MAT 1341 DGD

Introduction to Linear Algebra

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2013

- **TA:**
- **TA email:**

12. Find the polar form of

$$Z = \frac{1+i}{1-\sqrt{3}i}$$

$$Z_1 = 1 + i$$

$$Z_2 = 1 - \sqrt{3}i$$

$$Z = \frac{Z_1}{Z_2}$$

$$Z_1 = |Z_1|e^{i\theta_1} = |Z_1|(\cos\theta_1 + i\sin\theta_1)$$

$$Z_2 = |Z_2|e^{i\theta_2} = |Z_2|(\cos\theta_2 + i\sin\theta_2)$$

This is the definition of the polar form

- $Z_1 = 1 + i,\ |Z_1| = \sqrt{1^2 + 1^2} = \sqrt{2}$
- $Z = a + bi,\ |Z| = \sqrt{a^2 + b^2}$
- $Z_2 = 1 - \sqrt{3}i,\ |Z_2| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$

$$\left.\begin{aligned} \cos\theta_1 &= \frac{1}{Z_1} = \frac{1}{\sqrt{2}} \\ sin\theta_1 &= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \end{aligned}\right\} \Rightarrow \theta_2 = \frac{\pi}{4} \tag{1}$$

$$\left.\begin{aligned} \cos\theta_2 &= \frac{1}{|Z_2|} = \frac{1}{2} \\ \sin\theta_2 &= \frac{-\sqrt{3}}{|Z_2|} = -\frac{\sqrt{1}}{2} \end{aligned}\right\} \Rightarrow \theta_2 = -\frac{\pi}{3} \tag{2}$$

$$\begin{aligned} Z &= \frac{Z_1}{Z_2} \\ &= \frac{|Z_1|e^{i\theta_1}}{|Z_2|e^{i\theta_2}} \\ &= \frac{|Z_1|}{|Z_2|}e^{i\theta_1}e^{i\theta_2} \\ &= \frac{\sqrt{2}}{2}e^{i(\theta_1\theta_2)} \\ &= \frac{\sqrt{2}}{2}e^{i(\frac{\pi}{4}-\frac{-\pi}{3})} \\ &= \frac{\sqrt{2}}{2}e^{i\frac{\pi}{12}} \end{aligned}$$

7. A direction vector for the line of intersection of two planes with equations

   - $x - 2zy = 1$
   - $z + y - z = 0$

is?
$\vec{n_1} = (1, -2, 0)$
$\vec{n_2} = (1, 1, -1)$
Take the cross product of the two planes' normals:

$$\begin{aligned}
\vec{n} = \vec{n_1} \times \vec{n_2} &= \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \\
&= \hat{i} \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} - \hat{j} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} + \hat{k} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \\
&= \hat{i}(2 - 1 \times 0) - \hat{j}(-1 - 1 \times 0) + \hat{k}(1 - (1)(-2)) \\
&= \hat{i}(2) - \hat{j}(-1) + \hat{k}(3) \\
&= 2\hat{i} + \hat{j} + 3\hat{k} \\
n &= (2, 1, 3)
\end{aligned}$$

9. Find a scalar equation for the plane:

$$\begin{aligned}
H &= \{1 + \vec{s} + 2\vec{t}, 2 + \vec{t}, 1 + \vec{s} \mid \vec{s}, \vec{t} \in \diagdown\} \\
&= \{v_0 + \vec{s}\vec{d_1} + \vec{t}\vec{d_2} \mid \vec{s}, \vec{t} \in \mathbb{R}\} \\
\vec{d} &= \vec{d_1} \times \vec{d_2} \\
&= \{(1, 2, 1) + \vec{s}(1, 0, 1) + \vec{t}(2, 1, 0) \mid \vec{s}, \vec{t} \in \mathbb{R}\} \\
\vec{d} &= (1, 0, 1) \times (2, 1, 0) \\
&= (\hat{i} + 0\hat{j} + \hat{k}) \times (2\hat{i} + \hat{j} + 0\hat{k}) \\
&= \hat{k} + 2\hat{j} \\
d &= (-1, 2, 1)
\end{aligned}$$

$-1(x-1) + 2(y-2) + 1(z-1) = 0$
$= -z + 1 + 2y - 4 + z - 1$
$= -x + 2y + z = 4$

1. Which of the sets

   - $U = \{(x, x+y, y) \mid x, y \in \mathbb{R}\} = \{x(1, 1, 0) + y(0, 1, 1)\}$
   - $V = \{(x+y, y, y) \mid x, y \in \mathbb{R}\} = \{x(1, 0, 0) + y(1, 1, 1)\}$
   - $W = \{(x^2, y, x) \mid x, y \in \mathbb{R}\}$

   ... are subspaces of $\mathbb{R}^3$?

