Control y Servomecanismos A

Control Automático I

Tema: Linealización de Sistemas Dinámicos

Cursada Virtual 2020
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Linealización de Sistemas Dinámicos

Sea el sistema de tres estados y dos entradas

$$\dot{x}_1 = f_1(x_1, x_2, x_3, u_1, u_2)$$

$$\dot{x}_2 = f_2(x_1, x_2, x_3, u_1, u_2)$$

$$\dot{x}_3 = f_3(x_1, x_2, x_3, u_1, u_2)$$

donde todas las funciones involucradas admiten derivada continua en todo el dominio de operación

Supongamos que yo quiero operar el sistema alrededor de los valores de las entradas dados por :

$$(u_1, u_2) = (\bar{u}_1, \bar{u}_2)$$

Para esos valores de entrada los valores de los estados:

$$(x_1, x_2, x_3) = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$$

Constituyen valores de equilibrio, es decir:

$$\begin{aligned}
\dot{x}_1 &= f_1(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{u}_1, \bar{u}_2) = 0 \\
\dot{x}_2 &= f_2(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{u}_1, \bar{u}_2) = 0 \\
\dot{x}_3 &= f_3(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{u}_1, \bar{u}_2) = 0
\end{aligned}$$

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Planteamos una traslación de coordenadas :

$$
\begin{aligned}
x_1^* &= x_1 - \bar{x}_1\\
x_2^* &= x_2 - \bar{x}_2\\
x_3^* &= x_3 - \bar{x}_3\\
u_1^* &= u_1 - \bar{u}_1\\
u_2^* &= u_2 - \bar{u}_2
\end{aligned}
$$

Luego, cada una de las funciones en torno al punto de operación puede escribirse como :

$$f_i(x_1, x_2, x_3, u_1, u_2) =$$

$$= f_i(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{u}_1, \bar{u}_2) + \left[\left.\frac{\partial f_i}{\partial x_1}\right|_{P_{op}} (x_1 - \bar{x}_1) + hot \right] + \left[\left.\frac{\partial f_i}{\partial x_2}\right|_{P_{op}} (x_2 - \bar{x}_2) + hot \right]$$

$$+ \left[\left.\frac{\partial f_i}{\partial x_3}\right|_{P_{op}} (x_3 - \bar{x}_3) + hot \right] + \left[\left.\frac{\partial f_i}{\partial u_1}\right|_{P_{op}} (u_1 - u_1) + hot \right] + \left[\left.\frac{\partial f_i}{\partial u_2}\right|_{P_{op}} (u_2 - \bar{u}_2) + hot \right]$$

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Luego, como:

$$\dot{x}_i^* = \dot{x}_i - \dot{\bar{x}}_i = \dot{x}_i$$

cada ecuación de estados en torno del punto de operación puede escribirse como:

$$\dot{x}_i^* = \overbrace{\cancel{f_i(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{u}_1, \bar{u}_2)}}^{=0} + \left[\left. \frac{\partial f_i}{\partial x_1} \right|_{P_{op}} (x_1 - \bar{x}_1) + \cancel{hot} \right] + \left[\left. \frac{\partial f_i}{\partial x_2} \right|_{P_{op}} (x_2 - \bar{x}_2) + \cancel{hot} \right]$$

$$+ \left[\left. \frac{\partial f_i}{\partial x_3} \right|_{P_{op}} (x_3 - \bar{x}_3) + \cancel{hot} \right] + \left[\left. \frac{\partial f_i}{\partial u_1} \right|_{P_{op}} (u_1 - u_1) + \cancel{hot} \right] + \left[\left. \frac{\partial f_i}{\partial u_2} \right|_{P_{op}} (u_2 - \bar{u}_2) + \cancel{hot} \right]$$

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$$\dot{x}_i^* = \left[\left.\frac{\partial f_i}{\partial x_1}\right|_{P_{op}} x_1^*\right] + \left[\left.\frac{\partial f_i}{\partial x_2}\right|_{P_{op}} x_2^*\right] + \left[\left.\frac{\partial f_i}{\partial x_3}\right|_{P_{op}} x_3^*\right] + \left[\left.\frac{\partial f_i}{\partial u_1}\right|_{P_{op}} u_1^*\right] + \left[\left.\frac{\partial f_i}{\partial u_2}\right|_{P_{op}} u_2^*\right]$$

