

MAT 1341

Introduction to Linear Algebra

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Chapter 2: Complex Numbers

2.1 defining the complex numbers

The equation $x^2 + 1 = 0$ has no solutions in \mathbb{R} .

Let $i = \sqrt{-1}$ (\star)

Then $i^2 = -1$, hence $i^2 + 1 = 0$.

Hence i is a solution to (\star) .

Example: consider $x^2 + 2x + 2 = 0$

By the quadratic formula, the solutions are:

$$x = \frac{-2 \pm \sqrt{4 - 8}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2\sqrt{-1}}{2} = -1 \pm i$$

Check:

$$(-1 + i)^2 = (-1 + i)(-1 + i) = 1 - 2i - 1 = -2i$$

Hence:

$$(-1 + i)^2 + 2(-1 + i) + 2 = -2i - 2 + 2i + 2 = 0$$

Definition

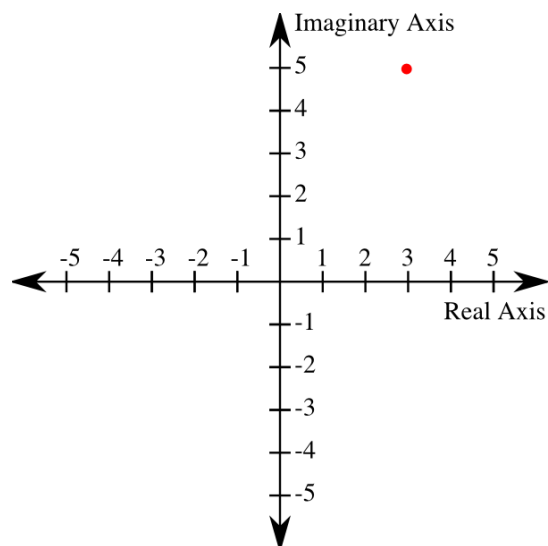
The set of complex numbers is $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$

When we write $z = a + bi$, where $a, b \in \mathbb{R}$:

- a is called the real part, and is denoted by $Re(z)$, and
- bi is called the imaginary part, and is denoted by $Im(z)$.

When $a = 0$, z is called *purely imaginary*.

When $b = 0$, $z \in \mathbb{R}$.



The complex plane. The red dot is $3+5i$.

Properties of complex numbers

If $z, w, y \in \mathbb{C}$:

- $z + w = w + z$
- $zw = wz$
- $1 \times z = z$
- $0 \times z = 0$
- $y(z + w) = yz + yw$
- $y(zw) = (yz)w$

Given any quadratic equation $ax^2 + bx + c$ where $\{a \neq 0 \mid a, b, c \in \mathbb{R}\}$, the solutions are found using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. If $b^2 - 4ac \geq 0$, the solutions $\in \mathbb{R}$. If $b^2 - 4ac < 0$, the solutions are complex.

2.2 Algebra of the complex numbers

For $a, b, c \in \mathbb{R}$:

- $a + bi = c + di \Leftrightarrow a = c \text{ and } b = d$
- $(a + bi) + (c + di) = (a + c) + (b + d)i$
- $(a + bi)(c + di) = ac - bd$

Definition

The **complex conjugate** of $z = a + bi$ is $\bar{z} = a - bi$.

Example

$$\begin{aligned}\overline{1 + 2i} &= 1 - 2i \\ z\bar{z} &= \bar{z}z \\ &= (a + bi)(a - bi) \\ &= a^2 + b^2\end{aligned}$$

z is a nonnegative, real number.

The absolute value of z is $|z| = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}}$

- $z = 0 \Leftrightarrow a = b = 0 \Leftrightarrow |z| = 0$
- $\frac{1}{z} = \frac{1}{z} \times \frac{\bar{z}}{\bar{z}}$
 $= \frac{\bar{z}}{|z|^2}$
 $= \frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$
 $= \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$

Example

$$\begin{aligned}\frac{1}{1+i} &= \frac{1}{1-i} \times \frac{1-i}{1-i} \\ &= \frac{1-i}{1^2+1^2} \\ &= \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i\end{aligned}$$

Example

$$\begin{aligned}\frac{2+i}{1-3i} &= \frac{2+i}{1-3i} \times \frac{1+3i}{1+3i} \\ &= \frac{(2-3)+(6+1)i}{1^2+3^2} \\ &= \frac{-1}{10} + \frac{7}{10}i\end{aligned}$$

2.3 Geometry of the complex numbers

Numbers on the complex plane may be treated as vectors when performing addition, with the real and complex parts corresponding to coordinates.

- Multiplication by a real corresponds to scaling
- $|z|$ = length of a vector, e.g. $|2+i| = \sqrt{2^2+1^2} = \sqrt{5}$

2.4 Polar form of complex numbers

$$z = a + bi$$

$$r = |z| = \sqrt{a^2 + b^2}$$

$$\cos \theta = \frac{a}{r}$$

$$\sin \theta = \frac{b}{r}$$

Polar form of z :

$$z = a + bi = (r \cos \theta) + (r \sin \theta)i$$

$$= r(\cos \theta + i \sin \theta)$$

Note that $\theta = \arg(z) \rightarrow$ argument of z is not uniquely determined since $\theta + 2n\pi$ also works for any $n \in \mathbb{Z}$. We usually pick $-\pi < \theta \leq \pi$ and write $\theta = \arg(z)$, principal argument of z .

Recall

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{4} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$e^{i\theta} = \cos \theta + i \sin \theta$$

So we can write

$$z = re^{i\theta}$$

Properties

$$re^{i\theta} = se^{i\phi} \Leftrightarrow r = s \text{ and } \theta = \phi + 2n \text{ for some integer } n.$$

$$\overline{re^{i\theta}} = re^{-i\theta}$$

$$|e^{i\theta}| = 1 \text{ for any } \theta$$

2.5 Multiplying complex numbers in polar form

If $z = re^{i\theta}$, $w = se^{i\phi}$

$$\begin{aligned}zw &= r(\cos \theta + i \sin \theta) \times s(\cos \phi + i \sin \phi) \\ &= rs[(\cos \theta \cos \phi - \sin \theta \sin \phi) + (\sin \theta \cos \phi + \cos \theta \sin \phi)i] \\ &= rs[\cos(\theta + \phi) + i \sin(\theta + \phi)] \\ &= rse^{i(\theta + \phi)}\end{aligned}$$

Example

$$1 + i = \sqrt{2}e^{-i\frac{\pi}{4}}$$

hence

$$\frac{1}{1+i} = \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}$$

2.6 Fundamental theorem of algebra

Every polynomial with coefficients in the complex numbers can be factored completely into linear factors of the form $zx + w$, with $z, w \in \mathbb{C}$.

Example

$$x^2 + 1 = (x + i)(x - i)$$

Every degree n polynomial with coefficients in the complex plane has n solutions (counting multiplicities).

Chapter 3: Vector geometry

3.1

Algebra \longleftrightarrow Geometry

\mathbb{R}	line
\mathbb{R}^2	plane
\mathbb{R}^3	3-plane
\mathbb{R}^n	n-space

Notations:

$$\vec{x} = (1, 2, 3)$$

$$\vec{x} = \underline{i}, \underline{2j}, \underline{3k}$$

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [1, 2, 3]$$

$$\mathbb{R}^n = \{(x_1, \dots, x_n \mid x_1, \dots, x_n \in \mathbb{R})\}$$

3.2 Properties

- $(x_1, \dots, x_n) = (y_1, \dots, y_n) \Leftrightarrow x_1 = y_1, x_n = y_n$
- $(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$
- $\vec{0} = (0, \dots, 0) \in \mathbb{R}^n$
- if $\vec{x} = (x_1, \dots, x_n)$, then $-\vec{x} = (-x_1, \dots, -x_n)$ and $\vec{x} + (-\vec{x}) = \vec{0}$
- if $r \in \mathbb{R}$, $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, then $r \cdot \vec{x} = (rx, rx_2, \dots, rx_n)$
- 2 vectors are equal \Leftrightarrow they have the same magnitude and same direction
- Head-to-tail rule
- $\vec{0}$ is the only vector with 0 magnitude.
- negative=reverse direction
- 2 vectors are parallel \Leftrightarrow they are multiples of each other

