frac{i=0}{\infty}(1-p)^i = \frac{1}{1-(1-p)} = \frac{1}{p}.$$

$E[X] = \frac{1}{p}$.
$P_r(X \geq 100 \cdot \frac{1}{p}) \leq \frac{E[X]}{\frac{1}{p}} = \frac{1}{100}$.

Definicija 1.2.3. $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{i=1}^{\infty} \frac{1}{i}$.

Izrek 1.2.4. $H_n \leq 1 + \ln(n)$.

Dokaz 1.2.5.

$$H_n = 1 + \sum_{i=2}^{n} \frac{1}{i} \overset{\text{integral}}{\leq} 1 + \int_1^n \frac{dx}{x} = 1 + \ln(x)|_1^n = 1 + \ln(n).$$

Poglavje 2

Quicksort, min-cut

2.1 Quicksort

```
Input: set (no equal element) (unordered list) S∈ ℝ
    (or whatever you can compare linearly)
Output: ordered list
Code:
  def Quicksort(S):
    if |S|= 0 or 1:
      return S
    else:
      a = uniformly at random from S
      S⁻ = {b ∈ S | b < a}
      S⁺ = {b ∈ S | a < b}
      return Quicksort(S⁻), a, Quicksort(S⁺)
```

$C(n)$ - random variable, the number of comparisons in evaluation of Quicksort with $|S| = n$.

Izrek 2.1.1. $E[C(n)] = O\left(N \log(n)\right)$.

Dokaz 2.1.2. $C(0) = C(1) = 0$.

$$
\begin{aligned}
E[C(n)] &= n - 1 + \sum_{i=1}^{n} \left(E[C(i-1)] + E[C(n-i)]\right) \cdot P_r(a \text{ is } i\text{-it element}) \leq \\
&\leq n + \frac{2}{n} \sum_{i=1}^{n-1} E[C(i)].
\end{aligned}
$$

Induction:

$n = 1 : \checkmark$

$n - 1 \to n$:

$$
\begin{aligned}
E[C(n)] &\leq n + \frac{2}{n} \sum_{i=1}^{n} E[C(i)] \leq \\
&\leq n + \frac{2}{n} \sum_{i=1}^{n} 5i \log i \leq \\
&\leq n + \frac{2}{n} \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} 5i \log i + \frac{2}{n} \sum_{i=1+\lfloor \frac{n}{2} \rfloor}^{n-1} 5i \log i \leq \\
&\leq n + \frac{2}{n} \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} 5i \log \frac{n}{2} + \frac{2}{n} \sum_{i=1+\lfloor \frac{n}{2} \rfloor}^{n-1} 5i \log n \leq \\
(\log \frac{n}{2} &= \log n - 1) \\
&\leq n + \frac{2}{n} \left(\sum_{i=1}^{n} 5i \log n - \sum_{i=1}^{\frac{n}{2}} 5i \right) = \\
&= n + \frac{10}{n} \left(\frac{n(n-1)}{2} \log n - \frac{\frac{n}{2}(\frac{n}{2}+1)}{2} \right) \leq \\
&\leq n + 5(n-1) \log n - n < \\
&< 5n \log n.
\end{aligned}
$$

$P\left(C(n) \geq b \cdot 5n \log n\right) \overset{\text{Markov}}{\leq} \frac{1}{b}$.

Dokaz 2.1.3.

2:

Let $S_1, S_2 \dots S_n$ sorted elements of S.

Define random variable $X_{ij} = \begin{cases} 1 \text{ if } S_i \text{ and } S_j \text{ are compared} \\ 0 \text{ else} \end{cases}$

$C(n) = \sum_{1 \le i < j \le n} E[X_{ij}]$.
$E[X_{ij}] = P(S_i$ and X_j compared).
S_{ij} - the last set including S_i and S_j.
$E[X_{ij}] = \frac{2}{|S_{ij}|} \le \frac{2}{j-i+1}$.
$|S_{ij}| \ge j - i + 1$.
S_{ij} has everything in between.

$$\begin{aligned}\implies E[C(n)] \le \sum_{1 \le i < j \le n} \frac{2}{j-i+1} = \\ \overset{k=j-i+1}{=} \sum_{i=1}^{n-1} \sum_{k=2}^{n-i+1} \frac{2}{k} \le \\ \le 2 \cdot n \cdot H_n \le \\ \le 2n(1 + \log n).\end{aligned}$$

2.2 Min-cut

G multigraph.
Cut: $U \subset V(G),\ U \neq \emptyset, V(g)$.
$(U, V(G) \setminus U) = \{uv \in E(G) \mid u \in U, v \in V(G) \setminus U\}$.
Problem min-cut:
Input: G.
Output: $\min |(U, V(G) \setminus U)|$ - cut size.

```
Algorithm 1:
  x ∈ V(G)
  Call maxFlow(G, x, y) ∀ y ∈ V(G)
  Take min
```

maxFlow is Edmonds-Karp algorithm $O\left(|V||E|^2\right)$.

```
Algorithm 2 (Stoer Wagner)
```

Is $O\left(|E||V| + |V|log|V|\right)$.

```
Algorithm randMinCut:
  G_0 = G
  i = 0
  while |V(G_i)| > 2:
    e_i = uniformly at random from G_i
    G_{i+1} = G_i / e_i
    i = i + 1
  u, v = V(G_{n-2}) // n = |V(G)|
  U = {w ∈ V(G) | w is merged into u}
  return (U, V(G) \ U)
```

Izrek 2.2.1. Algorithm *randMinCut* gives you a minimal cut with probability greater or equal to $\frac{2}{n(n-1)}$.