$$\begin{bmatrix} \dot{x}_1^* \\ \dot{x}_2^* \\ \dot{x}_3^* \end{bmatrix} = \underbrace{\begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_{P_{op}} & \left.\frac{\partial f_1}{\partial x_2}\right|_{P_{op}} & \left.\frac{\partial f_1}{\partial x_3}\right|_{P_{op}} \\ \left.\frac{\partial f_2}{\partial x_1}\right|_{P_{op}} & \left.\frac{\partial f_2}{\partial x_2}\right|_{P_{op}} & \left.\frac{\partial f_2}{\partial x_3}\right|_{P_{op}} \\ \left.\frac{\partial f_3}{\partial x_1}\right|_{P_{op}} & \left.\frac{\partial f_3}{\partial x_2}\right|_{P_{op}} & \left.\frac{\partial f_3}{\partial x_3}\right|_{P_{op}} \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} + \underbrace{\begin{bmatrix} \left.\frac{\partial f_1}{\partial u_1}\right|_{P_{op}} & \left.\frac{\partial f_1}{\partial u_2}\right|_{P_{op}} \\ \left.\frac{\partial f_2}{\partial u_1}\right|_{P_{op}} & \left.\frac{\partial f_2}{\partial u_2}\right|_{P_{op}} \\ \left.\frac{\partial f_3}{\partial u_1}\right|_{P_{op}} & \left.\frac{\partial f_3}{\partial u_2}\right|_{P_{op}} \end{bmatrix}}_{\boldsymbol{B}} \begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix}$$

Linealización de Sistemas Dinámicos

Si las salidas del sistema no lineal son:

$$y_1 = h_1(x_1, x_2, x_3\ )$$

$$y_2 = h_2(x_1, x_2, x_3\ )$$

Haciendo un procedimiento análogo al realizado anteriormente:

$$y_i = h_i(\bar{x}_1, \bar{x}_2, \bar{x}_3\ ) + \left[\left.\frac{\partial h_i}{\partial x_1}\right|_{P_{op}} (x_1 - \bar{x}_1) + \cancel{hot} \right] + \left[\left.\frac{\partial h_i}{\partial x_2}\right|_{P_{op}} (x_2 - \bar{x}_2) + \cancel{hot} \right]$$

$$+ \left[\left.\frac{\partial h_i}{\partial x_3}\right|_{P_{op}} (x_3 - \bar{x}_3) + \cancel{hot} \right]$$

$$\underbrace{y_i - h_i(\bar{x}_1, \bar{x}_2, \bar{x}_3\ )}_{y_i^*} = \left[\left.\frac{\partial h_i}{\partial x_1}\right|_{P_{op}} x_1^* \right] + \left[\left.\frac{\partial h_i}{\partial x_2}\right|_{P_{op}} x_2^* \right] + \left[\left.\frac{\partial h_i}{\partial x_3}\right|_{P_{op}} x_3^* \right]$$

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$$\begin{bmatrix}\dot{x}_1^*\\ \dot{x}_2^*\\ \dot{x}_3^*\end{bmatrix} = \underbrace{\begin{bmatrix}\left.\frac{\partial f_1}{\partial x_1}\right|_{P_{op}} & \left.\frac{\partial f_1}{\partial x_2}\right|_{P_{op}} & \left.\frac{\partial f_1}{\partial x_3}\right|_{P_{op}}\\ \left.\frac{\partial f_2}{\partial x_1}\right|_{P_{op}} & \left.\frac{\partial f_2}{\partial x_2}\right|_{P_{op}} & \left.\frac{\partial f_2}{\partial x_3}\right|_{P_{op}}\\ \left.\frac{\partial f_3}{\partial x_1}\right|_{P_{op}} & \left.\frac{\partial f_3}{\partial x_2}\right|_{P_{op}} & \left.\frac{\partial f_3}{\partial x_3}\right|_{P_{op}}\end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix}x_1^*\\ x_2^*\\ x_3^*\end{bmatrix} + \underbrace{\begin{bmatrix}\left.\frac{\partial f_1}{\partial u_1}\right|_{P_{op}} & \left.\frac{\partial f_1}{\partial u_2}\right|_{P_{op}}\\ \left.\frac{\partial f_2}{\partial u_1}\right|_{P_{op}} & \left.\frac{\partial f_2}{\partial u_2}\right|_{P_{op}}\\ \left.\frac{\partial f_3}{\partial u_1}\right|_{P_{op}} & \left.\frac{\partial f_3}{\partial u_2}\right|_{P_{op}}\end{bmatrix}}_{\boldsymbol{B}} \begin{bmatrix}u_1^*\\ u_2^*\end{bmatrix}$$

$$\begin{bmatrix}y_1^*\\ y_2^*\end{bmatrix} = \underbrace{\begin{bmatrix}\left.\frac{\partial h_1}{\partial x_1}\right|_{P_{op}} & \left.\frac{\partial h_1}{\partial x_2}\right|_{P_{op}} & \left.\frac{\partial h_1}{\partial x_3}\right|_{P_{op}}\\ \left.\frac{\partial h_2}{\partial x_1}\right|_{P_{op}} & \left.\frac{\partial h_2}{\partial x_2}\right|_{P_{op}} & \left.\frac{\partial h_2}{\partial x_3}\right|_{P_{op}}\end{bmatrix}}_{\boldsymbol{C}} \begin{bmatrix}x_1^*\\ x_2^*\\ x_3^*\end{bmatrix}$$

