

MAT 1341

**Introduction to Linear Algebra**

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## Chapter 2: Complex Numbers

### 2.1 defining the complex numbers

The equation  $x^2 + 1 = 0$  has no solutions in  $\mathbb{R}$ .

Let  $i = \sqrt{-1}$   $(\star)$

Then  $i^2 = -1$ , hence  $i^2 + 1 = 0$ .

Hence  $i$  is a solution to  $(\star)$ .

Example: consider  $x^2 + 2x + 2 = 0$

By the quadratic formula, the solutions are:

$$x = \frac{-2 \pm \sqrt{4 - 8}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2\sqrt{-1}}{2} = -1 \pm i$$

Check:

$$(-1 + i)^2 = (-1 + i)(-1 + i) = 1 - 2i - 1 = -2i$$

Hence:

$$(-1 + i)^2 + 2(-1 + i) + 2 = -2i - 2 + 2i + 2 = 0$$

#### Definition

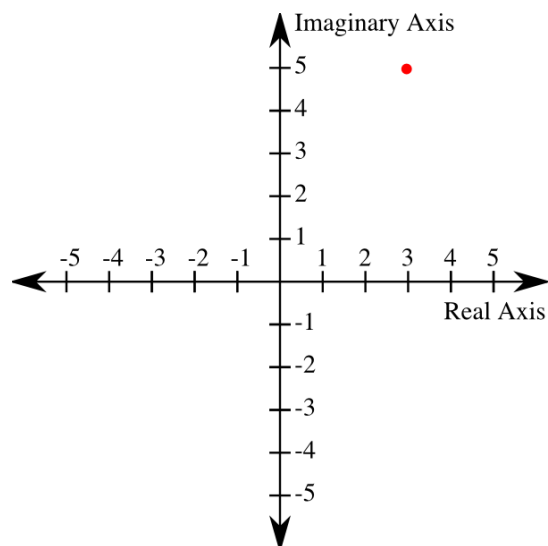
The set of complex numbers is  $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$

When we write  $z = a + bi$ , where  $a, b \in \mathbb{R}$ :

- $a$  is called the real part, and is denoted by  $Re(z)$ , and
- $bi$  is called the imaginary part, and is denoted by  $Im(z)$ .

When  $a = 0$ ,  $z$  is called *purely imaginary*.

When  $b = 0$ ,  $z \in \mathbb{R}$ .



*The complex plane. The red dot is  $3+5i$ .*

## Properties of complex numbers

If  $z, w, y \in \mathbb{C}$ :

- $z + w = w + z$
- $zw = wz$
- $1 \times z = z$
- $0 \times z = 0$
- $y(z + w) = yz + yw$
- $y(zw) = (yz)w$

Given any quadratic equation  $ax^2 + bx + c$  where  $\{a \neq 0 \mid a, b, c \in \mathbb{R}\}$ , the solutions are found using  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ . If  $b^2 - 4ac \geq 0$ , the solutions  $\in \mathbb{R}$ . If  $b^2 - 4ac < 0$ , the solutions are complex.

## 2.2 Algebra of the complex numbers

For  $a, b, c \in \mathbb{R}$ :

- $a + bi = c + di \Leftrightarrow a = c \text{ and } b = d$
- $(a + bi) + (c + di) = (a + c) + (b + d)i$
- $(a + bi)(c + di) = ac - bd$

### Definition

The **complex conjugate** of  $z = a + bi$  is  $\bar{z} = a - bi$ .

### Example

$$\begin{aligned}\overline{1 + 2i} &= 1 - 2i \\ z\bar{z} &= \bar{z}z \\ &= (a + bi)(a - bi) \\ &= a^2 + b^2\end{aligned}$$

$z$  is a nonnegative, real number.

The absolute value of  $z$  is  $|z| = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}}$

- $z = 0 \Leftrightarrow a = b = 0 \Leftrightarrow |z| = 0$
- $\frac{1}{z} = \frac{1}{z} \times \frac{\bar{z}}{\bar{z}}$   
 $= \frac{\bar{z}}{|z|^2}$   
 $= \frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$   
 $= \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$

### Example

$$\begin{aligned}\frac{1}{1+i} &= \frac{1}{1-i} \times \frac{1-i}{1-i} \\ &= \frac{1-i}{1^2+1^2} \\ &= \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i\end{aligned}$$

### Example

$$\begin{aligned}\frac{2+i}{1-3i} &= \frac{2+i}{1-3i} \times \frac{1+3i}{1+3i} \\ &= \frac{(2-3)+(6+1)i}{1^2+3^2} \\ &= \frac{-1}{10} + \frac{7}{10}i\end{aligned}$$

## 2.3 Geometry of the complex numbers

Numbers on the complex plane may be treated as vectors when performing addition, with the real and complex parts corresponding to coordinates.

- Multiplication by a real corresponds to scaling
- $|z|$  = length of a vector, e.g.  $|2+i| = \sqrt{2^2+1^2} = \sqrt{5}$

## 2.4 Polar form of complex numbers

$$z = a + bi$$

$$r = |z| = \sqrt{a^2 + b^2}$$

$$\cos \theta = \frac{a}{r}$$

$$\sin \theta = \frac{b}{r}$$

Polar form of  $z$ :

$$z = a + bi = (r \cos \theta) + (r \sin \theta)i$$

$$= r(\cos \theta + i \sin \theta)$$

Note that  $\theta = \arg(z) \rightarrow$  argument of  $z$  is not uniquely determined since  $\theta + 2n\pi$  also works for any  $n \in \mathbb{Z}$ . We usually pick  $-\pi < \theta \leq \pi$  and write  $\theta = \arg(z)$ , principal argument of  $z$ .

### Recall

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$e^{i\theta} = \cos \theta + i \sin \theta$$

So we can write

$$z = re^{i\theta}$$

### Properties

$$re^{i\theta} = se^{i\phi} \Leftrightarrow r = s \text{ and } \theta = \phi + 2n \text{ for some integer } n.$$

$$\overline{re^{i\theta}} = re^{-i\theta}$$

$$|e^{i\theta}| = 1 \text{ for any } \theta$$

## 2.5 Multiplying complex numbers in polar form

If  $z = re^{i\theta}$ ,  $w = se^{i\phi}$

$$\begin{aligned}zw &= r(\cos \theta + i \sin \theta) \times s(\cos \phi + i \sin \phi) \\ &= rs[(\cos \theta \cos \phi - \sin \theta \sin \phi) + (\sin \theta \cos \phi + \cos \theta \sin \phi)i] \\ &= rs[\cos(\theta + \phi) + i \sin(\theta + \phi)] \\ &= rse^{i(\theta + \phi)}\end{aligned}$$

**Example**

$$1 + i = \sqrt{2}e^{-i\frac{\pi}{4}}$$

hence

$$\frac{1}{1+i} = \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}$$

**2.6 Fundamental theorem of algebra**

Every polynomial with coefficients in the complex numbers can be factored completely into linear factors of the form  $zx + w$ , with  $z, w \in \mathbb{C}$ .

**Example**

$$x^2 + 1 = (x + i)(x - i)$$

Every degree  $n$  polynomial with coefficients in the complex plane has  $n$  solutions (counting multiplicities).

