

MAT 1320

**Calculus I**

2013

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## Notations and symbols

Let  $a \in \mathbb{R}$ . We defined the *absolute value* of  $a$  by:

$$|a| := \begin{cases} a & \text{if } a > 0 \\ -a & \text{if } a < 0 \end{cases}$$

*Note:  $:=$  is the "definition" sign.*

The result is always a positive number.

- $|0| = 0$
- $|3| = 3$
- $|-2| = 2$
- $|\sqrt{2} - 1| = \sqrt{2} - 1$

## Properties of the absolute value

- $|ab| = |a||b|$
- if  $b \neq 0$ , then  $|\frac{a}{b}| = \frac{|a|}{|b|}$

If  $a > 0$ , then

- Solve  $|2x - 5| = 3$ 
  - **Solution:** We have two cases to consider:  
 $2x - 5 = 3$ , which leads to  $x = 4$ ,  
 $-(2x - 5) = 3$ , which leads to  $x = 1$   
Hence the result:  $x = 1$  or  $4$
- Solve  $|3x + 2| \geq 4$ 
  - Either  $3x + 2 \geq 4$ , which leads to  $x \geq \frac{2}{3}$
  - Or  $3x + 2 \leq -4$ , which leads to  $x \leq -2$
  - Hence the result  $x \in ]-\infty, -2] \cup [\frac{2}{3}, +\infty[$

## Small remarks on the square root function:

By definition, its *domain*, i.e. the allowed numbers that can feed into the square root, is

$$x \in [0, +\infty[$$

While its *range*, i.e. what values the square root can "spit out", is

$$y \in [0, +\infty[$$

We usually encompass all this information in a neat, compact form, i.e.:

$$\sqrt{\cdot} : [0, +\infty[ \rightarrow [0, +\infty[$$

$$x \mapsto \sqrt{x}$$

Where the second line gives the actual transformation recipe.

We can extend this to all functions, i.e.

$$f : D \longrightarrow R$$

$$x \mapsto f(x)$$

Where we recover at once on the first line the name of the function  $f$ , its domain  $D$ , and its range  $R$ , while the second line gives us the explicit transformation rule.

Ex:

$$g : \mathbb{R} \longrightarrow \mathbb{R}$$

$$x \mapsto x^3 \cdot \sin(x) = g(x)$$

That being said, what is:

$$\sqrt{a^2} = |a|$$

The range of a square root function is always  $[0, +\infty[$ . Indeed...  $\sqrt{(-3)^2} = \sqrt{9} = 3 = |-3|$   
The square root function *is a one-to-one function*.

$$|a| = \sqrt{a^2}$$

**Recall**

We will denote a function  $f$  by

$$\begin{aligned} f : D &\rightarrow R \\ x &\mapsto f(x) \end{aligned}$$

Where  $D$  is the domain of  $f$ ,  $R$  is its range, and  $f(x)$  is the actual transform rule.

**Definition**

The *image* of  $f$ , denoted  $f(D)$ , is the set of all  $y \in R$  such that there exists at least one  $d \in D$  with

$$f(d) = y$$

Example: Consider the squaring function:

$$\begin{aligned} (\cdot)^2 : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto x^2 \end{aligned}$$

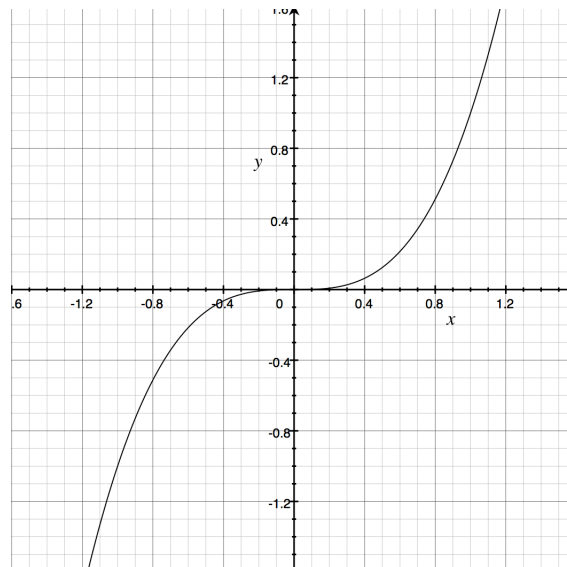
This leads to

**Definition**

A function is onto (or surjective) if its image is equal to its range.

Examples:

- $x^2$  is not onto
- $x^3$  is onto



*The function,  $x^3$*

**Definition**

A function is *one-to-one* or *injective* if whenever  $f(x_1) = f(x_2)$ , we then must have  $x_1 = x_2$ . In other words, a one-to-one function cannot produce more than once a given value.

Examples:

- The squaring function is **not** one-to-one because  $(-2)^2 = 2^2$  but  $-2 \neq 2$

- The cubic function is one-to-one.

If a function  $f$  is both onto *and* one-to-one, it's called a *bijection*.

**Theorem**

A function  $f : A \longrightarrow B$  is a bijection if and only if there exists a function  $f^{-1} : B \longrightarrow A$  such that:

- $f^{-1}(f(a)) = a$  for all  $a \in A$ , and
- $f(f^{-1}(b)) = b$  for all  $b \in B$

**Remarks:**

- If such a function exists, then it is unique. It is called the inverse of  $f$ .
- We must have

$$b = f(a) \iff a = f^{-1}(b)$$

- **WARNING:**

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

$$(f(x))^{-1} = \frac{1}{f(x)}$$

Example

Find the inverse of  $f(x) = x^3 + 2$ .

Solution

- Write  $y = x^3 + 2$
- Solve for  $x$  as a function of  $y$ , i.e.
  - $x^3 = y - 2$  hence  $x = (y - 2)^{1/3}$
- Switch  $x$  and  $y$ , i.e.  $y = (x - 2)^{1/3}$

Checking computations with...

