MAT 1341

Introduction to Linear Algebra

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Chapter 2: Complex Numbers
2.1 defining the complex numbers

The equation $x^2 + 1 = 0$ has no solutions in $\mathbb{R}$.
Let $i = \sqrt{-1} \qquad (\star)$
Then $i^2 = -1$, hence $i^2 + 1 = 0$.
Hence i is a solution to $(\star)$.

Example: consider $x^2 + 2x + 2 = 0$
By the quadratic formula, the solutions are:

$$x = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2\sqrt{-1}}{2} = \mathbf{-1 \pm i}$$

Check:

$$(-1+i)^2 = (-1+i)(-1+i) = 1 - 2i - 1 = \mathbf{-2i}$$

Hence:

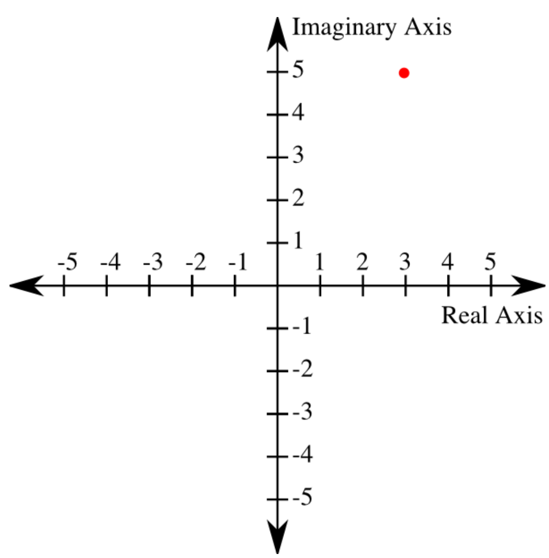
$$(-1+i)^2 + 2(-1+i) + 2 = -2i - 2 + 2i + 2 = \mathbf{0}$$

Definition

The set of complex numbers is $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$
When we write $z = a + bi$, where $a, b \in \mathbb{R}$:

- a is called the real part, and is denoted by $Re(z)$, and
- bi is called the imaginary part, and is denoted by $Im(z)$.

When $a = 0$, z is called *purely imaginary*.
When $b = 0$, $z \in \mathbb{R}$.



The complex plane. The red dot is 3+5i.

Properties of complex numbers
If z, w, $y \in \mathbb{C}$:

- $z + w = w + z$
- $zw = wz$
- $1 \times z = z$
- $0 \times z = 0$
- $y(z + w) = yz + yw$
- $y(zw) = (yz)w$

Given any quadratic equation $ax^2 + bx + c$ where $\{a \neq 0 \mid a, b, c \in \mathbb{R}\}$,
the solutions are found using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. If $b^2 - 4ac \geq 0$, the solutions $\in \mathbb{R}$.
If $b^2 - 4ac < 0$, the solutions are complex.

2.2 Algebra of the complex numbers

For a, b, $c \in \mathbb{R}$:

- $a + bi = c = di \Leftrightarrow a = c$ and $b = d$
- $(a + bi) + (c + di) = (a + c) + (b + d)i$
- $(a + bi)(c + di) = ac - bd$

Definition

The **complex conjugate** of $z = a + bi$ is $\overline{z} = a - bi$.

Example

$\overline{1 + 2i} = 1 - 2i$
$z\overline{z} = \overline{z}z$
$= (a + bi)(a - bi)$
$= a^2 + b^2$

z is a nonnegative, real number.
The absolute value of z is $|z| = \sqrt{a^2 + b^2} = \sqrt{z\overline{z}}$

- $z = 0 \Leftrightarrow a - b = 0 \Leftrightarrow |z| = 0$
- $\frac{1}{z} = \frac{1}{z} \times \frac{\overline{z}}{z}$
  $= \frac{\overline{z}}{|z^2|}$
  $= \frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$
  $= \frac{a}{a2+b2} - \frac{b}{a2+b2}i$

Example

$\frac{1}{1+i} = \frac{1}{1-i} \times \frac{1-i}{1-i}$

$= \frac{1-i}{1^2+1^2}$

$= \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i$

Example

$$\frac{2+i}{1-3i} = \frac{2+i}{1-3i} \times \frac{1+3i}{1+3i}$$

$$= \frac{(2-3)+(6+1)i}{1^2+3^2}$$

$$= \frac{-1}{10} + \frac{7}{10}i$$

2.3 Geometry of the complex numbers

Numbers on the complex plane may be treated as vectors when performing addition, with the real and complex parts corresponding to coordinates.

- Multiplication by a real corresponds to scaling
- $|z|$ = length of a vector, e.g. $|2+i| = \sqrt{2^2+1^2} = \sqrt{5}$

2.4 Polar form of complex numbers

$z = a + bi$
$r = |z| = \sqrt{a^2+b^2}$
$\cos\theta = \frac{a}{r}$
$\sin\theta = \frac{b}{r}$
Polar form of z:
$z = a + bi = (r\cos\theta) + (r\sin\theta)i$
$= r(\cos\theta + i\sin\theta)$

Note that $\theta = \arg(z) \rightarrow$ argument of z is not uniquely determined since $\theta + 2n\pi$ also works for any $n \in \mathbb{Z}$. We usually pick $-\pi < \theta \leq \pi$ and write $\theta = \arg(z)$, principal argument of z.

Recall

$e^z = i + z + \frac{z}{2} + \frac{z}{4} + ... = \sum_{n=0}^{\infty} \frac{z^n}{n!}$
$e^{i\theta} = \cos\theta + i\sin\theta$
So we can write
$z = re^{i\theta}$

Properties

$re^{i\theta} = se^{i\phi} \Leftrightarrow r = s$ and $\theta = \phi + 2n$ for some integer n.

$\overline{re^{i\theta}} = re^{-i\theta}$
$|e^{i\theta}| = 1$ for any θ

2.5 Multiplying complex numbers in polar form

If $z = re^{i\theta}$**,** $w = se^{i\phi}$

$zw = r(\cos\theta + i\sin\theta) \times s(\cos\phi + i\sin\phi)$
$= rs[(\cos\theta\cos\phi - \sin\theta\sin\phi) + (\sin\theta\cos\phi + \cos\theta\sin\phi)i]$
$= rs[\cos(\theta+\phi) + i\sin(\theta+\phi)]$
$= rse^{i(\theta+\phi)}$

Example

$1 + i = \sqrt{2}e^{-i\frac{\pi}{4}}$

hence

$\frac{1}{1+i} = \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}$

2.6 Fundamental theorem of algebra

Every polynomial with coefficients in the complex numbers can be factored completely into linear factors of the for $zx + w$, with z, $w \in \mathbb{C}$.

Example

$x^2 + 1 = (x + i)(x - i)$

Every degree n polynomial with coefficients in the complex plane has n solutions (counting multiplicities).

Chapter 3: Vector geometry

3.1

Algebra $\longleftrightarrow$ Geometry
$\mathbb{R}$ line
$\mathbb{R}^2$ plane
$\mathbb{R}^3$ 3-plane
$\mathbb{R}^n$ n-space

Notations:
$\vec{x} = (1, 2, 3)$
$\vec{x} = \underline{i}, \underline{2j}, \underline{3k}$
$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [1, 2, 3]$
$\mathbb{R}^n = \{(x_1, ...x_n \mid x_1, ..., x_n \in \mathbb{R}\}$

3.2 Properties

- $(x_1,\ ...,\ x_n) = (y_1,\ ...,\ y_n) \Leftrightarrow x_1 = y_1,\ x_n = y_n$
- $(x_1,\ ...,\ x_n) + (y_1,\ ...,\ y_n) = (x_1 + y_1,\ ...,\ x_n + y_n)$
- $\vec{0} = (0,\ ...,\ 0) \in \mathbb{R}^n$
- if $\vec{x} = (x_1,\ ...,\ x_n)$, then $-\vec{x} = (-x_1,\ ...,\ -x_n)$ and $\vec{x} + (-\vec{x}) = \vec{0}$
- if $r \in \mathbb{R},\ \vec{x} = (x_1,\ ...,\ x_n) \in \mathbb{R}^n$, then $r \cdot \vec{x} = (rx,\ rx_2,\ ...,\ rx_n)$
- 2 vectors are equal $\iff$ they have the same magnitude and same direction
- Head-to-tail rule
- $\vec{0}$ is te only vector with 0 magnitude.
- negative=reverse direction
- 2 vectors are parallel $\iff$ they are multiples of eachother

3.3 Definition:
If $r_1,\ ...,\ r_n \in \mathbb{R}$,
$\vec{x_1},\ ...,\ \vec{x_n} \in \mathbb{R}^n$
then $\vec{y} = r_1\vec{x_1} + ... + r_n\vec{x_n}$ is a *linear combination* of $\vec{x_1},\ ..., \vec{x_n}$
We are looking for two scalars, $r_1,\ r_2 \in \mathbb{R}$, such that

$$\begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} = r_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + r_2 \begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix}$$

$$\begin{cases} 3 = r_1 + r_2 \\ 3 = 2r_1 + \frac{1}{2}r_2 \\ 4 = 3r_1 \end{cases}$$