3.3 Definition:

If $r_1, \dots, r_n \in \mathbb{R}$,

$\vec{x}_1, \dots, \vec{x}_n \in \mathbb{R}^n$

then $\vec{y} = r_1\vec{x}_1 + \dots + r_n\vec{x}_n$ is a *linear combination* of $\vec{x}_1, \dots, \vec{x}_n$

We are looking for two scalars, $r_1, r_2 \in \mathbb{R}$, such that

$$\begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} = r_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + r_2 \begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix}$$

$$\begin{cases} 3 = r_1 + r_2 \\ 3 = 2r_1 + \frac{1}{2}r_2 \\ 4 = 3r_1 \end{cases}$$

But $3 \neq 2(\frac{4}{3}) + \frac{1}{5}(\frac{5}{3})$

3.4 More properties

$r, s \in \mathbb{R}, \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$

- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $\vec{u} + \vec{0} = \vec{u}$
- $\vec{u} + (-\vec{u}) = \vec{0}$
- $r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$
- $(r + s)\vec{u} = r\vec{u} + s\vec{u}$
- $(rs)\vec{u} = r(s\vec{u})$
- $1 \cdot \vec{u} = \vec{u}$

3.5 Definition

The dot product of $\vec{x} = (x_1, \dots, x_n)$ and $\vec{y} = (y_1, \dots, y_n)$ is

$$\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + \dots + x_ny_n$$

And the norm of \vec{x} is $\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$

$$= \sqrt{x_1^2 + \dots + x_n^2} \text{ Note that } \|\vec{x}\| = 0 \iff \vec{x} = \vec{0}$$

3.6 Definition

if $\vec{x} \cdot \vec{y} \in \mathbb{R}^n$

then \vec{x} and \vec{y} are said to be *orthogonal* (perpendicular), and $\vec{x} \cdot \vec{y} = 0$

3.7: The Cauchy-Schwarz Inequality

let $\vec{u}, \vec{v} \in \mathbb{R}^n$

then $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$

$$\|\vec{u} + \vec{v}\|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$$

$$= \|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2$$

$$\leq \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2$$

$$\leq \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2$$

$$= (\|\vec{u}\| + \|\vec{v}\|)^2$$

This implies $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$, triangle inequality.

Definition

let $\vec{u}, \vec{v} \in \mathbb{R}^n$, $\vec{u}, \vec{v} \neq \vec{0}$

the angle between \vec{u} and \vec{v} is defined by:

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

With $0 \leq \theta \leq \pi$

$$\cos \theta = \left(\frac{\vec{u}}{\|\vec{u}\|} \right) \cdot \left(\frac{\vec{v}}{\|\vec{v}\|} \right)$$

Example

Compute the angle between

$\vec{v} = (0, 2, 1, \sqrt{3})$ and $\vec{v} = (\sqrt{3}, 1, 2, 0)$

$$\cos \theta = \frac{0+2+2+0}{\sqrt{4+1+3+\sqrt{3+1+4}}} = \frac{4}{8} = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

Remark: \vec{u} and \vec{v} are orthogonal:

$$\iff \vec{u} \cdot \vec{v} = 0$$

$$\iff \cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = 0$$

$$\iff \theta = \frac{\pi}{2}$$

\vec{u} and \vec{v} are parallel:

$$\iff \theta = 0 \text{ or } \pi$$

$$\iff \cos \theta = 1 \text{ or } -1$$

$$\iff \vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \text{ or } -\|\vec{u}\| \|\vec{v}\|$$

i.e. $|\vec{u} \cdot \vec{v}| = \|\vec{u}\| \|\vec{v}\|$

This means that $|\vec{u} \cdot \vec{v}|$ attains its maximum value given by Cauchy-Schwarz Inequality.

Definition

let $\vec{u}, \vec{v} \in \mathbb{R}^n$, $\vec{u}, \vec{v} \neq \vec{0}$

The the projection of \vec{u} onto \vec{v} is

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$$

$$= \left(\vec{u} \frac{\vec{v}}{\|\vec{v}\|} \right) \frac{\vec{v}}{\|\vec{v}\|}$$

Properties

- $\text{proj}_{\vec{v}}(\vec{u})$ is parallel to \vec{v}
- $\vec{u} - \text{proj}_{\vec{v}}(\vec{u})$ is perpendicular to \vec{v}
- $\vec{u} = (\vec{u} - \text{proj}_{\vec{v}}(\vec{u})) + \text{proj}_{\vec{v}}(\vec{u})$

Example

$\vec{v} = (1, 0, 0)$, $\vec{v} = (2, 4, 6)$

Then $\text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2} \vec{v}$

$$= \frac{2}{1^2} (1, 0, 0)$$

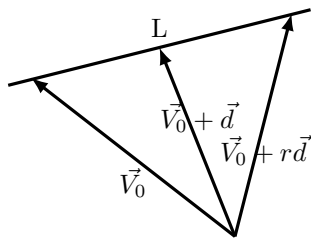
$$= (2, 0, 0)$$

$\vec{w} = (5, 0, 0)$

$\text{proj}_{\vec{w}}(\vec{v}) = (2, 0, 0)$

Chapter 4: Lines and planes

4.1



Equation of a line: $y = mx + c$

$$L = \{\vec{V}_0 + r\vec{d} \mid r \in \mathbb{R}\}$$

i.e. Any point on L can be written as $\vec{V}_0 + r\vec{d}$.

Ex: The line $y = 2x + 1$ in \mathbb{R}^2 can be written as follows:

Let $x = r \leftarrow$ parameter

$$y = 2r + 1 \text{ Then: } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r \\ 2r + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\vec{V}_0} + r \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\vec{d}}$$

Vector form or parametric form.

Ex: Use different letters to represent parameters.

Find the point of intersection of

$$L_1 = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} + r \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mid r \in \mathbb{R} \right\}$$

and

$$L_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}$$

Use t for the parameter of L_2 , and we solve:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} + r \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + r \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{cases} 0 + r = 1 + 0 \Rightarrow r = 1 \\ 1 + 2r = 1 + r \Rightarrow t = 2 \end{cases}$$

So the point of intersection is

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

4.2

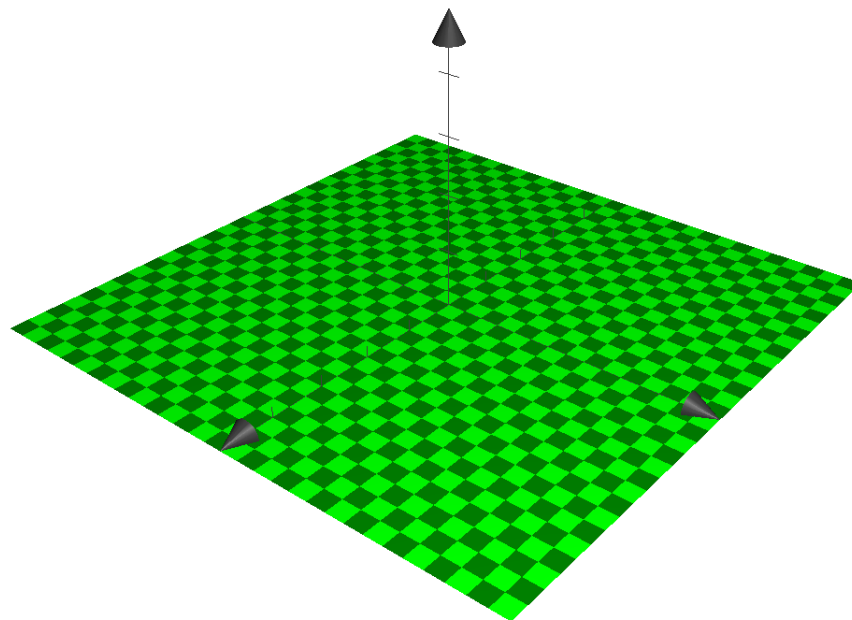
In \mathbb{R} : there is only one line.

In \mathbb{R}^2 : 2 distinct lines, either parallel or intersecting.

In \mathbb{R}^3 : 2 distinct lines, either parallel, intersecting, or skew (neither parallel nor intersecting). In the first two cases, there is a unique plane containing both lines.

If they are skew, there is no plane containing both, but there are two *parallel* planes, each containing one line.