- $f(f^{-1}(x)) = f((x - 2)^{1/3})$   
 $= ((x - 2)^{1/3})^3 + 2$   
 $= x - 2 + 2 = x$

The function  $\sqrt{x}$  is not the inverse of  $x^2$ .

Recall that  $\sqrt{x^2} = |x|$  and not  $x$  is neither one-to-one nor onto, and hence is not invertible!

But we can consider a new function with a smaller domain and a smaller range but with the same transformation rule.

$$h : [0, +\infty[ \longrightarrow [0, +\infty[$$

$$x \mapsto x^2$$

Now that function is a bijection and its inverse is  $\sqrt{x}$ , i.e.  $\sqrt{h} = x$  and  $(\sqrt{x}) = x$

# Logarithms

## Remarks:

1.  $f_a$  is hence one-to-one, and
2. If we restrict the range of  $f_a$  to  $]0, +\infty[$ , then  $f_a$  becomes a bijection.

We denote its inverse by

$$\log_a(x)$$

The number  $a$  is called the base

If the base is omitted as in  $\log 5$ , it is assumed to be base 10

When  $a = e \approx 2.71828$ , the log is written

$$\ln(x)$$

Recall that for  $1 \neq a > 0$  we have the exponential function:

$$f_a : \mathbb{R} \longrightarrow \mathbb{R}$$

$$x \mapsto a^x$$

which is

- strictly decreasing if  $a > 1$ ; and
- strictly increasing if  $a < 1$ .

## Remarks:

1.  $f_a$  is hence one-to-one, and
2.  $f_a(x) > 0$ .

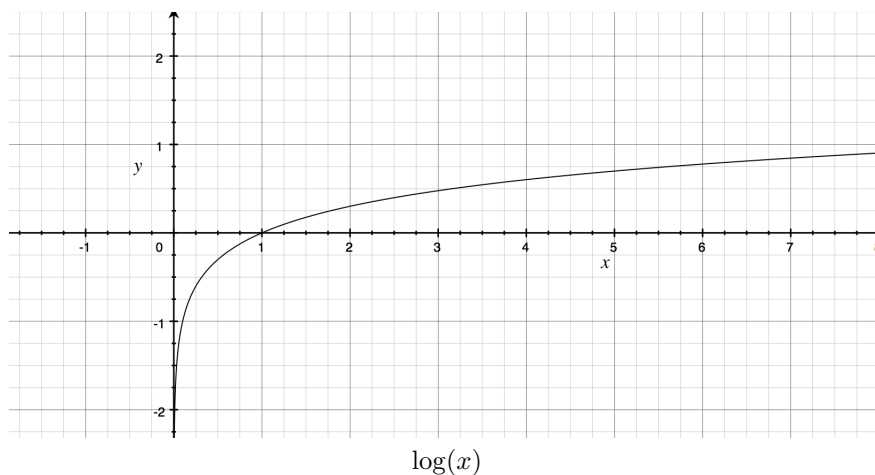
Hence if we restrict the range of  $f_a$  to  $]0, +\infty[$ , then  $f_a$  becomes a bijection.

We denote its inverse by  $\log_a(x)$

This function is characterized by:

$$\log_a(x) = y \iff a^y = x$$

- The number  $a$  is called the *base*
- If the base is omitted as in  $\log(5)$ , it is assumed to be 10.
- When  $a = e \approx 2.71828$ , we write  $\ln(x)$ , and we call it the natural logarithm.



Compute  $\log_2(80) - \log_2(5)$

$$\begin{aligned}\log_2(80) - \log_2(5) &= \log_2 \frac{80}{5} \\ &= \log_2(16) \\ &= 4\end{aligned}$$

Theorem:

Let  $1 \neq a > 0$  and  $x > 0$ . Then:

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}.$$

Why is this true?

If  $y = \log_a(x)$ , then  $a^y = x$ . Applying  $\ln$  to both sides we get:

$$\ln(x) = \ln(a^y) = y \ln a.$$

Hence  $y = \frac{\ln(x)}{\ln(a)}$ .

Compute with a precision of six decimal places

$$\log_8(5).$$

Solution:  $\log_8(5) = \frac{\ln(5)}{\ln(8)} \approx 0.773976$

Solve  $e^{5-3x} = 10$

Solution:

Apply  $\ln$  to both sides

Solve for  $x$ , i.e.  $x = \frac{5-\ln(10)}{3} \approx 0.899138$

# Trigonometry

## Inverse trig functions

Consider a circle of radius  $r$  and some arc of length  $a$  and  $c$  its circumference.

On a unit circle,

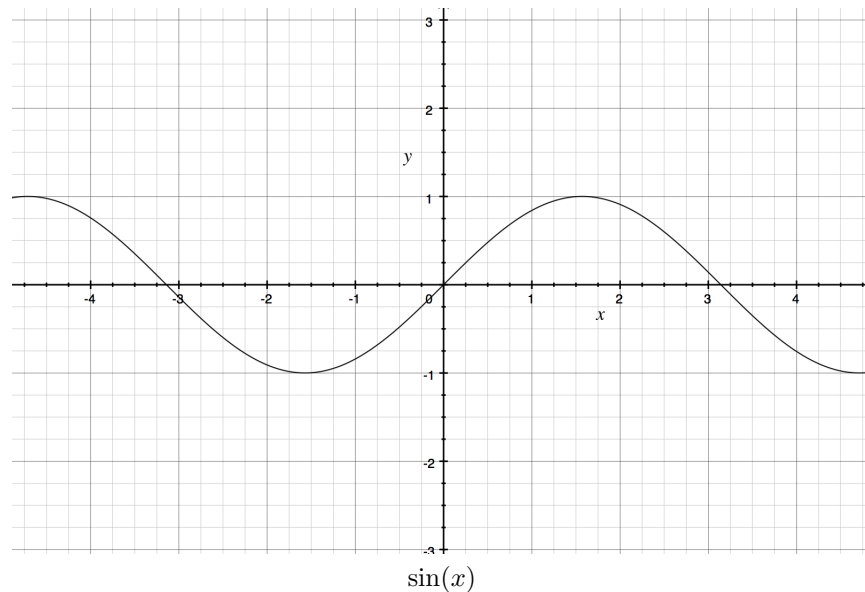
- $A(0)=(1,0)$  implies  $\cos(0) = 1$  and  $\sin(0) = 0$