Dokaz 2.2.2.

Fact 1: $minCut(G_i) \leq minCut(G_i)$;

$\ngtr$: $minCut$ remains.

Fact 2: $minCut(G) \leq \delta(G)$.
$k := minCut(G)$.
Let $(A{,}B)$ be an optimal cut.
ϵ_i not in $(A{,}B)$.

$$\begin{aligned}
&P_r(\text{Algorithm not returning } (A{,}B)) \\
&= P_r(\epsilon_0 \cap \cdots \cap \epsilon_{n-3}) \\
&= P_r(\epsilon_0 \cap \cdots \cap \epsilon_{n-4}) \cdot P_r(\epsilon_{n-3} \mid \epsilon_0 \cap \cdots \cap \epsilon_{n-4}) \\
&= P_r(\epsilon_{n-3} \mid \cap_{i=0}^{n-4} \epsilon_i) \cdot P_r(\epsilon_{n-3} \mid \cap_{i=0}^{n-4} \epsilon_i) \\
&\ldots P_r(\epsilon_1 \mid \epsilon_0) \cdot P_r(\epsilon_0).(*)
\end{aligned} \tag{2.1}$$

$$P_r(\overline{\epsilon_i} \mid \epsilon_{i-1} \cap \cdots \cap \epsilon_0) = \frac{k}{|E(G_i)|} \overset{(**)}{\leq} \frac{k}{\frac{(n-i)k}{2}} = \frac{2}{n-i}$$

$$|E(G_i)| \geq \frac{(n-i)\delta(G)}{2} \geq \frac{(n-i)k}{2}.(**) \tag{2.2}$$

$$P_r(\epsilon_i \mid \epsilon_{i-1} \cap \cdots \cap \epsilon_0) \geq 1 - \frac{2}{n-i} = \frac{n-2-i}{n-i}.$$
$$(*) \geq \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdots \frac{1}{3} = \frac{2}{n(n-1)}.$$

Izrek 2.2.3. Running *randMinCut* $n(n-1)$ times and taking best output gives correct solution with probability ≥ 0.86.

Dokaz 2.2.4. A_i - event that i-th run gives sub-optimal solution.

$$\begin{aligned} P_r(\text{solution not correct}) &= P_r(A_1 \cap \cdots \cap A_{n(n-1)}) \\ &= \prod_{i=1}^{n(n-1)} P_r(A_i) \leq (1 - \frac{2}{n(n-1)})^{n(n-1)} \\ &\leq e^{-\frac{2}{n(n-1)} \cdot n(n-1)} = e^{-2} \leq 0.14. \end{aligned}$$

$1 - x \leq e^x \; \forall x \in \mathbb{R}$.

If we run $n(n-1)log(n)$ times $\to O\left(\frac{1}{n}\right)$.
$O\left(n^2 \log n \cdot n\right)$.
Improved: $O\left(n^2 \log^3 n\right)$.

Poglavje 3

Complexity classes

Decision problem - yes/no question on a set of inputs = asking $w \in \Pi$.
Randomized algorithms:

- Las Vegas algorithms: always gives correct solution, example: *Quicksort*.
- Monte Carlo algorithms: it can give wrong answers. Monte Carlo algorithms subtypes:
  - type(1): $\begin{cases} \text{if } \omega \in \Pi \implies & \text{algorithm returns „}\omega \in \Pi\text{“ with probability } \geq \frac{1}{2} \\ \text{if } \omega \notin \Pi \implies & \text{algorithm returns „}\omega \in \Pi\text{“ with probability } = 0 \end{cases}$
  - type(2): $\begin{cases} \text{if } \omega \in \Pi \implies & \text{algorithm returns „}\omega \in \Pi\text{“ with probability } = 1 \\ \text{if } \omega \notin \Pi \implies & \text{algorithm returns „}\omega \in \Pi\text{“ with probability } \leq \frac{1}{2} \end{cases}$
  - type(3): $\begin{cases} \text{if } \omega \in \Pi \implies & \text{algorithm returns „}\omega \in \Pi\text{“ with probability } \geq \frac{3}{4} \\ \text{if } \omega \notin \Pi \implies & \text{algorithm returns „}\omega \in \Pi\text{“ with probability } \leq \frac{1}{2} \end{cases}$

  type(1) and type(2): one-sided error, type(3): 2-sided error.
  $\frac{1}{2}$, $\frac{3}{4}$ and $\frac{1}{4}$ arbitrary numbers, can be something different (for type(3) better than coin flip).

Primer. Decisional problem: does a graph G have $minCut \leq k$?

Run $randMinCut(G)$ $n(n-1)$ times.

```
Algorithm randMinCut:
  if one of runs gives |(A,B)| ≤ k:
    return true
  else:
    return false
```

Complexity classes:

- RP (randomized polynomial time): decisional problems for which there exists Monte Carlo algorithm of type(1) with polynomial time complexity (worst case).
- co-RP: decisional problems for which there exists Monte Carlo algorithm of type(2) with polynomial time complexity (worst case).
- BRP (bounded-error probabilistic polynomial time): decisional problems for which there exists Monte Carlo algorithm of type(3) with polynomial time complexity (worst case).
- ZPP (zero-error probabilistic polynomial time): decisional problems for which there exists Las Vegas algorithm with expected polynomial time complexity (worst case).

ZPP = RP $\cap$ co-RP.