4.3



$$W = \{\vec{v} \in \mathbb{R}^3 \mid (\vec{v} - \vec{v}_0) \cdot \vec{n} = 0\}$$

For example, the plane with $\vec{n} = (1, 2, 3)$ and containing the point $(0, 1, 1)$ is:

$$\{(x, y, z) \in \mathbb{R}^3 \mid [(x, y, z) - (0, 1, 1)] \cdot (1, 2, 3) = 0\}$$

$$(x, y - 1, z - 1) \cdot (1, 2, 3) = 0$$

$$x + 2y - 2 + 3z - 3 = 0$$

$$x + 2y + 3z = 5$$

Ex: find the distance from the point $(3, 3, 3)$ to the plane $x + 2y + 3z = 5$

$$\begin{aligned} D &= \|\text{proj}_{(1,2,3)}((3, 3, 3) - (0, 1, 1))\| \\ &= \|\text{proj}_{(1,2,3)}(3, 2, 2)\| \\ &= \left\| \frac{3 + 4 + 6}{1 + 4 + 9}(1, 2, 3) \right\| = \frac{13}{14}\sqrt{14} = \frac{13}{\sqrt{14}} \end{aligned}$$

4.4

Def: The angle between 2 planes in \mathbb{R}^3 is the angle between their normal vectors.

In \mathbb{R}^2 , there is only one plane.

In \mathbb{R}^3 , there may be 2 distinct planes, either are parallel, or intersect.

... But what about \mathbb{R}^4 ?

n	equations in \mathbb{R}^n	geometric object	dimension
1	$ax = b$	point	0
2	$ax + by = c$	line	1
3	$ax + by + cz = d$	plane	2
4	$ax + by + cz + dw = e$?	3

Idea: One equation in \mathbb{R}^n will cut down the dimension by 1. The resulting \mathbb{R}^n object is called a hyperplane.

4.5

If $\vec{x} = (x_1, x_2, x_3)$

$\vec{y} = (y_1, y_2, y_3)$

Then the cross product of \vec{x} and \vec{y} is:

$$\vec{x} \times \vec{y} = \begin{bmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = (x_2y_3 - y_2x_3)i - (x_1y_3 - y_1x_3)j + (x_1y_2 - y_1x_2)k$$

Ex: $(0, 1, 2) \times (-3, 4, 1)$

$$= \begin{bmatrix} i & j & k \\ 0 & 1 & 2 \\ -3 & 4 & 1 \end{bmatrix} = (-7, -6, 3)$$

Properties

- $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
- $(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$
- $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$
- $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
- $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin \theta$, where $0 \leq \theta \leq \pi$ is the angle between \vec{u} and \vec{v} .

Remark:

- $\|\vec{u} \times \vec{v}\|$ = area of the parallelogram with sides \vec{u} and \vec{v}
- Area of the triangle with sides \vec{u} and \vec{v} is $\frac{1}{2} \|\vec{u} \times \vec{v}\|$
- In general, $\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$
- Suppose $\vec{u}, \vec{v} \neq 0$. Then \vec{u}, \vec{v} parallel $\iff \vec{u} \times \vec{v} = \vec{0}$
- Direction of $\vec{u} \times \vec{v}$ is given by right hand rule.

4.6 Ex: find an equation of the plane:

- Containing the y-axis
- Perpendicular to the plane $4x - y + 3z = 5$

Normal vector of the given plane is perpendicular to $(0, 1, 0)$ and $(4, -1, 3)$.

$$(0, 1, 0) \times (4, -1, 3) = \begin{bmatrix} i & j & k \\ 0 & 1 & 0 \\ 4 & -1 & 3 \end{bmatrix} = (3, 0, -4)$$

Volume of the parallelepiped in \mathbb{R}^3 with sides \vec{u}, \vec{v} and \vec{w}

$$|(\vec{u} \times \vec{v}) \cdot \vec{w}| = |(\vec{v} \times \vec{w}) \cdot \vec{u}|$$

$$\text{Area of the base parallelogram} = \|\vec{u} \times \vec{v}\|$$

$$\text{Height} = \|\vec{w}\| \cos \theta$$

Ex: find the volume of the parallelepiped with sides:

$$\vec{u} = (2, 0, 3)$$

$$\vec{v} = (1, 1, -6)$$

$$\vec{w} = (-1, 2, 1)$$

Volume=

$$\begin{vmatrix} 2 & 0 & 3 \\ 1 & 1 & -6 \\ -1 & 2 & 1 \end{vmatrix} = 2(1 + 12) + 0 + 3(3) = 35$$

Chapter 5: Vector spaces

5.2 Definition

A vector space is:

- a set V (set of vectors) without a geometric representation (generally), with two operations:
 - addition of vectors
 - scalar multiplication

satisfying the following 10 axioms:

1. If $\vec{u}, \vec{v} \in V$, then $u + v \in V$
2. If $\vec{u} \in V, r \in \mathbb{R}$, then $r\vec{u} \in V$
3. There exists a vector, denoted $\vec{0}$, such that $\vec{0} + \vec{u} = \vec{u} \forall u \in V$
4. Given $\vec{u} \in V$, there exists a vector denoted $-\vec{u}$, such that $\vec{u} + (-\vec{u}) = \vec{0}$
5. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
6. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
7. $r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$
8. $(\vec{r} + \vec{s})\vec{u} = r\vec{u} + s\vec{u}$
9. $(\vec{r}\vec{s})\vec{u} = r(\vec{s}\vec{u})$
10. $1 \times \vec{u} = \vec{u}$

Remark: $0 \times \vec{u} = \vec{0}$
 $(-1)\vec{u} = -\vec{u}$

5.3 Example

1. \mathbb{R}^n with usual addition and scalar multiplication are vector spaces for every $n \in \mathbb{N}$

2. Spaces of linear equations

V = set of all linear equations in x, y, z

(a)

$$\begin{aligned} \text{if } u &= (ax + by + cz = d) \\ v &= (a'x + b'y + c'z = d') \end{aligned}$$

$$\text{Then } u + v = [(a + a')x + (b + b')y + (c + c')z = d + d']$$

(b) If $r \in \mathbb{R}$, $ru = (rax + rby + rcz = rd)$

$$\begin{aligned} \text{e.g. } u &= (-2x + y + 3z = 1) \\ v &= (x - y + z = 0) \end{aligned}$$

$$\text{Then } u + 2v = (-y + 5z = 1)$$

3. Spaces of functions

V = set of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$

(a) If $f, g \in V$,

then $f + g : \mathbb{R} \rightarrow \mathbb{R}$ is a function defined by $(f + g)(x) = f(x) + g(x) \forall x \in \mathbb{R}$

(b) If $f \in V$, $r \in \mathbb{R}$, then $rf : \mathbb{R} \rightarrow \mathbb{R}$ is a function defined by $(rf)(x) = r(f(x)) \forall \mathbb{R}$

Verification of axioms:

- (1), (2) ✓
- (3) the zero vector is the function $h : \mathbb{R} \rightarrow \mathbb{R}$ defined by $h(x) = 0 \forall \mathbb{R}$
- (4) Given $f : \mathbb{R} \rightarrow \mathbb{R}$, $-f$ is the function defined by $(-f)(x) = -(f(x))$
- (5)-(10) ✓

4. $V = \{0\}$

(a) $\vec{0} + \vec{0} = \vec{0}$

(b) $r\vec{0} = \vec{0} \forall r \in \mathbb{R}$

is a vector space, called the *zero vector space*. It corresponds to one-dimensional space.

5. $V = \{(x, 2x) \mid x \in \mathbb{R}\}$, With usual addition and scalar multiplication as in \mathbb{R}^2

Verification of axioms:

- (1) If $u = (a, 2a)$, $v = (b, 2b) \in V$ where $a, b \in \mathbb{R}$
then $u + v = (a + b, 2a + 2b)$
- (2) If $u = (a, 2a) \in V$, $r \in \mathbb{R}$, then

$$\begin{aligned} ru &= (ra, r2a) \\ &= (ra, 2(ra)) \in V \end{aligned}$$

- (3) $0 = (0, 0) \in V$
- (4) If $u = (a, 2a)$, then $-u = (-a, -2a) \in V$

6. $V = \{(x, x + a) \mid x \in \mathbb{R}\}$ with usual addition and scalar multiplication in \mathbb{R}^2 .
 $y = x + 1$ is **NOT** a vector space

- (a) If $u = (a, a + 1) \in V$
 $v = (b, b + 1) \in V$
then $u + v = (a + b, a + b + 2) \notin V$
- (b) If $u = (a, a + 1)$, $r \in \mathbb{R}$
then $ru = (ra, ra + r) \notin V \forall r \neq 1$
- (c) $(0, 0) \notin V$

Definition

An $m \times n$ matrix is a table of numbers with m rows and n columns.

e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is a 2×3 matrix

$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ is called a 3×1 column matrix

$[1 \quad 2 \quad 0 \quad 5]$ is a 1×4 row matrix

2 matrices of the same size can be added componentwise:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 4 \\ 6 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 \\ 10 & 13 & 16 \end{bmatrix}$$

Matrices can also be multiplied by a scalar componentwise:

$$-4 \times \begin{bmatrix} 2 & 0 \\ 5 & 6 \\ 7 & 1 \end{bmatrix} = \begin{bmatrix} -8 & 0 \\ -20 & -24 \\ -28 & -4 \end{bmatrix}$$

Examples

1. $V = M_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$ with the above addition and scalar multiplication is a vector space.