**Properties** The function  $A(\theta)$  is periodic of a period of  $2\pi$ , i.e.  $A(\theta) = A(\theta + 2\pi)$ . Hence:

$$\cos(\theta) = \cos(\theta + 2\pi) \text{ and } \cos(\theta) = \cos(\theta - 2\pi)$$

As one can deduce by either their respective definitions or their associated graphs,

- The function  $\sin \theta$  is an odd function
- The function  $\cos \theta$  is an even function



## Other trig functions

$$\tan(\theta) := \frac{\sin(\theta)}{\cos \theta} \text{ and } \cot \theta = \frac{\cos(\theta)}{\sin(\theta)}$$

Secant and cosec

From  $\cos^2(\theta) + \sin^2(\theta) = 1$  one can define:

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

**To do at home:** Show that:

- $\sin(2x) = 2 \sin(x) \cos(x)$
- 

$$\begin{aligned} \cos(2x) &= \cos^2(x) - \sin^2(x) \\ &= 2 \cos^2(x) - 1 \\ &= 1 - 2 \sin^2(x) \end{aligned}$$



Find all  $x \in [0, 2\pi]$  such that  $\sin(x) = \sin(2x)$

Solution: Recall that  $\sin(2x) = 2\sin(x)\cos(x)$ , hence we need to solve  $\sin(x) = 2\sin(x)\cos(x)$ . Equivalently,  $\sin(x)(1 - 2\cos(x)) = 0$ .

We have two cases:

- $\sin(x) = 0$ , which implies that  $x = 0, \pi, \text{ or } 2\pi$ , or
- $1 - 2\cos(x) = 0$ , which implies  $\cos(x) = \frac{1}{2}$ , or equivalently  $x = \frac{\pi}{3}$  or  $\frac{5\pi}{3}$

Conclusion: The solutions are  $x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ , or  $2\pi$

We want to "invert" the trigonometric functions. Recall that the domain and the range of  $\sin(x)$  and  $\cos(x)$  are the real numbers. Two problems arise:

- both functions are periodic, hence they are not one-to-one
- both functions are not onto as their images are  $[-1, 1]$

We then need to consider *restricted* versions of those functions if we want to proceed.

In the case of  $y = \sin(x)$ , we consider the classical domain

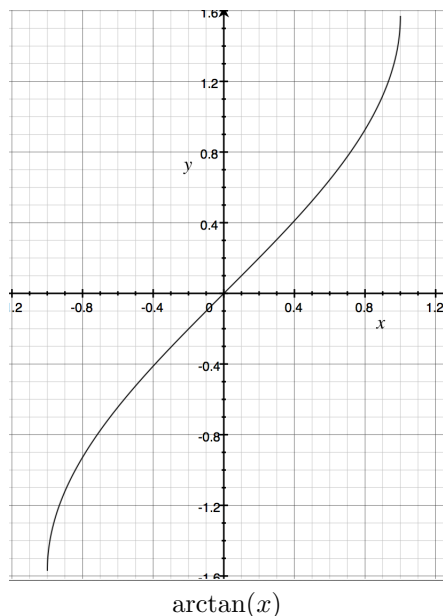
$$\begin{aligned} \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] &\longrightarrow [-1, 1] \\ x &\mapsto \sin(x) \end{aligned}$$

Hence the function  $y = \text{Sin}(x)$  is now onto and one-to-one and thus admits an inverse function

$$\arcsin : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

In particular,

- $\arcsin(\text{Sin}(x)) = x \ \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\text{Sin}(\arcsin(x)) = x \ \forall x \in [-1, 1]$
- $y = \arcsin(x) \iff \text{Sin}(y) = x$



NOTE: trig functions with caps first letter are informal notation to denote processed sine and cosine functions

**Exercises**

Simplify the expression  $\cos(\arctan(x))$

Solution:

$$y = \arctan(x) \iff \tan(y) = x \text{ for } y \in ]-\frac{\pi}{2}, \frac{\pi}{2}[$$

On the other hand, we have the trigonometric identity:

$$\begin{aligned}\sec^2(x) &= 1 + \tan^2(y) \\ &= 1 + x^2 \text{ why?}\end{aligned}$$

Finally,

$$\cos(\arctan(x)) = \cos(y) = \frac{1}{\sec(y)} = \frac{1}{\sqrt{1+x^2}} \dots \text{ why?}$$

Compute exactly  $\tan(\arcsin(1/3))$

Solution

$$\theta = \arcsin(1/3) \iff \sin(\theta) = 1/3 \text{ when } \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}],$$

On the other hand, we have the trigonometric identity  $\cos^2 \theta = 1 - \sin^2 \theta$

## Limits

Let  $f : D \rightarrow \mathbb{R}$  be a function,  $L \in \mathbb{R}$  and  $a \in \mathbb{R}$  a point of either  $D$  or a boundary point of  $D$ .

Examples of acceptable  $a$ :

- If  $D = [1, 2]$ , then  $a$  could be 1.5
- If  $D = ]1, 3[$  then  $a$  could also be 3 or -1, even though they are not in the domain of  $f$ .