## Chapter 3: Vector geometry

### 3.1

Algebra  $\longleftrightarrow$  Geometry

$\mathbb{R}$	line
$\mathbb{R}^2$	plane
$\mathbb{R}^3$	3-plane
$\mathbb{R}^n$	n-space

Notations:

$$\vec{x} = (1, 2, 3)$$

$$\vec{x} = \underline{i}, \underline{2j}, \underline{3k}$$

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [1, 2, 3]$$

$$\mathbb{R}^n = \{(x_1, \dots, x_n \mid x_1, \dots, x_n \in \mathbb{R})\}$$

### 3.2 Properties

- $(x_1, \dots, x_n) = (y_1, \dots, y_n) \Leftrightarrow x_1 = y_1, x_n = y_n$
- $(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$
- $\vec{0} = (0, \dots, 0) \in \mathbb{R}^n$
- if  $\vec{x} = (x_1, \dots, x_n)$ , then  $-\vec{x} = (-x_1, \dots, -x_n)$  and  $\vec{x} + (-\vec{x}) = \vec{0}$
- if  $r \in \mathbb{R}$ ,  $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ , then  $r \cdot \vec{x} = (rx, rx_2, \dots, rx_n)$
- 2 vectors are equal  $\Leftrightarrow$  they have the same magnitude and same direction
- Head-to-tail rule
- $\vec{0}$  is the only vector with 0 magnitude.
- negative=reverse direction
- 2 vectors are parallel  $\Leftrightarrow$  they are multiples of each other

### 3.3 Definition:

If  $r_1, \dots, r_n \in \mathbb{R}$ ,

$\vec{x}_1, \dots, \vec{x}_n \in \mathbb{R}^n$

then  $\vec{y} = r_1\vec{x}_1 + \dots + r_n\vec{x}_n$  is a *linear combination* of  $\vec{x}_1, \dots, \vec{x}_n$

We are looking for two scalars,  $r_1, r_2 \in \mathbb{R}$ , such that

$$\begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} = r_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + r_2 \begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix}$$

$$\begin{cases} 3 = r_1 + r_2 \\ 3 = 2r_1 + \frac{1}{2}r_2 \\ 4 = 3r_1 \end{cases}$$

But  $3 \neq 2(\frac{4}{3}) + \frac{1}{5}(\frac{5}{3})$

### 3.4 More properties

$r, s \in \mathbb{R}, \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$

- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $\vec{u} + \vec{0} = \vec{u}$
- $\vec{u} + (-\vec{u}) = \vec{0}$
- $r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$
- $(r + s)\vec{u} = r\vec{u} + s\vec{u}$
- $(rs)\vec{u} = r(s\vec{u})$
- $1 \cdot \vec{u} = \vec{u}$

### 3.5 Definition

The dot product of  $\vec{x} = (x_1, \dots, x_n)$  and  $\vec{y} = (y_1, \dots, y_n)$  is

$$\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + \dots + x_ny_n$$

And the norm of  $\vec{x}$  is  $\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$

$$= \sqrt{x_1^2 + \dots + x_n^2} \text{ Note that } \|\vec{x}\| = 0 \iff \vec{x} = \vec{0}$$

### 3.6 Definition

if  $\vec{x} \cdot \vec{y} \in \mathbb{R}^n$

then  $\vec{x}$  and  $\vec{y}$  are said to be *orthogonal* (perpendicular), and  $\vec{x} \cdot \vec{y} = 0$

### 3.7: The Cauchy-Schwarz Inequality

let  $\vec{u}, \vec{v} \in \mathbb{R}^n$

then  $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$

$$\|\vec{u} + \vec{v}\|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$$

$$= \|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2$$

$$\leq \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2$$

$$\leq \|\vec{u}\|^2 + 2\|\vec{u}\| \|\vec{v}\| + \|\vec{v}\|^2$$

$$= (\|\vec{u}\| + \|\vec{v}\|)^2$$

This implies  $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ , triangle inequality.

**Definition**

let  $\vec{u}, \vec{v} \in \mathbb{R}^n$ ,  $\vec{u}, \vec{v} \neq \vec{0}$

the angle between  $\vec{u}$  and  $\vec{v}$  is defined by:

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

With  $0 \leq \theta \leq \pi$

$$\cos \theta = \left( \frac{\vec{u}}{\|\vec{u}\|} \right) \cdot \left( \frac{\vec{v}}{\|\vec{v}\|} \right)$$

**Example**

Compute the angle between

$\vec{v} = (0, 2, 1, \sqrt{3})$  and  $\vec{v} = (\sqrt{3}, 1, 2, 0)$

$$\cos \theta = \frac{0+2+2+0}{\sqrt{4+1+3+\sqrt{3+1+4}}} = \frac{4}{8} = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

**Remark:**  $\vec{u}$  and  $\vec{v}$  are orthogonal:

$$\iff \vec{u} \cdot \vec{v} = 0$$

$$\iff \cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = 0$$

$$\iff \theta = \frac{\pi}{2}$$

$\vec{u}$  and  $\vec{v}$  are parallel:

$$\iff \theta = 0 \text{ or } \pi$$

$$\iff \cos \theta = 1 \text{ or } -1$$

$$\iff \vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \text{ or } -\|\vec{u}\| \|\vec{v}\|$$

i.e.  $|\vec{u} \cdot \vec{v}| = \|\vec{u}\| \|\vec{v}\|$

This means that  $|\vec{u} \cdot \vec{v}|$  attains its maximum value given by Cauchy-Schwarz Inequality.

**Definition**

let  $\vec{u}, \vec{v} \in \mathbb{R}^n$ ,  $\vec{u}, \vec{v} \neq \vec{0}$

The the projection of  $\vec{u}$  onto  $\vec{v}$  is

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$$

$$= \left( \vec{u} \frac{\vec{v}}{\|\vec{v}\|} \right) \frac{\vec{v}}{\|\vec{v}\|}$$

**Properties**

- $\text{proj}_{\vec{v}}(\vec{u})$  is parallel to  $\vec{v}$
- $\vec{u} - \text{proj}_{\vec{v}}(\vec{u})$  is perpendicular to  $\vec{v}$
- $\vec{u} = (\vec{u} - \text{proj}_{\vec{v}}(\vec{u})) + \text{proj}_{\vec{v}}(\vec{u})$

**Example**

$\vec{v} = (1, 0, 0)$ ,  $\vec{v} = (2, 4, 6)$

Then  $\text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2} \vec{v}$

$$= \frac{2}{1^2} (1, 0, 0)$$

$$= (2, 0, 0)$$

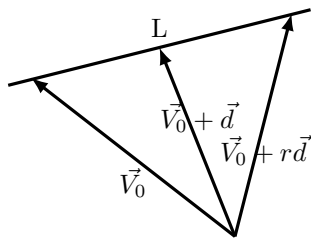
$\vec{w} = (5, 0, 0)$

$\text{proj}_{\vec{w}}(\vec{v}) = (2, 0, 0)$



## Chapter 4: Lines and planes

### 4.1



Equation of a line:  $y = mx + c$

$$L = \{\vec{V}_0 + r\vec{d} \mid r \in \mathbb{R}\}$$

i.e. Any point on L can be written as  $\vec{V}_0 + r\vec{d}$ .

Ex: The line  $y = 2x + 1$  in  $\mathbb{R}^2$  can be written as follows:

Let  $x = r \leftarrow$  parameter

$$y = 2r + 1 \text{ Then: } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r \\ 2r + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\vec{V}_0} + r \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\vec{d}}$$

Vector form or parametric form.

Ex: Use different letters to represent parameters.