Verification of the axioms:

- (1), (2) ✓
- (3) $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V$
- (4) ✓
- (5) If $u = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $v = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$
then $u + v = \begin{bmatrix} a + e & c + f \\ c + g & d + h \end{bmatrix}$
- (6)-(10) ✓

2. $M_{mn}(\mathbb{R})$ is a vector space for all $m, n \in \mathbb{N}$

Remark: In a vector space, $0u = \vec{0}$

$$1u = 0u = (1 + 0)u = 1u = u$$

$$\Rightarrow u + 0u + (-u) = u + (-u)$$

$$\Rightarrow 0u = \vec{0}$$

\mathbb{R}^3 has 3 dimensions.

Every vector space has a basis, and the number of vectors in a basis is the dimension.

Chapter 6: Subspaces and spanning sets

6.1 Definition

A subset W of a vector space V is called a subspace if it is a vector space itself, under the same addition and scalar multiplication of V .

Examples:

1. $W = \{(x, 2x) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$ is a subspace of \mathbb{R}^2

Theorem: If V is a vector space, $W \subseteq V$ subset then W is a subspace:

$$\iff \begin{cases} O \in W \\ \text{If } u, v \in W, \text{ then } u + v \in W \\ \text{If } u \in W, r \in \mathbb{R}, \text{ then } ru \in W \end{cases}$$

axioms of subspaces

6.2

Examples:

1. W is the plane $x + 2y + 3z = 0$ in \mathbb{R}^3

Verification:

- (a) $\vec{0} = (0, 0, 0) \in W$
- (b) If $\vec{u} = (a, b, c), \vec{v} = (a', b', c') \in W$
 $a + 2b + 3c = 0, a' + 2b' + 3c' = 0$
 $u + v = (a + a', b + b', c + c') \in W$

$$\begin{aligned} & (a + a') + 2(b + b') + 3(c + c') \\ &= (a + 2b + 3c) + (a' + 2b' + 3c') \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

- (c) If $u = (a, b, c) \in W, r \in \mathbb{R}$, then $ru = (ra, rb, rc) \in W$

$$\begin{aligned} & ra + 2(rb) + 3(rc) \\ &= r(a + 2b + 3c) \\ &= r \cdot 0 \\ &= 0 \end{aligned}$$

2. Any plane passing through the origin in \mathbb{R}^3 is a subspace.
Any plane *not* passing through the origin is *not* a subspace.

3. $v \in \mathbb{R}^n, v \neq 0$

$L = \{tv \mid t \in \mathbb{R}\}$ is a line in \mathbb{R}^n passing through the origin, with direction vector v .

Verification:

- (a) $\vec{0} = 0 \cdot v \in L$
- (b) If $u = tv, w = sv \in L$
 $u + v = tv + sv = (t + s)v \in L$

- (c) If $u = t\vec{v} \in L$, $r \in R$
 $ru = r(t\vec{v}) = (rt)\vec{v} \in L$
 So L is a subspace.
4. $W = \{(x, y) \mid x, y \geq 0\} \subseteq \mathbb{R}^2$
 Is this a subspace? **NO**, as multiplying a vector in the space by certain scalars will produce a vector not in the subspace.
5. Any line passing through the origin is a subspace in \mathbb{R}^n .
 Any line *not* passing through the origin is *not* a subspace in \mathbb{R}^n
6. $V =$ set of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$
 $W =$ the set of all polynomial functions $p : \mathbb{R} \rightarrow \mathbb{R}$
 i.e. $p(x) = a_0 + a_1x + \dots + a_nx^n$, n a non-negative integer, $a_0..a_n \in \mathbb{R}$
 Verification:
- (a) (1) The zero vector of V is the function $f(x) = 0 \forall x \in \mathbb{R}$. It is in W because $f(x)$ is the polynomial function with $n = 0$ and $a = 0$.
- (b) (2), (3) ✓
7. $V =$ set of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$
 $W =$ set of all continuous functions
 \therefore Multiplication of any continuous function by a scalar returns a continuous function
 $\therefore W$ is a subset of V .
8. V is the set of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$
 W is the set of all functions $f(x)$ such that $f(x) \in [-1, 1] \forall x$
 Is W a subspace of V ?
- (a) (1) ✓
- (b) (2) $f(x) = \cos(x)$
 $g(x) = \cos(x)$
 $(f + g)(x) = 2\cos(x) \notin W$

Matrices

Def: The transpose of an $m \times n$ matrix A , denoted A^T , is an $n \times m$ matrix whose columns are rows of A , e.g.:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ -2 & 5 \end{bmatrix}^T = \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 & 4 \end{bmatrix}$$

An $n \times n$ matrix (square matrix) is called *symmetric* if $A = A^T$, e.g. $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ is symmetric.

1. $V = M_{22}(\mathbb{R})$

$W =$ set of all 2×2 symmetric matrices

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \mid a, b, d \in \mathbb{R} \right\}$$

Verification:

(a) (1) $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$

(b) (2) If $u = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$, $v = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$,

$$u + v = \begin{bmatrix} a+e & b+f \\ b+g & d+h \end{bmatrix} \in W$$

(c) (3) If $u = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \in W$, $r \in \mathbb{R}$

$$ru = \begin{bmatrix} ra & rb \\ rb & rd \end{bmatrix} \in W$$

Let V be an arbitrary vector space.

V always has the following subspaces:

1. V is its own subspace
2. $\{0\}$, the zero subspace (or zero vector space)

Chapter 7: The span of vectors in a vector space

Definition

V is a vector space.

1. If $v_1, \dots, v_n \in V$ and $a_1, \dots, a_n \in \mathbb{R}$, then $u = a_1v_1 + \dots + a_nv_n$ is called a *linear combination* of v_1, \dots, v_n .
2. The set of all linear combinations of v_1, \dots, v_n is called the *span* if v_1, \dots, v_n .
 $\text{span}\{v_1, \dots, v_n\} = \{a_1v_1 + \dots + a_nv_n \mid a_1, \dots, a_n \in \mathbb{R}\}$
3. A vector (sub)space W is spanned by $v_1, \dots, v_n \in W$ if $W = \text{span}\{v_1, \dots, v_n\}$ (v_1, v_n spans W)

S = set of all 2×2 symmetric matrices

$$\begin{aligned} &= \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \mid a, b, d \in \mathbb{R} \right\} \\ &= \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mid a, b, d \in \mathbb{R} \right\} \\ &= \text{span}\{a, b, d\} \end{aligned}$$

A spanning set is *not* unique.

$$\text{span}\{0\} = \{0\}$$

Examples

1. $L = \{t\vec{v} \mid t \in \mathbb{R}\} \subseteq \mathbb{R}^n$ is a line
 $= \text{span } \vec{v}$
2. $\mathbb{R}^2 = \text{span}\{(1, 0), (0, 1)\}$ because any $(x, y) \in \mathbb{R}^2$ can be written as $x(1, 0) + y(0, 1)$.