But $3 \neq 2(\frac{4}{3}) + \frac{1}{5}(\frac{5}{3})$

3.4 More properties
$r,\ s \in \mathbb{R},\ \vec{u},\ \vec{v},\ \vec{w} \in \mathbb{R}^n$

- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $\vec{u} + \vec{0} = \vec{u}$
- $\vec{u} + (-\vec{u}) = \vec{0}$
- $r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$
- $(r + s)\vec{u} = r\vec{u} + s\vec{u}$
- $(rs)\vec{u} = r(s\vec{u})$
- $1 \cdot \vec{u} = \vec{u}$

3.5 Definition
The dot product of $\vec{x} = (x_1,\ ...,\ x_n)$ and $\vec{y} = (y_1,\ ...,\ y_n)$ is
$\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + ... + x_ny_n$
And the norm of $\vec{x}$ is $||\vec{x}|| = \sqrt{\vec{x} \cdot \vec{x}}$
$= \sqrt{x_1^2 + ... + x_n^2}$ Note that $||\vec{x}|| = 0 \iff \vec{x} = \vec{0}$

3.6 Definition
if $\vec{x} \cdot \vec{y} \in \mathbb{R}^n$
then $\vec{x}$ and $\vec{y}$ are said to be *orthogonal* (perpendicular), and $\vec{x} \cdot \vec{y} = 0$

3.7: The Cauchy-Schwarz Inequality
let $\vec{u},\ \vec{v} \in \mathbb{R}^n$
then $|\vec{u} \cdot \vec{v}| \leq ||\vec{u}||\ ||\vec{v}||$
$||\vec{u} + \vec{v}||^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$
$= ||\vec{u}||^2 + 2\vec{u}\vec{v} + ||\vec{v}||^2$
$\leq ||\vec{u}||^2 + 2|\vec{u}\vec{v}| + ||\vec{v}||^2$
$\leq ||\vec{u}||^2 + 2||\vec{u}||\ ||\vec{v}|| + ||\vec{v}||^2$
$= (||\vec{u}|| + ||\vec{v}||)^2$
This implies $||\vec{u} + \vec{v}|| \leq ||\vec{u}|| + ||\vec{v}||$, triangle inequality.

Definition
let $\vec{u},\ \vec{v} \in \mathbb{R}^n, \quad \vec{u},\ \vec{v} \neq \vec{0}$
the angle between $\vec{u}$ and $\vec{v}$ is defined by:

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\ ||\vec{v}||}$$

With $0 \leq \theta \leq \pi$

$$\cos\theta = (\frac{\vec{u}}{||\vec{u}||})\cdot(\frac{\vec{v}}{||\vec{v}||})$$

Example
Compute the angle between
$\vec{v} = (0, 2, 1, \sqrt{3})$ and $\vec{v} = (\sqrt{3}, 1, 2, 0)$
$\cos\theta = \frac{0+2+2+0}{\sqrt{4+1+3}+\sqrt{3+1+4}} = \frac{4}{8} = \frac{1}{2}$
$\theta = \frac{\pi}{3}$
Remark: $\vec{u}$ and $\vec{v}$ are orthogonal:
$\iff \vec{u}\cdot\vec{v} = 0$
$\iff \cos\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\ ||\vec{v}||} = 0$
$\iff \theta = \frac{\pi}{2}$

$\vec{u}$ and $\vec{v}$ are parallel:
$\iff \theta = 0$ or π
$\iff \cos\theta = 1$ or -1
$\iff \vec{u}\cdot\vec{v} = ||\vec{u}||\ ||\vec{v}||$ or $-||\vec{u}||\ ||\vec{v}||$
i.e. $|\vec{u}\cdot\vec{v}| = ||\vec{u}||\ ||\vec{v}||$
This means that $|\vec{u}\cdot\vec{v}|$ attains its maximum value given by Cauchy-Schwarz Inequality.

Definition
let $\vec{u}, \vec{v} \in \mathbb{R}^n, \quad \vec{u}, \vec{v} \neq \vec{0}$
The the projection of $\vec{u}$ onto $\vec{v}$ is
$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{||\vec{v}||^2}\vec{v}$
$= (\vec{u}\frac{\vec{v}}{||\vec{v}||})\frac{\vec{v}}{||\vec{v}||}$

Properties

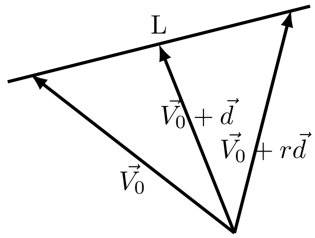
- $\text{proj}_{\vec{v}}(\vec{u})$ is parallel to $\vec{v}$
- $\vec{u} - \text{proj}_{\vec{v}}(\vec{u})$ is perpendicular to $\vec{v}$
- $\vec{u} = (\vec{u} - \text{proj}_{\vec{v}}(\vec{u})) + \text{proj}_{\vec{v}}(\vec{u})$

Example
$\vec{v} = (1, 0, 0),\ \vec{v} = (2, 4, 6)$
Then $\text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{v}\cdot\vec{u}}{||\vec{u}||^2}\vec{v}$
$= \frac{2}{1^2}(1, 0, 0)$
$= (2, 0, 0)$

$\vec{w} = (5, 0, 0)$
$\text{proj}_{\vec{w}}(\vec{v}) = (2, 0, 0)$

Chapter 4: Lines and planes

4.1



Equation of a line: $y = mx + c$

$L = \{\vec{V_0} + r\vec{d} \mid r \in \mathbb{R}\}$
i.e. Any point on L can be written as $\vec{V_0} + r\vec{d}$.
Ex: The line $y = 2x + 1$ in $\mathbb{R}^2$ can be written as follows:
Let $x = r \;\longleftarrow$ parameter

$y = 2r + 1$ Then: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r \\ 2r+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\vec{V_0}} + r \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\vec{d}}$

Vector form or parametric form.

Ex: Use different letters to represent parameters.
Find the point of intersection of

$$L_1 = \{\begin{bmatrix} 0 \\ 1 \end{bmatrix} + r \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mid \in \mathbb{R}\}$$

and

$$L_2 = \{\begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mid \in \mathbb{R}\}$$

Use t for the parameter of L_2, and we solve:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} + r \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + r \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{cases} 0 + r = 1 + 0 \Rightarrow r = 1 \\ 1 + 2r = 1 + r \Rightarrow t = 2 \end{cases}$$

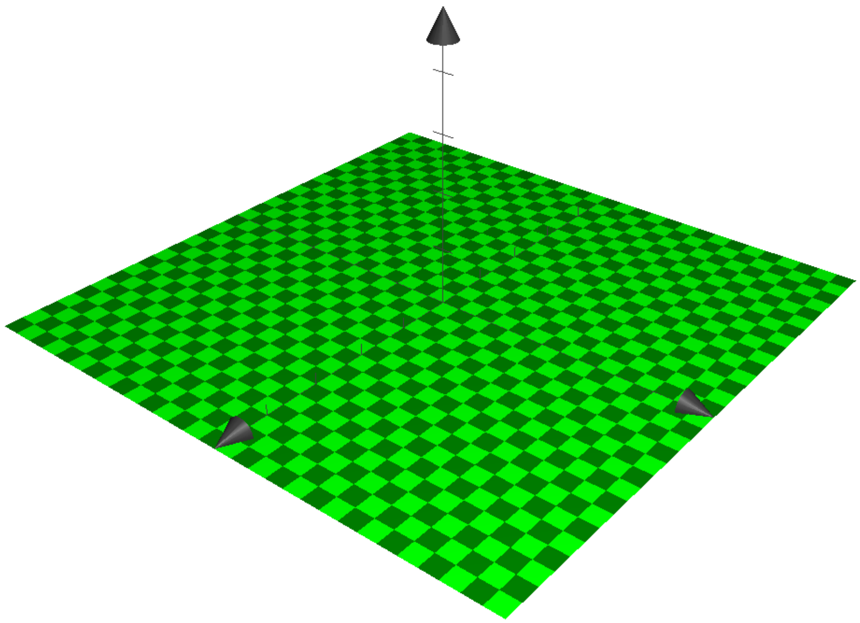
So the point of intersection is

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

4.2

In $\mathbb{R}$: there is only one line.
In $\mathbb{R}^2$: 2 distinct lines, either parallel or intersecting.
In $\mathbb{R}^3$: 2 distinct lines, either parallel, intersecting, or skew (neither parallel nor intersecting). In the first two cases, there is a unique plane containing both lines.
If they are skew, there is no plane containing both, but there are two *parallel* planes, each containing one line.