Poglavje 4

Chernoff bounds

Izrek 4.0.1. Let $X_1, X_2 \dots X_n$ independent random variables with image $\{0, 1\}$.
Let $p_i = P_r(X_i = x_i), X = \sum_{i=1}^{n} X_i$ and $\mu = E(X) = p_1 + \cdots + p_n$.
For every $\delta \in$ (0,1):

$$
\begin{aligned}
P_r(X - \mu \geq \delta\mu) &\leq e^{-\frac{\delta^2\mu}{3}} \\
P_r(\mu - X \leq \delta\mu) &\leq e^{-\frac{\delta^2\mu}{2}} \\
\implies P_r(|X - \mu| \geq \delta\mu) &\leq e^{-\frac{\delta^2\mu}{3}}.
\end{aligned}
$$

Probability falls extremely quickly after $E(X)$.

Dokaz 4.0.2.

$$\begin{aligned}
P_r(X-\mu \geq \delta\mu) &= P_r(X \geq \mu(1+\delta)) \\
&\overset{t\geq 0}{=} P_r(tX \geq t\mu(1+\delta)) \\
&\overset{e^y\geq 0}{=} P_r(e^{tX} \geq e^{t\mu(1+\delta)}) \\
&\overset{\text{Markov}}{\leq} \frac{E\left(e^{tX}\right)}{e^{t\mu(1+\delta)}} \\
&\overset{4.1}{\leq} \frac{e^{(e^t-1)\mu}}{e^{t\mu(1+\delta)}} \\
&\overset{4.3}{\leq} e^{-\mu\frac{\delta^2}{3}}.
\end{aligned}$$

$$\begin{aligned}
E(e^{tX}) &= E(e^{tX_1+\dots+tX_n}) \\
&= E(e^{tX_1}\dots e^{tX_n}) \\
&\overset{\text{independent}}{=} \prod_{i=1}^{n} E(e^{tX_i}) \\
&\overset{4.2}{\leq} \prod_{i=1}^{n} e^{p_i(e^t-1)} \\
&= e^{(e^t-1)\sum_{i=1}^n p_i} \\
&= e^{(e^t-1)\mu}. \qquad (4.1)
\end{aligned}$$

$$E(e^{tX_i}) = p_i\cdot e^t + (1-p_i)\cdot e^0 = 1 + p_i(e^t-1) \overset{1+x\leq e^x}{\leq} e^{p_i(e^t-1)}. \qquad (4.2)$$

Want:

$$e^t - 1 - t(1+\delta) \leq -\frac{\delta^2}{3}\ \forall\delta\in(0{,}1) \qquad (4.3)$$

$$t = \ln(1+\delta)$$

$$f(\delta) = 1+\delta-1-(1+\delta)\ln(1+\delta)+\frac{\delta^2}{3} \overset{?}{\leq} 0$$

$$f(0) = 0$$

$$f^{'}(\delta) = 1-\ln(1+\delta)-1+\frac{2}{3}\delta = \frac{2}{3}\delta-\ln(1+\delta) \overset{?}{\leq} 0$$

$$\frac{2}{3}\delta \leq \ln(1+\delta)$$

$$\delta = 1:\ \frac{2}{3} \overset{?}{\leq} \ln(2) \approx 0.69\checkmark$$

$$\begin{aligned} P_r(\mu - X \le \delta\mu) &= P_r(X \ge \mu(1-\delta)) \\ &\overset{t\ge 0}{=} P_r(tX \ge t\mu(1-\delta)) \\ &\overset{e^y \ge 0}{=} P_r(e^{tX} \ge e^{t\mu(1-\delta)}) \\ &\le \dots \le \frac{e^{(e^t-1)\mu}}{e^{t\mu(1-\delta)}}. \end{aligned}$$

Want: $e^t - 1 - t(1-\delta) \le -\frac{\delta^2}{2}\ \forall\delta \in (0{,}1)$:

$$\begin{aligned} &t = \ln(1-\delta) \\ &f(\delta) = 1 - \delta - 1 - (1-\delta)\ln(1-\delta) + \frac{\delta^2}{2} \overset{?}{\le} 0 \\ &f(0) = 0 \\ &f'(\delta) = -1 + 1 - \ln(1-\delta) + \delta \overset{?}{\le} 0 \\ &\frac{2}{3}\delta \le \ln(1+\delta) \\ &\ln(1-\delta) \overset{?}{\le} -\delta \checkmark \end{aligned}$$

■

$X_i \sim \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$
$X = \sum_{i=1}^n X_i$
$\mu = \frac{n}{2}$

$$\begin{aligned} P_r(|X-\mu| \ge \sqrt{\frac{3}{2}n\ln(n)}) &= P_r(|X-\mu| \ge \frac{n}{2}\sqrt{\frac{6}{n}\ln(n)}) \\ &\mu = \frac{n}{2}, \delta = \sqrt{\frac{6}{n}\ln(n)}, \\ &\text{for „big“ } n \delta \in (0{,}1) \\ &\overset{\text{Chernoff}}{\le} 2e^{-\frac{\frac{n}{2}\frac{6}{n}\ln(n)}{3}} = \frac{2}{n}. \end{aligned}$$

$d = \sqrt{\frac{3}{2}n\ln(n)}$

$$\implies P_r(X \in (\mu - \sqrt{\frac{3}{2}n\ln(n)}, \mu + \sqrt{\frac{3}{2}n\ln(n)})) \ge 1 - \frac{2}{n}.$$

Trditev 4.0.3.
Let $X_1, X_2 \dots$ independent random variables with image {0,1}.
$P_r(X_i = 1) = \frac{1}{2}\ \forall i$.
Let $X = \sum_{i=1}^{cm} X_i$ where $c \geq 4$.
Then $P_r(X \leq m) \leq e^{-\frac{cm}{16}}$.