Similarly, $\mathbb{R}^3 = \text{span}\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

Examples

1. $W = \text{xy-plane in } \mathbb{R}^3 = \{(x, y, 0) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$
 $= \text{span}\{(1, 0, 0), (0, 1, 0)\}$

7.4

Theorem:

Let V be a vector space such that

$$\{v_1, \dots, v_n\} \subseteq V \quad (\iff v_1, \dots, v_n \in V)$$

1. $U = \text{span}\{v_1, \dots, v_n\}$ is a subspace of V
2. If W is another subspace of V such that $v_1, \dots, v_n \in W$, then $U \subseteq W$
Therefore U is the smallest subspace containing v_1, \dots, v_n

Subset axioms:

1. $\vec{0} \in U$
2. If $u, v \in U$, then $u + v \in U$
3. If $u \in U$, $r \in \mathbb{R}$, then $ru \in U$

Proof of (1):

- $\vec{0} = 0v_1 + \dots + 0v_n \in \text{span}\{v_1, \dots, v_n\}$
- If $u, v \in U$,
then $u = a_1v_1 + \dots + a_nv_n$
and $w = b_1v_1 + \dots + b_nv_n$
For some $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$
 $u + w = (a_1 + b_1)v_1 + \dots + (a_n + b_n)v_n \in U$
- If $u \in U, r \in \mathbb{R}$
then $u = a_1v_1 + \dots + a_nv_n$
 $ru = (ra_1)v_1 + \dots + (ra_n)v_n \in U$

7.5

Definition:

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{1\ldots} & a_{1n} \\ a_{21} & \dots & \dots & \dots \\ a_{\ldots 1} & \dots & \dots & \dots \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix}$$

The trace of A , denoted $\text{Tr}(A)$ is defined as $\text{Tr}(A) = a_{11} + a_{22} + \dots + a_{nn}$.

Example:

$$\text{Tr} \left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \right) = 1 + 5 + 9 = 15$$

Example:

$$\begin{aligned} U &= \{A \in M_{22}(\mathbb{R}) \mid \text{Tr}(A) = 0\} \\ &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a + d = 0 \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\} \\ &= \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\} \\ &= \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} \end{aligned}$$

Example:

W = set of all 2×2 diagonal matrices

$$\begin{aligned} &= \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \mid a, d \in \mathbb{R} \right\} \\ &= \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ is a subspace of } M_{22}(\mathbb{R}) \end{aligned}$$

7.6 \mathbb{R} has only 2 subspaces: $\{0\}$ and \mathbb{R} . Proof: If $W \subseteq \mathbb{R}$ is a subspace and $W \neq \{0\}$, then there is a nonzero vector $\vec{v} \in W$

$$\begin{aligned} \mathbb{R} &= \text{span}\{w\} \subseteq W \\ &\Rightarrow \mathbb{R} = W \end{aligned}$$

The only subspaces of \mathbb{R}^2 are:

1. $\{0\}$
2. Lines through the origin
3. \mathbb{R}^2

Proof: If $W \subseteq \mathbb{R}^2$ is a subspace and $W \neq \{0\}$, $W \neq$ lines through the origin. Then we have to show that $W = \mathbb{R}^2$.

First, $W \neq \{0\}$. There exists a nonzero vector $v \in W \Rightarrow \text{span}\{v\} \subseteq W$

Since $W \neq$ lines through the origin, there exists $u \in W$ such that $u \notin \text{span}\{v\}$, i.e. u is not a multiple of v .
 $u, v \in W \Rightarrow \text{span}\{u, v\} \subseteq W$

Note that $\text{span}\{u, v\} = \mathbb{R}^2$ because any vector \mathbb{R}^2 can be written as a linear combination of u and v .

For any $w \in \mathbb{R}^2$, $w = \text{proj}_v(w) + (w - \text{proj}_v(w))$

$$\begin{aligned} W &= \text{proj}_v(w) + (w - \text{proj}_v(w)) \\ &= av + b(u - \text{proj}_v(u)) \\ &= av + bu - b(cv) \\ &= (a - bc)v + bu \in \text{span}\{u, v\} \end{aligned}$$

The only subspaces of \mathbb{R}^3 are

1. $\{0\}$
2. Lines through the origin
3. Planes through the origin
4. \mathbb{R}^3

7.7

Examples:

1. $\text{span}\{(1, 2, 1), (0, 1, 2)\} = \text{span}\{(1, 3, 3), (1, 1, -1)\}$
 Taking the cross products of each pair of vectors in these spans and showing that the resulting vectors are parallel is a proof.
2. $\text{span}\{(1, 2, 1), (0, 1, 2)\} = u$ and $\text{span}\{(1, 3, 3), (1, 1, -1)\} = v$
 $(1, 3, 3) = (1, 2, 1) + (0, 1, 2) \in \text{span}\{(1, 2, 1), (0, 1, 2)\} = u$

Chapter 8: Linear dependence and independence

8.2

2 vectors $\vec{u}, \vec{v} \in \mathbb{R}^3$ are collinear if there are non-zero $a, b \in \mathbb{R}$ such that $a\vec{u} + b\vec{v} = 0$.

e.g. $(1, 2, 0), (2, 4, 0)$ are collinear and $\text{span}\{(1, 2, 0), (2, 4, 0)\} = \text{span}\{(1, 2, 0)\}$

8.3

3 vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ are coplanar if there exists $a, b, c \in \mathbb{R}$ not all zero, such that $a\vec{u} + b\vec{v} + c\vec{w} = 0$.

e.g. $(1, 0, 0), (0, 1, 0), (3, 2, 0)$ are coplanar because $3(1, 0, 0) + 2(0, 1, 0) + (-1)(3, 2, 0) = 0$

$\text{span}\{(1, 0, 0), (0, 1, 0), (3, 2, 0)\} = \text{span}\{(1, 0, 0), (0, 1, 0)\}$

8.4

Definition:

- V is a vector space.
 $v_1, \dots, v_n \in V$
 $\{v_1, \dots, v_n\}$ is linearly dependent \iff there exists $a_1, \dots, a_n \in \mathbb{R}$ not all zero such that $a_1v_1 + \dots + a_nv_n = 0$.
- $\{v_1, \dots, v_n\}$ is linearly independent $\iff \{v_1, \dots, v_n\}$ is *not* linearly dependent \iff the only solution to the equation $a_1v_1 + \dots + a_nv_n = 0$ is the trivial solution $a_1 = a_2 = \dots = a_n = 0$.

8.7

Examples:

- $\{(1, 0), (0, 1)\} \subseteq \mathbb{R}^2$ is LI.

$$\text{Solve } a(1, 0) + b(0, 1) = (0, 0) \iff \begin{cases} a \cdot 1 + b \cdot 0 = 0 \\ a \cdot 0 + b \cdot 1 = 0 \end{cases}$$

So $a = b = 0$

- $\{(2, 2), (3, -1)\} \subseteq \mathbb{R}^2$ is LI.

$$\text{Solve } a(2, 2) + b(3, -1) = (0, 0) \iff \begin{cases} 2a + 3b = 0 \\ 2a - b = 0 \end{cases}$$

$$4b = 0 \Rightarrow b = 0$$

$$\Rightarrow a = 0$$

- $\{(2, 2), (3, 2), (-1, -1)\} \subseteq \mathbb{R}^2$ is LD because $1(2, 2) + 0(3, 2) + 2(-1, -1) = (0, 0)$.

- $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \subseteq M_{22}(\mathbb{R})$

Solve:

$$a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$a = 0, b = 0, c = 0, d = 0.$$

- Is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \right\} \subseteq M_{22}(\mathbb{R})$

linearly dependent or linearly independent?

Solution: It is linearly dependent.

$$1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - 1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{cases} a + b = 0 \\ b + 4c = 0 \\ b + 4c = 0 \\ 3d = 0 \end{cases}$$

$$d = 0$$

$$a + b = 0 \quad a = -b$$

$$b + 4c = 0 \quad c = \frac{-b}{4}$$

- $V = F(\mathbb{R}) = \{f: \mathbb{R} \rightarrow \mathbb{R}\}$
 $\{1, x, x^2\}$ is LI.
Solve $a \cdot 1 + bx + cx^2 = 0$, where 0 is the zero function. This is an equation of functions.
- $\{1, \sin^2(x), \cos^2(x)\}$ is linearly dependent because $1 - \sin^2(x) - \cos^2(x) = 0$.
- $\{1, \sin(x), \cos(x)\}$ is linearly independent. Solve $a + b \sin(x) + c \cos(x) = 0$.

8.8

Properties

- $\{0\}$ is linearly dependent.
- $v \in V$, then $\{v\}$ is LI $\iff v \neq 0$
- $\{v_1, \dots, v_n\}$ is LD \iff there are some v_i such that v_i is in the span of the rest, i.e. $v_i \in \text{span}\{v_1, \dots, v_n\}$.
- If a set S is LD, then any set containing S is LD.
- Conversely, if a set S is linearly independent, then any subset of S is still linearly independent.
- any set containing 0 is linearly dependent.
- $\{u, v\}$ is LD \iff one vector is a multiple of the other.