4.3



$$W = \{\vec{v} \in \mathbb{R}^3 \mid (\vec{v} - \vec{v_0}) \cdot \vec{n} = 0\}$$

For example, the plane with $\vec{n} = (1, 2, 3)$ and containing the point (0,1,1) is:

$$\{(x, y, z) \in \mathbb{R}^3 \mid [(x, y, z) - (0, 1, 1)] \cdot (1, 2, 3) = 0\}$$

$$(x, y - 1, z - 1) \cdot (1, 2, 3) = 0$$

$$x + 2y - 2 + 3z - 3 = 0$$

$$x + 2y + 3z = 5$$

Ex: find the distance from the point (3,3,3) to the plane $x + 2y + 3z = 5$

$$\begin{aligned} D &= ||\text{proj}_{(1,2,3)}((3, 3, 3) - (0, 1, 1))|| \\ &= ||\text{proj}_{(1,2,3)}(3, 2, 2)|| \\ &= ||\frac{3 + 4 + 6}{1 + 4 + 9}(1, 2, 3)|| = \frac{13}{14}\sqrt{14} = \frac{13}{\sqrt{14}} \end{aligned}$$

4.4

Def: The angle between 2 planes in $\mathbb{R}^3$ is the angle between their normal vectors.
In $\mathbb{R}^2$, there is only one plane.
In $\mathbb{R}^3$, there may be 2 distinct planes, either are parallel, or intersect.
... But what about $\mathbb{R}^4$?

n	equations in $\mathbb{R}^n$	geometric object	dimension
1	$ax = b$	point	0
2	$ax + by = c$	line	1
3	$ax + by + cz = d$	plane	2
4	$ax + by + cz + dw = e$	?	3

Idea: One equation in $\mathbb{R}^n$ will cut down the dimension by 1. The resulting $\mathbb{R}^n$ object is called a hyperplane.

4.5
If $\vec{x} = (x_1, x_2, x_3)$
$\vec{y} = (y_1, y_2, y_3)$
Then the cross product of $\vec{x}$ and $\vec{y}$ is:

$$\vec{x} \times \vec{y} = \begin{bmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = (x_2y_3 - y_2x_3)i - (x_1y_3 - y_1x_3)j + (x_1y_2 + y_1x_2)k$$

Ex: $(0, 1, 2) \times (-3, 4, 1)$

$$= \begin{bmatrix} i & j & k \\ 0 & 1 & 2 \\ -3 & 4 & 1 \end{bmatrix} = (-7, -6, 3)$$

Properties

- $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
- $(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$
- $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$
- $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
- $||\vec{u} \times \vec{v}|| = ||\vec{u}||\ ||\vec{v}|| \sin\theta$, where $0 \leq \theta \leq \pi$ is the angle between $\vec{u}$ and $\vec{v}$.

Remark:

- $||\vec{u} \times \vec{v}||$ = area of the parallelogram with sides $\vec{u}$ and $\vec{v}$
- Area of the triangle with sides $\vec{u}$ and $\vec{v}$ is $\frac{1}{2}\ ||\vec{u} \times \vec{v}||$
- In general, $\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$
- Suppose $\vec{u}$, $\vec{v} \neq 0$. Then $\vec{u}, \vec{v}$ parallel $\iff$ $\vec{u} \times \vec{v} = \vec{0}$
- Direction of $\vec{u} \times \vec{v}$ is given by right hand rule.

4.6 Ex: find an equation of the plane:

- Containing the y-axis
- Perpendicular to the plane $4x - y + 3z = 5$

Normal vector of the given plane is perpendicular to $(0, 1, 0)$ and $(4, -1, 3)$.

$$(0, 1, 0) \times (4, -1, 3) = \begin{bmatrix} i & j & k \\ 0 & 1 & 0 \\ 4 & -1 & 3 \end{bmatrix} = (3, 0, -4)$$

Volume of the parallelepiped in $\mathbb{R}^3$ with sides $\vec{u}, \vec{v}$ and $\vec{w}$
$|(\vec{u} \times \vec{v}) \cdot \vec{w}| = |(\vec{v} \times \vec{w}) \cdot \vec{u}|$
Area of the base parallelogram$= ||\vec{u} \times \vec{v}||$
Height$= ||\vec{w}|| \cos\theta$

Ex: find the volume of the parallelepiped with sides:
$\vec{u} = (2, 0, 3)$
$\vec{v} = (1, 1, -6)$
$\vec{w} = -1, 2, 1)$
Volume=

$$\begin{bmatrix} 2 & 0 & 3 \\ 1 & 1 & -6 \\ -1 & 2 & 1 \end{bmatrix} = 2(1 + 12) + 0 + 3(3) = 35$$

Chapter 5: Vector spaces

5.2 Definition

A vector space is:

- a set V (set of vectors) without a geometric representation (generally), with two operations:
  - addition of vectors
  - scalar multiplication

  satisfying the following 10 axioms:

  1. If $\vec{u},\ \vec{v} \in V$, then $u + v \in V$
  2. If $\vec{u} \in V$, $r \in \mathbb{R}$, then $\vec{r}\vec{u} \in V$
  3. There exists a vector, denoted $\vec{0}$, such that $\vec{0} + \vec{u} = \vec{u}\ \forall\ u \in V$
  4. Given $\vec{u} \in V$, there exists a vector denoted $\vec{-u}$, such that $\vec{u} + (\vec{-u}) = \vec{0}$
  5. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
  6. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
  7. $\vec{r}(\vec{u} + \vec{v}) = \vec{r}\vec{u} + \vec{r}\vec{v}$
  8. $(\vec{r} + \vec{s})\vec{u} = \vec{r}\vec{u} + \vec{s}\vec{u}$
  9. $(\vec{r}\vec{s})\vec{u} = \vec{r}(\vec{s}\vec{u})$
  10. $1 \times \vec{u} = \vec{u}$

$$\textbf{Remark: } \begin{aligned} 0 \times \vec{u} &= \vec{0} \\ (-1)\vec{u} &= -\vec{u} \end{aligned}$$

5.3 Example

1. $\mathbb{R}^n$ with usual addition and scalar multiplication are vector spaces for every $n \in \mathbb{N}$

2. Spaces of linear equations
   V = set of all linear equations in x, y, z

   (a)
   $$\begin{aligned} \text{if } u &= (ax + by + cz = d) \\ v &= (a'x + b'y + c'z = d') \end{aligned}$$

   Then $u + v = [(a + a')x + (b + b')y + (c + c')z = d + d']$

   (b) If $r \in \mathbb{R}$, $ru = (rax + rby + rcz = rd)$

   $$\begin{aligned} \text{e.g. } u &= (-2x + y + 3z = 1) \\ v &= (x - y + z = 0) \\ \text{Then } u + 2v &= (-y + 5z = 1) \end{aligned}$$

3. Spaces of functions
   V = set of all functions $f : \mathbb{R} \to \mathbb{R}$

   (a) If $f,\ g \in V$,
   then $f + g : \mathbb{R} \to \mathbb{R}$ is a function defined by $(f + g)(x) = f(x) + g(x)\ \forall\ x \in \mathbb{R}$

(b) If $f \in V$, $r \in \mathbb{R}$, then $rf : \mathbb{R} \to \mathbb{R}$ is a function defined by $(rf)(x) = r(f(x)) \; \forall \; \mathbb{R}$

Verification of axioms:

- (1), (2) ✓
- (3) the zero vector is the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = 0 \; \forall \; \mathbb{R}$
- (4) Given $f : \mathbb{R} \to \mathbb{R}$, $-f$ is the function defined by $(-f)(x) = -(f(x))$
- (5)-(10) ✓

4. $V = \{0\}$

   (a) $\vec{0} + \vec{0} = \vec{0}$

   (b) $\vec{r}\vec{0} = \vec{0} \; \forall \; \vec{r} \in \mathbb{R}$

   is a vector space, called the *zero vector space*. It corresponds to one-dimensional space.

5. $V = \{(x, 2x) \mid x \in \mathbb{R}\}$, With usual addition and scalar multiplication as in $\mathbb{R}^2$
   Verification of axioms:

   - (1) If $u = (a,\ 2a),\ v = (b,\ 2b) \in V$ where $a,\ b \in \mathbb{R}$
     then $u + v = (a + b,\ 2a + 2b)$
   - (2) If $u = (a,\ 2a) \in V$, $r \in \mathbb{R}$, then

$$\begin{aligned} ru &= (ra,\ r2a) \\ &= (ra,\ 2(ra)) \in V \end{aligned}$$

   - (3) $0 = (0, 0) = V$
   - (4) If $u = (a, 2a)$, then $-u = (-a, -2a) \in V$

6. $V = \{(x,\ x + a) \mid x \in \mathbb{R}\}$ with usual addition and scalar multiplication in $\mathbb{R}^2$.
   $y = x + 1$ is **NOT** a vector space

   (a) If $u = (a,\ a + 1) \in V$
   $v = (b,\ b + 1) \in V$
   then $u + v(a + b,\ a + b + z)) \notin v$

   (b) If $u = (a,\ a + 1),\ r \in \mathbb{R}$
   then $ru = (ra,\ ra + r) \notin V \; \forall \; r \neq 1$

   (c) $(0, 0) \notin V$

Definition

An $m \times n$ matrix is a table of numbers with m rows and n columns.

$$\text{e.g. } \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ is a } 2 \times 3 \text{ matrix}$$

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \text{ is called a } 3 \times 1 \text{ column matrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 5 \end{bmatrix} \text{ is a } 1 \times 4 \text{ row matrix}$$

2 matrices of the same size can be added componentwise:

$$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix} + \begin{bmatrix}0 & 2 & 4\\6 & 8 & 10\end{bmatrix} = \begin{bmatrix}1 & 4 & 7\\10 & 13 & 16\end{bmatrix}$$