Find the point of intersection of

$$L_1 = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} + r \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mid r \in \mathbb{R} \right\}$$

and

$$L_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}$$

Use  $t$  for the parameter of  $L_2$ , and we solve:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} + r \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + r \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{cases} 0 + r = 1 + 0 \Rightarrow r = 1 \\ 1 + 2r = 1 + r \Rightarrow t = 2 \end{cases}$$

So the point of intersection is

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

### 4.2

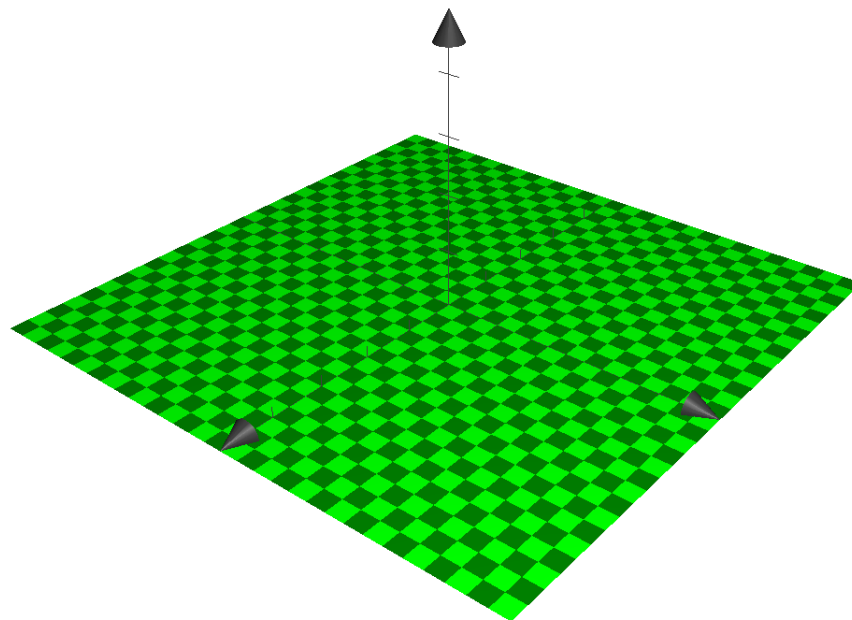
In  $\mathbb{R}$ : there is only one line.

In  $\mathbb{R}^2$ : 2 distinct lines, either parallel or intersecting.

In  $\mathbb{R}^3$ : 2 distinct lines, either parallel, intersecting, or skew (neither parallel nor intersecting). In the first two cases, there is a unique plane containing both lines.

If they are skew, there is no plane containing both, but there are two *parallel* planes, each containing one line.

### 4.3



$$W = \{\vec{v} \in \mathbb{R}^3 \mid (\vec{v} - \vec{v}_0) \cdot \vec{n} = 0\}$$

For example, the plane with  $\vec{n} = (1, 2, 3)$  and containing the point  $(0, 1, 1)$  is:

$$\{(x, y, z) \in \mathbb{R}^3 \mid [(x, y, z) - (0, 1, 1)] \cdot (1, 2, 3) = 0\}$$

$$(x, y - 1, z - 1) \cdot (1, 2, 3) = 0$$

$$x + 2y - 2 + 3z - 3 = 0$$

$$x + 2y + 3z = 5$$

Ex: find the distance from the point  $(3, 3, 3)$  to the plane  $x + 2y + 3z = 5$

$$\begin{aligned} D &= \|\text{proj}_{(1,2,3)}((3, 3, 3) - (0, 1, 1))\| \\ &= \|\text{proj}_{(1,2,3)}(3, 2, 2)\| \\ &= \left\| \frac{3 + 4 + 6}{1 + 4 + 9}(1, 2, 3) \right\| = \frac{13}{14} \sqrt{14} = \frac{13}{\sqrt{14}} \end{aligned}$$

### 4.4

Def: The angle between 2 planes in  $\mathbb{R}^3$  is the angle between their normal vectors.

In  $\mathbb{R}^2$ , there is only one plane.

In  $\mathbb{R}^3$ , there may be 2 distinct planes, either are parallel, or intersect.

... But what about  $\mathbb{R}^4$ ?

n	equations in $\mathbb{R}^n$	geometric object	dimension
1	$ax = b$	point	0
2	$ax + by = c$	line	1
3	$ax + by + cz = d$	plane	2
4	$ax + by + cz + dw = e$	?	3

Idea: One equation in  $\mathbb{R}^n$  will cut down the dimension by 1. The resulting  $\mathbb{R}^n$  object is called a hyperplane.

#### 4.5

If  $\vec{x} = (x_1, x_2, x_3)$

$\vec{y} = (y_1, y_2, y_3)$

Then the cross product of  $\vec{x}$  and  $\vec{y}$  is:

$$\vec{x} \times \vec{y} = \begin{bmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = (x_2y_3 - y_2x_3)i - (x_1y_3 - y_1x_3)j + (x_1y_2 - y_1x_2)k$$

Ex:  $(0, 1, 2) \times (-3, 4, 1)$

$$= \begin{bmatrix} i & j & k \\ 0 & 1 & 2 \\ -3 & 4 & 1 \end{bmatrix} = (-7, -6, 3)$$

#### Properties

- $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
- $(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$
- $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$
- $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
- $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin \theta$ , where  $0 \leq \theta \leq \pi$  is the angle between  $\vec{u}$  and  $\vec{v}$ .

#### Remark:

- $\|\vec{u} \times \vec{v}\|$  = area of the parallelogram with sides  $\vec{u}$  and  $\vec{v}$
- Area of the triangle with sides  $\vec{u}$  and  $\vec{v}$  is  $\frac{1}{2} \|\vec{u} \times \vec{v}\|$
- In general,  $\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$
- Suppose  $\vec{u}, \vec{v} \neq 0$ . Then  $\vec{u}, \vec{v}$  parallel  $\iff \vec{u} \times \vec{v} = \vec{0}$
- Direction of  $\vec{u} \times \vec{v}$  is given by right hand rule.

4.6 Ex: find an equation of the plane:

- Containing the y-axis
- Perpendicular to the plane  $4x - y + 3z = 5$

Normal vector of the given plane is perpendicular to  $(0, 1, 0)$  and  $(4, -1, 3)$ .

$$(0, 1, 0) \times (4, -1, 3) = \begin{bmatrix} i & j & k \\ 0 & 1 & 0 \\ 4 & -1 & 3 \end{bmatrix} = (3, 0, -4)$$

#### Volume of the parallelepiped in $\mathbb{R}^3$ with sides $\vec{u}, \vec{v}$ and $\vec{w}$

$$|(\vec{u} \times \vec{v}) \cdot \vec{w}| = |(\vec{v} \times \vec{w}) \cdot \vec{u}|$$

$$\text{Area of the base parallelogram} = \|\vec{u} \times \vec{v}\|$$

$$\text{Height} = \|\vec{w}\| \cos \theta$$

Ex: find the volume of the parallelepiped with sides:

$$\vec{u} = (2, 0, 3)$$

$$\vec{v} = (1, 1, -6)$$

$$\vec{w} = (-1, 2, 1)$$

Volume=

$$\begin{bmatrix} 2 & 0 & 3 \\ 1 & 1 & -6 \\ -1 & 2 & 1 \end{bmatrix} = 2(1 + 12) + 0 + 3(3) = 35$$

## Chapter 5: Vector spaces

### 5.2 Definition

A vector space is:

- a set  $V$  (set of vectors) without a geometric representation (generally), with two operations:
  - addition of vectors
  - scalar multiplication

satisfying the following 10 axioms:

1. If  $\vec{u}, \vec{v} \in V$ , then  $u + v \in V$
2. If  $\vec{u} \in V, r \in \mathbb{R}$ , then  $r\vec{u} \in V$
3. There exists a vector, denoted  $\vec{0}$ , such that  $\vec{0} + \vec{u} = \vec{u} \forall u \in V$
4. Given  $\vec{u} \in V$ , there exists a vector denoted  $-\vec{u}$ , such that  $\vec{u} + (-\vec{u}) = \vec{0}$
5.  $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
6.  $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
7.  $r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$
8.  $(\vec{r} + \vec{s})\vec{u} = r\vec{u} + s\vec{u}$
9.  $(\vec{r}\vec{s})\vec{u} = r(\vec{s}\vec{u})$
10.  $1 \times \vec{u} = \vec{u}$

**Remark:**  $0 \times \vec{u} = \vec{0}$   
 $(-1)\vec{u} = -\vec{u}$

### 5.3 Example

1.  $\mathbb{R}^n$  with usual addition and scalar multiplication are vector spaces for every  $n \in \mathbb{N}$

2. Spaces of linear equations

$V$  = set of all linear equations in  $x, y, z$

(a)

$$\begin{aligned} \text{if } u &= (ax + by + cz = d) \\ v &= (a'x + b'y + c'z = d') \end{aligned}$$

$$\text{Then } u + v = [(a + a')x + (b + b')y + (c + c')z = d + d']$$

(b) If  $r \in \mathbb{R}$ ,  $ru = (rax + rby + rcz = rd)$

$$\text{e.g. } u = (-2x + y + 3z = 1)$$

$$v = (x - y + z = 0)$$

$$\text{Then } u + 2v = (-y + 5z = 1)$$

3. Spaces of functions

$V$  = set of all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$

(a) If  $f, g \in V$ ,

then  $f + g : \mathbb{R} \rightarrow \mathbb{R}$  is a function defined by  $(f + g)(x) = f(x) + g(x) \forall x \in \mathbb{R}$

(b) If  $f \in V$ ,  $r \in \mathbb{R}$ , then  $rf : \mathbb{R} \rightarrow \mathbb{R}$  is a function defined by  $(rf)(x) = r(f(x)) \forall \mathbb{R}$

Verification of axioms:

- (1), (2) ✓
- (3) the zero vector is the function  $h : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $h(x) = 0 \forall \mathbb{R}$
- (4) Given  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $-f$  is the function defined by  $(-f)(x) = -(f(x))$
- (5)-(10) ✓

4.  $V = \{0\}$

(a)  $\vec{0} + \vec{0} = \vec{0}$

(b)  $r\vec{0} = \vec{0} \forall r \in \mathbb{R}$

is a vector space, called the *zero vector space*. It corresponds to one-dimensional space.

5.  $V = \{(x, 2x) \mid x \in \mathbb{R}\}$ , With usual addition and scalar multiplication as in  $\mathbb{R}^2$

Verification of axioms:

- (1) If  $u = (a, 2a)$ ,  $v = (b, 2b) \in V$  where  $a, b \in \mathbb{R}$   
then  $u + v = (a + b, 2a + 2b)$
- (2) If  $u = (a, 2a) \in V$ ,  $r \in \mathbb{R}$ , then

$$\begin{aligned} ru &= (ra, r2a) \\ &= (ra, 2(ra)) \in V \end{aligned}$$

- (3)  $0 = (0, 0) \in V$
- (4) If  $u = (a, 2a)$ , then  $-u = (-a, -2a) \in V$

6.  $V = \{(x, x + a) \mid x \in \mathbb{R}\}$  with usual addition and scalar multiplication in  $\mathbb{R}^2$ .  
 $y = x + 1$  is **NOT** a vector space

- (a) If  $u = (a, a + 1) \in V$   
 $v = (b, b + 1) \in V$   
then  $u + v = (a + b, a + b + 2) \notin V$
- (b) If  $u = (a, a + 1)$ ,  $r \in \mathbb{R}$   
then  $ru = (ra, ra + r) \notin V \forall r \neq 1$
- (c)  $(0, 0) \notin V$

## Definition

An  $m \times n$  matrix is a table of numbers with  $m$  rows and  $n$  columns.

e.g.  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$  is a  $2 \times 3$  matrix

$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$  is called a  $3 \times 1$  column matrix

$[1 \quad 2 \quad 0 \quad 5]$  is a  $1 \times 4$  row matrix

2 matrices of the same size can be added componentwise:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 4 \\ 6 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 \\ 10 & 13 & 16 \end{bmatrix}$$

Matrices can also be multiplied by a scalar componentwise:

$$-4 \times \begin{bmatrix} 2 & 0 \\ 5 & 6 \\ 7 & 1 \end{bmatrix} = \begin{bmatrix} -8 & 0 \\ -20 & -24 \\ -28 & -4 \end{bmatrix}$$

### Examples

1.  $V = M_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$  with the above addition and scalar multiplication is a vector space.

Verification of the axioms:

- (1), (2) ✓
- (3)  $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V$
- (4) ✓
- (5) If  $u = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ,  $v = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$   
then  $u + v = \begin{bmatrix} a + e & c + f \\ c + g & d + h \end{bmatrix}$
- (6)-(10) ✓

2.  $M_{mn}(\mathbb{R})$  is a vector space for all  $m, n \in \mathbb{N}$

Remark: In a vector space,  $0u = \vec{0}$

$$1u = 0u = (1 + 0)u = 1u = u$$

$$\Rightarrow u + 0u + (-u) = u + (-u)$$

$$\Rightarrow 0u = \vec{0}$$

$\mathbb{R}^3$  has 3 dimensions.

Every vector space has a basis, and the number of vectors in a basis is the dimension.

## Chapter 6: Subspaces and spanning sets

### 6.1 Definition

A subset  $W$  of a vector space  $V$  is called a subspace if it is a vector space itself, under the same addition and scalar multiplication of  $V$ .

Examples:

1.  $W = \{(x, 2x) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$  is a subspace of  $\mathbb{R}^2$

**Theorem:** If  $V$  is a vector space,  $W \subseteq V$  subset then  $W$  is a subspace:

$$\iff \begin{cases} O \in W \\ \text{If } u, v \in W, \text{ then } u + v \in W \\ \text{If } u \in W, r \in \mathbb{R}, \text{ then } ru \in W \end{cases}$$

*axioms of subspaces*

### 6.2

Examples:

1.  $W$  is the plane  $x + 2y + 3z = 0$  in  $\mathbb{R}^3$

Verification:

- (a)  $\vec{0} = (0, 0, 0) \in W$
- (b) If  $\vec{u} = (a, b, c), \vec{v} = (a', b', c') \in W$   
 $a + 2b + 3c = 0, a' + 2b' + 3c' = 0$   
 $u + v = (a + a', b + b', c + c') \in W$

$$\begin{aligned} & (a + a') + 2(b + b') + 3(c + c') \\ &= (a + 2b + 3c) + (a' + 2b' + 3c') \\ &= 0 + 0 \\ &= \vec{0} \end{aligned}$$

- (c) If  $u = (a, b, c) \in W, r \in \mathbb{R}$ , then  $ru = (ra, rb, rc) \in W$

$$\begin{aligned} & ra + 2(rb) + 3(rc) \\ &= r(a + 2b + 3c) \\ &= r \cdot 0 \\ &= \vec{0} \end{aligned}$$

2. Any plane passing through the origin in  $\mathbb{R}^3$  is a subspace.  
Any plane *not* passing through the origin is *not* a subspace.