   - $u = (x_1, y_1, z_1) \in U \Rightarrow (x_1, x_1 + y_1, y_1)$
   - $v = (x_2, y_2, z_2) \in U \Rightarrow (x_2, x_2 + y_2, y_2)$

2. Which of the following are subsets of M_{22}?

- $\left\{ \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in M_{22} \mid x, y, z \in \mathbb{R} \right\} = \operatorname{span} \left\{ \begin{matrix} 1 \\ 1 \\ 0 \end{matrix} \right\}$
- $\left\{ \begin{bmatrix} 0 & y \\ y & 0 \end{bmatrix} \in M_{22} \mid y \in \mathbb{R} \right\}$
- $\left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in M_{22} \mid xw - zy = 0 \in \mathbb{R} \right\}$
- $\left\{ \begin{bmatrix} x & y \\ z & x \end{bmatrix} \in M_{22} \mid x, y, z \in \mathbb{R} \right\}$

1. Which two of the following are not subspaces of $\mathbb{R}^4$?
   - $V = \{(a, b, c, d) \mid c = a + b, d = a - 3b\}$
   - $S = \{(a, b, c, d) \mid c = 0, d = 0\}$
   - $T = \{(a, b, c, d) \mid a + b = 1, c = d\}$
   - $U = \{(a, b, c, d) \mid b \geq 0, c \leq 0\}$

$$\begin{aligned} V &= \{(a, b, a + b, a - 3b)\} \\ &= \{a(1, 0, 1, 1) + b(0, 1, 1, -3)\} \\ &= \text{span}\{(1, 0, 1, 1), (0, 1, 1, -3)\} \checkmark \end{aligned}$$

So, V is a subspace of $\mathbb{R}^4$.

$$\begin{aligned} S &= \{(a, b, 0, 0\} \\ &= \{a(1, 0, 0, 0) + b(0, 1, 0, 0)\} \\ &= \text{span}\{(1, 0, 0, 0), (0, 1, 0, 0)\} \end{aligned}$$

Therefore, S is a subspace of $\mathbb{R}^4$.

2. **8** Given that the set $U = \{(x, x + 2) \mid x \in \mathbb{R}\}$:
$(x, y) \oplus (x', y') = (x + x', y + y' - 2)$ (vector addition)
$K \odot (x, y) = (Kx, Ky - 2K + 2)$ (multiplication by a scalar)

   (a) Prove that U is closed under $\oplus$.
   - $u_1 u_2 \in U \implies u_1 \oplus u_2 \in U$
   - $u_1 \in U \implies u_1 = (a, a + 2)$
   - $u_2 \in U \implies u_2 = (b, b + 2)$

$$\begin{aligned} u_1 \oplus u_2 &= (a, a + 2) \oplus (ub, b + 2) \\ &= (a + b, a + 2 + b + 2 - 2) \\ &= (a + b, a + b + 4 - 2) \\ &= (a + b, a + b + 2) \text{ let x=a+b} \\ &= (x, x + 2) \\ &\Rightarrow \text{ U is closed under } \oplus \end{aligned}$$

3. **6** Let $u = (1, 0, 1)$ and $X = \{w \in \mathbb{R}^3 \mid w \times u = 0\}$

   (a) Is X a subspace of R^3?