Matrices can also be multiplies by a scalar componentwise:

$$-4 \times \begin{bmatrix}2 & 0\\5 & 6\\7 & 1\end{bmatrix} = \begin{bmatrix}-8 & 0\\-20 & -24\\-28 & -4\end{bmatrix}$$

Examples

1. $V = M_2 2(\mathbb{R}) = \left\{ \begin{bmatrix}a & b\\c & d\end{bmatrix} | a,b,c,d \in \mathbb{R} \right\}$ with the above addition and scalar multiplication is a vector space.
   Verification of the axioms:
   - (1), (2) ✓
   - (3) $0 = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} \in V$
   - (4) ✓
   - (5) If $u = \begin{bmatrix}a & b\\c & d\end{bmatrix}$, $v = \begin{bmatrix}e & f\\g & h\end{bmatrix}$
     then $u + v = \begin{bmatrix}a+e & c+f\\c+g & d+h\end{bmatrix}$
   - (6)-(10) ✓
2. $M_{mn}(\mathbb{R})$ is a vector space for all $m,\ n \in \mathbb{N}$
   Remark: In a vector space, $0u = \vec{0}$
   $1u = 0u = (1+0)u = 1u = u$
   $\Rightarrow \quad u + 0u + (-u) = u + (-u)$
   $\Rightarrow \quad 0u = \vec{0}$

$\mathbb{R}^3$ has 3 dimensions.
Every vector space has a basis, and the number of vectors in a basis is the dimension.

Chapter 6: Subspaces and spanning sets

6.1 Definition

A subset W of a vector space V is called a subspace if it is a vector space itself, under the same addition and scalar multiplication of V.

Examples:

1. $W = \{(x, 2x) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$ is a subspace of $\mathbb{R}^2$

Theorem: If V is a vector space, $W \subseteq V$ subset then W is a subspace:

$$\iff \begin{cases} O \in W \\ \text{If } u, v \in W, \text{ then } u + v \in W \\ \text{If } u \in W,\ r \in \mathbb{R},\ \text{then } ru \in W \end{cases}$$

axioms of subspaces

6.2

Examples:

1. W is the plane $x + 2y + 3z = 0$ in R^3
   Verification:
   (a) $\vec{0} = (0, 0, 0) \in W$
   (b) If $\vec{u} = (a, b, c)$, $\vec{v} = (a', b', c') \in W$
   $a + 2b + 3c = 0$, $a' + 2b' + 3c' = 0$
   $u + v = (a + a',\ b + b',\ c + c') \in W$

$$\begin{aligned} &(a + a') + 2(b + b') + 3(c + c') \\ &= (1 + 2b + 3c) + (a' + 2b' + 3c') \\ &= 0 + 0 \\ &= \vec{0} \end{aligned}$$

   (c) If $u = (a, b, c) \in W,\ r \in \mathbb{R}$, then $ru = (ra, rb, rc) \in W$

$$\begin{aligned} &ra + 2(rb) + 3(rc) \\ &= r(a + 2b + 3c) \\ &= r \cdot 0 \\ &= \vec{0} \end{aligned}$$

2. Any plane passing through the origin in $\mathbb{R}^3$ is a subspace.
   Any plane *not* passing through the origin is *not* a subspace.

3. $v \in \mathbb{R}^n,\ v \neq 0$
   $L = \{tv \mid t \in \mathbb{R}\}$ is a line in $\mathbb{R}^n$ passing through the origin, with direction vector v.
   Verification:
   (a) $\vec{0} = 0 \cdot v \in L$
   (b) If $u = tv$, $w = sv \in L$
   $u + v = tv + sv = (t + s)v \in L$

(c) If $u = t\vec{v} \in L$, $r \in R$
$ru = r(t\vec{v}) = (rt)\vec{v} \in L$
So L is a subspace.

4. $W = \{(x, y) \mid x, y \geq 0\} \subseteq \mathbb{R}^2$
Is this a subspace? **NO,** as multiplying a vector in the space by certain scalars will produce a vector not in the subspace.

5. Any line passing through the origin is a subspace in $\mathbb{R}^n$.
Any line *not* passing through the origin is *not* a subspace in $\mathbb{R}^n$

6. $V =$ set of all functions $f : \mathbb{R} \to \mathbb{R}$
$W =$ the set of all polynomial functions $p : \mathbb{R} \to \mathbb{R}$
i.e. $p(x) = a_0 + a_1 x + ... + a_n x^n$, n a non-negative integer, $a_0..a_n \in \mathbb{R}$
Verification:

   (a) (1) The zero vector of V is the function $f(x) = 0 \ \forall \ x \in \mathbb{R}$. It is in W because $f(x)$ is the polynomial function with $n = 0$ and $a = 0$.

   (b) (2), (3) ✓

7. $V =$ set of all functions $f : \mathbb{R} \to \mathbb{R}$
$W =$ set of all continuous functions
$\because$ Multiplication of any continuous function by a scalar returns a continuous function
$\therefore W$ is a subset of V.

8. V is the set of all functions $f : \mathbb{R} \to \mathbb{R}$
W is the set of all functions $f(x)$ such that $f(x) \in [-1, 1] \ \forall \ x$
Is W a subspace of V?

   (a) (1) ✓

   (b) (2) $f(x) = \cos(x)$
   $g(x) = \cos(x)$
   $(f + g)(x) = 2\cos(x) \neq W$

Matrices

Def: The transpose of an $m \times n$ matrix A, denoted A^T, is an $n \times m$ matrix whose columns are rows of A, e.g.:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ -2 & 5 \end{bmatrix}^T = \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 & 4 \end{bmatrix}$$

An $n \times n$ matrix (square matrix) is called *symmetric* if $A = A^T$, e.g. $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ is symmetric.

1. $V = M_{22}(\mathbb{R})$
   $W =$ set of all 2×2 symmetric matrices

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R},\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right\}$$
$$= \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \mid a, b, d \in \mathbb{R} \right\}$$

   Verification:

   (a) (1) $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$

   (b) (2) If $u = \begin{bmatrix} a & b \\ b & d \end{bmatrix},\ v = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$,

$$u + v = \begin{bmatrix} a+e & b+f \\ b+g & d+h \end{bmatrix} \in W$$

   (c) (3) If $u = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \in W,\ r \in \mathbb{R}$

$$ru = \begin{bmatrix} ra & rb \\ rb & rd \end{bmatrix} \in W$$

Let V be an arbitrary vector space.
V always has the following subspaces:

1. V is its own subspace

2. $\{0\}$, the zero subspace (or zero vector space)

Chapter 7: The span of vectors in a vector space

Definition

V is a vector space.

1. If $v_1, ..., v_n \in V$ and $a_1, ..., a_n \in \mathbb{R}$, then $u = a_1v_1 + ... + a_nv_n$ is called a *linear combination* of $v_1, ..., v_n$.
2. The set of all linear combinations of $v_1, ..., v_n$ is called the *span* if $v_1, ..., v_n$.
   $\text{span}\{v_1, ..., v_n\} = \{a_1v_1 + ... + a_nv_n \mid a_1, ..., a_n \in \mathbb{R}\}$
3. A vector (sub)space W is spanned by $v_1, ..., v_n \in W$ if $W = \text{span}\{v_1, ..., v_n\}$ (v_1, v_n spans W)

S = set of all 2×2 symmetric matrices

$$\begin{aligned}
&= \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \mid a, b, d \in \mathbb{R} \right\} \\
&= \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & \\ 0 & 1 \end{bmatrix} \mid a, b, d \in \mathbb{R} \right\} \\
&= \text{span}\{a, b, d\}
\end{aligned}$$

A spanning set is *not* unique.
$\text{span}\{0\} = \{0\}$

Examples

1. $L = \{t\vec{v} \mid t \in \mathbb{R}\} \subseteq \mathbb{R}^n$ is a line
   $= \text{span } \vec{v}$
2. $\mathbb{R}^2 = \text{span}\{(1,0), (0,1)\}$ because any $(x, y) \in \mathbb{R}$ can be written as $x(1,0) + y(0,1)$.