### Definition:

We will write:

$$\lim_{x \rightarrow a} f : D$$

Remark: That value  $L$  must be independent of the way we approach  $a$ .

We can expand the definition of limits and ask, "is  $f(x)$  approaching a definite value as  $|x|$  grows arbitrarily?" We write in this case either:

$$\lim_{x \rightarrow \infty} f(x) \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x)$$

In the case of  $y = \frac{1}{x}$  we have:

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = 0$$

### Directional limits

In some problems, we might be interested in only a certain direction of approach.

If we approach  $a$  from the right, we write:

$$\lim_{x \rightarrow a^+} f(x)$$

Conversely, if we approach  $a$  from the left, we write:

$$\lim_{x \rightarrow a^-} f(x)$$

### Theorem:

The limit  $\lim_{x \rightarrow a} f(x)$  exists iff both the right and left limit exist and are equal.

Suppose both  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  are numbers and let  $c \in \mathbb{R}$ .

- $\lim_{x \rightarrow a} (f(x) \pm g(x)) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
- $c \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} c(f(x))$
- $\lim_{x \rightarrow a} f(x)g(x) = \left( \lim_{x \rightarrow a} f(x) \right) \left( \lim_{x \rightarrow a} g(x) \right)$
- $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$  if  $\lim_{x \rightarrow a} g(x) \neq 0$

**Definition** Let  $f : D \rightarrow \mathbb{R}$  be a function and  $a \in D$ . We say that  $f$  is *continuous at  $a$*  if:

$$f(a) = \lim_{x \rightarrow a} f(x)$$

### Examples of continuous functions

- All polynomial functions
- All trig functions

- All exponential and log functions
- All rational functions

**Warning:** One checks continuity only on  $x$  belonging to the domain of a function.

Question: Is  $\frac{1}{x}$  continuous? Yes, since it is not defined on  $x = 0$

Compute:

$$\lim_{x \rightarrow 0} \frac{(3+x)^2 - 9}{x}$$

Solution: First of all we notice that it is a rational function but 0 is not in its domain. Hence we can't simply substitute 0 in the expression. Thus:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{(3+x)^2 - 9}{x} &= \lim_{x \rightarrow 0} \frac{9 + 6x + x^2 - 9}{x} \\ &= \lim_{x \rightarrow 0} \frac{x(6+x)}{x} \end{aligned}$$

As we are interested in the limit when approaching 0, we are purposely avoiding  $x = 0$ , hence:

$$\lim_{x \rightarrow 0} \frac{(3+x)^2 - 9}{x} = \lim_{x \rightarrow 0} (6+x)$$

And now we notice that the function  $6+x$  is a line (hence continuous) and that 0 is in its domain.

Conclusion:  $\lim_{x \rightarrow 0} \frac{(3+x)^2 - 9}{x} = \lim_{x \rightarrow 0} (6+x) = 6+0 = 6$

Let  $f, g : D \rightarrow \mathbb{R}$  be two continuous functions (when we do not specify at a particular point we mean continuous everywhere on their domain).

- $f \pm g$  is continuous;
- $f \cdot g$  is continuous, and
- $\frac{f}{g}$  is continuous whenever the quotient is defined, i.e. it is continuous on all  $x \in D$  such that  $g(x) \neq 0$ .

Let  $f : D \rightarrow \mathbb{R}$  be a continuous function and suppose that there is an interval  $[a, b] \subseteq D$

**Theorem**

For each  $y$  between  $f(a)$  and  $f(b)$ , there is  $x_0 \in [a, b]$  such that  $y = f(x_0)$

Suppose that we have a continuous function  $f : [0, 1] \rightarrow [0, 1]$

Then one knows that there is  $x_0 \in [0, 1]$  such that  $f(x_0) = x_0$ , i.e.  $f$  admits a fixed point.

Why?

Consider the new function

$$\begin{aligned} g : [0, 1] &\longrightarrow \mathbb{R} \\ x &\mapsto f(x) - x \end{aligned}$$

On one hand, if  $f(1) = 1$ , then we are done as we have found a fixed point. Else by construction

$$g(1) < 0 \text{ since } f(1) < 1$$

On the other hand, if  $f(0) = 0$  then again we are done. Else by construction

$$g(0) > 0 \text{ since } f(0) > 0$$

**General culture**

This result is true in higher dimensions!

It is known as the *Brouwer fixed-point theorem*.

**Theorem:**

Let  $f : [0, 1]^n \rightarrow [0, 1]^n$  be a continuous function for  $n = 1, 2, 3, \dots$ . There exists  $x_0 \in [0, 1]^n$  such that:

$$f(x_0) = x_0$$

**Second hypothesis**

Let  $p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$  be a polynomial of odd degree greater than or equal to 1, i.e.  $n$  can only be an odd positive integer greater than or equal to one.

**Theorem**

One automatically knows that there is at least one  $x_0 \in \mathbb{R}$  such that

$$p(x_0) = 0$$

Why?

Intuitively, we know that

$$\lim_{x \rightarrow \pm\infty} p(x) = \lim_{x \rightarrow \pm\infty} x^n$$

Moreover, on one hand:

$$\lim_{x \rightarrow \infty} x^n = \infty$$

i.e. there is a  $b \in \mathbb{R}$  such that  $p(b) > 0$ .

On the other hand, we also know that since  $n$  is odd,

$$\lim_{x \rightarrow -\infty} x^n = -\infty$$

i.e. there is a  $a \in \mathbb{R}$  such that  $p(a) < 0$

Conclusion: By the intermediate value theorem there must exist  $x_0 \in [a, b]$  such that:

$$p(x_0) = 0$$

# The derivative

## Outline

1. Geometric interpretation
2. The derivative as a function
3. Computation

## Recall:

Let  $f : D \rightarrow \mathbb{R}$  be a function and  $x_0 \in D$ .