Dokaz 4.0.4.

$$\begin{aligned}
P_r(X \leq m) &= P_r(\frac{cm}{2} - X \geq \frac{cm}{2} - m) \\
&= P_r(\frac{cm}{2} - X \geq \frac{cm}{2}(1 - \frac{2}{c})) \\
&\overset{\text{Chernoff}}{\leq} e^{-\frac{\frac{cm}{2}(1-\frac{2}{c})^2}{2}} \\
&1 - \frac{2}{c} \geq \frac{1}{2} \text{ if } c \geq 4 \\
&\leq e^{-\frac{\frac{cm}{2}\frac{1}{4}}{2}} = e^{-\frac{cm}{16}}.
\end{aligned}$$

■

Back to Quicksort.

Izrek 4.0.5.
With probability $\geq 1 - \frac{1}{n}$ Quicksort uses at most $48n\ln(n)$ comparisons.

Dokaz 4.0.6.
For $s \in S$ define $S_1^S \dots S_{t_s}^S \neq \emptyset$ sets that include s, t_s - number of comparisons with s where s is not a pivot +1.
Define: iteration i is successful if $|S_{i+1}| \leq \frac{3}{4}|S_i|$ ($\frac{1}{2}$ is too strict).

$$X_i = \begin{cases} 1 \text{ if iteration } i \text{ is successful} \\ 0 \text{ else} \end{cases}$$

$P_r(X_i = 1) \geq \frac{1}{2}$
$S_i : n \to \frac{3}{4}n \to (\frac{3}{4})^2 n \to \dots \to 1$.
Notice: max number of iteration is $\log_{\frac{4}{3}}(n) = \frac{\ln(n)}{\ln(4)-\ln(3)}$.

Probability that we haven't succeeded in $\log_{\frac{4}{3}}(n)$ steps:

$$P_r\Big(\sum_{i=1}^{c\log_{\frac{4}{3}}(n)} X_i < \log_{\frac{4}{3}}(n)\Big) \leq P_r\Big(\sum_{i=1}^{c\log_{\frac{4}{3}}(n)} Y_i < \log_{\frac{4}{3}}(n)\Big) \tag{4.4}$$

$$\overset{\text{Chernoff}}{<} e^{-\frac{c\log_{\frac{4}{3}}(n)}{24}} \tag{4.5}$$

$$= e^{-\frac{c\ln(n)\log_{\frac{4}{3}}(e)}{24}} \tag{4.6}$$

$$= \frac{1}{n}\frac{c\log_{\frac{4}{3}}(e)}{24} \tag{4.7}$$

$$\log_{\frac{4}{3}}(e) \approx 3.4,\ c = 14 \tag{4.8}$$

$$\leq \left(\frac{1}{n}\right)^2 \tag{4.9}$$

4.4 because X_i not independent, $Y_i \sim \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ independent.

$P_r(t_s \geq c\log_{\frac{4}{3}}(n)) \geq \left(\frac{1}{n}\right)^2$ for one s.

$c = 14 \implies$ at least $48\ln(n)$ iterations with probability $\leq \left(\frac{1}{n}\right)^2$.

With probability as least $1 - \frac{1}{n}$ for all $s \in S$ it holds that s has $\leq 48\ln(n)$ comparisons with a pivot.

$\implies$ total number of comparisons $n \cdot 48\ln(n)$ with probability as least $1 - \frac{1}{n}$. ■

Poglavje 5

Monte Carlo methods

5.1 Example 1

Area of circle $= \frac{\pi}{4}$.

$X_i = \begin{cases} 1 \text{ if you hit the area of circle} \\ 0 \text{ else} \end{cases}$

$P_r(X_i = 1) = \frac{\frac{\pi}{2}}{1} = \frac{\pi}{4}$.

$E(X_i) = \frac{\pi}{4}$.

$X = \frac{\sum_{i=1}^{n} X_i}{n}$.

$E(X) = \frac{n \cdot E(X_i)}{n} = E(X_i)$.

5.2 Example 2

$I = \int_\Omega f(x)dx$ - volume.

$X_i = \begin{cases} 1 \; F(x_i, y_i) \leq z_i \\ 0 \text{ otherwise} \end{cases}$

$v \cdot E\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) = I$.

5.3 (ϵ,δ)-approximation

Definicija 5.3.1 **(**(ϵ,δ)-approximation**).** A random algorithm gives a (ϵ,δ)-approximation for value v if the output X satisfies:

$$P_r(|X-v| \leq \epsilon v) \geq 1-\delta.$$

Izrek 5.3.2. Let $X_1 \dots X_n$ be independent and identically distributed indicator variables. Let $\mu = E(X_i)$, $Y = \frac{\sum_{i=1}^m X_i}{m}$. If $m \geq \frac{3\ln\left(\frac{2}{\delta}\right)}{\epsilon^2\mu}$, then $P_r(|Y-\mu| \geq \epsilon\mu) \leq \delta \implies Y$ is (ϵ,δ)-approximation for μ.