Similarly, $\mathbb{R}^3 = \text{span}\{(1,0,0), (0,1,0), (0,0,1)\}$

Examples

1. W = xy-plane in $\mathbb{R}^3 = \{(x, y, o) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$
   $= \text{span}\{(1,0,0), (0,1,0)\}$

7.4

Theorem:
Let V be a vector space such that
$\{v_1, ..., v_n\} \subseteq V$ ($\iff v_1, ...v_n \in V$)

1. $U = \text{span}\{v_1, ..., v_n\}$ is a subspace of V
2. If W is another subspace of V such that $v_1, ..., v_n \in W$, then $U \subseteq W$
   Therefore U is the smallest subspace containing $v_1, ..., v_n$

Subset axioms:

1. $\vec{0} \in U$
2. If $u, v \in U$, then $u + v \in U$
3. If $u \in U$, $r \in \mathbb{R}$, then $ru \in U$

Proof of (1):

- $\vec{0} = 0v_1 + ... + 0v_n \in \text{span}\{v_1, ..., v_n\}$
- If $u, v \in U$,
  then $u = a_1v_1 + ... + a_nv_n$
  and $w = b_1v_1 + ... + b_nv_n$
  For some $a_1, ..., a_n,\ b_1, ..., b_n \in \mathbb{R}$
  $u + w = (a_1 + b_1)v_1 + ... + (a_n + b_n)v_n \in U$
- If $u \in U,\ r \in \mathbb{R}$
  then $u = a_1v_1 + ... + a_nv_n$
  $ru = (ra_1)v_1 + ... + (ra_n)v_n \in U$

7.5

Definition:
Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{1...} & a_{1n} \\ a_{21} & ... & ... & ... \\ a_{...1} & ... & ... & ... \\ a_{n1} & ... & ... & a_{nn} \end{bmatrix}$$

The trace of A, denoted $\text{Tr}(A)$ is defined as $\text{Tr}(A) = a_{11} + a_{22} + ... + a_{nn}$.
Example:

$$Tr\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\right) = 1 + 5 + 9 = 15$$

Example:

$$\begin{aligned} U &= \{A \in M_{22}(\mathbb{R}) \mid \text{Tr}(A) = 0\} \\ &= \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a + d = 0\right\} \\ &= \left\{\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mid a, b, c \in \mathbb{R}\right\} \\ &= \left\{a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mid a, b, c \in \mathbb{R}\right\} \\ &= \text{span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\} \end{aligned}$$

Example:
W = set of all 2×2 diagonal matrices

$$\begin{aligned} &= \left\{\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \mid a, d \in \mathbb{R}\right\} \\ &= \text{span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\} \text{ is a subspace of } M_{22}(\mathbb{R}) \end{aligned}$$

7.6

$\mathbb{R}$ has only 2 subspaces: $\{0\}$ and $\mathbb{R}$. Proof: If $W \subseteq \mathbb{R}$ is a subspace and $W \neq \{0\}$, then there is a nonzero vector $\vec{v} \in W$

$$\begin{aligned} \mathbb{R} = \text{span}\{w\} &\subseteq W \\ &\Rightarrow \mathbb{R} = W \end{aligned}$$

The only subspaces of $\mathbb{R}^2$ are:

1. $\{0\}$
2. Lines through the origin
3. $\mathbb{R}^2$

Proof: If $W \subseteq \mathbb{R}^2$ is a subspace and
$W \neq \{0\}$, $W \neq$ lines through the origin.
The we have to show that $W = \mathbb{R}^2$.
First, $W \neq \{0\}$. There exists a nonzero vector $v \in W \Rightarrow \text{span}\{v\} \subseteq W$

Since $W \neq$ lines through the origin, there exists $u \in W$ such that $u \notin \text{span}\{v\}$, i.e. u is not a multiple of v.
$u, v \in W \Rightarrow \text{span}\{u, v\} \subseteq W$

Note that $\text{span}\{u, v\} = \mathbb{R}^2$ because any vector $\mathbb{R}^2$ can be written as a linear combination of u and v.
For any $w \in \mathbb{R}^2, w = \text{proj}_v(w) + (w - \text{proj}_v(w)$
$W = \text{proj}_v(w) + (w - \text{proj}_v(w))$
$= av + b(u - \text{proj}_v(u))$
$= av + bu - b(cv)$
$= (a - bc)v + bu \in \text{span}\{u, v\}$ The only subspaces of $\mathbb{R}^3$ are

1. $\{\ 0\ \}$
2. Lines through the origin
3. Planes through the origin
4. $\mathbb{R}^3$

7.7

Examples:

1. $\text{span}\{(1,2,1),(0,1,2)\} = \text{span}\{(1,3,3),(1,1,-1)\}$
   Taking the cross products of each pair of vectors in these spans and showing that the resulting vectors are parallel is a proof.
2. $\text{span}\{(1,2,1),(0,1,2)\} = u$ and $\text{span}\{(1,3,3),(1,1,-1)\} = v$
   $(1,3,3) = (1,2,1) + (0,1,2) \in \text{span}\{(1,2,1),(0,1,2)\} = u$

Chapter 8: Linear dependence and independence

8.2

2 vectors $\vec{u}, \vec{v} \in \mathbb{R}^3$ are collinear if there are non-zero $a, b \in \mathbb{R}$ such that $a\vec{u} + b\vec{v} = 0$.
e.g. $(1,2,0), (2,4,0)$ are collinear and span $\{(1,2,0),(2,4,0)\} = \text{span}\{(1,2,0)\}$

8.3

3 vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ are coplanar if there exists $a, b, c \in \mathbb{R}$ not all zero, such that $a\vec{u} + b\vec{v} + c\vec{w} = 0$.
e.g. $(1,0,0), (0,1,0), (3,2,0)$ are coplanar because $3(1,0,0) + 2(0,1,0) + (-1)(3,2,0) = 0$
$\text{span}\{(1,0,0),(0,1,0),(3,2,0)\} = \text{span}\{(1,0,0),(0,1,0)\}$

8.4

Definition:

- V is a vector space.
  $v_1, ..., v_n \in V$
  $\{v_1, ..., v_n\}$ is linearly dependent $\iff$ there exists $a_1, ..., a_n \in \mathbb{R}$ not all zero such that $a_1v_1 + ... + a_nv_n = 0$.
- $\{v_1, ..., v_n\}$ is linearly independent $\iff$ $\{v_1, ..., v_n\}$ is *not* linearly dependent $\iff$ the only solution to the equation $a_1v_1 + ... + a_nv_n = 0$ is the trivial solution $a_1 = a_2 = ... = a_n = 0$.

8.7

Examples:

- $\{(1,0),(0,1)\} \subseteq \mathbb{R}^2$ is LI.

  Solve $a(1,0) + b(1,0) = (0,0) \iff \begin{cases} a \cdot 1 + b \cdot 0 = 0 \\ a \cdot 0 + b \cdot 1 = 0 \end{cases}$

  So $a = b = 0$
- $\{(2,2),(3,-1)\} \subseteq \mathbb{R}^2$ is LI.

  Solve $a(2,2) + b(3,-1) = (0,0) \iff \begin{cases} 2a + 3b = 0 \\ 2a - b = 0 \end{cases}$

  $$\begin{aligned} 4b = 0 &\Rightarrow b = 0 \\ &\Rightarrow a = 0 \end{aligned}$$
- $\{(2,2),(3,2),(-1,-1)\} \subseteq \mathbb{R}^2$ is LD because $1(2,2) + 0(3,2) + 2(-1,-1) = (0,0)$.
- $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \subseteq M_{22}(\mathbb{R})$
  Solve:
  $$a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
  $a = 0, b = 0, c = 0, d = 0$.
- Is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \right\} \subseteq M_{22}(\mathbb{R})$
  linearly dependent or linerarly independent?
  Solution: It is linearly dependent.
  $$1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - 1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$a\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}+b\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}+c\begin{bmatrix}0 & 4\\4 & 0\end{bmatrix}+d\begin{bmatrix}0 & 0\\0 & 3\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$$\begin{cases}a+b=0\\b+4c=0\\b+4c=0\\3d=0\end{cases}$$

$$d=0$$

$$a+b=0 \qquad a=-b$$

$$b+4c=0 \qquad c=\frac{-b}{4}$$

- $V=F(\mathbb{R})=\{f\mathbb{R}\to\mathbb{R}\}$
  $\{1,x,x^2\}$ is LI.
  Solve $a\cdot 1+bx+cx^2=0$, where 0 is the zero function. This is an equation of functions.
- $\{1,\sin^2(x),\cos^2(x)\}$ is linearly dependent because $1-\sin^2(x)-\cos^2(x)=0$.
- $\{1,\sin(x),\cos(x)\}$ is linearly independent. Solve $a+b\sin(x)+c\cos(x)=0$.

8.8

Properties

- $\{0\}$ is linearly dependent.
- $v\in V$, then $\{v\}$ is LI $\iff v\neq 0$
- $\{v_1,...,v_n\}$ is LD $\iff$ there are some v_i such that v_i is in the span of the rest, i.e. $v_i\in\operatorname{span}\{v_1,...,v_n\}$.
- If a set S is LD, then any set containing S is LD.
- Conversely, if a set S is linearly independent, then any subset of S is still linearly independent.
- any set containing 0 is linearly dependent.
- $\{u,v\}$ is LD $\iff$ one vector is a multiple of the other.

Chapter 9: Linear independence and spanning sets

9.2 **Theorem:**

If $v_1 \in \text{span}\{v_1, ..., v_n\}$, then $\text{span}\{v_1, ..., v_n\}$ =$\text{span}\{v_1, ..., v_n\}$

Therefore if $\{v_1, ..., v_n\}$ is LD, then $\text{span}\{v_1, ..., v_n\}$ =$\text{span}\{v_1, ..., v_{i-1}...v_n\}$ for some i.

<u>Recall:</u>

$\{v_1, ..., v_n\}$ is LD $\iff$ there exists a space v_i such that $v_i \in \text{span}\{v_i, v_{i-1}, ..., v_n$.

In this case, $\text{span}\{v_1, ..., v_n\} = \text{span}\{v_i, v_{i-1}, ..., v_n\}$.