## Definition:

We say that  $f$  is differentiable at  $x_0 \in D$  if the following limit:

$$\lim_{h \rightarrow 0} \frac{f(h + x_0) - f(x_0)}{h}$$

is a number, i.e. it exists.

In that case we denote the number  $f'(x_0)$ .

The question is, what does it mean to differentiate a function at  $x_0$ ?

Recall how one computes the slope of a line:

$$m = \frac{\Delta y}{\Delta x}$$

Hence in the case of the line  $\mathbb{D}$ , we have

$$m_{\mathbb{D}} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Now as  $\Delta x$  goes towards 0, the slope  $m_{\mathbb{D}}$  goes towards the slope of the red line.

## Continuity?

Suppose  $f : D \rightarrow \mathbb{R}$  is differentiable at  $x_0 \in \mathbb{R}$ .

Pick an  $x \in D$  but such that  $x \neq x_0$ . Then

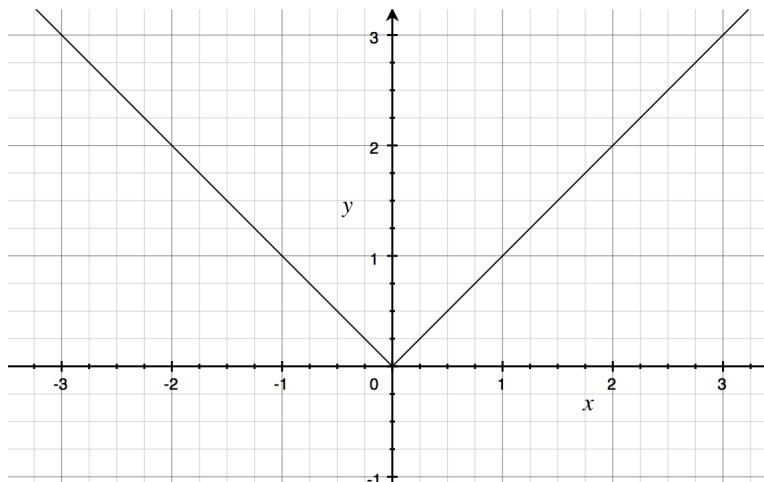
$$\begin{aligned} f(x) &= f(x) + 0 \\ &= f(x) - f(x_0) + f(x_0) \\ &= (f(x) - f(x_0)) \cdot 1 + f(x_0) \\ &= (f(x) - f(x_0)) \frac{x - x_0}{x - x_0} + f(x_0) \\ &= \frac{f(x) - f(x_0)}{x - x_0} (x - x_0) + f(x_0) \end{aligned}$$

And now we can take the limit on each side, i.e.

$$\begin{aligned} \lim_{x \rightarrow x_0} f(x) &= \lim_{x \rightarrow x_0} \left( \frac{f(x) - f(x_0)}{x - x_0} (x - x_0) + f(x_0) \right) \\ &= \left( \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \rightarrow x_0} (x - x_0) + \lim_{x \rightarrow x_0} f(x_0) \right) \\ &= f'(x_0) \cdot 0 + f(x_0) \\ &= f(x_0) \end{aligned}$$

**Conclusion:**  $f$  is continuous at  $x_0$ .

Warning: The converse is false, i.e. a continuous function is not necessarily differentiable.



$y = |x|$  is continuous, but not differentiable.

Let  $f : D \rightarrow \mathbb{R}$  be a function. Consider the subset

$$C = \{a \in D \mid f'(a) \text{ exists}\}$$

Then one can construct a new function:

$$\begin{aligned} f' : C &\rightarrow \mathbb{R} \\ x &\mapsto f'(x) \end{aligned}$$

If  $f(x) = c$ , i.e.,  $f$  is a constant, then

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{c - c}{h} \\ &= \lim_{h \rightarrow 0} \frac{0}{h} = 0 \end{aligned}$$

Let  $n = 1, 2, 3, \dots$  and consider the function  $f(x) = x^n$

Recall the binomial formula:

$$\begin{aligned} (a+b)^n &= \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \dots + \binom{n}{n-1} ab^{n-1} + \binom{n}{n} b^n \\ &= \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k \end{aligned}$$

Where  $\binom{n}{k} = \frac{n!}{(n-k)!k!}$  Then

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\left[ \sum_{k=0}^n \binom{n}{k} x^{n-k} h^k \right] - x^n}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\left[ \sum_{k=1}^n \binom{n}{k} x^{n-k} h^k \right]}{h} \\
 &= \lim_{h \rightarrow 0} \left[ \sum_{k=1}^n \binom{n}{k} x^{n-k} h^{k-1} \right] \\
 &= \binom{n}{1} x^{n-1} = nx^{n-1}
 \end{aligned}$$



## Trigonometry

How do you compute the derivative of  $y = \sin(x)$ ?

$$\begin{aligned}\frac{d \sin}{dx}(x) &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(x) \cos(h) + \sin(h) \cos(x) - \sin(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(x)(\cos(h) - 1)}{h} + \lim_{h \rightarrow 0} \frac{\sin(h) \cos(x)}{h}\end{aligned}$$

To help us go beyond those examples, we need "recipes:"

Let  $f$  and  $g$  two differentiable functions  $c \in \mathbb{R}$ .