Dokaz 5.3.3.
$X = \sum_{i=1}^n X_i$
$E(X) = mE(x_i) = m\mu$
$m \geq \frac{3\ln\left(\frac{2}{\delta}\right)}{\epsilon^2\mu}$

$$\begin{aligned} P_r(|Y-\mu| \geq \epsilon\mu) &= P_r(\left|\frac{X}{m}-\mu\right| \geq \epsilon\mu) \\ &= P_r(\frac{1}{m}\left|X-E(X)\right| \geq \frac{1}{m}\epsilon E(x)) \\ &\overset{\text{Chernoff}}{\leq} 2e^{-\frac{\epsilon^2 E(x)}{3}} \\ &= 2e^{-\frac{\epsilon^2\mu m}{3}} \\ &\leq 2e^{-\frac{\epsilon^2\mu}{3}\cdot\frac{3\ln\left(\frac{2}{\delta}\right)}{\epsilon^2\mu}} = \delta. \end{aligned}$$

Back to example 1:
$E(Y) = \frac{\pi}{4}, \delta = \frac{1}{1000}$ (99.9% sure), $\epsilon = \frac{1}{10000}$
$\implies M = \frac{3\ln\left(\frac{2}{\frac{1}{1000}}\right)4}{\pi\left(\frac{1}{10000}\right)^2} \approx 29106.$
Problems for MC (Monte-Carlo):

- rare events, e.g. $X \sim \begin{pmatrix} 0 & 10^{100} \\ 1-10^{-20} & 10^{-20} \end{pmatrix}$, $E(X) = 10^{80}$

5.4 DNF counting

CNF: $(X_{i_1} \vee \overline{X_{i_2}} \vee X_{i_4}) \wedge (X_{i_1} \vee \overline{X_{i_3}}) \wedge \dots$

DNF: $(\overline{X_{i_1}} \wedge X_{i_2} \vee \overline{X_{i_4}}) \vee \dots$ - easy to determine if solution exists.

Question: number of solutions to a given DNF?

Observation: CNF F has a solution $\iff$ DNF $\neg F$ has less than 2^n solutions, n is number of samples.

```
ALG_1(F):
  x = 0
  for i in range(1,m+1):
    x_1 ... x_n uniformly random from {0,1}^n
    if F(x_1 ... x_n) = 1:
      x += 1
  return x/m · 2^n
```

$Y = \frac{\sum_{i=1}^{m} X_i}{m}$

(ϵ,δ)-approximation for Y

$E(Y) = \frac{\text{number of solutions of } F}{2^n} = \frac{c(F)}{2^n}$

$m \geq \frac{3\ln\left(\frac{2}{\delta}\right)}{\epsilon^2 E(X)} = \frac{3\ln\left(\frac{2}{\delta}\right)}{\epsilon^2} \cdot \frac{2^n}{x(F)}$

$c(F)$ very small $\rightarrow$ m exponentially big $\rightarrow$ not good (we need a lot of samples).

Definicija 5.4.1.

$SC_i = \{(a_1 \dots a_n) \in \{0,1\}^n$ such that $F = F_1 \vee \dots \vee F_t,\ F_i(a_1 \dots a_n) = 1\}$.

$|SC_i| = 2^{n-l_i}$, l_i: number of values in F_i

$U = \{(i,a) \mid i \in \{1,2 \dots t\}, a \in SC_i\}$

$U = \sum_{i=1}^{t} |SC_i|$ - $O(tn)$ (space smaller than $\{0,1\}^n$)

$S = \{(i,a) \in U \mid a \in SC_i,\ a \notin SC_j\ 1 \leq j < i\}$

$|S| = |SC_1| + \dots + |SC_t| = c(F)$.

```
ALG_2(F):
  x = 0
  for i in range(1,m+1):
```

```
    (i, a) uniformly random from U (**)
    if (i, a) ∈ S: (*)
      x += 1
  return x/m · |U|
```

$(*)\ a \in SC_i \to O(n),\ a \notin SC_j\ j = 1 \dots i-1 \to O(tn) \implies O(tn), m$ times.

$(**)$: watch for details on how to, e.g. $x_2, x_2 \wedge x_3$: x_2 is more probable than $x_2 \wedge x_3 \to O(1)$.

Izrek 5.4.2. For $m = \lceil \frac{3t\ln\left(\left(\frac{2}{\delta}\right)\right)}{\epsilon^2} \rceil$ algorithm returns (ϵ,δ)-approximation in $O\left(\frac{t^n n \ln\left(\frac{2}{\delta}\right)}{\epsilon^2}\right)$ time.

Dokaz 5.4.3. $O(t \cdot n \cdot m)$.

Insert $m = ...$

Prove

$$P_r(Y|U| - c(F) > \epsilon c(F)) < \delta :$$

$c(F) = |S|, E(Y) = \frac{|S|}{|U|}$

$$P_r(Y|U| - c(F) > \epsilon c(F)) = P_r(|U|(Y - E(Y)) > \epsilon |U| E(Y)) \leq \delta$$

if

$$m \geq \frac{3\ln\left(\frac{2}{\delta}\right)}{\epsilon^2 E(Y)} \geq \frac{3\ln\left(\frac{2}{\delta}\right)t}{\epsilon^2}$$

where

$$E(Y) = \frac{|S|}{|U|} \geq \frac{1}{t}$$

(= if disjoint).

In new space $E(Y)$ much larger $\implies$ m smaller.

Poglavje 6

Polynomials

Let $\mathbb{F}$ be a field.
$\mathbb{F}$ can be $\mathbb{R}, \mathbb{C}, \mathbb{Z}_p, \mathbb{F}_{p^n}$.
$\mathbb{F}[x_1 \dots x_n]$ algebra of polynomials with values $x_1 \dots x_n$.
$f \in \mathbb{F}[x_1 \dots x_n]$
$deg(f[x_1 \dots x_n]) := deg(f[x \dots x])$.

Izrek 6.0.1. Let $p(x_1 \dots x_n) \in \mathbb{F}[x_1 \dots x_n]$ have the degree $d \geq 0$ and $p \neq 0$. Let $s \subset \mathbb{F}$ be finite. If $(r_1 \dots r_n)$ is uniformly at random element from S^n. Then $P_r(p(r_1 \dots r_n) = 0) \leq \frac{d}{|S|}$.