Chapter 9: Linear independence and spanning sets

9.2 Theorem:

If $v_1 \in \text{span}\{v_1, \dots, v_n\}$, then $\text{span}\{v_1, \dots, v_n\} = \text{span}\{v_1, \dots, v_n\}$

Therefore if $\{v_1, \dots, v_n\}$ is LD, then $\text{span}\{v_1, \dots, v_n\} = \text{span}\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$ for some i .

Recall:

$\{v_1, \dots, v_n\}$ is LD \iff there exists a space v_i such that $v_i \in \text{span}\{v_i, v_{i-1}, \dots, v_n\}$.

In this case, $\text{span}\{v_1, \dots, v_n\} = \text{span}\{v_i, v_{i-1}, \dots, v_n\}$.

Examples

1. $\text{span}\{(1, 0, 0), (0, 1, 0), (3, -2, 0)\} = \text{span}\{(1, 0, 0), (0, 1, 0)\}$
2. $\text{span}\{(1, 0, 0), (0, 1, 1), (0, 4, 4)\} = \text{span}\{(1, 0, 0), (0, 1, 1)\} \neq \text{span}\{(0, 1, 1), (0, 4, 4)\}$

9.3

Suppose $\{v_1, \dots, v_n\}$ is LI, $w \in V$.

Then $\{w, v_1, \dots, v_n\}$ is LI $\iff w \notin \text{span}\{v_1, \dots, v_n\}$ **Example:**

1. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ is L
and $\begin{bmatrix} 0 & 0 \\ 3 & 2 \end{bmatrix} \notin \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$
then $\left\{ \begin{bmatrix} 0 & 0 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ is LI.
2. $\begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} \in \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$
Then $\left\{ \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ is LD.

Chapter 10: Basis and dimension

10.1 Theorem:

If a vector space V can be spanned by n vectors, then any LI subset of V has at most n vectors. Equivalently, if V has a LI subset of m vectors, then any spanning set of V has at least m vectors.

$$\text{size of any spanning set of } V \geq \text{size of any LI subset of } V$$

Examples

1. $\mathbb{R}^3 = \text{span}\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ can be spanned by 3 vectors. Any subset of \mathbb{R}^3 containing 4 or more vectors is LD.
2. $M_{22}(\mathbb{R})$ can be spanned by 4 vectors. Any subset of $M_{22}(\mathbb{R})$ containing 5 or more vectors is LD.

10.2 Definition

$\{v_1, \dots, v_n\} \subseteq V$ is called a basis of V if

1. $\{v_1, \dots, v_n\}$ is LI
 2. $\text{span}\{v_1, \dots, v_n\} = V$
- A basis is a largest possible LI subset of a set V , i.e. if you add any vector to the set, it will become LD.
 - A basis is a smallest possible spanning set of V . If you remove any vector from the set, then it will not span V .

Examples:

1. $\{(1, 0), (0, 1)\}$ is a basis of \mathbb{R}^2 (the standard basis).
2. $\{(1, 1), (1, -1)\}$ is also a basis of \mathbb{R}^2 .
3. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is the standard basis of $M_{22}(\mathbb{R})$.
4. $\mathbb{P}_2 =$ vector space of polynomials of a degree ≤ 2 is a closed subspace under addition (as adding any polynomials can only reduce the degree), multiplication, and the zero vector test.
 $\{1, x, x^2\}$ is a basis of \mathbb{P}_2 .

Any polynomial of $\deg \leq 2$ is of the form $a + bx + cx^2$. **Examples**

- (a) $\{(1, 0, 0), (0, 1, 1)\} \in \mathbb{R}^3$ is NOT a basis as $\text{span}\{(1, 0, 0), (0, 1, 1)\} \neq \mathbb{R}^3$.
- (b) $V = \{a \in M_{22}(\mathbb{R}) \mid \text{Tr}(A) = 0\}$
 $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 3 & -2 \end{bmatrix} \right\}$ is NOT a basis of V because it is LD.
 $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis of V .

Theorem

If $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$ are 2 bases for V , then $n = m$.

Proof:

$\{v_1, \dots, v_n\}$ spans V and $\{w_1, \dots, w_m\}$ is LI.

$\Rightarrow n \geq m$.

$\{v_1, \dots, v_n\}$ is LI and $\{w_1, \dots, w_m\}$ spans V .

$n \leq m$ The only case in which both $n \geq m$ and $n \leq m$ are true is if $n = m$. QED.

10.3 Definition

If V has a finite basis $(\{v_1, \dots, v_n\})$ then we say that the dimension $\dim(V) = n$. In this case, V is called a finite dimensional space.

If V does not have a finite basis, then V is an infinite dimensional space.

Examples:

- $\dim(\mathbb{R}^2) = 2$
- $\dim(\mathbb{R}^n) = n$
- $\dim(M_{22}(\mathbb{R})) = 4$
- $\dim(M_{mn}(\mathbb{R})) = mn$
Standard basis: $\{E_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$
 E_{ij} is the matrix with (i, j) - entry is 1, other entries are 0.
- $\dim(\mathbb{P}_2) = 3$ because a basis is $\{1, x, x^2\}$
- $\dim(\mathbb{P}_n) = n + 1$
- $\dim(F(\mathbb{R})) = \infty$
- $\dim(V) = 3$ where $V = \{A \in M_{22}(\mathbb{R}) \mid \text{Tr}(A) = 0\}$
- $W =$ vector space of all 2×2 diagonal matrices. $\dim(W) = 2$.
- $W =$ the plane with equation $x + 3y + 5z = 0 \subseteq \mathbb{R}^3$.
 $\dim(W) = 2$. A basis of W is $\{(3, -1, 0), (5, 0, -1)\}$
Solve $a(3, -1, 0) + b(5, 0, 1) = (0, 0, 0)$
Then $a = b = 0$
 $\dim\{0\} = 0$. A basis for the zero vector space is \emptyset

$$V = M_{22}(\mathbb{R})$$

$$W = \left\{ \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

This set does not contain $\vec{0}$. It is therefore not a subspace.

$$W = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

This set contains $\vec{0}$.

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid ab = cd \right\}$$

This set is not closed under addition, and is not a subspace.

Chapter 11: Dimension theorems

11.1

V vector space with $\dim(V) = n < \infty$

- If $\{v_1, \dots, v_m\} \subseteq V$ is LI, $m \leq n$ and there exists $v_{m+1}, \dots, v_n \in V$ such that $\{v_1, \dots, v_m, \dots, v_n\}$ is a basis of V .
- If $\{w_1, \dots, w_r\}$ is a spanning set of V , $r \geq n$ and there is a subset of S that is a basis of V
 - Size of LI subset \leq size of spanning set

$V = \mathbb{R}^3$

$\{(1, 2, 0), (1, -1, 0)\}$

Because the 2 vectors are *not* multiples of each other, this set is LI. This is a shortcut for sets of 2 vectors.

$\{(1, 2, 0), (1, -1, 0), (0, 0, 1)\}$ is a basis of \mathbb{R}^3

It is LI because $(0, 0, 1) \notin \text{span}\{(1, 2, 0), (1, -1, 0)\}$

$\{(1, 0, 0), (0, 1, 0), (0, 1, 1), (0, 2, 3)\}$ is LD.

We see that $(0, 2, 3) = -(0, 1, 0) + 3(0, 1, 1)$

so $\{(1, 0, 0), (0, 1, 0), (0, 1, 1)\}$ is a basis of \mathbb{R}^3 .

Theorem

V vector space with $\dim(V) = n < \infty$

Then:

1. Any LI subset of n vectors is a basis of V
2. Any spanning set of n vectors is also a basis

Axioms:

1. LI
2. Spanning
3. Consisting of n vectors

Only need to check any 2 of these conditions to see whether a subset is a basis.

Examples

1. $W = \{(x, y, z) \in \mathbb{R}^3 \mid 3x - y + 5z = 0\}$
 $\dim(W) = 2$
 $(1, 3, 0), (5, 0, -3) \in W$
 $\{(1, 3, 0), (5, 0, -3)\}$ is LI
 $\Rightarrow \{(1, 3, 0), (5, 0, -3)\}$ is a basis of W (spanning is mathematically satisfied)
2. Expand $\{(1, 3, 0), (5, 0, -3)\}$ to a basis of \mathbb{R}^3 .
 $\{(1, 3, 0), (5, 0, -3), (3, -1, 5)\}$ is a basis of \mathbb{R}^3 .
 $(3, -1, 5)$ is a normal vector of W . In particular, $(3, -1, 5) \notin W$.
 $\Rightarrow \{(1, 3, 0), (5, 0, -3), (3, 1, 5)\}$ is LI.