- $(cf)'(x) = cf'(x)$
- $(f \pm g)'(x) = f'(x) \pm g'(x)$
- $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
- $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

Let  $f(x) = a^x$  for some  $a \in ]0, 1[ \cup ]1, +\infty[$

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^x(a^h - 1)}{h} \\ &= a^x \left( \lim_{h \rightarrow 0} \frac{a^h - 1}{h} \right)\end{aligned}$$

## The number $e$

Let's consider two possible  $a$ , i.e.  $a = 2$  and  $a = 3$ , and let's approximate in those two cases the limit

$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h}$$

$h$	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$
0.1	0.7177	1.1612
0.01	0.6956	1.1047
0.001	0.6934	1.0992
$\vdots$	$\vdots$	$\vdots$

### Definition:

The number  $e$  is a number between 2 and 3 such that

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

If  $f(x) = e^x$ , then  $f'(x) = e^x$ .

1. Tangent line
2. Velocity

### 3. Higher derivatives

#### Equation of a line

Recall that the equation of a line in the plane is  $y = mx + b$ , where  $m$  is the slope and  $b$  is the height on the y-axis at which the line crosses.

To find the equation of a line one needs the coordinates of two distinct points on that line or a point on the line and its slope. Now if one is given  $f'(x_0)$  and the point  $(x_0, f(x_0))$ , the equation of the tangent line to the graph of  $f$  at the point  $x_0$  is given by

$$y = f'(x_0)x + f(x_0) - x_0f'(x_0)$$

Consider  $f(x) = x^2 - \sin x$  and give the equation of the tangent line of  $f$  at the point  $x_0 = 0$ .

We need to compute:

- $f(0) = 0$
- $f'(0)$ . In general,  $f'(x^2) = 2x$

velocity Suppose a particle is moving along a straight line. Hence we have a function:

$$\begin{aligned} s : [a, b] &\rightarrow \mathbb{R} \\ t &\mapsto s(t) \end{aligned}$$

that tracks its position along that line. Its position with respect to the origin at some time  $t_0 \in [a, b]$  is

$$s(t_0)$$

#### Definition

The *average speed* at which it travelled during the interval of time  $[a, b]$  is the ratio of the total displacement  $\Delta s = s(b) - s(a)$  by the total time of travel, i.e.

$$\bar{v} = \frac{\Delta s}{\Delta t} = \frac{s(b) - s(a)}{b - a}$$

#### Definition

If the position of a particle is modelled along a differentiable function  $s$ , then the instantaneous velocity at time  $t_0$  is

$$s'(t_0)$$

#### Example

Suppose a man throws a stone upward and that the position once it leaves the hand of the stone with respect to the ground is given by  $s(t) = -5t^2 + 10t + 1$

- At what height does the stone leave the hand?
- What is the initial velocity of the stone?
- What is the maximum height that the stone reaches?
- At what velocity does the stone hit the ground?
- What is the average velocity of the stone from the time it leaves the hand to when it hits the ground?

Solutions:

- Set  $t = 0$  and compute.
- $v(t) = s'(t) = -10t + 10$ , hence  $v(0)=10$
- Find x-intercept of  $v(t)$  and plug x-value into  $s(t)$ .
- Find the root  $x_2$  of  $s(t)$  and the slope of the tangent at that point.
- Find  $s(t)$  at  $t = 0$ , and at  $t = x_2$ , and average.

### Higher derivatives

We have seen that from a differentiable function  $f$  one can compute its derivative  $f'$  which is yet another function. It is then reasonable to ask, "is that new function differentiable?" Let  $y = \sin(x)$ , then  $y' = \cos(x)$ . Then,  $y'' = -\sin(x)$ . Recall that the derivative at a point measures the instantaneous variation at that point of the original function. That can now be applied to the derivative, i.e. the second derivative at a point measures the instantaneous variation at that point of the derivative.

1. Compute the derivative of  $f(x) = \frac{1}{x^n}$  for  $n = 1, 2, 3$

$$\begin{aligned} f'(x) &= \frac{0 \cdot x^n - 1 \cdot nx^{n-1}}{(x^n)^2} \\ &= -\frac{nx^{n-1}}{x^{2n}} \\ &= -\frac{n}{x^{2n-n+1}} \\ &= -\frac{n}{x^{n+1}} \end{aligned}$$

Consider  $y = x^a$ .

In the case,  $a \in \mathbb{Z}$  we have  $y' = ax^{a-1}$ . This is true, in general.

### Theorem

For any  $a \in \mathbb{R}$ , we have

$$y' = ax^{a-1}$$

### Warning

Even though  $\sqrt{x}$  is continuous on its domain, i.e. on  $[0, +\infty]$ , it is *not* differentiable at 0.

2. Compute the derivative of  $y'$  when  $y = x^2 \cos(x)$  **Definition:**

We say that a value  $M \in \mathbb{R}$  is the maximum of  $f$  if there is an  $x_0 \in D$  such that

$$M = f(x_0) \geq f(x) \quad \forall x \in D$$

Clearly we have its counterpart, i.e.,  $m \in \mathbb{R}$  is the minimum of  $f$  if there is  $x_1 \in D$  such that

$$m = f(x_1) \leq f(x) \quad \forall x \in D$$

Let  $f : D \rightarrow \mathbb{R}$  be a function. Suppose that  $f$  is differentiable on an open interval. Pick a point  $a$  on that open interval.

### Theorem

If  $f$  achieves either a local maximum or a local minimum at  $a$ , then  $f'(a) = 0$ .

### Warnings

- The converse is false!
- We need  $f$  to be differentiable on an open interval  $a$ .

### Critical point

Let  $f : D \rightarrow \mathbb{R}$  be a function.