Dokaz 6.0.2. Induction on n.
$n = 1$:

$$\begin{aligned}
&p(x) = (x - z_1)(x - z_2) \dots (x - z_j)q(z) \\
&\text{number of zeros } \leq \text{ degree - fact} \\
&P_r(p(r_1) = 0) = \frac{\text{number of zeros}}{|S|} \leq \frac{d}{|S|}.
\end{aligned}$$

$n-1 \to n$:

$$\begin{aligned}
&\text{rewrite } p: \\
&p(x_1 \dots x_n) = \sum_{i=0}^{j} x^i p_i(x_2 \dots x_n) \\
&j \leq d
\end{aligned}$$

$$\begin{aligned}
P_r(p(r_1 \dots r_n) = 0) &= P_r(p(r_1 \dots r_n = 0) \mid p_j(r_2 \dots r_n) = 0) \cdot P_r(p_j(r_2 \dots r_n) = 0) \\
&+ P_r(p(r_1 \dots r_n = 0) \mid p_j(r_2 \dots r_n) \neq 0) \cdot P_r(p_j(r_2 \dots r_n) \neq 0) \\
&\leq 1 \cdot \frac{d-j}{|S|} + \frac{j}{|S|} \cdot 1,
\end{aligned}$$

because

$$\begin{aligned}
&P_r(p(r_1 \dots r_n = 0) \mid p_j(r_2 \dots r_n) \neq 0) \leq \frac{d-j}{|S|} \\
&P_r(p_j(r_2 \dots r_n) \neq 0) \leq \frac{j}{|S|}.
\end{aligned}$$

Problem:
Let A,B,$C \in \mathbb{F}^{n \times n}$, is $A \cdot B = C$?
Computing $A \cdot B$:

- school-book algorithm: $O\left(n^3\right)$,
- Strassen algorithm: $O\left(n^{2{,}807\dots}\right)$,
- galactic algorithm: $O\left(n^{2.372\dots}\right)$ - has enormous constants.

```
RAND_ACB(A,B,C):
  for i in range(1,k+1):
    x uniformly at random from {0,1}^n
    if A · (B · x) ≠ x:
      return false
  return true
```

$O\left(kn^2\right)$.

If $A \cdot B = C$, algorithm returns true.
If $A \cdot B \neq C$:

$$\begin{aligned} P_r(ABx = Cx) &= P_r((AB - C)x = 0) \\ &= P_r(||(AB - C)x||^2 = 0) \overset{\text{Poly}}{\leq} \frac{2}{3}. \end{aligned}$$

$||(AB_C)x||^2$ - polynomial in $x_1 \dots x_n$ of degree 2.
If $A \cdot B \neq C$, then algorithm return false with probability at least $1 - \left(\frac{2}{3}\right)^k$.
Problem:
1-factor in bipartite graphs.
$|V(g)| = 2n$.
Represent G with $n \times n$ matrix $Z = (Z_{ij})_{i,j=1}^n$

$$Z_{ij} = \begin{cases} X_{ij} \text{ if } a_i b_j \in E(x) & \text{(X: variable)} \\ 0 \text{ else} \end{cases}$$

$$\begin{aligned} detZ(x_{11} \dots x_{nn}) &= \sum_{\pi \in S_n} sign(\pi) z_{1,\pi(1)} \dots z_{n,\pi(n)} \\ &= \sum_{\pi \in S_n, \pi \text{ defines 1-factor}} sign(\pi) x_{1,\pi(1)} \dots x_{n,\pi(n)}. \end{aligned}$$

$detZ \neq 0 \iff G$ has 1-factor.

```
Rand_1factor(G):
  construct Z with variables x11 ... xnn
  for i in range(1,k+1):
    u <- uniformly at random from  1,2..2n-1^(n^2) (r11 ... rnn)
    compuze d = det Z(r11 ... rnn)
    if d != 0:
      return true
  return false
```

Complexity: $k\cdot$ computing determinant: $O\left(n^3\right)$ (Gaussian elimination).
or apply approximation algorithm:

- if G has no 1-factor it always returns false,
- if G has 1-factor, it returns true with probability at least $1-\left(\frac{n}{2n}\right)^k = 1-\left(\frac{1}{2}\right)^k$ (k konstant, larger set $\implies$ smaller k needed).

Poglavje 7

Random graphs

7.1 G(n,p) model

G is a random Erdös-Rény graph if it has n vertices and each pair of vertices is connected with probability p.

Primer. $G\left(5, \frac{1}{2}\right)$.

$E($ edges in G fron $G(n,p)) = \sum_{1 \leq i < j \leq n} E(X_{ij}) = \binom{n}{2} p$.

$X_{ij} = \begin{cases} 1 \text{ if } i \text{ and } j \text{ have edge} \\ 0 \text{ otherwise} \end{cases}$

p can be function of n.

Y_v : degree of v.

$E(Y_v) = (n-1)p$.

Definicija 7.1.1.

We say that a random graph has some property almost surely (A.S.) if $P_r(G \in G(n,p)$ has property$) \overset{n \to \infty}{\to} 1$.

Trditev 7.1.2.

Let p be constant. Then $G \in G(n,p)$ has diameter 2 A.S.

Dokaz 7.1.3.