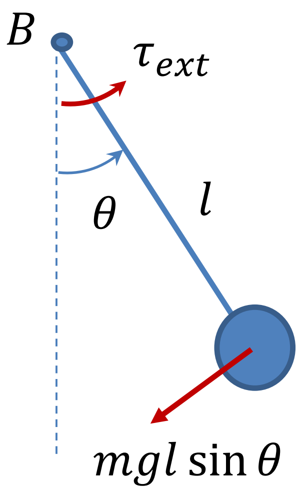
Linealización de Sistemas Dinámicos

Ejemplo: péndulo simple en equilibrio no natural

Consideremos un péndulo simple de masa M sostenida por un 'hilo' rígido, sobre el cual de alguna manera se pueden ejercer torques externos τ_{ext}.

Planteando la segunda ley de Newton:

$$ml^2\ddot{\theta} = \left(-mgl\sin\theta + B\dot{\theta}\right) + \tau_{ext}$$



Renombrando variables:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 + \frac{B}{ml^2}x_2 + \frac{\tau_{ext}}{ml^2}$$

Supongamos que queremos operar el péndulo en los alrededores de su posición horizontal derecha. En ese caso el equilibrio no es natural si no impuesto por un torque externo de valor mgl que determina que el punto de operación se encuentre en

$$(\bar{x}_1, \bar{x}_2, \bar{\tau}_{ext}) = (\pi/2 \quad 0 \quad mgl)$$

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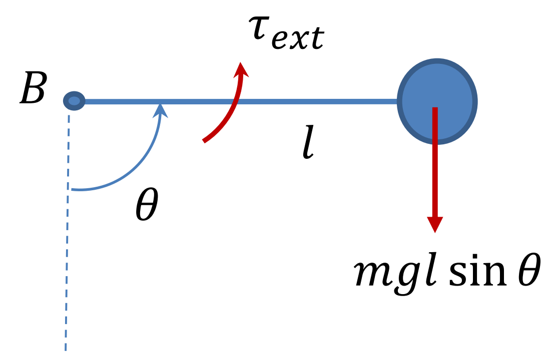

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 + \frac{B}{ml^2}x_2 + \frac{\tau_{ext}}{ml^2}$$

$$A = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{(\pi/2 \quad 0 \quad mgl)} = \left.\begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos\theta & \frac{B}{ml^2} \end{bmatrix}\right|_{(\pi/2 \quad 0 \quad mgl)} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{B}{ml^2} \end{bmatrix}$$

$$B = \left.\begin{bmatrix} \frac{\partial f_1}{\partial \tau_{ext}} \\ \frac{\partial f_2}{\partial \tau_{ext}} \end{bmatrix}\right|_{(\pi/2 \quad 0 \quad mgl)} = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}$$

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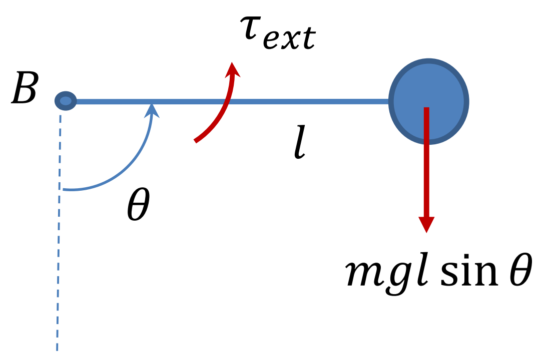
Ejemplo: péndulo simple en equilibrio no natural



$$\begin{bmatrix}\dot{x}_1^*\\ \dot{x}_2^*\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & \dfrac{B}{ml^2}\end{bmatrix}\begin{bmatrix}x_1^*\\ x_2^*\end{bmatrix} + \begin{bmatrix}0\\ \dfrac{1}{ml^2}\end{bmatrix}\tau^*$$

$$x_1 = x_1^* + \frac{\pi}{2}$$

$$x_2 = x_2^*$$

$$\tau_{total} = \tau_{ext} + \tau^*$$