### Definition

A point  $x_0 \in D$  is a critical point of  $f$  if either

$$f'(x_0) = 0 \text{ or } f'(x_0) \text{ does not exist.}$$

Example: In the following example,  $y = (1 - x)x^{1/3} + 1$

$$y' = -x^{1/3} + (1 - x)\frac{1}{3x^{2/3}} = \frac{1 - 4x}{3x^{2/3}}$$

### Outline

1. The chain rule
2. Implicit differentiation
3. Differentiating inverse functions

### The chain rule

We need to find a convenient way to differentiate functions of the form

$$\sqrt{x^2 + 1} \text{ and } e^{-x^2}.$$

Let  $f : D \rightarrow \mathbb{R}$  and  $g : E \rightarrow \mathbb{R}$  be two functions such that

$$g(E) \subset D.$$

Moreover, suppose that  $g$  is differentiable at  $x_0 \in E$  and that  $f$  is differentiable at  $g(x_0)$ .

### Theorem

$$(f \circ g)'(x_0) = f'(g(x_0)) \cdot g'(x_0)$$

### Examples

- We can finally compute the derivative of  $y = a^x$  for  $0 < a < 1$  or  $a > 1$ .  
Recall that  $x = e^{\ln x}$ .  
So, by the rule of exponents we have  $y = e^{x \ln a}$ .  
Consider  $g(x) = x \ln a$ . Then  $y = e^{g(x)}$ , hence

$$\begin{aligned} g' &= e^{g(x)} \cdot g'(x) \\ &= y \cdot \ln a \\ &= a^x \cdot \ln a \end{aligned}$$

- Compute the derivative of  $y = (x^3 - 1)^{100}$ .  
 $f(x) = x^3 - 1$ ,  $g(x) = f(x)^{100}$   
 $y' = 100(x^3 - 1)^{99} \cdot 3x^2$

- Compute the derivative of  $y = \left(\frac{t-2}{2t+1}\right)^9$   
 $f(x) = \frac{t-2}{2t+1}$ ,  $g(x) = f(x)^9$

$$y' = 9 \left(\frac{t-2}{2t+1}\right)^8 \cdot \frac{5}{(2t+1)^2}$$

- Compute the derivative of  $y = \sin(\cos(\tan(x)))$

$$y' = -\cos(\cos(\tan(x))) \cdot \sin(\tan(x)) \cdot \sec^2(x)$$

### Implicit differentiation

If a function  $y = f(x)$  verifies a certain relation, then its derivative  $y'$  should verify the derivative of the relation.

Remarks:

- In such a case it is not uncommon that the actual function is unknown
- The existence problem of such a function is beyond this course, i.e., given a relation, is there a function that verifies it? This course assumes that this is the case.

### Examples

- Suppose we know that a certain function  $y = f(x)$  satisfies the relation  $x^3 + y^3 = 6xy$ , i.e.

$$x^3 + (f(x))^3 = 6xf(x)$$

**Question:** What is the slope of the tangent line at the point,  $(x, y) = (3, 3)$ ?

We first differentiate the relation (1), i.e

$$3x^3 + 3(f(x))^2 \cdot f'(x) = 6f(x) + 6xf'(x)$$

Using chain rule on the left-hand side and product rule on the right-hand side.

$$f'(x) = \frac{6f(x) - 3x^3}{3(f(x))^2 - 6x} = \frac{2f(x) - x^2}{(f(x))^2 - 2x}$$

### Inverse trig functions

What is the derivative of  $\arcsin(x)$ ?

Solution: Recall that

$$y = \arcsin(x) \iff \sin(y) = x \text{ AND } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

- Since  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ , we have  $\cos(y) \geq 0$ . Moreover,

$$\cos(y) = +\sqrt{1 - \sin^2(y)} = \sqrt{1 - x^2}$$

- We now differentiate implicitly the relation  $\sin(y) = x$  with respect to  $x$ , i.e.

$$\cos(y) \cdot y' = 1$$

Conclusion:

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$$

- Homework: prove that the derivative of  $y = \arccos(x)$  is

$$\frac{-1}{\sqrt{1-x^2}}$$

- What about the derivative of  $y = \arctan(x)$ ?

Solution: Recall that

$$y = \arctan(x) \iff \tan(y) = x \text{ AND } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

– Recall that  $\tan^2(y) + 1 = \sec^2(y)$ . Hence

$$\sec^2(y) = x^2 + 1$$

- We now know how to differentiate implicitly the relation  $\tan(y) = x$  with respect to  $x$ , i.e.

$$\sec^2(y) \cdot y' = 1 \text{ (using chain rule)}$$

Suppose the  $f : A \rightarrow B$  and  $f^{-1} : B \rightarrow A$  are differentiable functions and inverse to each other, i.e.

$$f^{-1}(f(x)) = x \text{ and } f(f^{-1}(y)) = y$$

for all  $x \in A$  and  $y \in B$ .

If one differentiates the first of these two identities, one gets

$$(f^{-1})'(f(x)) \cdot f'(x) = 1$$

**Example:**  $\ln(y)$

We know that  $\ln(e^x) = x$  and that the derivative of  $y = e^x$  is itself.

Compute the derivative of  $y = \ln(\cos(x))$

Solution: It is a composition of functions. Hence we apply the chain rule:

$$\begin{aligned} y' &= \frac{1}{\cos(x)}(-\sin(x)) \\ &= -\tan(x) \end{aligned}$$

## Logarithmic differentiation

Compute the derivative of

$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$$

Solution: We will combine logarithmic laws and implicit differentiation to compute this. Start by taking  $\ln$  on both sides, i.e.

$$\begin{aligned}\ln y &= \ln \left( \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5} \right) \\ &= \frac{3}{4} \ln x + \frac{1}{2} \ln(x^2+1) - 5 \ln(3x+2) \\ \frac{1}{y} \frac{dy}{dx} &= \frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{2x}{x^2+1} - 5 \cdot \frac{3}{3x+2}\end{aligned}$$

