

CSC418 Assignment #1

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1. (a) To compute the implicit form, we'll rearrange $x(t)$ for t and then plug that into the equation for y .

$$x = 2\sin(t) \rightarrow t = \arcsin(\frac{x}{2}) \rightarrow y = 5\sin(\arcsin(\frac{x}{2}))\cos(\arcsin(\frac{x}{2}))$$

Then we can simply subtract y from both sides to obtain the implicit form:

$$0 = 5\sin(\arcsin(\frac{x}{2}))\cos(\arcsin(\frac{x}{2})) - y$$

- (b) The tangent vector can be found by defining the curve as $p(t) = (2\sin t, 5\sin t \cos t)$ and then finding $p'(t)$.

$$x'(t) = 2\cos t, y'(t) = 5\cos(2t) \rightarrow p'(t) = (2\cos t, 5\cos(2t))$$

The normal vector is the tangent vector with the elements inverted as such:

$$p_n(t) = (-5\cos(2t), 2\cos t)$$

- (c) The equation is symmetric about the x axis when $-y(t) = y(-t)$ and vice versa. Note: $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos(\theta)$

$$y(-t) = 5\sin(-t)\cos(-t) = 5(-\sin t)\cos t = -5\sin t \cos t = -y(t)$$

$$x(-t) = 2\sin(-t) = 2(-\sin t) = -2\sin t = -x(t)$$

Thus, the equation is symmetric in both the x and y axes.

- (d) To find the area, we need to solve $\int y dx$. Since the bowtie is symmetric, we can integrate from 0 to $\frac{\pi}{2}$ and multiply the result by 4. y is already given and $dx = 2\cos t$ from the tangent equations.

$$4 \int_0^{\frac{\pi}{2}} (5\sin t \cos t)(2\cos t) dt \rightarrow 40 \int_0^{\frac{\pi}{2}} \sin t \cos^2 t dt = 40(\frac{1}{3}) = \frac{40}{3} \approx 13.3$$

- (e) Approximating the perimeter is the same as finding the arc length, and since the bowtie is both x and y symmetric, we only need to integrate from 0 to $\frac{\pi}{2}$ and multiply the result by 4. The formula for finding arc length is $L = \int_a^b \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2} dt$. We already know dx and dy from a previous question:

$$4 \int_0^{\frac{\pi}{2}} \sqrt{(2\cos t)^2 + (5\cos(2t))^2} dt = 4 \int_0^{\frac{\pi}{2}} \sqrt{4\cos^2 t + 25\cos^2(2t)} dt \approx 4(5.62) \approx 22.48$$

2. Two transformations commute if $A \cdot B = B \cdot A$, so that's what we'll check.

- (a) Translation + translation:

$$\begin{bmatrix} 1 & 0 & a_1 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a_1 + b_1 \\ 0 & 1 & a_2 + b_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & a_1 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & b_1 + a_1 \\ 0 & 1 & b_2 + a_2 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus they commute.

(b) Translation + rotation:

$$\begin{bmatrix} 1 & 0 & a_1 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos b & \sin b & 0 \\ -\sin b & \cos b & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos b & \sin b & a_1 \\ -\sin b & \cos b & a_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos b & \sin b & 0 \\ -\sin b & \cos b & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & a_1 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos b & \sin b & a_1 \cos b + a_2 \sin b \\ -\sin b & \cos b & -a_1 \sin b + a_2 \cos b \\ 0 & 0 & 1 \end{bmatrix}$$

Thus they do not commute.

(c) These do not commute, but it is easier to show via an example.

We start with a square, $p_1, p_2, p_3, p_4 = (0, 0), (1, 0), (0, 1), (1, 1)$. Our scaling transform will simply be a $\times 2$ scale with fixed point p_1 , and our rotation transform will be a 90 degree clockwise rotation around fixed point p_2 .

Scaling then rotation:

$$\begin{bmatrix} (0, 0) \\ (1, 0) \\ (0, 1) \\ (1, 1) \end{bmatrix} \rightarrow \begin{bmatrix} (0, 0) \\ (2, 0) \\ (0, 2) \\ (2, 2) \end{bmatrix} \rightarrow \begin{bmatrix} (2, 2) \\ (2, 0) \\ (4, 2) \\ (4, 0) \end{bmatrix}$$

Rotating then scaling:

$$\begin{bmatrix} (0, 0) \\ (1, 0) \\ (0, 1) \\ (1, 1) \end{bmatrix} \rightarrow \begin{bmatrix} (1, 1) \\ (1, 0) \\ (2, 1) \\ (2, 0) \end{bmatrix} \rightarrow \begin{bmatrix} (1, 1) \\ (1, -1) \\ (3, 1) \\ (3, -1) \end{bmatrix}$$

Thus, they do not commute.

(d) Scaling + scaling:

$$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1 b_1 & 0 & 0 \\ 0 & a_2 b_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 a_1 & 0 & 0 \\ 0 & b_2 a_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus they commute. When scaling with a fixed point, all the other points are locked into the line they form with the fixed point, scaling the point only scales the distance between it and the fixed point along that line. This is equivalent to the fixed point being the origin, thus the above matrix proof applies to any fixed point scale, as long as the fixed point is the same for all scaling transforms.

3. First, noticing that x values of 0 map to 7 and x values of 1 map to 6, it seems like the square formed by these points flips horizontally. To achieve this, we first need to scale the x -axis by -1:

$$T_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, by subtracting the mapped points to the new points, we find every point translates by $(7, 2)$, so we apply a simple translation:

$$T_2 = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow T_2 T_1 = \begin{bmatrix} -1 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying these two transforms in that order give the correct mapping.

The point $(2, 5)$ under these transforms would get mapped to $(5, 7)$.

4. First, we calculate the lines between all of the vertices, so v_0v_1 , v_1v_2 and v_2v_0 . Then we calculate the area of the triangle by these lines, obtaining $v_0v_1v_2$. We then take point p and calculate new triangles substituting one vertex for p each time: v_0v_1p , v_0pv_2 and pv_1v_2 . If the sum of the areas of all these p triangles equals the area of the full triangle, it lies within the triangle.

To find the area of the triangle, we can take v_2 and subtract it from v_0 and v_1 in order to translate the triangle to the origin. Then, assuming $v_0 = (a, b)$ and $v_1 = (c, d)$, the area of the triangle is $A = \frac{|ad-bc|}{2}$. Any point can be subtracted, as long as one point is $(0, 0)$ and the other two are (a, b) and (c, d) .

To find the centroid, $q = (x', y')$, we simply take the vertices' x and y values, sum them and divide by 3. So, $x' = \frac{v_{x0}+v_{x1}+v_{x2}}{3}$, and similar for y' .