3.  $v \in \mathbb{R}^n, v \neq 0$

$L = \{tv \mid t \in \mathbb{R}\}$  is a line in  $\mathbb{R}^n$  passing through the origin, with direction vector  $v$ .

Verification:

- (a)  $\vec{0} = 0 \cdot v \in L$
- (b) If  $u = tv, w = sv \in L$   
 $u + v = tv + sv = (t + s)v \in L$



- (c) If  $u = t\vec{v} \in L$ ,  $r \in R$   
 $ru = r(t\vec{v}) = (rt)\vec{v} \in L$   
 So  $L$  is a subspace.
4.  $W = \{(x, y) \mid x, y \geq 0\} \subseteq \mathbb{R}^2$   
 Is this a subspace? **NO**, as multiplying a vector in the space by certain scalars will produce a vector not in the subspace.
5. Any line passing through the origin is a subspace in  $\mathbb{R}^n$ .  
 Any line *not* passing through the origin is *not* a subspace in  $\mathbb{R}^n$
6.  $V =$  set of all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$   
 $W =$  the set of all polynomial functions  $p : \mathbb{R} \rightarrow \mathbb{R}$   
 i.e.  $p(x) = a_0 + a_1x + \dots + a_nx^n$ ,  $n$  a non-negative integer,  $a_0..a_n \in \mathbb{R}$   
 Verification:
- (a) (1) The zero vector of  $V$  is the function  $f(x) = 0 \forall x \in \mathbb{R}$ . It is in  $W$  because  $f(x)$  is the polynomial function with  $n = 0$  and  $a = 0$ .
- (b) (2), (3) ✓
7.  $V =$  set of all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$   
 $W =$  set of all continuous functions  
 $\therefore$  Multiplication of any continuous function by a scalar returns a continuous function  
 $\therefore W$  is a subset of  $V$ .
8.  $V$  is the set of all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$   
 $W$  is the set of all functions  $f(x)$  such that  $f(x) \in [-1, 1] \forall x$   
 Is  $W$  a subspace of  $V$ ?
- (a) (1) ✓
- (b) (2)  $f(x) = \cos(x)$   
 $g(x) = \cos(x)$   
 $(f + g)(x) = 2\cos(x) \notin W$

## Matrices

**Def:** The transpose of an  $m \times n$  matrix  $A$ , denoted  $A^T$ , is an  $n \times m$  matrix whose columns are rows of  $A$ , e.g.:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ -2 & 5 \end{bmatrix}^T = \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 & 4 \end{bmatrix}$$

An  $n \times n$  matrix (square matrix) is called *symmetric* if  $A = A^T$ , e.g.  $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$  is symmetric.

1.  $V = M_{22}(\mathbb{R})$

$W =$  set of all  $2 \times 2$  symmetric matrices

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \mid a, b, d \in \mathbb{R} \right\}$$

Verification:

(a) (1)  $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$

(b) (2) If  $u = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ ,  $v = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ ,

$$u + v = \begin{bmatrix} a+e & b+f \\ b+g & d+h \end{bmatrix} \in W$$

(c) (3) If  $u = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \in W$ ,  $r \in \mathbb{R}$

$$ru = \begin{bmatrix} ra & rb \\ rb & rd \end{bmatrix} \in W$$

Let  $V$  be an arbitrary vector space.

$V$  always has the following subspaces:

1.  $V$  is its own subspace
2.  $\{0\}$ , the zero subspace (or zero vector space)

## Chapter 7: The span of vectors in a vector space

### Definition

$V$  is a vector space.

1. If  $v_1, \dots, v_n \in V$  and  $a_1, \dots, a_n \in \mathbb{R}$ , then  $u = a_1v_1 + \dots + a_nv_n$  is called a *linear combination* of  $v_1, \dots, v_n$ .
2. The set of all linear combinations of  $v_1, \dots, v_n$  is called the *span* if  $v_1, \dots, v_n$ .  
 $\text{span}\{v_1, \dots, v_n\} = \{a_1v_1 + \dots + a_nv_n \mid a_1, \dots, a_n \in \mathbb{R}\}$
3. A vector (sub)space  $W$  is spanned by  $v_1, \dots, v_n \in W$  if  $W = \text{span}\{v_1, \dots, v_n\}$  ( $v_1, v_n$  spans  $W$ )

$S$  = set of all  $2 \times 2$  symmetric matrices

$$\begin{aligned} &= \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \mid a, b, d \in \mathbb{R} \right\} \\ &= \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mid a, b, d \in \mathbb{R} \right\} \\ &= \text{span}\{a, b, d\} \end{aligned}$$

A spanning set is *not* unique.

$$\text{span}\{0\} = \{0\}$$

### Examples

1.  $L = \{t\vec{v} \mid t \in \mathbb{R}\} \subseteq \mathbb{R}^n$  is a line  
 $= \text{span } \vec{v}$
2.  $\mathbb{R}^2 = \text{span}\{(1, 0), (0, 1)\}$  because any  $(x, y) \in \mathbb{R}^2$  can be written as  $x(1, 0) + y(0, 1)$ .

Similarly,  $\mathbb{R}^3 = \text{span}\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

### Examples

1.  $W = \text{xy-plane in } \mathbb{R}^3 = \{(x, y, 0) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$   
 $= \text{span}\{(1, 0, 0), (0, 1, 0)\}$

## 7.4

### Theorem:

Let  $V$  be a vector space such that

$$\{v_1, \dots, v_n\} \subseteq V \quad (\iff v_1, \dots, v_n \in V)$$

1.  $U = \text{span}\{v_1, \dots, v_n\}$  is a subspace of  $V$
2. If  $W$  is another subspace of  $V$  such that  $v_1, \dots, v_n \in W$ , then  $U \subseteq W$   
Therefore  $U$  is the smallest subspace containing  $v_1, \dots, v_n$

Subset axioms:

1.  $\vec{0} \in U$
2. If  $u, v \in U$ , then  $u + v \in U$
3. If  $u \in U$ ,  $r \in \mathbb{R}$ , then  $ru \in U$

**Proof of (1):**

- $\vec{0} = 0v_1 + \dots + 0v_n \in \text{span}\{v_1, \dots, v_n\}$
- If  $u, v \in U$ ,  
then  $u = a_1v_1 + \dots + a_nv_n$   
and  $w = b_1v_1 + \dots + b_nv_n$   
For some  $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$   
 $u + w = (a_1 + b_1)v_1 + \dots + (a_n + b_n)v_n \in U$
- If  $u \in U, r \in \mathbb{R}$   
then  $u = a_1v_1 + \dots + a_nv_n$   
 $ru = (ra_1)v_1 + \dots + (ra_n)v_n \in U$

7.5

**Definition:**

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{1\ldots} & a_{1n} \\ a_{21} & \dots & \dots & \dots \\ a_{\ldots 1} & \dots & \dots & \dots \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix}$$

The trace of  $A$ , denoted  $\text{Tr}(A)$  is defined as  $\text{Tr}(A) = a_{11} + a_{22} + \dots + a_{nn}$ .