11.3 Theorem

V vector space, $\dim(V) = n < \infty$

$W \subseteq V$ subspace. Then:

1. $0 \leq \dim(W) \leq \dim(V)$
2. $\dim(W) = \dim(V) \iff W = V$
3. $\dim(W) = 0 \iff W = \{0\}$

Proof

1. Suppose $\{w_1, \dots, w_m\}$ is a basis of W ($\dim(W) = m$).
Then, $\{w_1, \dots, w_m\}$ is an LI subset of W , hence also a subset of V .
Therefore, $m \leq \dim(V)$.
2. If $W = V$, then $\dim(W) = \dim(V)$.
If $\dim(W) = \dim(V)$, let $S = \{w_1, \dots, w_n\}$ be a basis of W .
(LI + consisting of n vectors) \Rightarrow also a basis of V .

Examples

1. Any subspaces of $M_{22}(\mathbb{R})$ has dimensions 0, 1, 2, 3 or 4 and the only subspace of $\dim(4)$ is $M_{22}(\mathbb{R})$. The only subspace of $\dim(0)$ is $\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$
2. Any subspace of \mathbb{R}^n has dimension 0, 1, 2, ..., or n and the only subspace with $\dim(n)$ is \mathbb{R}^n . The only subspace with $\dim(0)$ is $\{(0, 0, \dots, 0)\}$

11.6 Theorem:

Let $B = \{v_1, \dots, v_n\}$ be a basis of V .

Then for any $v \in V$, there exist *unique* scalars $a, \dots, a_n \in \mathbb{R}$ such that $v = a_1v_1, \dots, a_nv_n$.

Examples:

1. $\{(1, 0), (0, 1)\}$ is a basis of \mathbb{R}^2 .
2. $\{(1, 0), (0, 1), (0, 3)\}$ is NOT a basis of \mathbb{R}^3 s

Application

We can identify V with \mathbb{R}^n

$$v = a_1v_1 + \dots + a_nv_n \longleftrightarrow (a_1, \dots, a_n)$$

$$\text{e.g. } V = M_{22}(\mathbb{R}), B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\text{Then } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longleftrightarrow (a, b, c, d)$$

Chapter 12: solving systems of linear equations

12.1 Examples

1.

$$\begin{cases} x + 3y + 2z = 6 \\ 0x + y - z = 0 \end{cases}$$

is a linear system of 2 equations.

Definition

The general solution to a linear system is the set of all solutions, e.g. $\{(6 - 5t, t, t) \mid t \in \mathbb{R}\}$ is the general solution to the above example (t is the parameter of the general solution).

2.

$$\begin{cases} x + y + z = 2 \\ x + y + z = 3 \end{cases}$$

Has no solution. The general solution is \emptyset , the empty set.

Definition

- A linear system with NO solution is called *inconsistent*.
- A linear system with at least one solution is called *consistent*.
- A linear system in which all the constants on the right are zero is called *homogenous*.
- A linear system in which at least one of the constants is nonzero is called *inhomogenous*.

Remark

- Homogenous system are always consistent.
This is because $(0, 0, 0)$ is always a solution (the trivial solution).

Definition

- A linear equation is called *degenerate* if all the coefficients are zero.

Remark

- Any linear system containing a *degenerate*, *homogenous* equation is *inconsistent*.

Theorem

Any linear system (with real coefficients) has either

1. NO solutions
2. exactly one solution
3. infinitely many solutions

Examples

1.

$$x + y + 2z = 3 \quad (1)$$

$$x - y + z = 2 \quad (2)$$

$$y = z = 1 \quad (3)$$

$$(2) - (1) = x + y + 2z = 3$$

$$-2y - z = -1$$

$$y - z = 1$$

$$(2) - (3) = x + y + 2z = 3$$

$$y = z = 1$$

$$-2y = -1$$

$$(3) + 2(2) = x + y + 2z = 3$$

$$y - z = 1$$

$$-3z = 1$$

$$-\frac{1}{3}(3)$$

$$x + y + 2z = 3$$

$$z = -\frac{1}{3}$$

$$(2) + (3)$$

$$x + y + 2z = 3$$

$$y + \frac{2}{3}$$

etc.

This method is inefficient. We can do this with matrices.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & 1 \end{array} \right]$$

$$x = 3$$

$$y = \frac{2}{3}$$

$$z = \frac{1}{3}$$

We used the following operations:

- Add a multiple of one row to another row
- Interchange two rows
- Multiply one row by a nonzero scalar

These are called *elementary row operations* (ERO) and performing these operations to a linear system will not change the general solution.

12.3 Definition

- A matrix is in *Row Echelon Form* (REF) if the following conditions are satisfied:
 - All zero rows are at the bottom
 - The first nonzero entry of each row is 1 (leading 1)
 - Each leading 1 is to the right of the leading 1s in the rows above

- Each leading 1 is the only nonzero entry in its column, then the matrix is in *reduced row echelon form* (RREF).

e.g.

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -5 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

is REF but not RREF

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array} \right]$$

is RREF, and the general solution is $\{(7, 4, 5)\}$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

General solution: \emptyset

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_3 = s$$

$$x_4 = t \quad (s, t \in \mathbb{R})$$

$$x_2 = -2s - t$$

$$x_1 = 1 - 2s$$

Chapter 13: Solving systems of linear equations (cont'd)

Def:

Two linear systems are equivalent if they have the same general solution.

Thm:

If an ERO is performed on the augmented matrix of a linear system, the resulting linear system is equivalent to the original one.

Def:

Two matrices A and B are row equivalent if

- $A \sim B$, if B can be obtained from A by a finite sequence of ERO.

Thm:

Every matrix is row equivalent to a unique matrix in RREF. This statement is false if we replace RREF by REF).

$$\begin{array}{ccc} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} & \xrightarrow{R_1 - R_2} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \text{REF} & & \text{RREF} \end{array}$$

13.1

- If the RREF of the augmented matrix has a row like

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & b \end{array} \right]$$

where $b \neq 0$, then the system is inconsistent.

Otherwise, the system is consistent.

- If every column has leading 1, then there is a unique solution, e.g.

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array} \right]$$

- Otherwise, there are infinitely many solutions.
 1. Set each non-leading variable equal to a different parameter
 2. Solve for the leading variables

e.g.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Set $z = t$, $t \in \mathbb{R}$

Then

$$y = 4 + 4t$$

$$x = 3 - 2t$$

$$\{3 - 2t, 4 + 4t, t \mid t \in \mathbb{R}\}$$

e.g.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 7 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & -1 & 6 \end{array} \right]$$

Set $x_2 = s$

$x_5 = t$

$s, t \in \mathbb{R}$

Then $x_4 = 6 + t$

$x_3 = 5$

$x_1 = 7 - 2s - 3t$

13.2 Gaussian elimination

1. If the matrix is 0 (the zero matrix), done.
2. Locate the leftmost nonzero column.
Interchange rows if necessary, to bring a nonzero entry to the 1st row.
3. Multiply the first row with a scalar to get a leading 1.
4. **Annihilate** the rest of the column.
5. Repeat step 1-4 ignoring the first row.

Iteration 2:

1. Same as above
2. Locate the leftmost column with a nonzero entry in the 2nd-last row, e.g.

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 4 & 7 & -4 \\ 0 & 0 & 6 & 8 & 3 \end{array} \right]$$

Interchange rows if necessary to bring a nonzero entry to the 2nd row.