$$\begin{aligned} w &= (x, y, z) \in \mathbb{R}^3 \\ w \times u &= (x, y, z) \times (1, 0, 1) \\ &= (x\hat{i} + y\hat{j} z\hat{k}) \times (1\hat{i} + 0\hat{j} + 1\hat{k}) \\ &= x\hat{j} - y\hat{k} + y\hat{i} + z\hat{k} \\ &= (y, x + z, -y) \\ &= k\hat{j} \end{aligned}$$

$$= \hat{i} \begin{vmatrix} y & z \\ 0 & 1 \end{vmatrix} - \hat{j} \begin{vmatrix} x & z \\ 1 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} y & y \\ 1 & 0 \end{vmatrix}$$

$$
\begin{aligned}
&= \hat{i}(y \cdot 1 - 0 \cdot z) - \hat{j}(x \cdot 1 - 1 \cdot z) + \hat{k}(x \cdot 0 - 1 \cdot y) \\
&= y\hat{i} - (x - z)\hat{j} - y\hat{k} \\
X &= (y, -x + z, -y) \\
&= \{x(0, -1, 0) + y(1, 0, -1) + z(0, 1, 0)\} \\
X &= \text{span}\{(0, -1, 0), (1, 0, -1), (0, 1, 0)\}
\end{aligned}
$$

$$\vec{w}.\vec{u} = 0$$

$$(x, y, z).(1, 0, 1) = 0$$

1. 8. Give the set $U = \{(x, x + 2) \mid x \in \mathbb{R}\}$
$(x, y) \oplus (x'y') = (x + x', y + y' - 2)$ and
$K \odot (x, y) = (Kx, Ky - 2K - 2)$

   b. Show that there is a zero vector in U.
$(0, 0) \in U$ if $(0, 0 + 2) = (0, 2)$
$(x, y) \oplus (o, 2) = (x + 0, y + 2 - 2)$
$= (x, y) \in U$
$K \odot (0, 2) = (K0, K2 - 2K + 2)$
$= (0, 2) \in U$

   c. Show that every element $(x, x + 2) \in X$ has a negative in U (i.e. If $v = (x; x + 2) \in U$, what is $-V \in U$?
$(x, x + 2) \oplus (-x, -x + 2)$
$(x - x, x + 2 - x + 2 - 2)$
$= (0, 2) \in U$

2. 1. For what value of C is the set of vectors $\{(1, C, 1), (0, 1, -1), (1, 0, 2)\}$ linearly dependent? $au + bv + cw = 0$
$a(0, 1, -1) + b(1, 0, 2) = (1, C, 1)$

$$\begin{cases} a0 + b1 = 1 \\ a + b0 = C \\ a(-1) + 2b = 1 \end{cases}$$

$$\begin{cases} a + b = 1 \\ a + 0 = C \\ -a + 2b = 1 \end{cases}$$

   (i) $b = 1$ (iv)
(iv) in (iii)$-a + 2 = 1 \Rightarrow a = 1$ (v)
(v) in (ii) $c = 1$

3. 4. Let $F[0, \pi] = \{f \mid f : [0, \pi] \in \mathbb{R}\}$ be the vector space of real-valued functions defined on $[0, \pi]$. Define the functions in $F[0, \pi]$ by $f(x) = \sin(2x), g(x) = \cos(x)$ and $h(x) = 1$.
$\forall x \in [0, \pi]$ and let $U = \text{span}\{f, g, h\}$:
a. Show that $\{f, g, h\}$ is linearly independent:
$\{f, g, h\}$ is linearly independent if:

- $af + bg + ch = 0 \Rightarrow a = b = c = 0$
- $a\sin(2x) + b\cos(x) + c1 = 0$

At 0: $a0 + b1 + c = 0$
At $\pi/2$: $a0 + b\cos(\pi/2)$
At π: $a\sin(2\pi) + b\cos(2\pi) + c = 0$

$$\Rightarrow \begin{cases} 0 + b + c = 0 \\ a + b\frac{\sqrt{2}}{2} + c = 0 \\ 0 - b + c = 0 \end{cases}$$

(i)+(iii): $0 + b - b + c + c - 0$
$\Rightarrow 2c = 0$
$\Rightarrow c = 0$ (iv)
(iv) in (i): $0 + b + 0 = 0 \Rightarrow b = 0$ (v)
(iv) and (v) in (ii): $a + 0\frac{\sqrt{2}}{2} + 0 = 0$
$\therefore a = b = c = 0$

4. a. Show that $\{f, g, h\}$ is linearly independent:
   $\{f, g, h\}$ is linearly independent if:

   - $af + bg + ch = 0 \Rightarrow a = b = c = 0$
   - $a\sin(2x) + b\cos(x) + c1 = 0$

   From (a) $af + bg + ch = 0$