**Example:**

$$\text{Tr} \left( \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \right) = 1 + 5 + 9 = 15$$

**Example:**

$$\begin{aligned} U &= \{A \in M_{22}(\mathbb{R}) \mid \text{Tr}(A) = 0\} \\ &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a + d = 0 \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\} \\ &= \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\} \\ &= \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} \end{aligned}$$

**Example:**

$W$  = set of all  $2 \times 2$  diagonal matrices

$$\begin{aligned} &= \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \mid a, d \in \mathbb{R} \right\} \\ &= \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ is a subspace of } M_{22}(\mathbb{R}) \end{aligned}$$

7.6  $\mathbb{R}$  has only 2 subspaces:  $\{0\}$  and  $\mathbb{R}$ . Proof: If  $W \subseteq \mathbb{R}$  is a subspace and  $W \neq \{0\}$ , then there is a nonzero vector  $\vec{v} \in W$

$$\begin{aligned} \mathbb{R} &= \text{span}\{w\} \subseteq W \\ &\Rightarrow \mathbb{R} = W \end{aligned}$$

The only subspaces of  $\mathbb{R}^2$  are:

1.  $\{0\}$
2. Lines through the origin
3.  $\mathbb{R}^2$

**Proof:** If  $W \subseteq \mathbb{R}^2$  is a subspace and  $W \neq \{0\}$ ,  $W \neq$  lines through the origin. Then we have to show that  $W = \mathbb{R}^2$ .

First,  $W \neq \{0\}$ . There exists a nonzero vector  $v \in W \Rightarrow \text{span}\{v\} \subseteq W$

Since  $W \neq$  lines through the origin, there exists  $u \in W$  such that  $u \notin \text{span}\{v\}$ , i.e.  $u$  is not a multiple of  $v$ .  
 $u, v \in W \Rightarrow \text{span}\{u, v\} \subseteq W$

Note that  $\text{span}\{u, v\} = \mathbb{R}^2$  because any vector  $\mathbb{R}^2$  can be written as a linear combination of  $u$  and  $v$ .

For any  $w \in \mathbb{R}^2$ ,  $w = \text{proj}_v(w) + (w - \text{proj}_v(w))$

$$\begin{aligned} W &= \text{proj}_v(w) + (w - \text{proj}_v(w)) \\ &= av + b(u - \text{proj}_v(u)) \\ &= av + bu - b(cv) \\ &= (a - bc)v + bu \in \text{span}\{u, v\} \end{aligned}$$

The only subspaces of  $\mathbb{R}^3$  are

1.  $\{0\}$
2. Lines through the origin
3. Planes through the origin
4.  $\mathbb{R}^3$

## 7.7

### Examples:

1.  $\text{span}\{(1, 2, 1), (0, 1, 2)\} = \text{span}\{(1, 3, 3), (1, 1, -1)\}$   
 Taking the cross products of each pair of vectors in these spans and showing that the resulting vectors are parallel is a proof.
2.  $\text{span}\{(1, 2, 1), (0, 1, 2)\} = u$  and  $\text{span}\{(1, 3, 3), (1, 1, -1)\} = v$   
 $(1, 3, 3) = (1, 2, 1) + (0, 1, 2) \in \text{span}\{(1, 2, 1), (0, 1, 2)\} = u$

## Chapter 8: Linear dependence and independence

### 8.2

2 vectors  $\vec{u}, \vec{v} \in \mathbb{R}^3$  are collinear if there are non-zero  $a, b \in \mathbb{R}$  such that  $a\vec{u} + b\vec{v} = 0$ .

e.g.  $(1, 2, 0), (2, 4, 0)$  are collinear and  $\text{span}\{(1, 2, 0), (2, 4, 0)\} = \text{span}\{(1, 2, 0)\}$

### 8.3

3 vectors  $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$  are coplanar if there exists  $a, b, c \in \mathbb{R}$  not all zero, such that  $a\vec{u} + b\vec{v} + c\vec{w} = 0$ .

e.g.  $(1, 0, 0), (0, 1, 0), (3, 2, 0)$  are coplanar because  $3(1, 0, 0) + 2(0, 1, 0) + (-1)(3, 2, 0) = 0$

$\text{span}\{(1, 0, 0), (0, 1, 0), (3, 2, 0)\} = \text{span}\{(1, 0, 0), (0, 1, 0)\}$

### 8.4

#### Definition:

- $V$  is a vector space.  
 $v_1, \dots, v_n \in V$   
 $\{v_1, \dots, v_n\}$  is linearly dependent  $\iff$  there exists  $a_1, \dots, a_n \in \mathbb{R}$  not all zero such that  $a_1v_1 + \dots + a_nv_n = 0$ .
- $\{v_1, \dots, v_n\}$  is linearly independent  $\iff \{v_1, \dots, v_n\}$  is *not* linearly dependent  $\iff$  the only solution to the equation  $a_1v_1 + \dots + a_nv_n = 0$  is the trivial solution  $a_1 = a_2 = \dots = a_n = 0$ .

### 8.7

#### Examples:

- $\{(1, 0), (0, 1)\} \subseteq \mathbb{R}^2$  is LI.

$$\text{Solve } a(1, 0) + b(0, 1) = (0, 0) \iff \begin{cases} a \cdot 1 + b \cdot 0 = 0 \\ a \cdot 0 + b \cdot 1 = 0 \end{cases}$$

So  $a = b = 0$

- $\{(2, 2), (3, -1)\} \subseteq \mathbb{R}^2$  is LI.

$$\text{Solve } a(2, 2) + b(3, -1) = (0, 0) \iff \begin{cases} 2a + 3b = 0 \\ 2a - b = 0 \end{cases}$$

$$4b = 0 \Rightarrow b = 0$$

$$\Rightarrow a = 0$$

- $\{(2, 2), (3, 2), (-1, -1)\} \subseteq \mathbb{R}^2$  is LD because  $1(2, 2) + 0(3, 2) + 2(-1, -1) = (0, 0)$ .

- $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \subseteq M_{22}(\mathbb{R})$

Solve:

$$a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$a = 0, b = 0, c = 0, d = 0.$$

- Is  $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \right\} \subseteq M_{22}(\mathbb{R})$

linearly dependent or linearly independent?

Solution: It is linearly dependent.

$$1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - 1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{cases} a + b = 0 \\ b + 4c = 0 \\ b + 4c = 0 \\ 3d = 0 \end{cases}$$

$$d = 0$$

$$a + b = 0 \quad a = -b$$

$$b + 4c = 0 \quad c = \frac{-b}{4}$$

- $V = F(\mathbb{R}) = \{f: \mathbb{R} \rightarrow \mathbb{R}\}$   
 $\{1, x, x^2\}$  is LI.  
Solve  $a \cdot 1 + bx + cx^2 = 0$ , where 0 is the zero function. This is an equation of functions.
- $\{1, \sin^2(x), \cos^2(x)\}$  is linearly dependent because  $1 - \sin^2(x) - \cos^2(x) = 0$ .
- $\{1, \sin(x), \cos(x)\}$  is linearly independent. Solve  $a + b \sin(x) + c \cos(x) = 0$ .

## 8.8

### Properties

- $\{0\}$  is linearly dependent.
- $v \in V$ , then  $\{v\}$  is LI  $\iff v \neq 0$
- $\{v_1, \dots, v_n\}$  is LD  $\iff$  there are some  $v_i$  such that  $v_i$  is in the span of the rest, i.e.  $v_i \in \text{span}\{v_1, \dots, v_n\}$ .
- If a set  $S$  is LD, then any set containing  $S$  is LD.
- Conversely, if a set  $S$  is linearly independent, then any subset of  $S$  is still linearly independent.
- any set containing 0 is linearly dependent.
- $\{u, v\}$  is LD  $\iff$  one vector is a multiple of the other.

## Chapter 9: Linear independence and spanning sets

### 9.2 Theorem:

If  $v_1 \in \text{span}\{v_1, \dots, v_n\}$ , then  $\text{span}\{v_1, \dots, v_n\} = \text{span}\{v_1, \dots, v_n\}$

Therefore if  $\{v_1, \dots, v_n\}$  is LD, then  $\text{span}\{v_1, \dots, v_n\} = \text{span}\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$  for some  $i$ .