Examples

1. $\text{span}\{(1,0,0),(0,1,0),(3,-2,0)\} = \text{span}\{(1,0,0),(0,1,0)\}$
2. $\text{span}\{(1,0,0),(0,1,1),(0,4,4)\} = \text{span}\{(1,0,0),(0,1,1)\} \neq \text{span}\{(0,1,1),(0,4,4)\}$

9.3

Suppose $\{v_1, ..., v_n\}$ is LI, $w \in V$.

Then $\{w, v_1, ..., v_n\}$ is LI $\iff$ $w \notin \text{span}\{v_1, ..., v_n\}$ **Example:**

1. $\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right\}$ is L

   and $\begin{bmatrix}0 & 0\\3 & 2\end{bmatrix} \notin \text{span}\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right\}$

   then $\left\{\begin{bmatrix}0 & 0\\3 & 2\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right\}$ is LI.

2. $\begin{bmatrix}0 & 4\\0 & 0\end{bmatrix} \in \text{span}\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right\}$

   Then $\left\{\begin{bmatrix}0 & 4\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right\}$ is LD.

Chapter 10: Basis and dimension

10.1 Theorem:

If a vector space V can be spanned by n vectors, then any LI subset of V has at most n vectors. Equivalently, if V has a LI subset of m vectors, then any spanning set of V has at least m vectors.

$$\text{size of any spanning set of } V \geq \text{size of any LI subset of } V$$

Examples

1. $\mathbb{R}^{\nVdash} = \text{span}\{(1,0,0),(0,1,0),(0,0,1)\}$ can be spanned by 3 vectors. Any subset of $\mathbb{R}^3$ containing 4 or more vectors is LD.
2. $M_{22}(\mathbb{R})$ can be spanned by 4 vectors. Any subset of $M_{22}(\mathbb{R})$ containing 5 or more vectors is LD.

10.2 Definition $\{v_1,...,v_n\} \subseteq V$ is called a basis of V if

1. $\{v_1,...,v_n\}$ is LI
2. $\text{span}\{v_1,...,v_n\} = V$

- A basis is a largest possible LI subset of a set V, i.e. if you add any vector to the set, it will become LD.
- A basis is a smallest possible spanning set of V. If you remove any vector from the set, then it will not span V.

Examples:

1. $\{(1,0),(0,1)\}$ is a basis of $\mathbb{R}^2$ (the standard basis).
2. $\{(1,1),(1,-1)\}$ is also a basis of $\mathbb{R}^2$.
3. $\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$ is the standard basis of $M_{22}(\mathbb{R})$.
4. $\mathbb{P}_2 =$ vector space of polynomials of a degree ≤ 2 is a closed subspace under addition (as adding any polynomials can only reduce the degree), multiplication, and the zero vector test.
   $\{1, x, x^2\}$ is a basis of $\mathbb{P}_2$.
   Any polynomial of deg≤ 2 is of the form $a + bx + cx^2$. **Examples**
   (a) $\{(1,0,0),(0,1,1)\} \in \mathbb{R}^3$ is NOT a basis as $\text{span}\{(1,0,0),(0,1,1)\} \neq \mathbb{R}^3$.
   (b) $V = \{a \in M_{22}(\mathbb{R}) \mid \text{Tr}(A) = 0\}$
   $\left\{\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}2 & 0\\3 & -2\end{bmatrix}\right\}$ is NOT a basis of V because it is LD.
   $\left\{\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right\}$ is a basis of V.

Theorem

If $\{v_1,...,v_n\}$ and $\{w_1,...,w_m\}$ are 2 bases for V, then $n = m$.
Proof:
$\{v_1,...,v_n\}$ spans V and $\{w_1,...,w_m\}$ is LI.
$\Rightarrow n \geq m$.
$\{v_1,...,v_n\}$ is LI and $\{w_1,...,w_m\}$ spans V.
$n \leq m$ The only case in which both $n \geq m$ and $n \leq m$ are true is if $n = m$. QED.

10.3 Definition

If V has a finite basis ($\{v_1, ..., v_n\}$) then we say that the dimension $\dim(V) = n$. In this case, V is called a finite dimensional space.
If V does not have a finite basis, then V is an infinite dimensional space.
Examples:

- $\dim(\mathbb{R}^2) = 2$
- $\dim(\mathbb{R}^n) = n$
- $\dim(M_{22}(\mathbb{R})) = 4$
- $\dim(M_{22}(\mathbb{R})) = mn$
  Standard basis: $\{E_{ij} \mid 1 \leq i \leq m, \quad 1 \leq j \leq n\}$
  E_{ij} is the matrix with (i, j) - entry is 1, other entries are 0.
- $\dim(\mathbb{P}_2) = 3$ because a basis is $\{1, x, x^2\}$
- $\dim(\mathbb{P}_n) = n + 1$
- $\dim(F(\mathbb{R})) = \infty$
- $\dim(V) = 3$ where $V = \{A \in M_{22}(\mathbb{R}) \mid \mathrm{Tr}(A) = 0\}$
- $W =$ vector space of all 2×2 diagonal matrices. $\dim(W) = 2$.
- $W =$ the plane with equation $x + 3y + 5z = 0 \subseteq \mathbb{R}^3$.
  $\dim(W) = 2$. A basis of W is $\{(3, -1, 0), (5, 0, -1)\}$
  Solve $a(3, -1, 0) + b(5, 0, 1) = (0, 0, 0)$
  Then $a = b = 0$
  $\dim\{0\} = 0$. A basis for the zero vector space is $\varnothing$

$V = M_{22}(\mathbb{R})$

$$W = \left\{ \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

This set does not contain $\vec{0}$. It is therefore not a subspace.

$$W = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

This set contains $\vec{0}$.

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid ab = cd \right\}$$

This set is not closed under addition, and is not a subspace.

Chapter 11: Dimension theorems

11.1

V vector space with $\dim(V) = n < \infty$

- If $\{v_1, ..., v_m\} \subseteq V$ is LI, $m \leq n$ and there exists $v_{m+1}, ..., v_n \in V$ such that $\{v_1, ..., v_m, ..., v_n\}$ is a basis of V.
- If $\{w_1, ..., w_r\}$ is a spanning set of V, $r \geq n$ and there is a subset of S that is a basis of V
  - Size of LI subset $\leq$ size of spanning set

$V = \mathbb{R}^3$
$\{(1, 2, 0), (1, -1, 0)\}$
Because the 2 vectors are *not* multiples of each other, this set is LI. This is a shortcut for sets of 2 vectors.
$\{(1, 2, 0), (1, -1, 0), (0, 0, 1)\}$ is a basis of $\mathbb{R}^3$
It is LI because $(0, 0, 1) \notin \text{span}\{(1, 2, 0), (1, -1, 0)\}$
$\{(1, 0, 0), (0, 1, 0), (0, 1, 1), (0, 2, 3)\}$ is LD.
We see that $(0, 2, 3) = -(0, 1, 0) + 3(0, 1, 1)$
so $\{(1, 0, 0), (0, 1, 0), (0, 1, 1)\}$ is a basis of $\mathbb{R}^3$.

Theorem

V vector space with $\dim(V) = n < \infty$
Then:

1. Any LI subset of n vectors is a basis of V
2. Any spanning set of n vectors is also a basis

Axioms:

1. LI
2. Spanning
3. Consisting of n vectors

Only need to check any 2 of these conditions to see whether a subset is a basis.
Examples

1. $W = \{(x, y, z) \in \mathbb{R}^3 \mid 3x - y + 5z = 0\}$
   $\dim(W) = 2$
   $(1, 3, 0), (5, 0, -3) \in W$
   $\{(1, 3, 0), (5, 0, -3)\}$ is LI
   $\Rightarrow \{(1, 3, 0), (5, 0, -3)\}$ is a basis of W (spanning is mathematically satisfied)
2. Expand $\{(1, 3, 0), (5, 0, -3)\}$ to a basis of $\mathbb{R}^3$.
   $\{(1, 3, 0), (5, 0, 3), (3, -1, 5)\}$ is a basis of $\mathbb{R}^3$.
   $(3, -1, 5)$ is a normal vector of W. In particular, $(3, -1, 5) \notin W$.
   $\Rightarrow \{(1, 3, 0), (5, 0, -5), (3, 1, 5)\}$ is LI.

11.3 Theorem

V vector space, $\dim(V) = n < \infty$
$W \subseteq V$ subspace. Then:

1. $0 \leq \dim(W) \leq \dim(V)$
2. $\dim(W) = \dim(V) \iff W = V$
3. $\dim(W) = 0 \iff W = \{0\}$

Proof

1. Suppose $\{w_1, ..., w_m\}$ is a basis of W ($\dim(W) = m$).
   Then, $\{w_1, .., w_m\}$ is an LI subset of W, hence also a subset of V.
   Therefore, $m \leq \dim(V)$.

2. If $W = V$, then dim(W)=dim(V).
   If dim(W)=dim(V), let $S = \{w_1, ...w_n\}$ be a basis of W.
   (LI + consisting of n vectors)$\Rightarrow$ also a basis of V.