Let $u,v \in V(G)$
$X_w = \begin{cases} 1 \text{ if } uw \in E(G) \text{ in } vw \in E(G) \end{cases}$
$P_r(X_w = 1) = p^2$
$P_r(X_w = 0 \text{ for all } w \neq u,v) = (1-p^2)^{n-2}$.
$P_r(G \text{ has diameter} > 2)$
$= P_r(X_w = 0 \text{ for all } w \notin u,v \text{ for some } u,v)$
$\leq \binom{n}{2}(1-p^2)^{n-2} \overset{n\to\infty}{\to} 0;$
$\binom{n}{2}$ - polynomial, $e^{\cdots}$ - exponent.

$p = f(n)$
$\frac{1}{n}, \frac{1}{n^3}, \frac{\log n}{n}$

Izrek 7.1.4. (without proof)
Let p be a function of n: let $G \in G(n,p)$:

- $np < 1$ - G A.S. disconnected with connected components of size $O(\log n)$
- $np = 1$ - G A.S. has 1 large component of size $O\left(n^{\frac{2}{3}}\right)$
- $np = c > 1$ - G A.S. has giant component of size $dn,\ d \in (0,1)$
- $np \leq (1-\epsilon)\ln n$ - G A.S. disconnected with isolated vertices
- $np > (1-\epsilon)\ln n$ - G A.S. connected.

Izrek 7.1.5.
Let $np = \omega(n)\ln(n)$ for $\omega(n) \to \infty$ „very slowly“ think of $\omega(n) = \log(\log n)$, then $diam(G)$ in $\Theta\left(\frac{\ln n}{\ln(np)}\right)$ for G in $G(n,p)$.

Lema 7.1.6.
Let $S \subset V(G), |S| = cn$ for $c \in (0,1]$ and $v \notin S$.
then $cnp(1-\omega^{-\frac{1}{3}}) \leq N_S(v) \leq cnp(1+\omega^{-\frac{1}{3}})$ A.S. ($\omega^{-\frac{1}{3}} \to 0$ very slowly).

Dokaz 7.1.7. (Lemma):
$E(N_s(v)) = c \cdot n \cdot p, \delta = \omega^{-\frac{1}{3}}$

$$\begin{aligned} P_r(|N_s(v) - cnp| \geq \delta cnp) &\overset{\text{Chernoff}}{\leq} 2e^{-\frac{\omega^{-\frac{2}{3}} cnp}{3}} \\ &= 2e^{-\frac{cnp}{3\omega(n)^{\frac{2}{3}}}} \overset{n\to\infty}{\to} 0. \end{aligned}$$

For all v: $n \cdot 2e^{-\frac{cnp}{3\omega(n)^{\frac{2}{3}}}} \overset{n\to\infty}{\to} 0$.

Dokaz 7.1.8. (Theorem):
k be such that $\sum_{i=0}^{k-1} |N_i| \leq \frac{n}{2}, \sum_{i=0}^{k} |N_i| > \frac{n}{2}$.
$|N_0| = 1$
$|N_i| \leq |N_{i-1}| \cdot n \cdot p \cdot (1 + \omega^{-\frac{1}{3}})$:
$|S| \leq n,\ np(1 + \omega^{-\frac{1}{3}})$-each element.
$k = \frac{\log\left(\frac{n}{3}\right)}{\log\left(n \cdot p \cdot \left(1 + \omega^{-\frac{1}{3}}\right)\right)}$
$= \log_{np(1+\omega^{-\frac{1}{3}})} \frac{n}{3} = \Theta\left(\frac{\ln(n)}{\ln(np)}\right)$.
$|N_{\leq k}| = |N_1 \cup \cdots \cup N_k|$.

$$\begin{aligned} |N_{\leq k}| &\leq \sum_{i=0}^{k} (np(1 + \omega^{-\frac{1}{3}}))^i \\ &= \frac{(np(1 + \omega^{-\frac{1}{3}}))^{k+1} - 1}{np(1 + \omega^{-\frac{1}{3}}) - 1} \\ &< \frac{np(1 + \omega^{-\frac{1}{3}})^{k+1}}{\frac{1}{2} np(1 + \omega^{-\frac{1}{3}})} \\ &= 2np(1 + \omega^{-\frac{1}{3}})^k \\ &\overset{k}{=} 2 \cdot \frac{n}{3} \text{ haven't covered all} \\ &\implies diam(G) > k \text{ bound from below.} \end{aligned}$$

$N_i \subseteq S$
$\frac{1}{2} np \left(1 - \omega^{-\frac{1}{3}}\right) \cdot |N_{i-1}| \leq |N_i|$

$$\begin{aligned} n &\geq \sum_{i=0}^{k} |N_i| \\ &\geq \sum_{i=0}^{k} \left(\frac{1}{2} np \left(1 - \omega^{-\frac{1}{3}}\right)\right)^i \\ &= \frac{\left(\frac{1}{2} np \left(1 - \omega^{-\frac{1}{3}}\right)\right)^{k+1} - 1}{\frac{1}{2} np \left(1 - \omega^{-\frac{1}{3}}\right) - 1} \\ &\geq \left(\frac{1}{2} np \left(1 - \omega^{-\frac{1}{3}}\right)\right)^k \qquad / \ln \end{aligned}$$

$\frac{\ln n}{\ln(np)} \approx \frac{\ln n}{\ln\left(\frac{1}{2} np \left(1 - \omega^{-\frac{1}{3}}\right)\right)} \geq k.$
$\implies w \in S'.$
Number of neighbors in N_k A.S. ≥ 1,
$|N_k| \geq \left(\frac{1}{2} np \left(1 - \omega^{-\frac{1}{3}}\right)\right)^k \approx c \cdot n$
$\implies diam(G) = k + 1$ A.S.