Recall:

$\{v_1, \dots, v_n\}$  is LD  $\iff$  there exists a space  $v_i$  such that  $v_i \in \text{span}\{v_i, v_{i-1}, \dots, v_n\}$ .

In this case,  $\text{span}\{v_1, \dots, v_n\} = \text{span}\{v_i, v_{i-1}, \dots, v_n\}$ .

### Examples

1.  $\text{span}\{(1, 0, 0), (0, 1, 0), (3, -2, 0)\} = \text{span}\{(1, 0, 0), (0, 1, 0)\}$
2.  $\text{span}\{(1, 0, 0), (0, 1, 1), (0, 4, 4)\} = \text{span}\{(1, 0, 0), (0, 1, 1)\} \neq \text{span}\{(0, 1, 1), (0, 4, 4)\}$

### 9.3

Suppose  $\{v_1, \dots, v_n\}$  is LI,  $w \in V$ .

Then  $\{w, v_1, \dots, v_n\}$  is LI  $\iff w \notin \text{span}\{v_1, \dots, v_n\}$  **Example:**

1.  $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$  is L  
and  $\begin{bmatrix} 0 & 0 \\ 3 & 2 \end{bmatrix} \notin \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$   
then  $\left\{ \begin{bmatrix} 0 & 0 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$  is LI.
2.  $\begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} \in \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$   
Then  $\left\{ \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$  is LD.



## Chapter 10: Basis and dimension

### 10.1 Theorem:

If a vector space  $V$  can be spanned by  $n$  vectors, then any LI subset of  $V$  has at most  $n$  vectors. Equivalently, if  $V$  has a LI subset of  $m$  vectors, then any spanning set of  $V$  has at least  $m$  vectors.

$$\text{size of any spanning set of } V \geq \text{size of any LI subset of } V$$

### Examples

1.  $\mathbb{R}^3 = \text{span}\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$  can be spanned by 3 vectors. Any subset of  $\mathbb{R}^3$  containing 4 or more vectors is LD.
2.  $M_{22}(\mathbb{R})$  can be spanned by 4 vectors. Any subset of  $M_{22}(\mathbb{R})$  containing 5 or more vectors is LD.

### 10.2 Definition $\{v_1, \dots, v_n\} \subseteq V$ is called a basis of $V$ if

1.  $\{v_1, \dots, v_n\}$  is LI
  2.  $\text{span}\{v_1, \dots, v_n\} = V$
- A basis is a largest possible LI subset of a set  $V$ , i.e. if you add any vector to the set, it will become LD.
  - A basis is a smallest possible spanning set of  $V$ . If you remove any vector from the set, then it will not span  $V$ .

### Examples:

1.  $\{(1, 0), (0, 1)\}$  is a basis of  $\mathbb{R}^2$  (the standard basis).
2.  $\{(1, 1), (1, -1)\}$  is also a basis of  $\mathbb{R}^2$ .
3.  $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$  is the standard basis of  $M_{22}(\mathbb{R})$ .
4.  $\mathbb{P}_2 =$  vector space of polynomials of a degree  $\leq 2$  is a closed subspace under addition (as adding any polynomials can only reduce the degree), multiplication, and the zero vector test.  
 $\{1, x, x^2\}$  is a basis of  $\mathbb{P}_2$ .

Any polynomial of  $\deg \leq 2$  is of the form  $a + bx + cx^2$ . **Examples**

- (a)  $\{(1, 0, 0), (0, 1, 1)\} \in \mathbb{R}^3$  is NOT a basis as  $\text{span}\{(1, 0, 0), (0, 1, 1)\} \neq \mathbb{R}^3$ .
- (b)  $V = \{a \in M_{22}(\mathbb{R}) \mid \text{Tr}(A) = 0\}$   
 $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 3 & -2 \end{bmatrix} \right\}$  is NOT a basis of  $V$  because it is LD.  
 $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$  is a basis of  $V$ .

### Theorem

If  $\{v_1, \dots, v_n\}$  and  $\{w_1, \dots, w_m\}$  are 2 bases for  $V$ , then  $n = m$ .

#### Proof:

$\{v_1, \dots, v_n\}$  spans  $V$  and  $\{w_1, \dots, w_m\}$  is LI.

$\Rightarrow n \geq m$ .

$\{v_1, \dots, v_n\}$  is LI and  $\{w_1, \dots, w_m\}$  spans  $V$ .

$n \leq m$  The only case in which both  $n \geq m$  and  $n \leq m$  are true is if  $n = m$ . QED.

### 10.3 Definition

If  $V$  has a finite basis  $(\{v_1, \dots, v_n\})$  then we say that the dimension  $\dim(V) = n$ . In this case,  $V$  is called a finite dimensional space.

If  $V$  does not have a finite basis, then  $V$  is an infinite dimensional space.

**Examples:**

- $\dim(\mathbb{R}^2) = 2$
- $\dim(\mathbb{R}^n) = n$
- $\dim(M_{22}(\mathbb{R})) = 4$
- $\dim(M_{mn}(\mathbb{R})) = mn$   
Standard basis:  $\{E_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$   
 $E_{ij}$  is the matrix with  $(i, j)$  - entry is 1, other entries are 0.
- $\dim(\mathbb{P}_2) = 3$  because a basis is  $\{1, x, x^2\}$
- $\dim(\mathbb{P}_n) = n + 1$
- $\dim(F(\mathbb{R})) = \infty$
- $\dim(V) = 3$  where  $V = \{A \in M_{22}(\mathbb{R}) \mid \text{Tr}(A) = 0\}$
- $W =$  vector space of all  $2 \times 2$  diagonal matrices.  $\dim(W) = 2$ .
- $W =$  the plane with equation  $x + 3y + 5z = 0 \subseteq \mathbb{R}^3$ .  
 $\dim(W) = 2$ . A basis of  $W$  is  $\{(3, -1, 0), (5, 0, -1)\}$   
Solve  $a(3, -1, 0) + b(5, 0, 1) = (0, 0, 0)$   
Then  $a = b = 0$   
 $\dim\{0\} = 0$ . A basis for the zero vector space is  $\emptyset$

$$V = M_{22}(\mathbb{R})$$

$$W = \left\{ \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

This set does not contain  $\vec{0}$ . It is therefore not a subspace.

$$W = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

This set contains  $\vec{0}$ .

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid ab = cd \right\}$$

This set is not closed under addition, and is not a subspace.

## Chapter 11: Dimension theorems

### 11.1

$V$  vector space with  $\dim(V) = n < \infty$

- If  $\{v_1, \dots, v_m\} \subseteq V$  is LI,  $m \leq n$  and there exists  $v_{m+1}, \dots, v_n \in V$  such that  $\{v_1, \dots, v_m, \dots, v_n\}$  is a basis of  $V$ .
- If  $\{w_1, \dots, w_r\}$  is a spanning set of  $V$ ,  $r \geq n$  and there is a subset of  $S$  that is a basis of  $V$ 
  - Size of LI subset  $\leq$  size of spanning set

$V = \mathbb{R}^3$

$\{(1, 2, 0), (1, -1, 0)\}$

Because the 2 vectors are *not* multiples of each other, this set is LI. This is a shortcut for sets of 2 vectors.