3. Multiply the 2nd row with a scalar to get a leading 1.

e.g.

$$\left[\begin{array}{cccc|c} 0 & 0 & 3 & 3 & 4 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & -1 & 0 & 1 & 1 \end{array} \right] \rightarrow R_1 \leftrightarrow R_2 \rightarrow \left[\begin{array}{cccc|c} 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 3 & 3 & 4 \\ 0 & -1 & 0 & 1 & 1 \end{array} \right] \rightarrow R_3 + R_1 \rightarrow \left[\begin{array}{cccc|c} 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 3 & 3 & 4 \\ 0 & 0 & 1 & 3 & 1 \end{array} \right] \rightarrow$$

$$\frac{1}{3}R_2 \rightarrow \left[\begin{array}{cccc|c} 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 4/3 \\ 0 & 0 & 1 & 3 & 1 \end{array} \right] \rightarrow R_1 - R_2, R_3 - R_2 \rightarrow \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & -4/3 \\ 0 & 0 & 1 & 1 & 4/3 \\ 0 & 0 & 0 & 2 & -1/3 \end{array} \right] \rightarrow$$

$$\frac{1}{2}R_3 \rightarrow \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & -4/3 \\ 0 & 0 & 1 & 1 & 4/3 \\ 0 & 0 & 0 & 1 & -1/6 \end{array} \right] \rightarrow$$

At this point, the matrix is already in REF. To find the RREF:

$$R_1 - R_3, R_2 - R_3 \rightarrow \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & -7/6 \\ 0 & 0 & 1 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & -1/6 \end{array} \right]$$

The general solution of $\{(t, -\frac{7}{6}, \frac{3}{2}, -\frac{1}{6}) \mid t \in \mathbb{R}\}$ **Ex:**

$$\left[\begin{array}{ccc|c} 2 & 2 & -3 & 1 \\ 1 & 0 & 1 & 5 \\ 3 & 4 & -7 & -3 \end{array} \right] \rightarrow \frac{1}{2}R_1 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -3/2 & 1/2 \\ 1 & 0 & 1 & 5 \\ 3 & 4 & -7 & -3 \end{array} \right] \rightarrow \frac{1}{3}R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -3/2 & 1/2 \\ 1 & 0 & 1 & 5 \\ 1 & 4/3 & -7/3 & -1 \end{array} \right] \rightarrow$$

$$R_2 \leftrightarrow R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -3/2 & 1/2 \\ 1 & 4/3 & -7/3 & -1 \\ 1 & 0 & 1 & 5 \end{array} \right] \rightarrow R_2 \leftrightarrow R_1 \rightarrow \left[\begin{array}{ccc|c} 1 & 4/3 & -7/3 & -1 \\ 1 & 1 & -3/2 & 1/2 \\ 1 & 0 & 1 & 5 \end{array} \right]$$

13.3 Definition

The rank of a matrix A , denoted $\text{rank}(A)$ is the number of leading 1s in any REF of A .

e.g.

$$\text{rank} \left(\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \right) = 2$$

Remark

$\text{rank}(A) \leq \text{number of columns of } A$

Chapter 14: Applications and examples of solving linear systems

Recall:

$\text{rank}(A)$ = number of leading 1s in any REF of A .

$\text{rank}(A) \leq \text{rank}[A \mid b] \leq \text{rank}(A) + 1$

e.g. For a homogeneous linear system,

$$\text{rank}(A) = \text{rank}[A \mid b]$$

then the system is consistent.

Remark

If $\text{rank}(A) = \text{rank}[A \mid b]$, then the system is consistent.

1. If $\text{rank}(A) (= \text{rank}[A \mid b]) < \text{number of columns of } A$, then there is at least one column with a leading 1.
2. If $\text{rank}(A) = \text{number of columns of } A$, then the system has a unique solution.

14.2

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 300 \\ 1 & 0 & -1 & 0 & 100 \\ 0 & 0 & 1 & 1 & 500 \\ 0 & 1 & 0 & 1 & 300 \end{array} \right] \rightarrow R_2 - R_1 \rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 300 \\ 0 & 1 & -1 & 0 & -200 \\ 0 & 0 & 1 & 1 & 500 \\ 0 & 1 & 0 & 1 & 300 \end{array} \right] \rightarrow R_1 + R_2; R_4 - R_2 \rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 300 \\ 1 & 0 & -1 & 0 & 100 \\ 0 & 0 & 1 & 1 & 500 \\ 0 & 1 & 0 & 1 & 300 \end{array} \right]$$

The general solution is $\{(600 - t, 300 - t, 500 - 2t, t) \mid t \in \mathbb{R}\}$

e.g. If the street x_2 is blocked, what is the effect?

$$x_2 = 0 \Rightarrow t = 300$$

14.3

Consider

$$kx + y + z = 1$$

$$x + ky + z = 1$$

$$x + y + kz = 1$$

with $k \in \mathbb{R}$

Find all values of k so that the above system has:

1. no solution
2. a unique solution
3. infinitely many solution

$$\left[\begin{array}{ccc|c} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \end{array} \right] \rightarrow R_1 \leftrightarrow R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & k & 1 & 1 \\ k & 1 & 1 & 1 \\ 1 & 1 & k & 1 \end{array} \right] \rightarrow R_2 - kR_1; R_3 - R_1 \rightarrow \left[\begin{array}{ccc|c} 1 & k & 1 & 1 \\ 0 & 1 - k^2 & 1 - k & 1 - k \\ 0 & 1 - k & k - 1 & 0 \end{array} \right]$$

Case $k = 1$ we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

14.4

Compute $\text{span}\{(1, 2, 3), (4, 5, 6), (7, 8, 9)\}$

What are the vectors $(x, y, z) \in \mathbb{R}^3$ such that there are scalars $a_1, a_2, a_3 \in \mathbb{R}$ satisfying

$$a_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + a_3 \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} ?$$

Chapter 15 Matrix Multiplication

15.1 Definition

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix.

The product AB is an $m \times p$ matrix whose (i, j) -entry is the dot product of the i -th row of A with the j -th column of B .

i.e. If $A = [a_{ij}]$, $B = [b_{ij}]$

Then $AB = [c_{ij}]$

Where

$$c_{ij} = \sum_{k=1}^n a_{ik}b_{kj} = a_{i1}b_{j1} + a_{i2}b_{j2} + \dots + a_{in}b_{jn}$$

Examples:

- $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32$, a 1×1 matrix.
- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$
- $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 4 & -1 \end{bmatrix}$ The result of this multiplication is undefined as the matrices have different numbers of columns.
- $\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 2 \\ -3 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 7 & -1 & -1 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 3 \\ -2 & -5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y \\ 4x - y + 3z \\ -2x - 5y + z \end{bmatrix}$

A system of linear equations

$$a_{11}x + a_{12}y + a_{13}z = b_1 \tag{4}$$

$$a_{21}x + a_{22}y + a_{23}z = b_2 \tag{5}$$

$$a_{31}x + a_{32}y + a_{33}z = b_3 \tag{6}$$

Can be expressed as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

15.2 Properties

- Sometimes AB is defined but BA is not (commutativity is not an inherent property of matrix multiplication)

e.g. $A = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

- Even if AB and BA are both defined, they may be different.

e.g. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$

$$AB = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$$

- It is possible that $A \neq 0$ and $B \neq 0$ but $AB = 0$.
- It is possible that $AC = BC$; $C \neq 0$ but $A \neq B$
 e.g. $A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
 $AC = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$ and $BC = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$

Examples:

- Let $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$. View $A, B \in \mathbb{R}^3$
 The dot product of A and B
 $= 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6$
 $= A^T B$

15.3

Properties of the transpose:

- $(A + B)^T = A^T + B^T$
- $(kA)^T = k(A^T)$, $k \in \mathbb{R}$
- $(A^T)^T = A$

For any positive integer n , define $I_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ an $n \times n$ matrix called the identity matrix.

Theorem:

Let A, B, C be matrices and $k \in \mathbb{R}$. Whenever defined,

- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(A + B)C = AC + BC$
- $k(AB) = (kA)B = A(kB)$
- $(AB)^T = B^T A^T$
- $AI = A$, $IA = A$
- If A is an $m \times n$ matrix, then $A0_{n \times p} = 0_{m \times p}$, $0_{q \times m}A = 0_{q \times n}$

Proof of $A(BC) = (AB)C$:

Write $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$

the (i, j) -entry of $BC = \sum_{k=1}^p b_{ik}c_{kj}$

the (i, j) -entry of $A(BC) = \sum_{l=1}^n (a_{il} \cdot \sum_{k=1}^p b_{lk}c_{kj})$

the (i, j) -entry of $(AB)C = \sum_{l=1}^n (\sum_{k=1}^p a_{ik}b_{kl}) c_{lj}$

Examples:

- $(A + B)(C + D) = AC + AD + BC + BD$
- $(A + B)^2 = AA + AB + BA + BB = A^2 + AB + BA + B^2$
- $(A + B)(A - B) = A^2 - AB + BA + B^2 \neq A^2 - B^2$