7.1.1 Scale free property

$G \in G(n,p)$.

In real world: $p(k)$ = proportion of degree k vertices.

$\log(p(k)) = -\gamma \cdot \log k$

$p(k) = k^{-\gamma}$.

Internet: $\gamma \approx 3.42$,

protein reactions: $\gamma \approx 2.89$.

7.2 Barbási-Albert Model

B.A. model.

Start with m modes.

Grow:

- add node v,

- add m edges from v (to u),
- for each new edge: $P(v \sim u) = \frac{degu}{\sum_x degx}$.

Izrek 7.2.1.

B.A. model has scale free property, in particular
$p_k = \frac{2m(m+1)}{k(k+1)(k+2)}$.

Definicija 7.2.2.

$p_n(k)$: expected proportion of degree k vertices in graph with k vertices,
$p_k := \lim_{n\to\infty} p_n(k)$.

Dokaz 7.2.3.

$p_n(k) \cdot n$: expected number of degree k vertices,
$p_n(k)n \cdot \frac{k}{\sum_u degu} m = p_n(k) \cdot \frac{k}{2}$: expected number of degree k vertices changing into degree $k+1$ vertices.
$\sum_u degu = 2|E|$
$p_{n+1}(k) \cdot (n+1) = p_n(k) \cdot n - p_n(k) \cdot \frac{k}{2} + p_n(k-1) \cdot \frac{k-1}{2}$, where
$p_n(k) \cdot n$: degree $k \to k$,
$p_n(k) \cdot \frac{k}{2} : k \to k+1$,
$p_n(k-1) \cdot \frac{k-1}{2} : k-1 \to k$.
For n very big (very close to limit):
$p_n \cdot (n+1) = p_k \cdot n - p_{k-1} \cdot \frac{k}{2} + p_{k-1} \cdot \frac{k-1}{2}$
$\implies p_k = \frac{k-1}{k+2} p_{k-1}$.
For degree m:
$(n+1) \cdot p_{n+1}(m) = p_n(m) \cdot n - p_n(m) \cdot \frac{m}{2} + 1$

$$
\begin{aligned}
p_m &= \frac{2}{m+2} \\
\implies p_{m+1} &= \frac{2}{m+2} \cdot \frac{m}{m+3} \\
\implies p_{m+2} &= \frac{2m(m+1)}{(m+2)(m+3)} \\
\implies p_k &= \frac{2m(m+1)}{k(k+1)(k+2)}.
\end{aligned}
$$

Poglavje 8

Markov chains

Ω: finite set (of states).

Definicija 8.0.1 (Markov chain)**.**
(Discrete time) Markov chain is a sequence of random variables $X = X_0, X_1, X_2 \ldots$ with image Ω and properties:

- $P(X_{i+1} = x \mid X_i = x_i, X_{i-1} = x_{i-1} \ldots X_0 = x_0) =$
  $P(X_{i+1} = x \mid X_i = x_i)$,
- $PX_{i+1} = x \mid X_i = y = P(X_1 = x \mid X_0 = y)$ - time is homogenous.

Primer.
$\Omega = \mathbb{Z}_5$
$P(X_{i+1} = x + 1 \mid X_i = x) = \frac{1}{2}$
$P(X_{i+1} = x - 1 \mid X_i = x) = \frac{1}{2}$.

Definicija 8.0.2 (Transition matrix)**.**
$\Omega = \{x_1 \ldots x_n\}$
$p_{ij} = P(X_{t+1} = j \mid X_t = i)$

$$\begin{bmatrix} p_{11} & \dots & \\ p_{1n} & & \\ \vdots & & \vdots \\ p_{n1} & \dots & p_{nn} \end{bmatrix}.$$

Definicija 8.0.3 (Transition graph)**.**

Edge between states i and j exists if $p_{ij} > 0$.

P is stochastic matrix:
$p_{ij} \in [0,1]$
$\sum_j p_{ij} = 1$.
We choose beginning state randomly.
$q(0) = (q_1(0) \dots q_n(0))$
$P(X_0 = i) = q_i(0)$.
Let $q(t) = (q_1(t) \dots q_n(t))$
$P(X_t = i) = q_i(t)$.
It holds: $q(t) = q(t-1) \cdot P = q(0) \cdot P^t$.

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$q(0) = (1, 0, 0, 0, 0)$
$q(1) = (1, \frac{1}{2}, 0, 0, \frac{1}{2})$
$q(2) = (\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{4}, 0)$
$\vdots$

Definicija 8.0.4.

- Distribution π is stationary if $\pi = \pi \cdot P$,
- f_{ij}: probability that $X_t = x_j$ for some t assuming $X_0 = x_i$,

- h_{ij}: expected number of steps needed to get to state X_j strting in X_i (hitting time),
- $N(i, t, q(0))$: expected number of times we visit x_i after t steps starting with distribution $q(0)$,
- $\forall f_{ij} > 0 \iff$ transition graph is strongly connected $\iff$ we say the chain is irreducible,
- M.C. is aperiodic if there is no $c \in \{2, 3, 4 \dots\}$ such that all lengths of cycled are divisible by c.

Izrek 8.0.5.
Let X be finite irreducible M.C. Then:

a) there exists unique stationary distribution $\pi = (\pi_1 \dots \pi_n)$,

b) $f_{ii} = 1, h_{ii} = \frac{1}{\pi_i}$,

c) $\lim_{t\to\infty} \frac{N(i,t,q(0))}{t} = \pi_i$ - approaches π regardless of $q(0)$,

d) if X is aperiodic: $\lim_{t\to\infty} q(0) \cdot P^t = \pi$.