$\{(1, 2, 0), (1, -1, 0), (0, 0, 1)\}$  is a basis of  $\mathbb{R}^3$

It is LI because  $(0, 0, 1) \notin \text{span}\{(1, 2, 0), (1, -1, 0)\}$

$\{(1, 0, 0), (0, 1, 0), (0, 1, 1), (0, 2, 3)\}$  is LD.

We see that  $(0, 2, 3) = -(0, 1, 0) + 3(0, 1, 1)$

so  $\{(1, 0, 0), (0, 1, 0), (0, 1, 1)\}$  is a basis of  $\mathbb{R}^3$ .

### Theorem

$V$  vector space with  $\dim(V) = n < \infty$

Then:

1. Any LI subset of  $n$  vectors is a basis of  $V$
2. Any spanning set of  $n$  vectors is also a basis

Axioms:

1. LI
2. Spanning
3. Consisting of  $n$  vectors

Only need to check any 2 of these conditions to see whether a subset is a basis.

### Examples

1.  $W = \{(x, y, z) \in \mathbb{R}^3 \mid 3x - y + 5z = 0\}$   
 $\dim(W) = 2$   
 $(1, 3, 0), (5, 0, -3) \in W$   
 $\{(1, 3, 0), (5, 0, -3)\}$  is LI  
 $\Rightarrow \{(1, 3, 0), (5, 0, -3)\}$  is a basis of  $W$  (spanning is mathematically satisfied)
2. Expand  $\{(1, 3, 0), (5, 0, -3)\}$  to a basis of  $\mathbb{R}^3$ .  
 $\{(1, 3, 0), (5, 0, -3), (3, -1, 5)\}$  is a basis of  $\mathbb{R}^3$ .  
 $(3, -1, 5)$  is a normal vector of  $W$ . In particular,  $(3, -1, 5) \notin W$ .  
 $\Rightarrow \{(1, 3, 0), (5, 0, -3), (3, 1, 5)\}$  is LI.

### 11.3 Theorem

$V$  vector space,  $\dim(V) = n < \infty$

$W \subseteq V$  subspace. Then:

1.  $0 \leq \dim(W) \leq \dim(V)$
2.  $\dim(W) = \dim(V) \iff W = V$
3.  $\dim(W) = 0 \iff W = \{0\}$

## Proof

1. Suppose  $\{w_1, \dots, w_m\}$  is a basis of  $W$  ( $\dim(W) = m$ ).  
Then,  $\{w_1, \dots, w_m\}$  is an LI subset of  $W$ , hence also a subset of  $V$ .  
Therefore,  $m \leq \dim(V)$ .
2. If  $W = V$ , then  $\dim(W) = \dim(V)$ .  
If  $\dim(W) = \dim(V)$ , let  $S = \{w_1, \dots, w_n\}$  be a basis of  $W$ .  
(LI + consisting of  $n$  vectors)  $\Rightarrow$  also a basis of  $V$ .

## Examples

1. Any subspaces of  $M_{22}(\mathbb{R})$  has dimensions 0, 1, 2, 3 or 4 and the only subspace of  $\dim(4)$  is  $M_{22}(\mathbb{R})$ . The only subspace of  $\dim(0)$  is  $\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$
2. Any subspace of  $\mathbb{R}^n$  has dimension 0, 1, 2, ..., or  $n$  and the only subspace with  $\dim(n)$  is  $\mathbb{R}^n$ . The only subspace with  $\dim(0)$  is  $\{(0, 0, \dots, 0)\}$

## 11.6 Theorem:

Let  $B = \{v_1, \dots, v_n\}$  be a basis of  $V$ .

Then for any  $v \in V$ , there exist *unique* scalars  $a, \dots, a_n \in \mathbb{R}$  such that  $v = a_1v_1, \dots, a_nv_n$ .

## Examples:

1.  $\{(1, 0), (0, 1)\}$  is a basis of  $\mathbb{R}^2$ .
2.  $\{(1, 0), (0, 1), (0, 3)\}$  is NOT a basis of  $\mathbb{R}^3$ s

## Application

We can identify  $V$  with  $\mathbb{R}^n$

$$v = a_1v_1 + \dots + a_nv_n \longleftrightarrow (a_1, \dots, a_n)$$

$$\text{e.g. } V = M_{22}(\mathbb{R}), B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\text{Then } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longleftrightarrow (a, b, c, d)$$

## Chapter 12: solving systems of linear equations

### 12.1 Examples

1.

$$\begin{cases} x + 3y + 2z = 6 \\ 0x + y - z = 0 \end{cases}$$

is a linear system of 2 equations.

#### Definition

The general solution to a linear system is the set of all solutions, e.g.  $\{(6 - 5t, t, t) \mid t \in \mathbb{R}\}$  is the general solution to the above example ( $t$  is the parameter of the general solution).

2.

$$\begin{cases} x + y + z = 2 \\ x + y + z = 3 \end{cases}$$

Has no solution. The general solution is  $\emptyset$ , the empty set.

#### Definition

- A linear system with NO solution is called *inconsistent*.
- A linear system with at least one solution is called *consistent*.
- A linear system in which all the constants on the right are zero is called *homogenous*.
- A linear system in which at least one of the constants is nonzero is called *inhomogenous*.

#### Remark

- Homogenous system are always consistent.  
This is because  $(0, 0, 0)$  is always a solution (the trivial solution).

#### Definition

- A linear equation is called *degenerate* if all the coefficients are zero.

#### Remark

- Any linear system containing a *degenerate*, *homogenous* equation is *inconsistent*.

#### Theorem

Any linear system (with real coefficients) has either

1. NO solutions
2. exactly one solution
3. infinitely many solutions

## Examples

1.

$$x + y + 2z = 3 \quad (1)$$

$$x - y + z = 2 \quad (2)$$

$$y = z = 1 \quad (3)$$

$$(2) - (1) = x + y + 2z = 3$$

$$-2y - z = -1$$

$$y - z = 1$$

$$(2) - (3) = x + y + 2z = 3$$

$$y = z = 1$$

$$-2y = -1$$

$$(3) + 2(2) = x + y + 2z = 3$$

$$y - z = 1$$

$$-3z = 1$$

$$-\frac{1}{3}(3)$$

$$x + y + 2z = 3$$

$$z = -\frac{1}{3}$$

$$(2) + (3)$$

$$x + y + 2z = 3$$

$$y + \frac{2}{3}$$

etc.

This method is inefficient. We can do this with matrices.

$$\left[ \begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & 1 \end{array} \right]$$

$$x = 3$$

$$y = \frac{2}{3}$$

$$z = \frac{1}{3}$$

We used the following operations:

- Add a multiple of one row to another row
- Interchange two rows
- Multiply one row by a nonzero scalar

These are called *elementary row operations* (ERO) and performing these operations to a linear system will not change the general solution.

### 12.3 Definition

- A matrix is in *Row Echelon Form* (REF) if the following conditions are satisfied:
  - All zero rows are at the bottom
  - The first nonzero entry of each row is 1 (leading 1)
  - Each leading 1 is to the right of the leading 1s in the rows above

- Each leading 1 is the only nonzero entry in its column, then the matrix is in *reduced row echelon form* (RREF).

e.g.

$$\left[ \begin{array}{cccc|c} 1 & 2 & 3 & 4 & -5 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

is REF but not RREF

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array} \right]$$

is RREF, and the general solution is  $\{(7, 4, 5)\}$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

General solution:  $\emptyset$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_3 = s$$

$$x_4 = t \quad (s, t \in \mathbb{R})$$

$$x_2 = -2s - t$$

$$x_1 = 1 - 2s$$