Now solve for  $\frac{dy}{dx}$ :

$$\begin{aligned}\frac{dy}{dx} &= y \left( \frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2} \right) \\ &= \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5} \left( \frac{3}{4x} \text{INCOMPLETE} \right)\end{aligned}$$

Compute the derivative of  $y = x^{\sqrt{x}}$

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x} \ln x$$

We now differentiate implicitly and use the product rule, namely:

$$\frac{y'}{y} = \frac{1}{2\sqrt{x}} \ln x + \sqrt{x} \frac{1}{x}$$

And solve for  $y'$ :

$$y' = y \left( \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \right)$$

Linear approximation

Suppose there is an engineer at the train station on the departing deck who wants to measure the length of the train as it is leaving the station. Of course, this engineer is equipped with a state of the art laser measuring tool. Einstein tells us that the length  $\tilde{\mathcal{L}}$  is scaled by a factor of

$$\sqrt{1 - \frac{v^2}{c^2}}$$

We can, with a high degree of confidence use linear approximation.

$$\begin{aligned}\tilde{\mathcal{L}} &= \mathcal{L} \sqrt{1 - \frac{v^2}{c^2}} \approx L \left( 1 - \frac{1}{2} \frac{v^2}{c^2} \right) \\ \text{INCOMPLETE}\end{aligned}$$

Suppose that the speed of the train is  $v = 0.95c$ , i.e. 95% of the speed of light. Then the ratio

$$\frac{v^2}{c^2} = (0.95)^2 = 0.9025$$

can no longer be considered close to 0.

If you insist on using the linear approximation you would predict a contraction of

$$\tilde{\mathcal{L}} \approx \mathcal{L} \left( 1 - \frac{1}{2} \frac{v^2}{c^2} \right)$$

Physics engineers when fine-tuning cyclotrons are faced with "heavy" electrons. These are electrons accelerated to near the speed of light. Their masses are given by

$$\tilde{m} = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Linear approximation (cont'd)

Suppose we have a differentiable function  $f : D \rightarrow \mathbb{R}$  such that both  $f(a)$  and  $f'(a)$  are known.

Then, the linear approximation of  $f$  at  $a$  is

$$L_a(x) = f'(a)(x - a) + f(a)$$

We often write

$$f(a + \Delta x) \approx L_a(a + \Delta x)$$

**Example:**

Find the linear approximation to  $f(x) = \sqrt{x+3}$  in the neighbourhood of  $a = 1$ . Then use that approximation to estimate  $\sqrt{3.98}$  and  $\sqrt{4.02}$ .

Solution: We need to compute both  $f(1)$  and  $f'(1)$ , i.e.

- $f(1) = \sqrt{1+3} = 2$
- $f'(x) = \frac{1}{2\sqrt{x+3}}$ . Hence,  $f'(1) = \frac{1}{4}$ .

Conclusion: The linear approximation is

$$L_1(x) = \frac{x-1}{4} + 2$$

Let's first compute an estimate of  $\sqrt{3.98}$ .

$$\begin{aligned} \sqrt{3.98} &= \sqrt{0.98+3} = f(0.98) \\ &\approx L_1(0.98) = \frac{0.98-1}{4} + 2 = 1.995 \end{aligned}$$

The real value is  $\sqrt{3.98} \approx 1.99499371$

Finally we proceed along the same line of thought to estimate 4.02.

$$\sqrt{4.02} = \sqrt{1.02+3} = f(1.02) \approx L_1(1.02) = \frac{1.02-1}{4} + 2 = 2.005$$

The actual value is  $\sqrt{4.02} \approx 2.00499376$

Errors under control

Idea: Suppose you make a measurement  $a$  (e.g. length, mass, ...). Clearly there is an associated error  $\Delta x$  coming from you, the human, and/or coming from the actual instrument. With this measurement you want to compute a value  $f(1)$  (e.g. area, volume, ...).

Question: given that error  $\Delta x$  on your measurement, what is the error on the computed



value  $f(a)$ ?

One hopes that you are a competent technician and that your company is providing good quality instruments, i.e.  $\Delta x$  is small!

Under that hypothesis, one can estimate the error as being INCOMPLETE

Suppose you measure the radius of a sphere and you obtain  $r = (21 \pm 0.05)cm$ . If you use that measurement to compute the volume of the sphere, estimate the error on the computed volume.

Solution: Here,  $\Delta r = 0.05$ . Now recall that the volume of a sphere is  $V = \frac{4}{3}\pi r^3$ .

The derivative is  $4\pi r^2$ . Hence the first error to first order is

$$\Delta V \approx |V'(21)| \cdot 0.05 = 277.1cm^3$$

Have we lost control over that error?

### Definition

The relative error on a measured or computed value  $y$  is

$$\frac{\Delta y}{y}$$

In our example, the relative error on the volume is

$$\frac{|V'(21)|\Delta r}{V(21)} = \frac{4\pi(21)^2 INCOMPLETE}{V(21)}$$

While the relative error on the measurement of the radius is  $\frac{\Delta r}{r} = \frac{0.05}{21} = 0.0024$ , or a 0.24% error.

WARNING: In general, the more computations you do with a single measurement, the more the error will increase.

Related Rates

1. One is inflating a spherical balloon at a rate of  $100cm^3/s$ . At what speed is the radius increasing when the diameter hits  $50cm$ ?

Solution: Both the volume,  $V(t)$ , and the radius  $r(t)$ , of the balloon are functions of time.

- What we know is that  $\frac{dV}{dt} = 100cm^3/s$
- What we are looking for is INCOMPLETE

The volume and radius are related obviously by

$$V = \frac{4}{3}\pi r^3$$

Hence,

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt} \text{ by the chain rule.}$$

Conclusion:

$$\left. \frac{dr}{dt} \right|_{r=25cm} = \frac{1}{4\pi(25)^2} \frac{dV}{dt}$$