Examples

1. Any subspaces of $M_{22}(\mathbb{R})$ has dimensions $0, 1, 2, 3$ or 4 and the only subspace of dim(4) is $M_{22}(\mathbb{R})$. The only subspace of dim(0) is $\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

2. Any subspace of $\mathbb{R}^n$ has dimension 0,1,2,..., or n and the only subspace with dim(n) is $\mathbb{R}^n$. The only subspace with dim(0) is $\{(0, 0, ..., 0)\}$

11.6 Theorem:

Let $B = \{v_1, ..., v_n\}$ be a basis of V.
Then for any $v \in V$, there exist *unique* scalars $a, ..., a_n \in \mathbb{R}$ such that $v = a_1v_1, ..., a_nv_n$.
Examples:

1. $\{(1, 0), (0, 1)\}$ is a basis of $\mathbb{R}^2$.

2. $\{(1, 0), (0, 1), (0, 3)\}$ is NOT a basis of $\mathbb{R}^3$s

Application
We can identify V with $\mathbb{R}^n$
$v = a_1v_1 + .. + a_nv_n \longleftrightarrow (a_1, ..., a_n)$
e.g. $V = M_{22}(\mathbb{R})$, $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$
Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longleftrightarrow (a, b, c, d)$

Chapter 12: solving systems of linear equations

12.1 Examples

1. $$\begin{cases} x + 3y + 2z = 6 \\ 0x + y - z = 0 \end{cases}$$

   is a linear system of 2 equations.
   Definition
   The general solution to a linear system is the set of all solutions, e.g. $\{(6-5t, t, t) \mid t \in \mathbb{R}\}$ is the general solution to the above example (t is the parameter of the general solution).

2. $$\begin{cases} x + y + z = 2 \\ x + y + z = 3 \end{cases}$$

   Has no solution. The general solution is $\varnothing$, the empty set.

Definition

- A linear system with NO solution is called *inconsistent*.
- A linear system with at least one solution is called *consistent*.
- A linear system in which all the constants on the right are zero is called *homogenous*.
- A linear system in which at least one of the constants is nonzero is called *inhomogenous*.

Remark

- Homogenous system are always consistent.
  This is because $(0, 0, 0)$ is always a solution (the trivial solution).

Definition

- A linear equation is called *degenerate* if all the coefficients are zero.

Remark

- Any linear system containing a *degenerate, homogenous* equation is *inconsistent*.

Theorem
Any linear system (with real coefficients) has either

1. NO solutions
2. exactly one solution
3. infinitely many solutions

Examples

1.

$$x + y + 2z = 3 \tag{1}$$
$$x - y + z = 2 \tag{2}$$
$$y = z = 1 \tag{3}$$

$(2) - (1) = x + y + 2z = 3$
$-2y - z = -1$
$y - z = 1$

$(2) - (3) = x + y + 2z = 3$
$y = z = 1$
$-2y = -1$

$(3) + 2(2) = x + y + 2z = 3$
$y - z = 1$
$-3z = 1$

$-\frac{1}{3}(3)$
$x + y + 2z = 3$
$z = -\frac{1}{3}$

$(2) + (3)$
$x + y + 2z = 3$
$y + \frac{2}{3}$
etc.
This method is inefficient. We can do this with matrices.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$x = 3$
$y = \frac{2}{3}$
$z = \frac{1}{3}$
We used the following operations:

- Add a multiple of one row to another row
- Interchange two rows
- Multiply one row by a nonzero scalar

These are called *elementary row operations* (ERO) and performing these operations to a linear system will not change these the general solution.
12.3 Definition

- A matrix is in *Row Echelon Form* (REF) if the following conditions are satisfied:
  - All zero rows are at the bottom
  - The first nonzero entry of each row is 1 (leading 1)
  - Each leading 1 is to the right of the leading 1s in the rows above

– Each leading 1 is the only nonzero entry int its column, then the matrix is in *reduced row echelon form* (RREF).
e.g.

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -5 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

is REF but not RREF

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

is RREF, and the general solution is $\{(7,4,5)\}$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

General solution: $\varnothing$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = s$
$x_4 = t \quad (s, t \in \mathbb{R})$
$x_2 = -2s - t$
$x_1 = 1 - 2s$

Chapter 13: Solving systems of linear equations (cont'd)

Def:
Two linear systems are equivalent if they have the same general solution.

Thm:
If an ERO is performed on the augmented matrix of a linear system, the resulting linear system is equivalent to the original one.

Def:
Two matrices A and B are row equivalent if

- $A \sim B$, if B can be obtained from A by a finite sequence of ERO.

Thm:
Every matrix is row equivalent to a unique matrix in RREF. This statement is false if we replace RREF by REF).

$$\underset{\text{REF}}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} \longrightarrow R_1 - R_2 \longrightarrow \underset{\text{RREF}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

13.1

- If the RREF of the augmented matrix has a row like

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & b \end{array}\right]$$

where $b \neq 0$, then the system is inconsistent.

Otherwise, the system is consistent.

- If every column has leading 1, then there is a unique solution, e.g.

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

- Otherwise, there are infinitely many solutions.
  1. Set each non-leading variable equal to a different parameter
  2. Solve for the leading variables

  e.g.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

  Set $z = t$, $t \in \mathbb{R}$
  Then

$$y = 4 + 4t$$
$$x = 3 - 2t$$
$$\{3 - 2t, 4 + 4t, t \mid t \in \mathbb{R}\}$$

e.g.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 7 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & -1 & 6 \end{array}\right]$$

Set $x_2 = s$
$x_5 = t$
$s, t \in \mathbb{R}$
Then $x_4 = 6 + t$
$x_3 = 5$
$x_1 = 7 - 2s - 3t$

13.2 Gaussian elimination

1. If the matrix is 0 (the zero matrix), done.
2. Locate the leftmost nonzero column.
   Interchange rows if necessary, to bring a nonzero entry to the 1^{st} row.
3. Multiply the first row with a scalar to get a leading 1.
4. ***Annihilate*** the rest of the column.
5. Repeat step 1-4 ignoring the first row.

Iteration 2:

1. Same as above
2. Locate the leftmost column with a nonzero entry in the 2^{nd}-last row, e.g.

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 4 & 7 & -4 \\ 0 & 0 & 6 & 8 & 3 \end{array}\right]$$

   Interchange rows if necessary to bring a nonzero entry to the 2^{nd} row.
3. Multiply the 2^{nd} row with a scalar to get a leading 1.
   e.g.

$$\left[\begin{array}{cccc|c} 0 & 0 & 3 & 3 & 4 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & -1 & 0 & 1 & 1 \end{array}\right] \to R_1 \leftrightarrow R_2 \to \left[\begin{array}{cccc|c} 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 3 & 3 & 4 \\ 0 & -1 & 0 & 1 & 1 \end{array}\right] \to R_3 + R_1 \to \left[\begin{array}{cccc|c} 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 3 & 3 & 4 \\ 0 & 0 & 1 & 3 & 1 \end{array}\right] \to$$

$$\frac{1}{3}R_2 \to \left[\begin{array}{cccc|c} 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 4/3 \\ 0 & 0 & 1 & 3 & 1 \end{array}\right] \to R_1 - R_2,\ R_3 - R_2 \to \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & -4/3 \\ 0 & 0 & 1 & 1 & 4/3 \\ 0 & 0 & 0 & 2 & -1/3 \end{array}\right] \to$$

$$\frac{1}{2}R_3 \to \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & -4/3 \\ 0 & 0 & 1 & 1 & 4/3 \\ 0 & 0 & 0 & 1 & -1/6 \end{array}\right] \to$$

   At this point, the matrix is already in REF. To find the RREF:

$$R_1 - R_3,\ R_2 - R_3 \to \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & -7/6 \\ 0 & 0 & 1 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & -1/6 \end{array}\right]$$

The general solution of $\{(t, -\frac{7}{6}, \frac{3}{2}, -\frac{1}{6}) \mid t \in \mathbb{R}\}$ **Ex:**

$$\left[\begin{array}{ccc|c} 2 & 2 & -3 & 1 \\ 1 & 0 & 1 & 5 \\ 3 & 4 & -7 & -3 \end{array}\right] \to \frac{1}{2}R_1 \to \left[\begin{array}{ccc|c} 1 & 1 & -3/2 & 1/2 \\ 1 & 0 & 1 & 5 \\ 3 & 4 & -7 & -3 \end{array}\right] \to \frac{1}{3}R_3 \to \left[\begin{array}{ccc|c} 1 & 1 & -3/2 & 1/2 \\ 1 & 0 & 1 & 5 \\ 1 & 4/3 & -7/3 & -1 \end{array}\right] \to$$

$$R_2 \leftrightarrow R_3 \to \left[\begin{array}{ccc|c} 1 & 1 & -3/2 & 1/2 \\ 1 & 4/3 & -7/3 & -1 \\ 1 & 0 & 1 & 5 \end{array}\right] \to R_2 \leftrightarrow R_1 \to \left[\begin{array}{ccc|c} 1 & 4/3 & -7/3 & -1 \\ 1 & 1 & -3/2 & 1/2 \\ 1 & 0 & 1 & 5 \end{array}\right]$$

13.3 Definition

The rank of a matrix A, denoted $\text{rank}(A)$ is the number of leading 1s in any REF of A.
e.g.

$$\text{rank}\left(\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\right) = 2$$

Remark
$\text{rank}(A) \leq$ number of columns of A

Chapter 14: Applications and examples of solving linear systems

Recall:

rank(A) =number of leading 1s in any REF of A.

rank$(A) \leq$ rank$[A \mid b] \leq$ rank$(A) + 1$

e.g. For a homogeneous linear system,

$$\text{rank}(A) = \text{rank}[A \mid b]$$

then the system is consistent.

Remark

If rank$(A) =$ rank$[A \mid b]$, then the system is consistent.

1. If rank(A)(= rank$[A \mid b]$ <number of columns of A, then there is at least one column with a leading 1.

2. If rank(A) =number of columns of A, then the system has a unique solution.

14.2

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 300 \\ 1 & 0 & -1 & 0 & 100 \\ 0 & 0 & 1 & 1 & 500 \\ 0 & 1 & 0 & 1 & 300 \end{array}\right] \to R_2 - R_1 \to \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 300 \\ 0 & 1 & -1 & 0 & -200 \\ 0 & 0 & 1 & 1 & 500 \\ 0 & 1 & 0 & 1 & 300 \end{array}\right] \to R_1 + R_2;\ R_4 - R_2 \to \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 300 \\ 1 & 0 & -1 & 0 & 100 \\ 0 & 0 & 1 & 1 & 500 \\ 0 & 1 & 0 & 1 & 300 \end{array}\right]$$

The general solution is $\{(600 - t, 300 - t, 500 - 2, t) \mid t \in \mathbb{R}\}$

e.g. If the street x_2 is blocked, what is the effect?

$x_2 = 0 \Rightarrow t = 300$

14.3

Consider

$$\begin{aligned} kx + y + z &= 1 \\ x + ky + z &= 1 \\ x + y + kz &= 1 \\ \text{with } k \in \mathbb{R} \end{aligned}$$

Find all values of k so that the above system has:

1. no solution

2. a unique solution

3. infinitely many solution

$$\left[\begin{array}{ccc|c} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \end{array}\right] \to R_1 \leftrightarrow R_2 \to \left[\begin{array}{ccc|c} 1 & k & 1 & 1 \\ k & 1 & 1 & 1 \\ 1 & 1 & k & 1 \end{array}\right] \to R_2 - kR_1;\ R_3 - R_1 \to \left[\begin{array}{ccc|c} 1 & k & 1 & 1 \\ 0 & 1-k^2 & 1-k & 1-k \\ 0 & 1-k & k-1 & 0 \end{array}\right]$$

Case $k = 1$ we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

14.4

Compute span$\{(1, 2, 3), (4, 5, 6), (7, 8, 9)\}$

What are the vectors $(x, y, z) \in \mathbb{R}^3$ such that there are scalars $a_1, a_2, a_3 \in \mathbb{R}$ satisfying

$$a_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + a_3 \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}?$$

Chapter 15 Matrix Multiplication

15.1 Definition

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix.

The product AB is an $m \times p$ matrix whose (i, j)-entry is the dot product of the i-th row of A with the j-th column of B.

i.e. If $A = [a_{ij}]$, $B = [b_{ij}]$

Then $AB = [c_{ij}]$

Where

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{j1} + a_{i2}b_{j2} + ... + a_{in}b_{jn}$$

Examples:

- $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32$, a 1×1 matrix.
- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$
- $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 4 & -1 \end{bmatrix}$ The result of this multiplication is undefined as the matrices have different numbers of columns.
- $\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 2 \\ -3 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 7 & -1 & -1 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 3 \\ -2 & -5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y \\ 4x - y + 3z \\ -2x - 5y + z \end{bmatrix}$

A system of linear equations

$$a_{11}x + a_{12}y + a_{13}z = b_1 \quad (4)$$
$$a_{21}x + a_{22}y + a_{23}z = b_2 \quad (5)$$
$$a_{31}x + 1_{32}y + a_{33}z = b_3 \quad (6)$$

Can be expressed as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

15.2 Properties

- Sometimes AB is defined but BA is not (commutativity is not an inherent property of matrix multiplication)

  e.g. $A = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
- Even if AB and BA are both defined, they may be different.

  e.g. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$

  $AB = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$

  $BA = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$

- It is possible that $A \neq 0$ and $B \neq 0$ but $AB = 0$.
- It is possible that $AC = BC$; $C \neq 0$ but $A \neq B$
  e.g. $A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
  $AC = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$ and $BC = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$

Examples:

- Let $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$. View $A,\ B \in \mathbb{R}^3$
  The dot product of A and B
  $= 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6$
  $= A^T B$

15.3

Properties of the transpose:

- $(A+B)^T = A^T + B^T$
- $(kA)^T = k(A^T)$, $k \in \mathbb{R}$
- $(A^T)^T = A$

For any positive integer n, define $I_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ an $n \times n$ matrix called the identity matrix.

Theorem:

Let A, B, C be matrices and $k \in \mathbb{R}$. Whenever defined,

- $A(BC) = (AB)C$
- $A(B+C) = AB + AC$
- $(A+B)C = AC + BC$
- $k(AB) = (kA)B = A(kB)$
- $(AB)^T = B^T A^T$
- $AI = A,\ IA = A$
- If A is an $m \times n$ matrix, then $A0_{n \times p} = 0_{m \times p}$, $O_{q \times m} A = O_{q \times n}$

Proof of $A(BC) = (AB)C$:
Write $A = [a_{ij}],\ B = [b_{ij}],\ C = [c_{ij}]$

the (i,j)-entry of $BC = \sum_{k=1}^{p} b_{ik} c_{kj}$

the (i,j)-entry of $A(BC) = \sum_{l=1}^{n} (a_{il} \cdot \sum_{k=1}^{p} b_{lk} c_{kj})$

the (i,j)-entry of $(AB)C = \sum_{l=1}^{n} (\sum_{k=1}^{p} a_{ikb_{kl}})\, c_{lj}$

Examples:

- $(A+B)(C+D) = AC + AD + BC + BD$
- $(A+B)^2 = AA + AB + BA + BB = A^2 + AB + BA + B^2$
- $(A+B)(A-B) = A^2 - AB + BA + B^2 \neq A^2 - B^2$

Matrix multiplication
Examples:

1. $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

   Identity matrix

2. $$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

3. $$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ c & d \end{bmatrix}$$

4. $$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 3c & 3d \end{bmatrix}$$

5. $$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$$

Remark:
Any ERO can be represented by multiplying a suitable matrix from the *left*.

$$\begin{aligned}
\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A &= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) A \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A + \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} A \\
&= A + \begin{bmatrix} 0 & 0 & 0 \\ -2 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

Block multiplication

Example:

1. $$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} = A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$$

   We want to compute A^2

   $$A^2 = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$$

15.5
Given a linear system $a_1x_1 + ... + a_nx_n$,
Let $A = [a_{ij}]$, $b = [b_{ij}]$

Coefficient matrix $\begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix}$

Then the linear system can be expressed as $Ax = b$.
Let C_i be the i-th column of A
So $A = [C_1, C_2, ..., C_n]$
$A_x = [C_1, C_2, ..., C_n] \begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix} = C_1x_1 + C_2x_2 + ... + C_nx_n$
$= x_1C_1 + x_2C_2 + ...x_nC_n$

- $Ax = b$ is consistent $\iff$ b is a linear combination of columns of A.
- $Ax = 0$ has a unique solution
  - $\iff$ the equation $x_1C_1 + x_2C_2 + ... + x_nC_n$ has only one solution
  - $\iff$ $C_1, ..., C_n$ are LI.

1. $[A \mid b] = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 4 & 8 & 0 \end{array}\right]$
   $\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Definition
Let $A = [C_1, ..., C_n]$ be an $m \times n$, C_1 i-th column of A.
Then C_i can be viewed as vectors in $\mathbb{R}^m$ and the colum space is defined as $\text{col}(A) = \text{span}\{C_1, ..., C_n\} \subseteq \mathbb{R}^m$.

- $\text{Col}(A) = \mathbb{R}^m \iff$ columns of A span $\mathbb{R}^m$
  $\iff$ for every $b \in \mathbb{R}^m$, the system $Ax = b$ is *consistent*
  $\iff$ RREF of A does not have a zero row

If RREF of A has a zero row,
$$[A \mid b] \longrightarrow \left[\begin{array}{c|c} * & * \\ 000000 & * \end{array}\right]$$

- Columns of A are LI
  $\iff$ $Ax = 0$ has a unique solution
  $\iff$ there is a leading 1 in every column of RREF of A
  $\iff$ $\text{rank}(A) =$ number of columns of A

For a square matrix A,

- Columns of A form a basis of $\mathbb{R}^n$
  $\iff$ columns of A are LI
  $\iff$ $\text{rank}(A) = n$
  $\iff$ $\text{Col}(A) = \mathbb{R}^n$
  $\iff$ RREF of A has no zero row
  $\iff$ RREF of A is an $n \times n$ matrix
  $\iff$ RREF of A is equal to I_n